Chapter 1
Fuzzy numbers

1.1 Fuzzy sets

1.1.1 Modeling imprecision

Sometimes we can precisely state if a certain object belongs or not to a given set. For example, let us consider a set A of people which are 40 or less years old. If χ_A is a characteristic function of A, i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A, \end{cases}$$

then we can say without any doubt that $\chi_A(x) = 1$ if x is a person 40 or less years old and $\chi_A(x) = 0$ if x is a person with older than 40. However, very often we cannot say for sure if an object belongs to a given set, especially if this set is described imprecisely or ambiguously, using words common in a natural language. For example, in such everyday language the opposite of the word "young" is the word "old". Therefore, the classical logic encourages us to split people in two categories: old and "young". Although in some cases it would be more or less obvious, e.g. 20 years old person is surely "young", while 80 years old man would be classified into the category of "old" persons, sometimes it would be difficult to decide into which category should belong a person 52 years old. It seems that the answer here is context dependent.

To overcome problems where the classical logic appears not sufficient to model situations under study the notion of **fuzzy set** was introduced by Lotfi Zadeh in 1965.

Definition 1.1. (Zadeh [267]) Let $\mathbb{X}$ be a universe of discourse. A **fuzzy set** A in $\mathbb{X}$ is characterized by a **membership function** $\mu_A : \mathbb{X} \to [0,1]$, which assigns to each object $x \in \mathbb{X}$ a real number in the interval $[0,1]$, so as $\mu_A(x)$ represents the degree of membership of x into A.

Keeping the notations as in the above definition, a fuzzy set A may be perceived as

$$A = \{(x, \mu_A(x)) : x \in \mathbb{X}, \mu_A(x) \in [0,1]\}.$$

The set of all fuzzy sets in $\mathbb{X}$ is denoted with $\mathbb{FS}(\mathbb{X})$. If for a fuzzy set $A \in \mathbb{FS}(\mathbb{X})$ we have $\mu_A(x) = 0$ for all $x \in \mathbb{X}$, then we say that A is an **empty set**and we write as usual $A = \emptyset$.If a set $\{x \in \mathbb{X} : \mu_A(x) > 0\}$ is finite then the corresponding fuzzy set A is called a **discrete fuzzy set**. In this we usually describe fuzzy set A by neglecting all the elements $x \in \mathbb{X}$ such that $\mu_A(x) = 0$. For instance, $A \in \mathbb{FS}(\mathbb{Z})$ given by

$$A = \{(-3, 0.2), (0, 0.5), (2, 1), (5, 0.7), (6, 0.3)\}$$

is an example of a discrete fuzzy set.

The interpretation of the grade of membership is very natural: if $\mu_A(x) = 1$ then we are sure that element x belongs to A, while in the case when $\mu_A(x) = 0$ then it surely does not belong to A. In all other cases, i.e. if $\mu_A(x) \in (0,1)$ then we have a partial membership (or partial belongingness to A). It means that if $\mu_A(x)$ is very close to 1 then the degree of membership of x in A is very high, while if $\mu_A(x)$ is very close to 0 then the degree of membership of x in A is very low. If $\mu_A(x) \in \{0,1\}$ for all $x \in \mathbb{X}$ then the fuzzy set A reduces to a set in the classical meaning. It means that each "usual" set is a fuzzy set whose membership function coincides with the characteristic function of that set. In fuzzy set theory such "usual" sets are usually called **crisp** sets.

The way we assign a degree of membership strongly depends on our perception regarding the objects that we are dealing with. It means that the particular shape of a fuzzy set might be designed in a very subjective way. Sometimes on may also apply some statistical methods for constructing membership functions (see, e.g. [94]). On the other hand in engineering, economics and other research areas where fuzzy sets are used, there are many examples of fuzzy sets with commonly accepted membership functions.

Please note, that since the membership function μ_A describes completely a corresponding fuzzy set A, many authors - to simplify and reduce the notation - denote the membership function of A by $A(x)$, instead of $\mu_A(x)$.

1.1.2 Basic operations on fuzzy sets

Basic operations on crisp sets (equality, complement, inclusion, union, intersection) can be extended in a natural way to fuzzy sets. In what follows we list the definitions of these basic operations (see Zadeh [267]). A more detailed discussion, including also other approaches, is made in Hanss [162] (see also the references cited there).

Definition 1.2. Let $A, B \in \mathbb{FS}(\mathbb{X})$.

(i) A and B are equal (and we write $A = B$) if $\mu_A(x) = \mu_B(x)$ for all $x \in \mathbb{X}$.
(ii) A is included in B (and we write $A \subseteq B$) if $\mu_A(x) \leq \mu_B(x)$ for all $x \in \mathbb{X}$.

(iii) The complement of A, denoted $\neg A(x)$, is characterized by the membership function $\mu_{\neg A(x)} : \mathbb{X} \to [0,1]$ such that

$$\mu_{\neg A}(x) = 1 - \mu_A(x),$$

for all $x \in \mathbb{X}$.

(iv) The union of A and B, denoted $A \cup B$, is characterized by the membership function $\mu_{A\cup B} : \mathbb{X} \to [0,1]$ given by

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

for all $x \in \mathbb{X}$.

(v) The intersection of A and B, denoted $A \cap B$, is characterized by the membership function $\mu_{A\cap B} : \mathbb{X} \to [0,1]$ given by

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

for all $x \in \mathbb{X}$.

The above presented operations satisfy some remarkable identities such as De Morgan's laws:

$$\neg(A \cup B) = \neg A \cap \neg B$$
$$\neg(A \cap B) = \neg A \cup \neg B,$$

associativity:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C),$$

distributivity:

$$C \cap (A \cup B) = (C \cap A) \cup (C \cap B),$$
$$C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$$

and commutativity:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A,$$

which holds for all $A, B, C \in \mathbb{FS}(\mathbb{X})$. But they do not satisfy the law of contradiction and the law excluded middle, i.e. there exist $A \in \mathbb{FS}(\mathbb{X})$ such that $A \cap \neg A \neq \emptyset$ and $A \cup \neg A \neq \mathbb{X}$.

Apart from these classical operations on fuzzy sets, proposed by Zadeh, there are many other ways one may generalize operations defined on crisp sets into fuzzy domain. In fact we may obtain many interesting operations on fuzzy sets using so-called t-norms and t-conorms (also called s-norms). For more details we refer the reader to Section 1.4.2.

1.1.3 Height, core, support and $\alpha-$cut of a fuzzy set

The **height** of a fuzzy set $A \in \mathbb{FS}(\mathbb{X})$ is defined by

$$\mathrm{hgt}(A) = \sup_{x \in \mathbb{X}} \mu_A(x). \tag{1.1}$$

From Definition 1.1 it is immediate that for any a fuzzy set A we have $\mathrm{hgt}(A) \le 1$. If there exists $x_0 \in \mathbb{X}$ such that $\mathrm{hgt}(A) = \mu_A(x_0) = 1$ then A is called **normal**.

The **core** of $A \in \mathbb{FS}(\mathbb{X})$ is denoted by $\mathrm{core}(A)$ and is given by

$$\mathrm{core}(A) = \{x \in \mathbb{X} : \mu_A(x) = 1\}. \tag{1.2}$$

It is immediate that $\mathrm{core}(A) \neq \emptyset$ if and only if A is normal.

The **support** of a fuzzy set $A \in \mathbb{FS}(\mathbb{X})$ is denoted by $\mathrm{supp}(A)$ and represents the set of all elements of $\mathbb{X}$ with a nonzero degree of membership, i.e.

$$\mathrm{supp}(A) = \{x \in \mathbb{X} : \mu_A(x) > 0\}. \tag{1.3}$$

It is easy to check that $A \neq \emptyset$ if and only if $\mathrm{supp}(A) \neq \emptyset$.

Another notion that plays an important role in the theory of fuzzy sets is the so-called **α-cut**. For $\alpha \in [0,1]$ the α-cut of a fuzzy set $A \in \mathbb{FS}(\mathbb{X})$, denoted A_α, is given by

$$A_\alpha = \{x \in \mathbb{X} : \mu_A(x) \ge \alpha\}. \tag{1.4}$$

It is immediate that $A_0 = \mathbb{X}$ and $A_1 = \mathrm{core}(A)$.

It is clear that knowing the membership function od a given fuzzy set we can find all its α-cuts by formula (1.4). But, what is interesting, knowing all α-cuts of a fuzzy number we can also reconstruct its membership function.

Lemma 1.1. *(see e.g. Hanss [162], p. 20) If $A \in \mathbb{FS}(\mathbb{X})$ then*

$$\mu_A(x) = \sup_{\alpha \in [0,1]} \alpha \cdot \chi_{A_\alpha}(x), \tag{1.5}$$

for every $x \in \mathbb{X}$, where χ_{A_α} is the characteristic function of the set A_α.

The above result proves that each fuzzy set is completely determined by its $\alpha-$cuts. We illustrate this by the following example.

Example 1.1. Let $A \in \mathbb{FS}(\mathbb{R})$ such that $A_\alpha = [\alpha + 1, 5 - \alpha]$ for $\alpha \in [0,1]$. By (1.5) after some elementary calculus we obtain

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty, 1] \cup [5, \infty), \\ x+1 & \text{if} \quad x \in [1, 2], \\ 1 & \text{if} \quad x \in [2, 4], \\ 5-x & \text{if} \quad x \in [4, 5]. \end{cases}$$

A membership function of the fuzzy number A is given in Figure 1.1. □

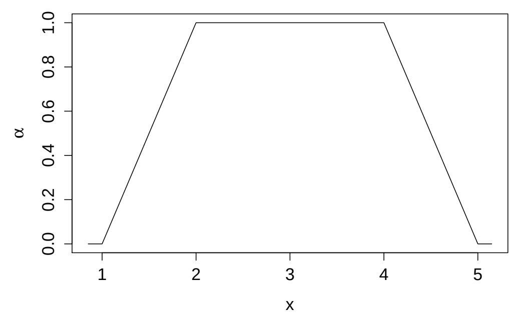

Fig. 1.1 A membership functions of the fuzzy number A (see Example 1.1).

