Dynamic Bayesian Networks II

Particle filters

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Inference in general DBNs

- We saw that various types of inference can be performed efficiently on Hidden Markov Models
- What about general Dynamic Bayesian Networks?
- Do those algorithms still work?

- In principle: yes
- But the hidden state of a DBN is described by several variables
- Exact inference can only possible when those variables have some independence structure
- but...

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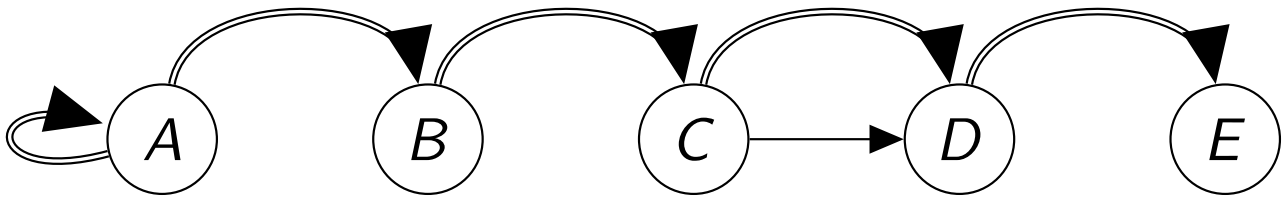
Inference in general DBNs – the entanglement problem

The entaglement problem

As we move with time more and more dependencies appear between variables. They become entangled

- Consider a rather sparse DBN with few dependencies within/between time slices



- Let's unroll it

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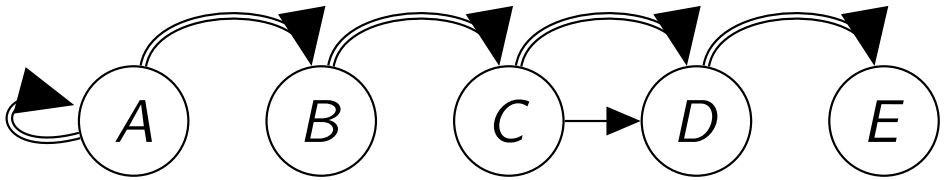

