Space-Filling Curves

A continuous curve $f\colon [0,1] \to \mathbb{R}^d$, where $d \geq 2$ is called a ***space-filling curve*** iff $f\big[[0,1]\big]$ contains a ball $B_r(x)$. This possibility was realized by Peano, so it might be called a Peano curve. (We will see later that Heighway's dragon is a space-filling curve, Proposition 2.5.9.) Here is an easy example of a Peano curve. It can be done as a dragon curve. To go from one stage to the next, each line segment is replaced by nine segments with 1/3 the length, as in Fig. 2.4.12. Figure 2.4.13 shows a few stages, as usual.

Exercise 2.4.16. Prove that the limit curve is space filling.

2.5 The Hausdorff Metric

Felix Hausdorff devised a way to describe convergence of sets. It is (for some purposes) better than the one discussed in the previous section, since it does not require finding an appropriate parameter space and parameterizations for the sets. It is simply a definition of a metric that applies to sets.

Convergence of Sets

Let S be a metric space. Let A and B be subsets of S. We say that A and B are within ***Hausdorff distance*** r of each other iff every point of A is within distance r of some point of B, and every point of B is within distance r of some point of A.

This idea can be made into a metric, called the ***Hausdorff metric***, D. If A is a set and $r > 0$, then the ***open*** r***-neighborhood*** of A is

$$N_r(A) = \{\, y : \; \varrho(x,y) < r \text{ for some } x \in A \,\}\,.$$

The definition of the Hausdorff metric D:

$$D(A,B) = \inf\{\, r > 0 : \; A \subseteq N_r(B) \text{ and } B \subseteq N_r(A) \,\}\,.$$

By convention, $\inf \varnothing = \infty$.

This definition D does not define a metric, however. There are various problems. For example, in $\mathbb{R}$, what is the distance between $\{0\}$ and $[0,\infty)$? It is infinite. That is not allowed in the definition of metric. Therefore, we will restrict the use of D to bounded sets. What is the distance $D(\varnothing, \{0\})$? Again, infinite. So we will restrict the use of D to nonempty sets. What is the distance $D\big((0,1),[0,1]\big)$? Now the distance is 0, even though the two sets are not equal. Therefore we will restrict the use of D to closed sets. In fact for the purposes of this book, we will apply D only to nonempty compact sets.

If S is a metric space, we will write $\mathbb{H}(S)$ for the collection of all nonempty compact subsets of S. This is called the ***hyperspace*** for S.

Theorem 2.5.1. *Let S be a metric space. The Hausdorff function D is a metric on the set $\mathbb{H}(S)$.*

Proof. First, clearly $D(A,B) \geq 0$ and $D(A,B) = D(B,A)$. Since A and B are compact, they are bounded, so $D(A,B) < \infty$.

If $A = B$, then for every $\varepsilon > 0$ we have $A \subseteq N_\varepsilon(B)$; therefore $D(A,B) = 0$. Conversely, suppose $A, B \in \mathbb{H}(S)$ satisfy $D(A,B) = 0$. If $x \in A$, then for every $\varepsilon > 0$, we have $x \in N_\varepsilon(B)$, so $\operatorname{dist}(x,B) = 0$. Now B is compact, hence closed, so $x \in B$. This shows $A \subseteq B$. Similarly $B \subseteq A$, so $A = B$.

Finally we have the triangle inequality. Let $A, B, C \in \mathbb{H}(S)$. Let $\varepsilon > 0$. If $x \in A$, then there is $y \in B$ with $\varrho(x,y) < D(A,B) + \varepsilon$. Then there is $z \in C$ with $\varrho(y,z) < D(B,C) + \varepsilon$. This shows that A is contained in the $(D(A,B) + D(B,C) + 2\varepsilon)$-neighborhood of C. Similarly, C is contained in the $(D(A,B) + D(B,C) + 2\varepsilon)$-neighborhood of A. Therefore $D(A,C) \leq D(A,B) + D(B,C) + 2\varepsilon$. This is true for all $\varepsilon > 0$, so $D(A,C) \leq D(A,B) + D(B,C)$. □

Here is one way to describe the limit.