1.1.4 Convex fuzzy sets

The notion of convexity for fuzzy sets is introduced (see Zadeh [267]) in a way which allows to preserve the properties of the ordinary convex sets. Convexity is useful both from the theoretical point of view as well as in practice, especially in pattern classification, optimization, etc.

Definition 1.3. Let $\mathbb{X}$ be a convex subset of a real vector space. We say that $A \in \mathbb{FS}(\mathbb{X})$ is **convex** if $A_\alpha = \{x \in \mathbb{X} : \mu_A(x) \geq \alpha\}$ is a convex subset of $\mathbb{X}$ for all $\alpha \in [0, 1]$.

It is immediate that $A \in \mathbb{FS}(\mathbb{X})$ is convex if and only if the membership function μ_A is a quasi-concave function, i.e.

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}, \tag{1.6}$$

for any $x_1, x_2 \in \mathbb{X}$ and $\lambda \in [0, 1]$. In many textbooks and papers formula (1.6) is used as a definition of a convex fuzzy set.

The following lemma devoted to the particular case when $\mathbb{X} = \mathbb{R}$ is useful.

Lemma 1.2. *Suppose* $A \in FS(\mathbb{R})$ *has a continuous membership function and* $\operatorname{supp}(A)$ *is bounded. Then* A *is convex if and only if there exist* $a, b, c \in \mathbb{R}, a \leq c \leq b$ *such that*

a) $\mu_A = 0$ *outside the interval* $[a,b]$,
b) μ_A *is nondecreasing on the interval* $[a,c]$,
c) μ_A *is nonincreasing on the interval* $[c,b]$.

In this way the interpretation of a convex fuzzy set is more clear and, additionally, we can easily find examples of fuzzy sets that are non-convex.

1.1.5 Extension principle

The **extension principle**, introduced by Zadeh in [267], allows to extend basic mathematical concepts for fuzzy quantities. The n-dimensional case of Zadeh's extension principle, especially for $n = 2$, is important because it allows us to extend operations between real numbers.

Definition 1.4. (see, e.g., Hanss [162], p. 41) Let $\mathbb{X}_1,\ldots,\ \mathbb{X}_n,\ \mathbb{Y}$ be non-empty sets and let us consider the function $f : \mathbb{X} \to \mathbb{Y}$ where $\mathbb{X}$ is the product space $\mathbb{X} = \mathbb{X}_1 \times \ldots \times \mathbb{X}_n$. Furthermore, we consider $A_i \in \mathbb{FS}(\mathbb{X}_i)$ for all $i \in \{1,\ldots,n\}$. Using function f we can define a fuzzy set $C = f(A_1,\ldots,A_n) \in \mathbb{FS}(\mathbb{Y})$ characterized by the membership function $\mu_C : \mathbb{Y} \to [0,1]$

$$\mu_C(y) = \begin{cases} \sup\limits_{(x_1,\ldots,x_n)\in f^{-1}(y)} \min\{\mu_{A_1}(x_1),\ldots,\mu_{A_n}(x_n)\} & \text{if} \quad y \in f(\mathbb{X}), \\ 0 & \text{otherwise}. \end{cases} \tag{1.7}$$

The above formula indicates a straightforward way how to extend various operations defined on crisp sets into fuzzy environment.

Example 1.2. If $A, B \in FS(\mathbb{Z})$ are given by

$$\begin{aligned} A &= \{(-1,0.2),(0,0.5),(2,1),(3,0.6),(6,0.2)\}, \\ B &= \{(-2,0.3),(-1,0.5),(0,0.8),(1,1),(3,0.7),(5,0.4)\} \end{aligned}$$

and we consider a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f(x,y) = x + y$. Then by (1.7) we get $f(A,B) = A + B$, where

$$\begin{aligned} A + B = \{&(-3,0.2),(-2,0.3),(-1,0.5),(0,0.5),(1,0.5),(2,0.8),(3,1), \\ &(4,0.6),(5,0.7),(6,0.6),(7,0.4),(8,0.4),(9,0.2),(11,0.2)\}. \end{aligned}$$

□

1.2 Fuzzy numbers - definitions and representations

Fuzzy numbers are fuzzy sets in $\mathbb{R}$ which satisfy some additional properties. Since they generalize real numbers they are basic for theoretical development of fuzzy set theory (fuzzy analysis, fuzzy differential equations, etc.) and very useful in numerous applications related to the representation and handling of uncertainty and incomplete information in decision making, linguistic controllers, biotechnological systems, expert systems, data mining, pattern recognition, etc.

Definition 1.5. (see [119]) A **fuzzy number** A is a fuzzy set in $\mathbb{R}$ which satisfies the following properties:

(i) A is normal,
(ii) A is convex,
(iii) μ_A is upper semicontinuous in every $x_0 \in \mathbb{R}$ (i.e. $\forall \varepsilon > 0, \exists \delta > 0$ such that $\mu_A(x) - \mu_A(x_0) < \varepsilon$, whenever $|x - x_0| < \delta$),
(iv) $cl\,\{x \in \mathbb{R} : \mu_A(x) > 0\}$ is bounded, where cl denotes the closure operator.

A family of all fuzzy numbers will be denoted by $\mathbb{F}(\mathbb{R})$.

Example 1.3. A fuzzy set $A \in \mathbb{FS}(\mathbb{R})$ given by

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \le x < 2 \\ \frac{1}{2}x^2 - \frac{7}{2}x + 6 & \text{if} \quad 2 \le x < 4 \\ 0 & \text{if} \quad x \ge 4 \end{cases}$$

is a fuzzy number. Its membership function is given in Figure 1.2. □

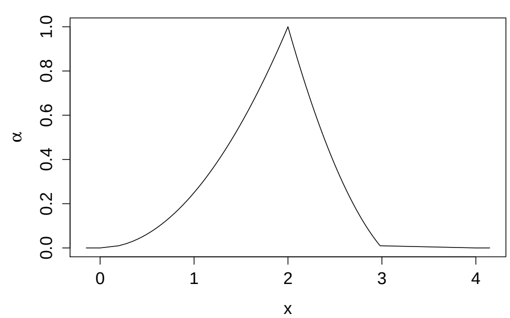

Fig. 1.2 A membership functions of the fuzzy number A (see Example 1.3).

Please note that any real number x_0 is a fuzzy number with the membership function equal to the characteristic function $\chi_{\{x_0\}}$. Similarly, any real interval $[a,b]$ is a fuzzy number with with the membership function equal to the characteristic function $\chi_{[a,b]}$.

The α-cuts A_α of a fuzzy number A are given by $A_\alpha = \{x \in \mathbb{R} : \mu_A(x) \ge \alpha\}$ for $\alpha \in (0,1]$ and $A_0 = cl\,\{x \in \mathbb{R} : \mu_A(x) > 0\}$.

The following result, known as the Stacking Theorem, gives us important information about α-cuts.

Theorem 1.1. *(Negoiţă-Ralescu [207]) Let $A \in F(\mathbb{R})$ with its α-cuts A_α, $\alpha \in [0,1]$. Then*

a) A_α is a closed interval, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, for any $\alpha \in [0,1]$,
b) if $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then $A_{\alpha_2} \subseteq A_{\alpha_1}$,
c) for any sequence $\{\alpha_n\}$ which converges from below to $\alpha \in (0,1]$ we have $\bigcap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha$,
d) for any sequence $\{\alpha_n\}$ which converges from above to 0 we have $cl\,(\bigcup_{n=1}^{\infty} A_{\alpha_n}) = A_\alpha$.

The endpoints of each α-cut A_α, $\alpha \in [0,1]$, are given by

$$A_L(\alpha) = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\} \tag{1.8}$$

and

$$A_U(\alpha) = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}. \tag{1.9}$$

It is easily seen that the following definition of a fuzzy number is equivalent to Definition 1.5.

Definition 1.6. A fuzzy number A is a fuzzy set characterized by a membership function $\mu_A : \mathbb{R} \to [0,1]$ of the form

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x \leq a_1, \\ l_A(x) & \text{if} \quad a_1 \leq x \leq a_2, \\ 1 & \text{if} \quad a_2 \leq x \leq a_3, \\ r_A(x) & \text{if} \quad a_3 \leq x \leq a_4, \\ 0 & \text{if} \quad a_4 \leq x, \end{cases} \tag{1.10}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$, $l_A : [a_1, a_2] \longrightarrow [0,1]$ is a nondecreasing upper semicontinuous function, $l_A(a_1) = 0$, $l_A(a_2) = 1$, called the left side of the fuzzy number and $r_A : [a_3, a_4] \longrightarrow [0,1]$ is a nonincreasing upper semicontinuous function, $r_A(a_3) = 1, r_A(a_4) = 0$, called the right side of the fuzzy number.

If the sides of the fuzzy number A are strictly monotone then one can see easily that A_L and A_U are inverse functions of l_A and r_A respectively. Moreover, it can be proved that the functions A_L and A_U are left continuous.

By (1.5) we can define a fuzzy number using its α-cut representation. Consequently, we obtain the following definition of a fuzzy number introduced by Goetschel and Voxman in the paper [136].

Definition 1.7. A fuzzy number A is an ordered pair of left continuous functions $[A_L(\alpha), A_U(\alpha)]$, $0 \leq \alpha \leq 1$, which satisfy the following requirements:

(i) A_L is nondecreasing on $[0,1]$,
(ii) A_U is nonincreasing on $[0,1]$,

(iii) $A_L(1) \leq A_U(1)$.

If a fuzzy number A is defined by Definition 1.6 we say that A is given in *L-R* form. Otherwise, if A is defined using Definition 1.7, we say that A is given in *L-U* form.