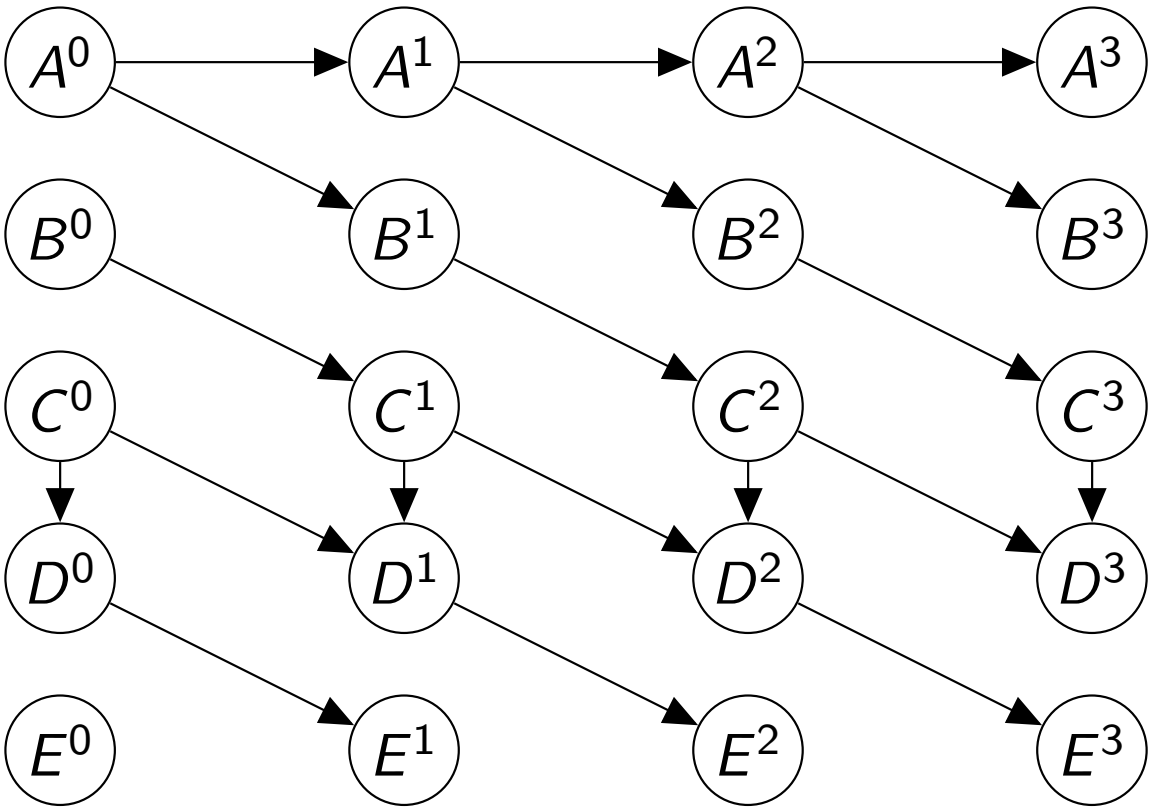

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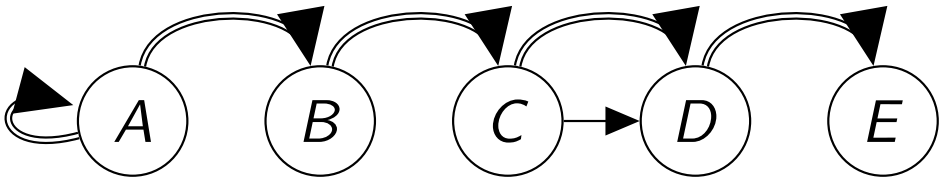
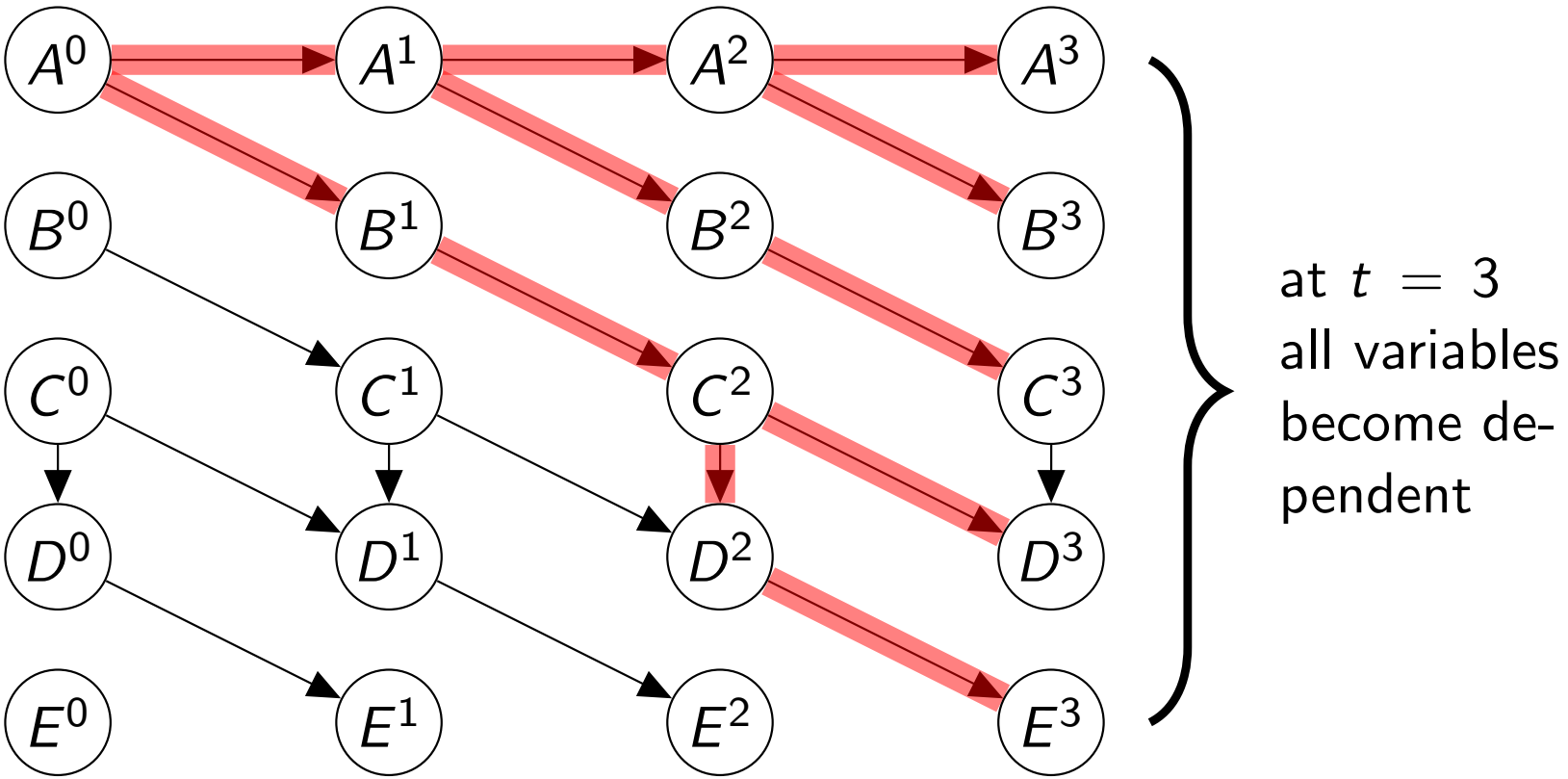
Inference in general DBNs – the entanglement problem





