Zad. 1

$$H = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & h_{22} & \dots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \dots & h_{nn} \end{bmatrix} \qquad \boxed{\begin{matrix} H^T = H \\ H^2 = H \end{matrix}}$$

Z symetrii i idempotentnosci macierzy H wynika, że:

$$\begin{cases} h_{11} = h_{11}^2 + h_{12}^2 + \dots + h_{1n}^2 \\ h_{22} = h_{12}^2 + h_{22}^2 + \dots + h_{2n}^2 \\ \vdots \\ h_{nn} = h_{1n}^2 + h_{n2}^2 + \dots + h_{nn}^2 \end{cases}$$

Czyli $h_{ii} = \sum_{j=1} h_{ij}^2$

$$h_{ii} = \sum_{j \neq i} h_{ij}^2 + h_{ii}^2$$

$$\sum_{j \neq i} h_{ij}^2 = h_{ii} - h_{ii}^2 .$$

A dla dowolnego $x \in \mathbb{R}$ zachodzi:

$$x - x^2 \leq \frac{1}{4}$$

Dow: $x - x^2 \leq \frac{1}{4}$

$$x^2 - x + \frac{1}{4} \geq 0 \quad / \cdot 4$$

$$4x^2 - 4x + 1 \geq 0$$

$$(2x-1)^2 \geq 0 \quad \text{ok.} \quad \blacksquare$$

Tak więc:

$$\sum_{j \neq i} h_{ij}^2 = h_{ii} - h_{ii}^2 \leq \frac{1}{4}$$

Czyli w szczegolnosci dla $j \neq i$:

$$h_{ij}^2 \leq \frac{1}{4}$$

Nierownosc $h_{ij}^2 \geq 0$ jest oczywista. $\blacksquare$

Zad. 2

$$\boxed{\begin{aligned} Y &= X\beta + \mathcal{E} \\ \hat{Y} &= X\hat{\beta} \\ \hat{\beta} &= (X'X)^{-1}X'Y \\ \hat{Y} &= HY, \quad \text{gdzie} \quad H = X(X'X)^{-1}X' \\ e &= Y - \hat{Y} = (I-H)Y \end{aligned}}$$

1. $e'Y \overset{?}{=} Y'(I-H)Y$

$$e'Y = (Y-\hat{Y})'Y = (Y-HY)'Y = ((I-H)Y)'Y =$$

$$= Y'(I-H)'Y = Y'(I-H)Y$$

$\{$bo $(I-H)$ jest symetryczna$\}$. ∎

$$\boxed{\begin{aligned} Cov(A,B) &= \mathbb{E}[(A-\mathbb{E}A)(B-\mathbb{E}B)'] \\ Cov(aA, bB) &= a\,Cov(A,B)\,b' \end{aligned}}$$

własność pomocnicza

Dow: (powyższej równości)

$$Cov(aA, bB) = \mathbb{E}[(aA - a\mathbb{E}A)(bB - b\mathbb{E}B)'] =$$

$$= \mathbb{E}[a(A-\mathbb{E}A)(B-\mathbb{E}B)'b'] =$$

$$= a\,\mathbb{E}[(A-\mathbb{E}A)(B-\mathbb{E}B)']\,b' = a\,Cov(A,B)\,b'$$ ∎

2. $Cov(Y,e) \overset{?}{=} \sigma^2(I-H)$

$$Cov(Y,e) = Cov(Y, (I-H)Y) =$$

$$= Cov(Y,Y)(I-H)' = Var(Y)(I-H)' =$$

$$= \sigma^2 I (I-H)' = \sigma^2(I-H)$$

$\{$bo $(I-H)$ jest symetryczna$\}$

∎

④ $e'X \stackrel{?}{=} 0$

$$e'X = (Y - \hat{Y})'X = (Y' - \hat{Y}')X = Y'X - \hat{Y}'X =$$
$$= Y'X - (X\hat{\beta})'X = Y'X - \hat{\beta}'X'X =$$
$$= Y'X - [(X'X)^{-1}X'Y]'X'X =$$
$$= Y'X - Y'X[(X'X)^{-1}]'X'X \xrightarrow{} \{\text{bo } (A^{-1})' = (A')^{-1}\}$$
$$= Y'X - Y'X\underbrace{(X'X)^{-1}X'X}_{I} = Y'X - Y'X = 0$$

■

⑤ $Cov(e, \hat{Y}) \stackrel{?}{=} 0$

$$Cov(e, \hat{Y}) = Cov((I-H)Y, HY) = (I-H)\,Var\,Y\,H' =$$
$$= (I-H)\sigma^2 I H = \sigma^2(I-H)H =$$
$$= \sigma^2[IH - H^2] = \sigma^2(H-H) = 0$$

{bo H - idempotentna}

■

③ $\bar{e} \stackrel{?}{=} 0$

Skorzystam tu z własności (4): $e'X = 0$

tzn:

$$[e_1 \dots e_n]\begin{bmatrix} 1 & x_{1,1} & \dots & x_{1,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n,1} & \dots & x_{n,p-1} \end{bmatrix} = \left[\sum_{i=1}^{n} e_i, \sum_{i=1}^{n} e_i x_{i,1}, \dots, \sum_{i=1}^{n} e_i x_{i,p-1}\right]$$

No a skoro to jest równe 0, to w szczególności pierwszy wyraz tego wektora będzie równy zero.

Tak więc:

$$\sum_{i=1}^{n} e_i = 0 \quad / \cdot \frac{1}{n}$$
$$\frac{1}{n}\sum_{i=1}^{n} e_i = 0$$
$$\bar{e} = 0$$

■