Even if Definitions 1.6 and 1.7 are equivalent, we cannot always pass from *L-R* representation to *L-U* representation. This is easily observed since the passing from *L-R* representation to *L-U* requires the calculus of the inverses of the side functions which cannot always be performed. For this reason, in some situations such like the approximation of fuzzy numbers we need to use the same type of representation for all fuzzy numbers.

An important class of fuzzy numbers often used in practice is a family of symmetric fuzzy numbers defined as follows.

Definition 1.8. A fuzzy number A is called a **symmetric fuzzy number** if $A_L(1) - A_L(\alpha) = A_U(\alpha) - A_U(1)$ for all $\alpha \in [0,1]$.

At the end of this section we will discuss about the equality of two fuzzy numbers. Due to the fact that most of the main results of the book are in relation with L_p-type metrics we adopt the following definition.

Definition 1.9. We say that fuzzy numbers A and B are equal (and we denote $A = B$) if $A_L = B_L$ and $A_U = B_U$ for almost every $\alpha \in [0,1]$.

The above definition applies only when we work with L_p-types metrics on the space of fuzzy numbers.

1.3 Basic families of fuzzy numbers

Many types of fuzzy numbers can be found in the literature. Below we mention the most important families of fuzzy numbers. Some other types, like the Gaussian fuzzy numbers or the quadratic fuzzy numbers, which are sometimes successfully applied in engineering, are considered in e.g. [162].

1.3.1 Crisp fuzzy numbers

We say that the fuzzy number A is a **crisp fuzzy number** if there exists $c \in \mathbb{R}$ such that $\mu_A(c) = 1$ and $\mu_A(x) = 0$ for all $x \in \mathbb{R} \setminus \{c\}$. It is immediate that $A_L(\alpha) = A_U(\alpha) = c$, for every $\alpha \in [0,1]$. For simplicity, if A is a crisp fuzzy number then we usually write $A = c$. The graph of a such crisp fuzzy number is the pair $(c,1)$ which suggests the notion of singleton, the other way a crisp fuzzy number can be called.

If $B \in \mathbb{F}(\mathbb{R})$ with α-cuts $B_\alpha = [B_L(\alpha), B_U(\alpha)]$, $\alpha \in [0,1]$, and c is a crisp fuzzy number then the sum $B + c$ is a fuzzy number with α-cuts

$$(B+c)_\alpha = [B_L(\alpha)+c, B_U(\alpha)+c].$$

Crisp fuzzy numbers are the reason why real numbers are particular cases of fuzzy numbers. It is elementary to find a bijection between the set of real numbers and the set of crisp fuzzy numbers. For this reason we will make no distinction between crisp fuzzy numbers and real numbers.

1.3.2 Interval fuzzy numbers

A fuzzy number A is called an **interval fuzzy number** if there exist the reals $a, b \in \mathbb{R}$, $a \le b$, such that $\mu_A(x) = 1$ for all $x \in [a,b]$ and $\mu_A(x) = 0$ for all $x \in \mathbb{R} \setminus [a,b]$. It is immediate that $A_L(\alpha) = a$ and $A_U(\alpha) = b$, for every $\alpha \in [0,1]$. The notation $A = [a,b]$ used there suggests the terminology of interval fuzzy number. A family of all interval fuzzy numbers will be denoted by $\mathbb{F}^I$.

If $B \in \mathbb{F}(\mathbb{R})$ with α-cuts $B_\alpha = [B_L(\alpha), B_U(\alpha)]$, $\alpha \in [0,1]$, and $A = [a,b] \in \mathbb{F}^I$ then $A+B$ is a fuzzy number with α-cuts

$$(A+B)_\alpha = [B_L(\alpha)+a, B_U(\alpha)+b].$$

1.3.3 Triangular fuzzy numbers

A fuzzy number A is called a **triangular fuzzy number** if there exist $t_1 \le t_2 \le t_3$ such that

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ \frac{x-t_1}{t_2-t_1} & \text{if} \quad t_1 \le x \le t_2, \\ \frac{t_3-x}{t_3-t_2} & \text{if} \quad t_2 \le x \le t_3, \\ 0 & \text{if} \quad t_3 < x. \end{cases} \tag{1.11}$$

Since by (1.11) is represented completely by those three real values t_1,t_2 and t_3, we usually denote such triangular fuzzy number by $A = (t_1, t_2, t_3)$.

It is easily seen that the α-cuts of such triangular fuzzy number are given by

$$A_\alpha = [t_1 + (t_2 - t_1)\alpha, t_3 - (t_3 - t_2)\alpha]. \tag{1.12}$$

The family of all triangular fuzzy numbers will be denoted by $\mathbb{F}^\Delta(\mathbb{R})$. Since crisp fuzzy numbers and interval fuzzy numbers can be regarded as particular cases of triangular fuzzy numbers, we have

$$\mathbb{R} \subset \mathbb{F}^I \subset \mathbb{F}^\Delta(\mathbb{R}).$$

1.3.4 Trapezoidal fuzzy numbers

A generalization of the triangular fuzzy number is the **trapezoidal fuzzy number**. A trapezoidal fuzzy number T is completely determined by four real parameters $t_1 \le t_2 \le t_3 \le t_4$ such that

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ \frac{x-t_1}{t_2-t_1} & \text{if} \quad t_1 \le x \le t_2, \\ 1 & \text{if} \quad t_2 \le x \le t_3, \\ \frac{t_4-x}{t_4-t_3} & \text{if} \quad t_3 \le x \le t_4, \\ 0 & \text{if} \quad t_4 < x. \end{cases} \tag{1.13}$$

We use here the notation $T = (t_1,t_2,t_3,t_4)$. When $t_2 = t_3$, T becomes a triangular fuzzy number. If $t_2 - t_1 = t_4 - t_3$ we obtain a symmetric trapezoidal fuzzy number. One can easily verify that the α-cut of such trapezoidal fuzzy number are given by

$$T_\alpha = [t_1 + (t_2 - t_1)\alpha, t_4 - (t_4 - t_3)\alpha]. \tag{1.14}$$

A family of all trapezoidal fuzzy numbers will be denoted with $\mathbb{F}^T(\mathbb{R})$ and a family of all symmetric trapezoidal fuzzy numbers will be denoted with $\mathbb{F}^{ST}(\mathbb{R})$. Finally, we mention that naturally we have $\mathbb{F}^\Delta(\mathbb{R}) \subset \mathbb{F}^T(\mathbb{R})$.

A family of trapezoidal fuzzy numbers is the most important subset of fuzzy sets. It is caused by the simplicity of representation which makes easier all transformations and calculations made on trapezoidal fuzzy numbers, simplifies computer applications and usually gives more intuitive and more natural interpretation. This is also the reason why the trapezoidal approximation of fuzzy numbers is a matter of great importance. The extensive study of the above mentioned approximation is given in this monograph.

1.3.5 Semi-trapezoidal fuzzy numbers

The so called parametric fuzzy numbers were introduced in the paper [206] mainly to generalize the trapezoidal approximation problem. A **parametric fuzzy number** of type (s_L, s_R) is a fuzzy number A with α-cuts $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ given by

$$A_L(\alpha) = a - \sigma(1-\alpha)^{1/s_L}, \tag{1.15}$$

$$A_U(\alpha) = b + \beta(1-\alpha)^{1/s_R}, \tag{1.16}$$

where $a, b, \sigma, \beta, s_L, s_R \in \mathbb{R}$, $a \le b$, $\sigma \ge 0$, $\beta \ge 0$, $s_L > 0$, $s_R > 0$. Note that the condition $a \le b$ is imposed in order to obtain a proper parametric fuzzy number. We use the notation $A = (a, b, \sigma, \beta)_{s_L, s_R}$. A membership function of a fuzzy number $A = (2,3,1,2)_{2,0.5}$ is given in Figure 1.3.

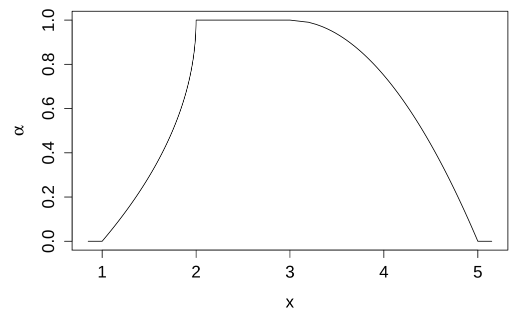

Fig. 1.3 A membership functions of a semi-trapezoidal fuzzy number $A = (2,3,1,2)_{2,0.5}$.

When $s_L = s_R = 1$ then A becomes a trapezoidal fuzzy number. A family of all (s_L, s_R) fuzzy numbers will be denoted with $\mathbb{F}^{s_L,s_R}(\mathbb{R})$. Recently (see [262]) parametric fuzzy numbers are also called **semi-trapezoidal fuzzy numbers**.

1.3.6 Bodjanova type fuzzy numbers

Another important type of fuzzy numbers were introduced by Bodjanova in [68] to generalize trapezoidal fuzzy numbers. A membership of a fuzzy number from the proposed class has the following form

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1, \\ \left(\frac{x-a_1}{a_2-a_1}\right)^r & \text{if} \quad a_1 \le x \le a_2, \\ 1 & \text{if} \quad a_2 \le x \le a_3, \\ \left(\frac{a_4-x}{a_4-a_3}\right)^r & \text{if} \quad a_3 \le x \le a_4, \\ 0 & \text{if} \quad a_4 < x, \end{cases} \tag{1.17}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that $a_1 < a_2 \le a_3 < a_4$ and $r > 0$. From (1.17) and (1.13) it is easily seen that for $r = 1$ we obtain a trapezoidal fuzzy number. A Bodjanova type fuzzy number is denoted by $A = (a_1, a_2, a_3, a_4)_r$. It is immediate that α-cuts $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ of such fuzzy number are given by

$$A_L(\alpha) = a_1 + \alpha^{1/r}(a_2 - a_1),$$
$$A_U(\alpha) = a_4 - \alpha^{1/r}(a_4 - a_3).$$

A membership function of a fuzzy number $A = (-3,-1,1,3)_{0.4}$ is given in Figure 1.4.