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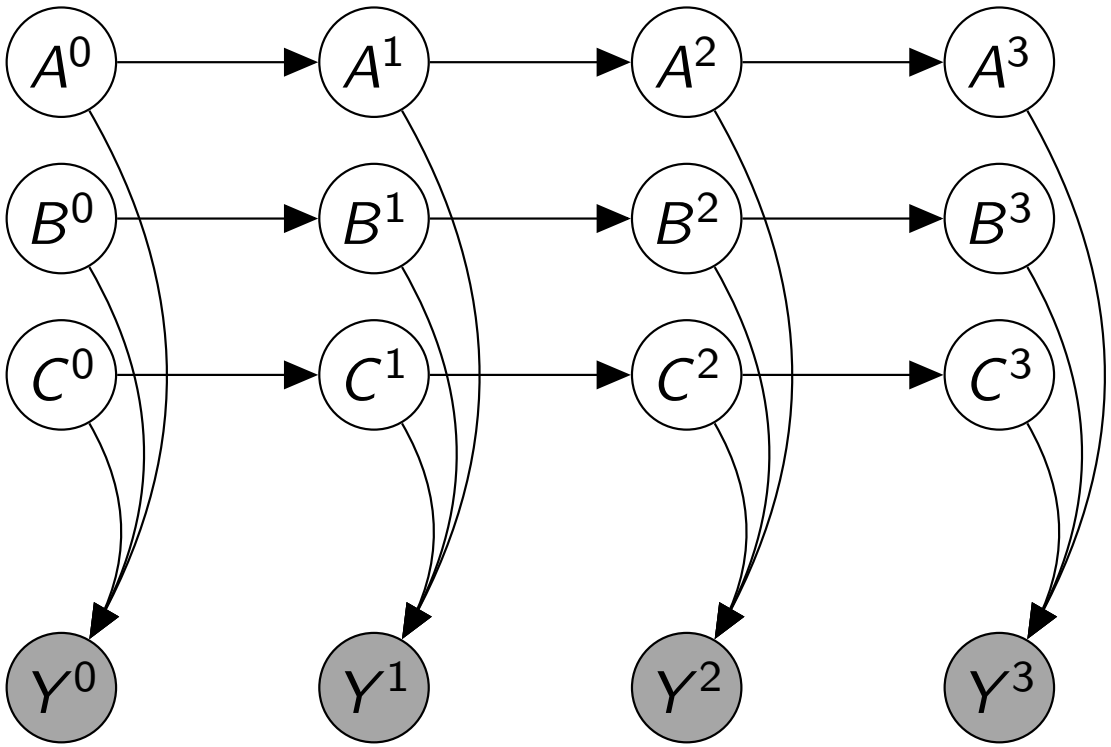
Inference in general DBNs – the entanglement problem