Exercise 2.5.2. Let A_n be a sequence of nonempty compact subsets of S and let A be a nonempty compact subset of S. If A_n converges to A in the Hausdorff metric, then

$$A = \{\, x : \text{ there is a sequence } (x_n) \text{ with } x_n \in A_n \text{ and } x_n \to x \,\}.$$

We may ask about the metric properties of the metric space $\mathbb{H}(S)$. The most important one will be completeness.

Theorem 2.5.3. *Suppose S is a complete metric space. Then the space $\mathbb{H}(S)$ is complete.*

Proof. Suppose (A_n) is a Cauchy sequence in $\mathbb{H}(S)$. I must show that A_n converges. Let $A = \{\, x : \text{ there is a sequence } (x_k) \text{ with } x_k \in A_k \text{ and } x_k \to x \,\}$. I must show that $D(A_n, A) \to 0$ and A is nonempty and compact.

Let $\varepsilon > 0$ be given. Then there is $N \in \mathbb{N}$ so that $n, m \geq N$ implies $D(A_n, A_m) < \varepsilon/2$. Let $n \geq N$. I claim that $D(A_n, A) \leq \varepsilon$.

If $x \in A$, then there is a sequence (x_k) with $x_k \in A_k$ and $x_k \to x$. So, for large enough k, we have $\varrho(x_k, x) < \varepsilon/2$. Thus, if $k \geq N$, then (since $D(A_k, A_n) < \varepsilon/2$) there is $y \in A_n$ with $\varrho(x_k, y) < \varepsilon/2$, and we have $\varrho(y,x) \leq \varrho(y, x_k) + \varrho(x_k, x) < \varepsilon$. This shows that $A \subseteq N_\varepsilon(A_n)$.

Now suppose $y \in A_n$. Choose integers $k_1 < k_2 < \cdots$ so that $k_1 = n$ and $D(A_{k_j}, A_m) < 2^{-j}\varepsilon$ for all $m \geq k_j$. Then define a sequence (y_k) with $y_k \in A_k$ as follows: For $k < n$, choose $y_k \in A_k$ arbitrarily. Choose $y_n = y$. If y_{k_j} has been chosen, and $k_j < k \leq k_{j+1}$, choose $y_k \in A_k$ with $\varrho(y_{k_j}, y_k) < 2^{-j}\varepsilon$. Then y_k is a Cauchy sequence, so it converges. Let x be its limit. So $x \in A$. We have $\varrho(y,x) = \lim_k \varrho(y, y_k) < \varepsilon$. So $y \in N_\varepsilon(A)$. This shows that $A_n \subseteq N_\varepsilon(A)$. Note that, taking $\varepsilon = 1$ in this argument, I have also proved that $A \neq \varnothing$.

So we have $D(A, A_n) \leq \varepsilon$. This concludes the proof that (A_n) converges to A.

Next I show that A is "totally bounded": that is, for every $\varepsilon > 0$, there is a finite ε-net in A. Choose n so that $D(A_n, A) < \varepsilon/3$. By Proposition 2.3.5, there is a finite $(\varepsilon/3)$-net for A_n, say $\{y_1, y_2, \cdots, y_m\}$. Now for each y_i, there is $x_i \in A$ with $\varrho(x_i, y_i) < \varepsilon/3$. The finite set $\{x_1, x_2, \cdots, x_m\}$ is an ε-net for A.

Now I will show that A is a closed subset of S. Let x belong to the closure $\overline{A}$ of A. Then there exists a sequence (y_n) in A with $\varrho(x, y_n) < 2^{-n}$. For each n there is a point $z_n \in A_n$ with $\varrho(z_n, y_n) < D(A_n, A) + 2^{-n}$. Now

$$\varrho(z_n, x) \leq \varrho(z_n, y_n) + \varrho(y_n, x) < D(A_n, A) + 2^{-n} + 2^{-n}.$$

This converges to 0, so $z_n \to x$. Thus $x \in A$. This shows that A is closed.