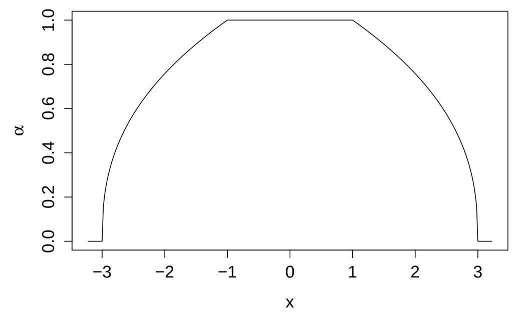

Fig. 1.4 A membership functions of a Bodjanova type fuzzy number $A = (-3, -1, 1, 3)_{0.4}$.

1.3.7 L-R fuzzy numbers

Another type of fuzzy numbers, that appears broadly in the literature, was introduced by Dubois and Prade [119] and is known as a family of *L-R***fuzzy numbers**.

Definition 1.10. Let $L, R : [0,1] \to [0,1]$ be two continuous strictly increasing functions such that $L(0) = R(0) = 0$ and $L(1) = R(1) = 1$. Moreover, consider $t_1, t_2, t_3, t_4 \in \mathbb{R}$ such that $t_1 \le t_2 \le t_3 \le t_4$. Then a fuzzy number A given by

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ L\left(\frac{x-t_1}{t_2-t_1}\right) & \text{if} \quad t_1 \le x \le t_2, \\ 1 & \text{if} \quad t_2 \le x \le t_3, \\ R\left(\frac{t_4-x}{t_4-t_3}\right) & \text{if} \quad t_3 \le x \le t_4, \\ 0 & \text{if} \quad t_4 < x, \end{cases} \tag{1.18}$$

is called an *L-R* fuzzy number.

The set of all *L-R* fuzzy numbers will be denoted by $\mathbb{F}_{L,R}(\mathbb{R})$ and an element of $\mathbb{F}_{L,R}(\mathbb{R})$ as above by $A = (t_1, t_2, t_3, t_4)_{L,R}$. The α-cuts $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ of a fuzzy number (1.18) are given by

$$A_L(\alpha) = t_1 + (t_2 - t_1)L^{-1}(\alpha),$$
$$A_U(\alpha) = t_4 - (t_4 - t_3)R^{-1}(\alpha).$$

It is worth noting that if $L = R = 1_{[0,1]}$ then $\mathbb{F}_{L,R}(\mathbb{R}) = \mathbb{F}^T(\mathbb{R})$.

Example 1.4. Suppose that $L(x) = R(x) = x^3$, $x \in [0,1]$ and let $t_1 = 0$, $t_2 = 2$, $t_3 = 3$ and $t_4 = 5$. We obtain

$$\mu_A(x) = \begin{cases} \frac{x^3}{8} & \text{if} \quad 0 \le x \le 2, \\ 1 & \text{if} \quad 2 \le x \le 3, \\ \left(\frac{5-x}{2}\right)^3 & \text{if} \quad 3 \le x \le 5, \\ 0 & \text{if} \quad x \notin [0,5]. \end{cases}$$

Moreover, $A_\alpha = [2\sqrt[3]{\alpha}, 5 - \sqrt[3]{\alpha}]$ for $\alpha \in [0,1]$. A membership function of A is given in Figure 1.5. □

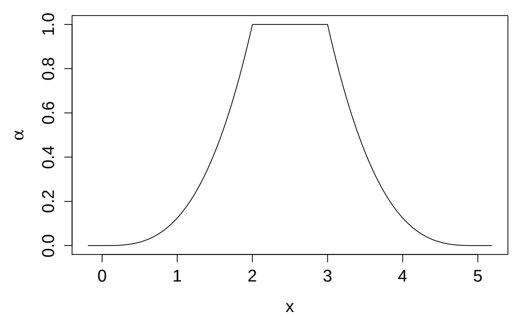

Fig. 1.5 A membership functions of the *L-R* fuzzy number *A* (see Example 1.4).

1.3.8 Piecewise linear fuzzy numbers

Trapezoidal fuzzy numbers might be generalized also by considering fuzzy numbers with piecewise linear side functions each consisting of two segments.

Definition 1.11. ([102]) An **α_0-piecewise linear** 1-**knot fuzzy number** A, where $\alpha_0 \in (0,1)$, is a fuzzy number with the following membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < a_1 \\ \alpha_0 \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \le x < a_2 \\ \alpha_0 + (1-\alpha_0)\,\frac{x-a_2}{a_3-a_2} & \text{if } a_2 \le x < a_3 \\ 1 & \text{if } a_3 \le x \le a_4 \\ \alpha_0 + (1-\alpha_0)\,\frac{a_5-x}{a_5-a_4} & \text{if } a_4 < x \le a_5 \\ \alpha_0 \frac{a_6-x}{a_6-a_5} & \text{if } a_4 < x \le a_6 \\ 0 & \text{if } x > a_6, \end{cases} \tag{1.19}$$

where $a_1 \le ... \le a_6$.

A set of all α_0-piecewise linear 1-knot fuzzy numbers will be denoted by $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. An element $A \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ as above can be denoted as $A = (\alpha_0, (a_1, a_2, a_3, a_4, a_5, a_6))$, while the α-cuts $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ of fuzzy number (1.19) are given by

$$A_L(\alpha) = \begin{cases} a_1 + (a_2 - a_1)\frac{\alpha}{\alpha_0} & \text{if } \alpha \in [0, \alpha_0), \\ a_2 + (a_3 - a_2)\frac{\alpha - \alpha_0}{1 - \alpha_0} & \text{if } \alpha \in [\alpha_0, 1], \end{cases}$$

$$A_U(\alpha) = \begin{cases} a_5 + (a_6 - a_5)\frac{\alpha_0 - \alpha}{\alpha_0} & \text{if } \alpha \in [0, \alpha_0), \\ a_4 + (a_5 - a_4)\frac{1 - \alpha}{1 - \alpha_0} & \text{if } \alpha \in [\alpha_0, 1]. \end{cases}$$

A membership function of a fuzzy number $A = (0.75, (1, 1.5, 2, 3, 4.5, 5))$ is given in Figure 1.6.

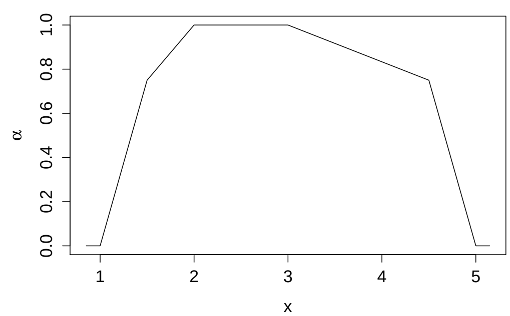

Fig. 1.6 A membership functions of the α_0-piecewise linear 1-knot fuzzy number $A = (0.75, (1, 1.5, 2, 3, 4.5, 5))$.

1.4 Operations on fuzzy numbers

1.4.1 Basic standard operations on fuzzy numbers

We start with a very important result of Nguyen ([208]) which gives sufficient conditions for closed operations on the set of fuzzy numbers.

Theorem 1.2. *([208], Proposition 5.1) Let us consider a continuous function* $f : \mathbb{R}^n \to \mathbb{R}$ *and suppose that* $A_1, \ldots, A_n$ *are fuzzy numbers. Then* $Z = f(A_1, \ldots, A_n)$ *obtained by the extension principle (1.7) is a fuzzy number. Moreover, we have*

$$Z_\alpha = f\left((A_1)_\alpha, (A_2)_\alpha, \ldots, (A_n)_\alpha\right), \quad \alpha \in [0, 1].$$

Note that Nguyen considered only binary operations in the above theorem but by mathematical induction the above generalization is obvious.

Since fuzzy numbers extend real numbers it would be natural to introduce basic arithmetic operations such as addition, subtraction, multiplication or division on the space of fuzzy numbers as well. The natural way is to apply the Zadeh extension principle given in Definition 1.4.

Addition of fuzzy numbers

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,y) = x + y$ and take two arbitrary fuzzy numbers A and B. We denote $f(A,B) = A+B$ and by (1.7) we obtain a fuzzy set with the following membership function

$$\begin{aligned}\mu_{A+B}(z) &= \sup_{(x,y)\in\mathbb{R}^2 : x+y=z} (\min\{\mu_A(x), \mu_B(y)\}) \\ &= \sup_{x\in\mathbb{R}} (\min\{\mu_A(x), \mu_B(z-x)\})\,.\end{aligned} \tag{1.20}$$

We say that $A+B$ is the **sum of fuzzy numbers** A and B. Since f is continuous by Theorem [208] it results that $A+B$ is a fuzzy number with the following α-cuts

$$\begin{aligned}(A+B)_\alpha &= A_\alpha + B_\alpha \\ &= [A_L(\alpha)+B_L(\alpha), A_U(\alpha)+B_U(\alpha)],\end{aligned} \tag{1.21}$$

for each $\alpha \in [0,1]$. Therefore, in particular we have $\mathrm{supp}(A+B) = \mathrm{supp}(A) + \mathrm{supp}(B)$ and $\mathrm{core}(A+B) = \mathrm{core}(A) + \mathrm{core}(B)$. Please note, that i formula (1.20) we can take max instead of sup.

The following two lemmas, although immediate to prove, are useful in practice.