- When observed outputs are present, the situation is even worse



- 3 otherwise independent internal variables become immediately dependent when the output is observed

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Inference in general DBNs – the entanglement problem

- The only real option is to somehow represent the full joint distribution of states at every time step!
- If there are more than a few internal variables, there is no hope of being able to do exact inference
- We have to approximate!

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Inference in general DBNs – approximating the joint distribution

Great many approximations have been proposed

- Mixtures of Gaussians
- Product approximations (the Boyen-Koller filter)
- Sampling (particle filters)
- ...

We will briefly mention the Boyen-Koller filter, and spend more time on particle filters

But first let's talk about the filtering task in general BDNs

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Inference in general DBNs – filtering

- We performed several different types of inference on HMMs
- It is possible to do them on the DBNs also
- But the most frequently discussed problem is filtering, that is computing

$$P(\mathbf{X}^t|\mathbf{y}^{1..t})$$

  the estimate of current state based on all previous outputs
- Most algorithms are called filters

- Given the solution at time t we want to update it to time $t+1$
- This is in fact the forward algorithm for DBNs
- Practical motivation: tracking the system's hidden state in real time

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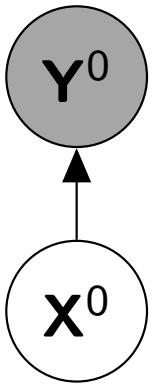
The forward algorithm for DBNs

- Main idea: start at $t = 0$ and proceed recursively to $t = 1, 2, \ldots$
- The time slice of the DBN at $t = 0$:



  where $\mathbf{X}$ and $\mathbf{Y}$ are sets of variables
- Joint distribution

$$P(\mathbf{Y}^0 = \mathbf{y}^0, \mathbf{X}^0) = P(\mathbf{X}^0)P(\mathbf{y}^0|\mathbf{X}^0)$$

- How to get $P(\mathbf{X}^0|\mathbf{y}^0)$? Just normalize
- So the base case is easy

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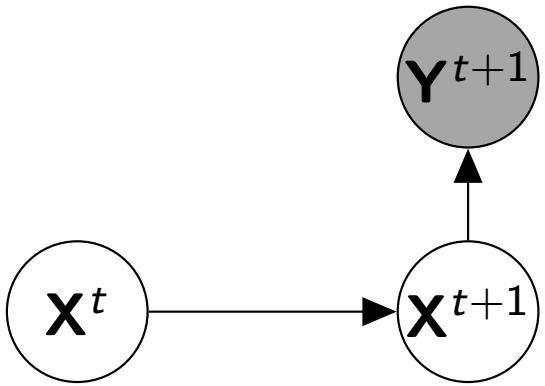
The forward algorithm for DBNs

- Suppose we have computed $P(\mathbf{X}^t|\mathbf{y}^{0..t})$
- How do we compute $P(\mathbf{X}^{t+1}|\mathbf{y}^{0..t+1})$?
- The relevant slice of the DBN:



- Joint distribution:

$$P(\mathbf{X}^t|\mathbf{y}^{0..t})P(\mathbf{X}^{t+1}|\mathbf{X}^t)P(\mathbf{y}^{t+1}|\mathbf{X}^{t+1})$$