Finally, to show that A is compact, I will show that it is countably compact. Let F be an infinite subset of A. There is a finite $(1/2)$-net B for A, so each element of F is within distance $1/2$ of some element of B. Now F is infinite and B is finite, so there is an element of B within distance $1/2$ of infinitely many elements of F. Let $F_1 \subseteq F$ be that infinite subset. The points of F_1 are all within distance 1 of each other; that is, $\operatorname{diam} F_1 \leq 1$. In the same way, there is an infinite set $F_2 \subseteq F_1$ with $\operatorname{diam} F_2 \leq 1/2$; and so on. There are infinite sets F_j with $\operatorname{diam} F_j \leq 2^{-j}$ and $F_{j+1} \subseteq F_j$ for all j. Now if x_j is chosen from F_j, we have $\varrho(x_j, x_k) \leq 2^{-j}$ if $j < k$, so (x_j) is a Cauchy sequence. Since S is complete, (x_j) converges, say $x_j \to x$. Since A is closed, $x \in A$. But then x is a cluster point of the set F. Therefore A is compact. ☐

Exercise 2.5.4. Under what conditions on S is $\mathbb{H}(S)$ compact?

Exercise 2.5.5. Under what conditions on S is $\mathbb{H}(S)$ ultrametric?

Convergence in the Examples

Whenever we have used the idea of "convergence" for a sequence of sets in a metric space, we have been talking about nonempty compact sets. The Hausdorff metric is the proper way to interpret it. In fact, it agrees with the other interpretations that have been used.

Proposition 2.5.6. *Let A_n be a sequence of nonempty compact sets, and suppose they decrease: $A_1 \supseteq A_2 \supseteq \cdots$. Then A_n converges to the intersection $A = \bigcap_{n\in\mathbb{N}} A_n$ in the Hausdorff metric.*

Proof. Let $\varepsilon > 0$ be given. Now $A \subseteq A_n$, so $A \subseteq N_\varepsilon(A_n)$. For the other direction, note that the ε neighborhood of A,

$$N_\varepsilon(A) = \{\, y : \ \varrho(x, y) < \varepsilon \text{ for some } x \in A \,\},$$

is an open set. The family $\{N_\varepsilon(A)\} \cup \{\, S \setminus A_n : n \in \mathbb{N} \,\}$ is an open cover of A_1. Since A_1 is compact, there is a finite subcover. This means that, for some $N \in \mathbb{N}$, we have $(S \setminus A_n) \cup N_\varepsilon(A) \supseteq A_1$ for all $n \geq N$. Therefore we have $A_n \subseteq N_\varepsilon(A)$. So $D(A, A_n) \leq \varepsilon$ for all $n \geq N$. This shows that $A_n \to A$. ☐

Proposition 2.5.7. *Suppose S is a compact metric space, T is a metric space and the sequence $f_n\colon S \to T$ converges uniformly to f. Then the image sets $f_n[S]$ converge to $f[S]$ according to the Hausdorff metric on $\mathbb{H}(T)$.*

Proof. This follows from the inequality $D\big(f[S], f_n[S]\big) \le \varrho_{\mathrm{u}}(f, f_n)$. □

Exercise 2.5.8. Let A_n be a sequence of nonempty compact subsets of S. If A is a cluster point of the sequence (A_n), then A is contained in the set of all points $x \in S$ for which there exists a sequence (x_n) such that $x_n \in A_n$ and x is a cluster point of (x_n).

Heighway's Dragon Tiles the Plane

Let us apply some of the properties of set convergence to study the Highway dragon fractal (p. 20).