Lemma 1.3. *If A and B are trapezoidal fuzzy numbers described as $A = (t_1,t_2,t_3,t_4)$ and $B = (s_1,s_2,s_3,s_4)$, respectively, then $A+B$ is also a trapezoidal fuzzy number and it could be denoted as*

$$A+B = (t_1+s_1, t_2+s_2, t_3+s_3, t_4+s_4)\,.$$

Example 1.5. Let us consider two triangular fuzzy numbers (i.e. a particular trapezoidal fuzzy numbers) $A = (1,2,3)$ and $B = (2,3,4)$. One can easily find that their sum $A+B = (3,5,7)$ is also a triangular fuzzy number. Membership functions of A, B and $A+B$ are given in Figure 1.7. □

Lemma 1.4. *If A and B are L-R fuzzy numbers described as $A = (t_1,t_2,t_3,t_4)_{L,R}$ and $B = (s_1,s_2,s_3,s_4)_{L,R}$, respectively, then $A+B$ is also a L-R fuzzy number and it could be denoted as*

$$A+B = (t_1+s_1, t_2+s_2, t_3+s_3, t_4+s_4)_{L,R}\,.$$

Subtraction of fuzzy numbers

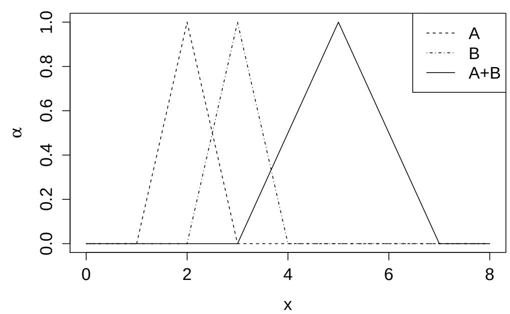

Fig. 1.7 Membership functions of A, B and $A+B$ (see Example 1.5).

One may define a subtraction of two fuzzy numbers A and B similarly as their sum. Finally, we receive that $A-B$ is a a fuzzy number, called the **difference of fuzzy numbers** A and B, with the membership function

$$\mu_{A-B}(z) = \sup_{(x,y)\in\mathbb{R}^2 : x-y=z} (\min\{\mu_A(x), \mu_B(y)\}) \tag{1.22}$$

and the following α-cuts

$$\begin{aligned}(A-B)_\alpha &= A_\alpha - B_\alpha \\ &= [A_L(\alpha) - B_U(\alpha), A_U(\alpha) - B_L(\alpha)],\end{aligned} \tag{1.23}$$

for each $\alpha \in [0,1]$.

Multiplication of fuzzy numbers

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,y) = x \cdot y$ and take two arbitrary fuzzy numbers A and B. We denote $f(A,B) = A \cdot B$ and by (1.7) we obtain a fuzzy set with the following membership function

$$\mu_{A\cdot B}(z) = \sup_{(x,y)\in\mathbb{R}^2 : x\cdot y=z} (\min\{\mu_A(x), \mu_B(y)\}) \tag{1.24}$$

We say that $A \cdot B$ is a **product of fuzzy numbers** A and B, and since f is continuous it results that $A \cdot B$ is a fuzzy number with the following α-cuts

$$\begin{aligned}A_\alpha &= A_\alpha \cdot B_\alpha \\ &= \big[\min\{A^L_\alpha B^L_\alpha, A^L_\alpha B^U_\alpha, A^U_\alpha B^L_\alpha, A^U_\alpha B^U_\alpha\}, \\ &\quad \max\{A^L_\alpha B^L_\alpha, A^L_\alpha B^U_\alpha, A^U_\alpha B^L_\alpha, A^U_\alpha B^U_\alpha\}\big],\end{aligned} \tag{1.25}$$

for each $\alpha \in [0,1]$.

Consider the following example showing that contrary to Lemma 1.3 the product of two trapezoidal fuzzy numbers may be not a trapezoidal one.

Example 1.6. Let us consider two trapezoidal fuzzy numbers $A = (1,2,4,6)$ and $B = (2,4,4,5)$. One can easily find that their α-cuts are as follows: $A_\alpha = [1+\alpha, 6-2\alpha]$ and $B_\alpha = [2+2\alpha, 5-\alpha]$, respectively. Then

$$\begin{aligned}(A \cdot B)_\alpha &= [1+\alpha, 6-2\alpha] \cdot [2+2\alpha, 5-\alpha] \\ &= [2+3\alpha+2\alpha^2, 30-16\alpha+2\alpha^2].\end{aligned}$$

It is clear that $A \cdot B$ is not a trapezoidal fuzzy number. □

Division of fuzzy numbers

Consider a function $f : \mathbb{R} \times (\mathbb{R} \smallsetminus \{0\}) \to \mathbb{R}$ such that $f(x,y) = x/y$, take two fuzzy numbers A and B, where B is called a nonzero fuzzy number, i.e. $0 \notin \mathrm{supp}(B)$. We denote $f(A,B) = A/B$ and by (1.7) we obtain a fuzzy set with the following membership function

$$\mu_{A/B}(z) = \sup_{(x,y)\in\mathbb{R}\times(\mathbb{R}\smallsetminus\{0\}): x/y=z} \left(\min\{\mu_A(x), \mu_B(y)\}\right). \tag{1.26}$$

We say that A/B is a **quotient of fuzzy numbers** A and B, and it is a fuzzy number with the following α-cuts

$$\begin{aligned}(A/B)_\alpha &= A_\alpha / B_\alpha \\ &= \big[\min\{A^L_\alpha/B^L_\alpha, A^L_\alpha/B^U_\alpha, A^U_\alpha/B^L_\alpha, A^U_\alpha/B^U_\alpha\}, \\ &\quad\ \max\{A^L_\alpha/B^L_\alpha, A^L_\alpha/B^U_\alpha, A^U_\alpha/B^L_\alpha, A^U_\alpha/B^U_\alpha\}\big],\end{aligned} \tag{1.27}$$

for each $\alpha \in [0,1]$ and B such that $0 \notin \mathrm{supp}(B)$.

Scalar multiplication

In the case of fuzzy numbers we distinguish not only multiplication of fuzzy numbers as defined in (1.24), but also the so called scalar multiplication, which is easily obtain from the general multiplication presented earlier. It will suffice to characterize this operations only by the α-cut representation. Therefore, if A is an arbitrary fuzzy number and $\lambda \in \mathbb{R}$ then the scalar multiplication between λ and A will be denoted $\lambda \cdot A$ and from (1.25) it results that for any $\alpha \in [0,1]$ we have

$$(\lambda \cdot A)_\alpha = \lambda A_\alpha = \begin{cases} [\lambda A_L(\alpha), \lambda A_U(\alpha)] & \text{if} \quad \lambda \geq 0, \\ [\lambda A_U(\alpha), \lambda A_L(\alpha)] & \text{if} \quad \lambda < 0. \end{cases} \tag{1.28}$$

In particular, considering a trapezoidal fuzzy number $T = (t_1, t_2, t_3, t_4)$ it results that $\lambda \cdot T = (\lambda t_1, \lambda t_2, \lambda t_3, \lambda t_4)$ if $\lambda \geq 0$ and $\lambda \cdot T = (\lambda t_4, \lambda t_3, \lambda t_2, \lambda t_1)$ if $\lambda < 0$.

The proof of the following proposition is standard.

Proposition 1.1. *The following equalities hold for any fuzzy* $A, B, C \in \mathbb{F}(\mathbb{R})$ *and any* $\lambda, \beta \in \mathbb{R}$*:*

a) $A + B = B + A$
b) $(A + B) + C = A + (B + C)$
c) $A + 0 = A$
d) $1 \cdot A = A$
e) $\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B$
f) $\lambda \cdot (\beta \cdot A) = \beta \cdot (\lambda \cdot A) = (\lambda\beta) \cdot A$
g) $A \cdot B = B \cdot A$
h) $A \cdot (B \cdot C) = (A \cdot B) \cdot C.$

It is easy to check that the only fuzzy numbers having opposite elements with respect to the addition of fuzzy numbers are the crisp numbers. Actually, it is evident that in general the property $A + (-A) = 0$ does not hold for fuzzy numbers. Therefore, the triple $(\mathbb{F}(\mathbb{R}), +, \cdot)$ is not a vector space. This lack of property causes real difficulties in some practical situations such as solving fuzzy systems of equations or when we consider for example the best approximation problem. It is well known that most of the existence results concerning the best approximation problem are given in normed vector spaces. However, as it will be seen in Chapter 3, the problem of approximating fuzzy numbers by fuzzy numbers with simpler form will be reduced to approximation problems in Hilbert spaces in the case of the L_2-type metrics. But as we look on the above proposition we observe that $(\mathbb{F}(\mathbb{R}), +, \cdot)$ is a semilinear space and therefore by Theorem 5.3 it is very important to notice that there exists a vector space$(\widetilde{\mathbb{F}(\mathbb{R})}, \oplus, \odot)$ and an injective application (inclusion) $i : \mathbb{F}(\mathbb{R}) \to \widetilde{\mathbb{F}(\mathbb{R})}$ and, regarding $\mathbb{F}(\mathbb{R})$ as a subset of $\widetilde{\mathbb{F}(\mathbb{R})}$ (that is adopting the convention $i(A) = A$ for all $A \in \mathbb{F}(\mathbb{R})$) we have

$$A \oplus B = A + B,$$
$$\lambda \odot A = \lambda \cdot A,$$

for all $A, B \in \mathbb{F}(\mathbb{R})$ and $\lambda \in [0, \infty)$.

1.4.2 Interactive operations on fuzzy numbers

In this section we will generalize the extension principle which will alow us to obtain more general formulas for the addition or multiplication of fuzzy numbers. These generalizations are important in some applications in which the extension principle may not give satisfactory results. For example, by Example 1.6 we know that standard multiplication of fuzzy numbers is not a closed operation. So, we can say that multiplication is not a shape preserving operations (in this case linear shape). Interestingly, it is possible to define a new formula for the multiplication such that the linear shape is preserved (see [167]). In order to generalize the extension principle we need the notion of a triangular norm.