- Eliminate $\mathbf{X}^t$ and normalize to get $P(\mathbf{X}^{t+1}|\mathbf{y}^{0..t+1})$
- Proceed to the next time step

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The forward algorithm for DBNs

- The elimination/normalization is a two step process

$$P(\mathbf{X}^{t+1}|\mathbf{y}^{0..t}) = \sum_{\mathbf{X}^t} P(\mathbf{X}^t|\mathbf{y}^{0..t})P(\mathbf{X}^{t+1}|\mathbf{X}^t)$$

$$P(\mathbf{X}^{t+1}|\mathbf{y}^{0..t+1}) = \frac{P(\mathbf{y}^{t+1}|\mathbf{X}^{t+1})P(\mathbf{X}^{t+1}|\mathbf{y}^{0..t})}{\sum_{\mathbf{X}^{t+1}} P(\mathbf{y}^{t+1}|\mathbf{X}^{t+1})P(\mathbf{X}^{t+1}|\mathbf{y}^{0..t})}$$

- Both sums a hard to compute if $|\mathbf{X}|$ is large

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The forward algorithm for DBNs

- DBNs are frequently used with continuous variables (tracking)
- For continuous variables the update steps become

$$P(\mathbf{X}^{t+1}|\mathbf{y}^{0..t}) = \int P(\mathbf{X}^{t}|\mathbf{y}^{0..t})P(\mathbf{X}^{t+1}|\mathbf{X}^{t})\, d\mathbf{X}^{t}$$

$$P(\mathbf{X}^{t+1}|\mathbf{y}^{0..t+1}) = \frac{P(\mathbf{y}^{t+1}|\mathbf{X}^{t+1})P(\mathbf{X}^{t+1}|\mathbf{y}^{0..t})}{\int P(\mathbf{y}^{t+1}|\mathbf{X}^{t+1})P(\mathbf{X}^{t+1}|\mathbf{y}^{0..t})\, d\mathbf{X}^{t+1}}$$

- P is now a probability density function
- Both integrals can be hard to compute even if $|\mathbf{X}|$ is small (depending on actual distributions)

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The forward algorithm for DBNs

- Due to the entaglement problem those sums/integrals cannot be computed using variable elimination
- It is necessary to approximate the state distribution $P(\mathbf{X})$
- We'll discuss three approaches
  - the Kalman filter – sketch only
  - product decomposition (the BK filter) – sketch only
  - particle filters

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The Kalman filter

- If for some matrices $\mathbf{A}$, $\mathbf{B}$
  - $\mathbf{X}^0 \sim N(\mu_0, \mathbf{\Sigma}_0)$
  - $\mathbf{X}^{t+1}|\mathbf{X}^t = \mathbf{A}\mathbf{X}^t + N(\mathbf{0}, \mathbf{\Sigma}_1)$
  - $\mathbf{Y}^t|\mathbf{X}^t = \mathbf{B}\mathbf{X}^t + N(\mathbf{0}, \mathbf{\Sigma}_2)$

  then the DBN is called a Kalman filter
- Exact inference is possible using linear algebra
- Kalman filters are very popular in control, video image analysis, etc.

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The Boyen-Koller (BK) filter – sketch

- Introduced in Xavier Boyen and Daphne Koller in *Tracable Inference for Complex Stochastic Processes*, UAI'98. The paper is here
- The approach: factorize the joint distribution of states even though this violates the dependencies between variables
- We thus approximate the full joint distribution by a product of smaller distributions
- How does this affect the error of the forward algorithm?
- One worries that it will get worse and worse as the algorithm proceeds
  ... and the results will quickly become useless

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The Boyen-Koller (BK) filter – sketch

A positive result

Under certain assumptions the error never grows above a certain threshold

- Main idea:
  - as time goes by the system forgets the initial state so the initial distribution and its approximation become closer and closer to each other
    - similar to Markov Chains converging to stationary distributions
  - if this convergence is fast enough, it will offset the growing approximation error