Proposition 2.5.9. *The Heighway dragon is a space-filling curve.*

Proof. The information and notation used here are from the proof of Proposition 1.5.7. Suppose, at some stage of the construction, the polygon P_n contains all of the sides of a square S of the lattice L_n together with all of the sides of the four adjoining squares, as in Fig. 2.5.9(a). Then the polygon P_{n+2}, shown in Fig. 2.5.9(b), contains all four sides of all of the squares contained in S, together with all of the sides of all their adjoining squares.

Now the polygons P_7 and P_8 each contain such a square S. By induction, all sides of all of the subsquares of S are contained in the polygons P_k, for $k \ge 7$. By Exercise 2.5.2, the limit P contains all points of the interior of the square S. □

We claim, in fact, that P tiles the plane. By this we mean: $\mathbb{R}^2$ is the union of countably many sets, each congruent to P, and any two of them intersect at most in their boundaries.

Start with the vertical lines $x = k$ with integer k and horizontal lines $y = j$ with integer j. They subdivide the plane into squares of side 1. Imagine these

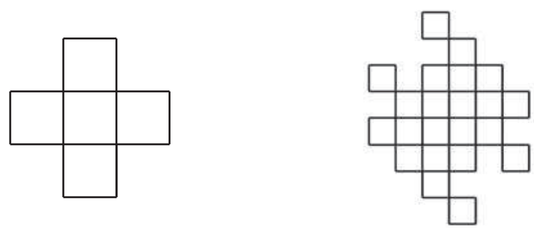

Fig. 2.5.9. (a) (b)

squares classified in the checkerboard pattern as "black" and "white" so that squares that share an edge have opposite colors. For our tiling we begin with the edges of this square subdivision. Each edge is a line-segment of length 1. Starting with the edge, the process described for producing the dragon can be carried out. We need a "direction" for each edge. Let's say the vertical edges have the black square on their left, and the horizontal edges have the black square on their right. Then each edge of this square lattice occurs once. When we carry out one step in the process, replacing each edge by two shorter ones, we will see (as in the proof of proof of Proposition 1.5.7) that the result will be a new square lattice in the plane, with segments $1/\sqrt{2}$ times the length, but still every segment in the lattice occurs exactly once. In the limit, each edge from the original square lattice gives rise to one congruent copy of Heighway's dragon.

See a few stages in Plate 16. One particular edge is shown in black at the center, and a few of its neighboring edges are shown in in other colors. We know (Proposition 1.5.5) that the approximating polygons for any given dragon remain in a bounded region of the plane. So: given any point (x, y) in the plane, the squares surrounding it in the stages of the process can only come from a finite number of original edges. (In the plate, only a finite number of colors can be involved.) So there is at least one color for which a sequence in that color converges to the point (x, y). So (x, y) belongs to at least one of the dragons. It could happen that a point (x, y) belongs to two or more dragons: these will be boundary points for two or more dragons. Plate 17 shows nine dragons of the tiling that result from the nine edges of Plate 16.

2.6 Metrics for Strings

In Sect. 2.1, we defined a metric $\varrho_{1/2}$ for the space $E^{(\omega)}$ of infinite strings from the two-letter alphabet $\{\mathtt{0}, \mathtt{1}\}$. There are other, equally good, metrics for the same space.

Metrics for 01-Strings

Let r be a real number satisfying $0 < r < 1$. A metric ϱ_r is defined on $E^{(\omega)}$ in the same way as the metric $\varrho_{1/2}$: if $\sigma = \alpha\sigma'$, $\tau = \alpha\tau'$, where the first character of σ' is different than the first character of τ', and if $k = |\alpha|$ is the length of α, then

$$\varrho_r(\sigma, \tau) = r^k.$$

Exercise 2.6.1. (1) ϱ_r is a metric on $E^{(\omega)}$.
(2) The basic set $[\alpha]$ has diameter $r^{|\alpha|}$, for all $\alpha \in E^{(*)}$.
(3) The space $\left(E^{(\omega)}, \varrho_r\right)$ is complete, compact, and separable.

Proposition 2.6.2. *The metric spaces constructed from $E^{(\omega)}$ using the different metrics ϱ_r are all homeomorphic to each other.*