Definition 1.12. (see e.g [139]) A **triangular norm** (**t-norm** for short) is a function $T : [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties:

(i) $T(x,1) = x$, for all $x \in [0,1]$ (identity)
(ii) $T(x,y) = T(y,x)$, for all $x,y \in [0,1]$ (commutativity)
(iii) $T(x,T(y,z)) = T(T(x,y),z)$, for all $x,y,z \in [0,1]$ (associativity)
(iv) if $x \le u$ and $y \le v$ then $T(x,y) \le T(u,v)$, for all $x,y,z \in [0,1]$ (monotonicity).

If T_1, T_2 are triangular norms such that $T_1(x,y) \le T_2(x,y)$, for all $x,y \in [0,1]$, then we say that T_1 is weaker than T_2 (or that T_2 is stronger than T_1) and we denote $T_1 \le T_2$ (or $T_2 \ge T_1$).

Note, that if T is a triangular norm then $T(x,x) \le T(x,1) = x$ for any $x \in [0,1]$. Therefore, for any $x \in [0,1]$ we have $T(0,x) \le T(0,1) = 0$ and hence $T(0,x) = 0$ for all $x \in [0,1]$.

Let us consider a few examples presenting three famous t-norms.

Example 1.7. Let $T_M : [0,1] \times [0,1] \to [0,1]$ denote a function defined as follows

$$T_M(x,y) = \min\{x,y\}. \tag{1.29}$$

It is immediate that T_M is a triangular norm. Moreover, if T is an arbitrary triangular norm then $T(x,y) \le T(x,1) = x$ and $T(x,y) = T(y,x) \le T(y,1) = y$. Thus $T(x,y) \le \min\{x,y\} = T_M(x,y)$ which implies that $T \le T_M$ for any triangular norm T. For this reason T_M is called the **strongest t-norm**. □

Example 1.8. Consider $T_w : [0,1] \times [0,1] \to [0,1]$ defined as follows

$$T_w(x,y) = \begin{cases} 0 & \text{if } \max\{x,y\} < 1, \\ \min\{x,y\} & \text{otherwise.} \end{cases} \tag{1.30}$$

It is easily seen that T_w is a triangular norm. Let T be an arbitrary triangular norm and let us choose $x,y \in [0,1]$. If $\max\{x,y\} < 1$ then $T_w(x,y) = 0 \le T(x,y)$. If $\max\{x,y\} = 1$ then $x = 1$ or $y = 1$. If $x = 1$ then $T(x,y) = T(1,y) = y \ge \min\{x,y\} = T_w(x,y)$. Similarly, if $y = 1$ then $T(x,y) \ge T_w(x,y)$. Therefore $T \ge T_w$ for any triangular norm T. For this reason T_w is called the **weakest t-norm**. □

Example 1.9. Define $T_L : [0,1] \times [0,1] \to [0,1]$ as follows

$$T_L(x,y) = \max\{x+y-1, 0\}. \tag{1.31}$$

Then T_L is a triangular norm known as the **Łukasiewicz t-norm**. □

Let A and B be two arbitrary fuzzy numbers. By (1.20) and (1.29) we can write

$$\begin{aligned} (A+B)(z) &= \sup_{x+y=z} (\min\{\mu_A(x), \mu_B(y)\}) \\ &= \sup_{x+y=z} T_M(\mu_A(x), \mu_B(y)) \end{aligned}$$

for any $z \in \mathbb{R}$. Replacing above T_M with an arbitrary triangular norm T we obtain a generalization of the standard addition called **T-norm based addition** (see e.g. [129, 164, 182, 201, 202]).

We denote the T-norm based addition of A and B by $A \oplus_T B$. The membership function of the sum $A \oplus_T B$ is given for any $z \in \mathbb{R}$ as follows

$$\mu_{A \oplus_T B}(z) = \sup_{(x,y) \in \mathbb{R}^2 : x+y=z} T(\mu_A(x), \mu_B(y)). \tag{1.32}$$

It can be proved that $A \oplus_T B$ is a fuzzy number and in addition, if T is upper semi-continuous then we can take max instead of sup in (1.32).

Since $T \leq T_M$ then $A \oplus_T B \leq A + B$. This implies that $\text{core}\,(A \oplus_T B) \subseteq \text{core}\,(A+B)$. On the other hand, if $z \in \text{core}\,(A+B)$ thus there exist $x_0 \in \text{core}\,(A)$ and $y_0 \in \text{core}\,(B)$ such that $x_0 + y_0 = z$. Hence we have

$$\begin{aligned} \mu_{A \oplus_T B}(z) &= \sup_{x+y=z} T(\mu_A(x), \mu_B(y)) \\ &\geq T(A(x_0), B(y_0)) = T(1,1) = 1. \end{aligned}$$

Therefore, we get $\text{core}\,(A+B) \subseteq \text{core}\,(A \oplus_T B)$. Thus finally, by the double inclusion we obtain

$$\text{core}\,(A+B) = \text{core}\,(A \oplus_T B)\,. \tag{1.33}$$

Similarly, we may define the **T-norm based multiplication** (see e.g. [167]) of A and B, denoted as $A \odot_T B$, where the membership function of the product $A \odot_T B$ is given by

$$\mu_{A \odot_T B}(z) = \sup_{(x,y) \in \mathbb{R}^2 : x \cdot y = z} T(\mu_A(x), \mu_B(y)). \tag{1.34}$$

It is worth noting that T_w is is the only triangular norm which induces a shape preserving multiplication of *L-L* fuzzy numbers (it was proved in [167]). Please notice, that in the case of addition the situation is different. Namely, the standard addition, i.e.based on the strongest triangular norm T_M, always preserves the shape of *L-R* fuzzy numbers (see Lemma 1.4). But for an arbitrary triangular norm it may not hold. Therefore, an interesting problem is to find those triangular norms which induce a shape preserving addition. Important results concerning this problem can be found in [164, 182, 202].

Let us now discuss in some sense even more general approach than operations based on triangular norms. Firstly, let us define a notion of the so called joint possibility distribution.

Definition 1.13. (see [130])Let us consider two arbitrary fuzzy numbers A and B. A function $C : \mathbb{R}^2 \to \mathbb{R}$ is called a **joint possibility distribution** of A and B if

$$\sup_{y \in \mathbb{R}} C(x,y) = \mu_A(x)$$

for all $x \in \mathbb{R}$ and

$$\sup_{x \in \mathbb{R}} C(x,y) = \mu_B(y)$$

for all $y \in \mathbb{R}$. We say that A and B are the marginal distributions of C.

If C is upper semicontinuous then we can take operator max instead of sup in the above definition.

Example 1.10. (see e.g. [72]) Let us consider two triangular fuzzy numbers $A = (0,0,1)$ and $B = (0,1,1)$. It can be shown that a function $C : \mathbb{R}^2 \to \mathbb{R}$ given by

$$C(x,y) = (y-x) \cdot \chi_S(x,y),$$

where χ_S denotes the characteristic function of the set $S = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \leq 1, y-x \geq 0\}$, is a joint possibility distribution of A and B. □

It is worth noting that any triangular norm T generates a joint possibility distribution C_T of A and B, where

$$C_T(x,y) = T(\mu_A(x), \mu_B(y)). \tag{1.35}$$

However, there exist joint possibility distributions which cannot be generated by triangular norms.

Now we are able to define operations on fuzzy numbers based on a joint possibility distribution. Actually, an **interactive addition of A and B based on their joint possibility distribution** C is denoted $A +_C B$ and the sum $A +_C B$ is described by its membership function given as follows (see [75])

$$\mu_{A +_C B}(z) = \sup_{(x,y) \in \mathbb{R}^2 : x+y=z} C(x,y). \tag{1.36}$$

The **interactive multiplication based on a joint possibility distribution** is denoted by $A \cdot_C B A \cdot_C B$, where the product $A \cdot_C B$ is a fuzzy set with the following membership function

$$\mu_{A \cdot_C B} = \sup_{(x,y) \in \mathbb{R}^2 : x \cdot y = z} C(x,y). \tag{1.37}$$

The interactive addition and multiplication from above are obtained using the so called interactive extension principle which introduced in [75]. Moreover, it can be proved (see [75]) that both $A +_C B$ and $A \cdot_C B$ are fuzzy numbers.

We end this section by mentioning that joint possibility distributions are used in many problems of rather statistical nature (see [72, 76, 101, 130]) but also in problems concerning fuzzy arithmetic (see [73, 75, 98, 100]). For example, in paper [75] the following problem was formulated: "Let C be a joint possibility distribution with marginal distributions A and B. At what conditions does the equality $A +_C B = A + B$ hold?" It is worth mentioning that the solution, i.e. necessary and sufficient conditions for this equality were given in [98].

1.5 Distances between fuzzy numbers

There are numerous metrics defined on the space of fuzzy numbers. In this section we will list only those metrics which are suitable to our investigation on the approximation of fuzzy numbers.

One of the most popular metric is the so called **Euclidean metric** ([141]) given by

$$d(A,B)=\left[\int_0^1 (A_L(\alpha)-B_L(\alpha))^2\,d\alpha+\int_0^1 (A_U(\alpha)-B_U(\alpha))^2\,d\alpha\right]^{1/2}. \tag{1.38}$$

As an application, it is immediate that for two trapezoidal fuzzy numbers $T=(t_1,t_2,t_3,t_4)$ and $T'=(t_1',t_2',t_3',t_4')$, after elementary calculus, we obtain

$$\begin{aligned}d^2(T,T')&=\frac{1}{3}(t_1-t_1')^2+\frac{1}{3}(t_2-t_2')^2+\frac{1}{3}(t_1-t_1')(t_2-t_2')\\&\quad+\frac{1}{3}(t_3-t_3')^2+\frac{1}{3}(t_4-t_4')^2+\frac{1}{3}(t_3-t_3')(t_4-t_4').\end{aligned} \tag{1.39}$$

Yeh ([263]) generalized metric (1.38) considering the **weighted L_2-type distance** d_λ, defined as follows

$$\begin{aligned}d_\lambda(A,B)=&\left[\int_0^1 (A_L(\alpha)-B_L(\alpha))^2\,\lambda_L(\alpha)d\alpha\right. \\&\left.+\int_0^1 (A_U(\alpha)-B_U(\alpha))^2\,\lambda_U(\alpha)d\alpha\right]^{1/2},\end{aligned} \tag{1.40}$$

where, in order to obtain indeed a metric, we suppose that weighting functions $\lambda_L,\lambda_U:[0,1]\to\mathbb{R}$ are strictly positive almost everywhere on $[0,1]$ and integrable. If $\lambda_L=\lambda_U$ and $\int_0^1\lambda_L(\alpha)d\alpha=1/2$, we rediscover the metric of Zeng and Li ([270]). Further on we use the notation $\lambda=(\lambda_L,\lambda_U)$.

More generally, considering $p\geq 1$ and a weight $\lambda=(\lambda_L,\lambda_U)$, the weighted L_p-type distance $\delta_{p,\lambda}$ is given by

$$\begin{aligned}\delta_{p,\lambda}(A,B)=&\left[\int_0^1 |(A_L(\alpha)-B_L(\alpha))|^p\,\lambda_L(\alpha)d\alpha\right. \\&\left.\int_0^1 |(A_U(\alpha)-B_U(\alpha))|^p\,\lambda_U(\alpha)d\alpha\right]^{1/p}.\end{aligned} \tag{1.41}$$

If $\lambda_L(\alpha)=\lambda_U(\alpha)=1$, $\alpha\in[0,1]$, we prefer the notation

$$d_p(A,B)=\left[\int\limits_0^1 |(A_L(\alpha)-B_L(\alpha))|^p\,d\alpha+\int\limits_0^1 |(A_U(\alpha)-B_U(\alpha))|^p\,d\alpha\right]^{1/p}. \quad (1.42)$$

Another class of distances between fuzzy numbers, introduced by Bertoluzza et al. [63], is given by

$$\widetilde{D}_{f,\varphi}(A,B)=\left(\int_0^1 \widetilde{D}_f^2(A_\alpha,B_\alpha)\,d\varphi(\alpha)\right)^{1/2},$$

where

$$\widetilde{D}_f^2([a,b],[c,d])=\int_0^1 (t\,|a-c|+(1-t)\,|b-d|)^2\,df(t),$$

and where f is a normalized weighting measure on $[0,1]$, while function φ satisfies usually the following conditions: $\varphi(\alpha)\geq 0$ for any $\alpha\in[0,1]$, $\alpha_1\leq\alpha_2$ implies $\varphi(\alpha_1)\leq\varphi(\alpha_2)$ and $\int_0^1\varphi(\alpha)\,d\alpha=1$.

Finally, let us consider the metric proposed by Trutschnig et al. ([236]) as follows:

$$D^*_{\Psi,\theta}(A,B)=\left(\int_0^1 (D^*_\theta(A_\alpha,B_\alpha))^2\,d\Psi(\alpha)\right)^{1/2}, \quad (1.43)$$

where $\theta\in(0,1]$, Ψ is a weighting probability measure on $[0,1]$ given by

$$(D^*_\theta([a,b],[c,d]))^2=(\mathrm{mid}\,[a,b]-\mathrm{mid}\,[c,d])^2+\theta\,(\mathrm{spr}\,[a,b]-\mathrm{spr}\,[c,d])^2, \quad (1.44)$$

while operators mid and spr correspond to the middle point of the interval under study and its spread (the half of its length), respectively, and their are defined as follows

$$\mathrm{mid}\,[a_1,a_2]=\frac{a_1+a_2}{2}, \quad (1.45)$$

$$\mathrm{spr}\,[a_1,a_2]=\frac{a_2-a_1}{2}. \quad (1.46)$$

Combining formulae (1.43)–(1.46) and substituting there α-cuts of fuzzy numbers A and B we get the following well-known formula for the Trutschnig distance

$$D^*_{\psi,\theta}(A,B)=\left(\int_0^1\left([\mathrm{mid}A_\alpha-\mathrm{mid}B_\alpha]^2+\theta\,[\mathrm{spr}A_\alpha-\mathrm{spr}B_\alpha]^2\right)\psi(\alpha)d\alpha\right)^2, \quad (1.47)$$

where $\psi:[0,1]\to[0,1]$ is a weighting function. It is worth noting that if $\psi\equiv 1$ and $\theta=1$ then such Trutschnig distance is equivalent with the Euclidean distance, i.e. $D^*_{1,1}(A,B)=\frac{1}{2}d(A,B)$.

1.6 Other notations for fuzzy numbers

Many authors examining fuzzy numbers introduce their own notation which for some specific reasons seem to be convenient in their considerations. For instance, given notation may be more easy for a particular type of fuzzy numbers (e.g. trapezoidal) or if one works with given type of metrics (say L_2-type metrics), while the same notation may appear troublesome or even inappropriate for another. In this section we present a new notation for trapezoidal fuzzy numbers introduced by Yeh in his papers [258] and [263]. Then we show a new notations introduced by Ban and Coroianu [42] which is convenient for calculations on semi-trapezoidal fuzzy numbers.

We start with notations for trapezoidal fuzzy numbers which are suitable with the Euclidean metric d as it will be seen later. One can easily verify that the α-cut of a trapezoidal fuzzy number $T = (t_1, t_2, t_3, t_4)$ can be written as follows

$$T_\alpha = [l + x(\alpha - \frac{1}{2}), u - y(\alpha - \frac{1}{2})]. \tag{1.48}$$

Therefore, by (1.14) we easily get that

$$l = \frac{t_1 + t_2}{2}, \tag{1.49}$$
$$u = \frac{t_3 + t_4}{2}, \tag{1.50}$$
$$x = t_2 - t_1, \tag{1.51}$$
$$y = t_4 - t_3 \tag{1.52}$$

or, equivalently,

$$t_1 = l - \frac{x}{2}, \tag{1.53}$$
$$t_2 = l + \frac{x}{2}, \tag{1.54}$$
$$t_3 = u - \frac{y}{2}, \tag{1.55}$$
$$t_4 = u + \frac{y}{2}. \tag{1.56}$$

Hence, a trapezoidal fuzzy number T with the α-cuts given as in (1.48) will be also denoted as $T = [l, u, x, y]$. It is easily seen that unlike the traditional notation describing a fuzzy number by four real numbers corresponding to the borders of its support and core, our new notation uses four parameters, which indicate the location of the fuzzy number (l and u) and the spread of its arms (x and y).

Now, if $T = [l, u, x, y]$ and $T' = [l', u', x', y']$ then by (1.38) and (1.48) after some simple calculations we get

$$d^2(T, T') = (l - l')^2 + (u - u')^2 + \frac{1}{12}(x - x')^2 + \frac{1}{12}(y - y')^2. \tag{1.57}$$

Clearly, the above expression may be perceived as more convenient than formula (1.39) because it shows directly that the Euclidean distance between two trapezoidal fuzzy numbers reduces to the distance between location parameters and characteristics of spread of those two fuzzy numbers. Other benefits will be seen in Chapter 3 where we investigate the approximations of fuzzy numbers by trapezoidal fuzzy numbers.

Now let us consider the space of fuzzy numbers endowed with a weighted metric d_λ given by formula (1.40). Let us introduce the following notations:

$$a = \int_0^1 \lambda_L(\alpha)d\alpha, \tag{1.58}$$

$$b = \int_0^1 \lambda_U(\alpha)d\alpha, \tag{1.59}$$

$$\omega_L = \frac{1}{a}\int_0^1 \alpha\lambda_L(\alpha)d\alpha, \tag{1.60}$$

$$\omega_U = \frac{1}{b}\int_0^1 \alpha\lambda_u(\alpha)d\alpha, \tag{1.61}$$

$$c = \int_0^1 (\alpha - \omega_L)^2\lambda_L(\alpha)d\alpha, \tag{1.62}$$

$$d = \int_0^1 (\alpha - \omega_U)^2\lambda_U(\alpha)d\alpha. \tag{1.63}$$

Next, let T be a trapezoidal fuzzy number with α-cuts given by

$$T_\alpha = [l + x(\alpha - \omega_L), u - y(\alpha - \omega_U)], \alpha \in [0,1]. \tag{1.64}$$

Such a trapezoidal fuzzy number will be denoted for simplicity by $T = [l,u,x,y]_\lambda$ (λ is a generic notation for the pair (λ_L, λ_U).

If $T = [l,u,x,y]_\lambda$ and $T' = [l',u',x',y']_\lambda$ then the weighted distance between T and T' becomes (see Proposition 2.2 in [263])

$$d_\lambda^2(T,T') = a(l-l')^2 + b(u-u')^2 + c(x-x')^2 + d(y-y')^2. \tag{1.65}$$

It is easilt seen that formula (1.57) is obtained from (1.65) by taking $\lambda_L = \lambda_U = 1$.

Now let us consider a semi-trapezoidal fuzzy number $A = (a,b,\sigma,\beta)_{s_L,s_R}$. By (1.15) and (1.16) we know that $A_L(\alpha) = a - \sigma(1-\alpha)^{1/s_L}$ and $A_U(\alpha) = b + \beta(1-\alpha)^{1/s_R}$ for $\alpha \in [0,1]$. Since we may express A_L and A_U

In what follows we introduce new notations for semi-trapezoidal fuzzy numbers. For this purpose let denote a in the following form

$$A_L(\alpha) = a - \sigma \frac{s_L}{s_L+1} - \sigma\left((1-\alpha)^{1/s_L} - \frac{s_L}{s_L+1}\right),$$
$$A_U(\alpha) = b + \beta \frac{s_R}{s_R+1} + \beta\left((1-\alpha)^{1/s_R} - \frac{s_R}{s_R+1}\right)$$

we obtain

$$A_L(\alpha) = l - x\left((1-\alpha)^{1/s_L} - \frac{s_L}{s_L+1}\right), \tag{1.66}$$

$$A_U(\alpha) = u + y\left((1-\alpha)^{1/s_R} - \frac{s_R}{s_R+1}\right), \tag{1.67}$$

which gives us another representation of the semi-trapezoidal fuzzy number A, where

$$l = a - \sigma \frac{s_L}{s_L+1},$$
$$x = \sigma,$$
$$u = b + \beta \frac{s_R}{s_R+1},$$
$$y = \beta.$$

Thus a semi-trapezoidal fuzzy number $A = (a,b,\sigma,\beta)_{s_L,s_R}$ could be represented equivalently as $A = [l,u,x,y]_{s_L,s_R}$. If $s_L = s_R = 1$ then we obtain the representation of a trapezoidal fuzzy number given above so the indices s_L, s_R might be omitted.

If $A = [l,u,x,y]_{s_L,s_R}$ and $B = [l',u',x',y']_{s_L,s_R}$ then the Euclidean distance between A and B becomes (see [42], Proposition 2)

$$d^2(A,B) = (l-l')^2 + (u-u')^2 + \frac{s_L}{(s_L+2)(s_L+1)^2}(x-x')^2 + \frac{s_R}{(s_R+2)(s_R+1)^2}(y-y')^2 \tag{1.68}$$

1.7 Characteristics of fuzzy numbers

Besides the membership function and α-cuts some numerical characteristics of fuzzy numbers are frequently used. They usually describe in a concise way some specific features of a fuzzy number like its location, dispersion, etc.

The **expected interval** of a fuzzy number was introduced independently by Dubois and Prade ([121]) and Heilpern ([163]). It is the following real interval

$$EI(A) = \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha \right]. \tag{1.69}$$

The expected interval is a very important characteristic of a fuzzy number having many interesting properties and useful in many situations, like defuzzification or approximation of fuzzy numbers (see, e.g., Chapter 3). Please, note that $EI(A)$ can also be regarded as a fuzzy number (more precisely, as an interval-type fuzzy number).

The middle point of the expected interval is called the **expected value** of the fuzzy number and is defined as follows

$$EV(A) = \frac{1}{2} \left[\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha \right]. \tag{1.70}$$

The expected value of a fuzzy number A is a characteristic of location, i.e. a such a point that indicates a value which is - in some sense - typical for a fuzzy notion modeled by A (see [121, 163]). Sometimes its generalization, called **weighted expected value**, might be interesting (see [141]). It is defined by

$$EV_q(A) = (1-q)\int_0^1 A_L(\alpha)d\alpha + q\int_0^1 A_U(\alpha)d\alpha, \tag{1.71}$$

where $q \in [0,1]$. Here, by the appropriate choice of the weight q one may draw more attention to the left or right side of a fuzzy number under study.

Another characteristic of location of a fuzzy number is called just a **value of a fuzzy number** and is defined by the following formula

$$Val_s(A) = \int_0^1 s(\alpha)(A_U(\alpha) + A_L(\alpha))d\alpha, \tag{1.72}$$

where $s : [0,1] \to [0,1]$ is a nondecreasing function satisfying $s(0) = 0$ and $s(1) = 1$, called a **reducing function** ([114]). More precisely, (1.72) is a value of A with respect to s.

The **ambiguity** of A with respect to s is

$$Amb_s(A) = \int_0^1 s(\alpha)(A_U(\alpha) - A_L(\alpha))d\alpha. \tag{1.73}$$

The ambiguity of A may be seen as the global spread of the membership function A with the reducing function s playing a weighting role. Hence ambiguity is a measure of vagueness of a fuzzy number A.

The value and ambiguity where introduced by Delgado et. all ([114]) to obtain a new and simple representation of fuzzy numbers (called a **canonical represen-**

tation) and to use them in decision-making. Since a value and ambiguity represent basic features of a fuzzy number, therefore according to Delgado et. all opinion two fuzzy numbers with the same ambiguity and value might be considered as equal.

If $s_k(\alpha) = \alpha^k$ for a fixed $k \in \mathbb{N}$, $\alpha \in [0,1]$, then for simplicity we denote $Val_{s_k}(A) = Val_k(A)$ and $Amb_{s_k}(A) = Amb_k(A)$ and which means that

$$Val_k(A) = \int_0^1 \alpha^k (A_U(\alpha) + A_L(\alpha)) d\alpha, \tag{1.74}$$

and

$$Amb_k(A) = \int_0^1 \alpha^k (A_U(\alpha) - A_L(\alpha)) d\alpha. \tag{1.75}$$

The most often used reducing function is $s_k(\alpha) = \alpha$ and hence in many paper by the value and ambiguity one simply considers $Amb_1(A) = Amb(A)$ and $Val_1(A) = Val(A)$ i.e.

$$Val(A) = \int_0^1 \alpha (A_U(\alpha) + A_L(\alpha)) d\alpha \tag{1.76}$$

and

$$Amb(A) = \int_0^1 \alpha (A_U(\alpha) - A_L(\alpha)) d\alpha, \tag{1.77}$$

respectively.

To describe the spread of the left-hand right-hand of a fuzzy number A with respect to the expected value, the **left-hand ambiguity** and **right-hand ambiguity** of A were introduced in [149] as follows:

$$Amb_L(A) = \int_0^1 \alpha (EV(A) - A_L(\alpha)) d\alpha, \tag{1.78}$$

$$Amb_U(A) = \int_0^1 \alpha (A_U(\alpha) - EV(A)) d\alpha. \tag{1.79}$$

Another useful parameter characterizing the nonspecifity of a fuzzy number is called the **width** of a fuzzy number (see [77]) and is defined by

$$w(A) = \int_0^1 (A_U(\alpha) - A_L(\alpha)) d\alpha. \tag{1.80}$$

One may easily prove that for a fuzzy number A with a membership function μ_A we have

$$w(A) = \int_{-\infty}^{\infty} \mu_A(x)dx. \tag{1.81}$$

In what follows we will give an interpretation for the expected interval of a fuzzy number and also we will generalize this concept. Grzegorzewski ([143]) proved that for any fuzzy number A, the expected interval $EI(A)$ is the nearest (with respect to the Euclidean distance d) interval fuzzy number to A, that is

$$d(A, EI(A)) = \min_{B \in \mathbb{F}^I(\mathbb{R})} d(A, B).$$

In addition, it can be easily proved that the expected value of A is the nearest (with respect to the Euclidean distance d) crisp fuzzy number to A, i.e.

$$d(A, EV(A)) = \min_{c \in \mathbb{R}} d(A, c).$$

The above considerations suggests that in the case of a weighted L_2-type metric we should adjust the definition of the expected interval so that the interpretation would be the same.

Definition 1.14. ([49], Definition 9) Let d_λ, $\lambda = (\lambda_L, \lambda_U)$, be a weighted L_2-type metric defined on $F(\mathbb{R})$. For a fuzzy number A the interval

$$EI^\lambda(A) = \left[\frac{1}{a} \int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha, \frac{1}{b} \int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha \right], \tag{1.82}$$

where a and b are introduced in relations (1.58)-(1.59), is calld the **λ-weighted expected interval** of A.

By (1.58) and (1.59) we have

$$\frac{1}{a}\int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha \le \frac{1}{a}\int_0^1 A_L(1)\lambda_L(\alpha)d\alpha = A_L(1)$$
$$\le A_U(1) = \frac{1}{b}\int_0^1 A_U(1)\lambda_U(\alpha)d\alpha \le \frac{1}{b}\int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha,$$

and therefore $EI^\lambda(A)$ is well-defined. The **λ-weighted expected value** of A is given by

$$EV^\lambda(A) = \frac{1}{a+b}\left[a\int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha + b\int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha \right].$$

It can be proved that in the case of the weighted expected interval and weighted expected value we have the same interpretation with respect to the weighted metric

d_λ as in the case of the usual expected interval. The extension of the weighted expected interval and of the weighted expected value for the case of extended fuzzy numbers is done in the same way as in the case of the usual ones.

Problems

1.1. Let $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x^2 + 1$ and $A \in \mathbb{FS}(\mathbb{Z})$ given by

$$A = \{(-2, 0.5), (-1, 0.4), (0, 0.7), (1, 0.6), (2, 0.3), (3, 0.6), (4, 0.8), (5, 0.7)\}.$$

Calculate $f(A)$.

1.2. Prove that a fuzzy set given by

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \le x < 2 \\ 1 & \text{if} \quad 2 \le x < 4 \\ x^2 - \frac{21}{2}x + 27 & \text{if} \quad 4 \le x < 6 \\ 0 & \text{if} \quad x \ge 6. \end{cases}$$

is a fuzzy number.

1.3. Let A denote an arbitrary fuzzy number. Prove that for any $\alpha \in [0, 1]$

$$\mu_A(A_L(\alpha)) \ge \alpha,$$
$$\mu_A(A_U(\alpha)) \ge \alpha.$$

1.4. Let A denote a continuous fuzzy number. Prove that A_L and A_U are strictly monotone and, in addition, we have

$$\mu_A(A_L(\alpha)) = \mu_A(A_U(\alpha)) = \alpha,$$

for all $\alpha \in [0, 1]$.

1.5. Let us consider a fuzzy set $A \in \mathbb{FS}(\mathbb{R})$ defined as

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 2x & \text{if} \quad 0 \le x < 0.3 \\ \frac{0.4x+0.3}{0.7} & \text{if} \quad 0.3 \le x < 1 \\ 1 & \text{if} \quad 1 \le x \le 2 \\ 1.4 - 0.2x & \text{if} \quad 2 < x \le 4 \\ 3 - 0.6x & \text{if} \quad 4 < x \le 5 \\ 0 & \text{if} \quad x > 5. \end{cases}$$

Prove that A is an α_0-piecewise linear 1-knot fuzzy number. Make a graph of its membership function and find its α-cuts.