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The Boyen-Koller (BK) filter – sketch

Definition

Let $P(\mathbf{X}^{t+1}|\mathbf{X}^t)$ be the transition probabilities for the internal state of the DBN. The minimum mixing rate of the DBN is

$$\gamma = \min_{\mathbf{x}_1,\mathbf{x}_2} \sum_{\mathbf{x}} \min\{P(\mathbf{x}|\mathbf{x}_1), P(\mathbf{x}|\mathbf{x}_2)\}$$

- It is the minimum probability that, starting in two different states at time t, we end up in the same state at $t+1$

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The Boyen-Koller (BK) filter – sketch

Weakly interacting set of processes

Suppose the internal state $\mathbf{X}$ can be partitioned into blocks $\mathbf{X}_1 \cup \ldots \cup \mathbf{X}_k$ such that

- there are no edges within the same time slice between $\mathbf{X}_i$ and $\mathbf{X}_j$, $i \neq j$
- each $\mathbf{X}_i$ influences at most q other blocks at time $t+1$
- at time $t+1$ each $\mathbf{X}_i$ is influenced by at most r other blocks

- Of course $\mathbf{X}_i^{t+1}$ usually depends on $\mathbf{X}_i^t$
- Intuition: each $\mathbf{X}_i$ forms a subprocess within the DBN and the amount of interaction between subprocesses is limited
- Idea: approximate $P(\mathbf{X})$ using $\tilde{P}(\mathbf{X}) = \prod P(\mathbf{X}_i)$

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The Boyen-Koller (BK) filter – sketch

Theorem

Suppose that:

- the internal state of a DBN can be decomposed into k weakly interacting subprocesses such that each subprocess has a minimum mixing rate at least γ
- for all t, using the approximation $\tilde{P}$ increases the error by at most ε:

$$KL(P : \tilde{P}) - KL(P : \hat{P}) \leq \varepsilon$$

where $\hat{P}$ is the unapproximated estimate of P

Then

$$\mathrm{E}KL(P : \tilde{P}) \leq \varepsilon/(\gamma/r)^q$$

$KL(P : Q)$ is the Kullback-Leibler divergence between distributions P and Q

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The Boyen-Koller (BK) filter – sketch

- Conclusion: the approximation error does not grow above certain threshold no matter how long we perform the filtering
- In other words: approximation errors do not accumulate above $\varepsilon/(\gamma/r)^q$

- Very nice theoretically
- In practice works well if the internal state can really be decomposed into weakly interacting subprocesses

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Particle filters

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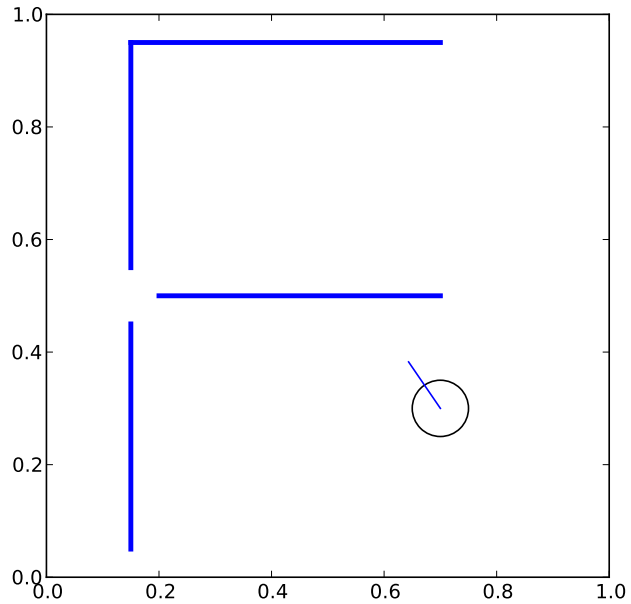
Example: robot localization problem

Let us begin with a motivating example

Given:

- the terrain map
- the history of moves a robot has made
- the history of robot's sensor readings



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Example: robot localization problem

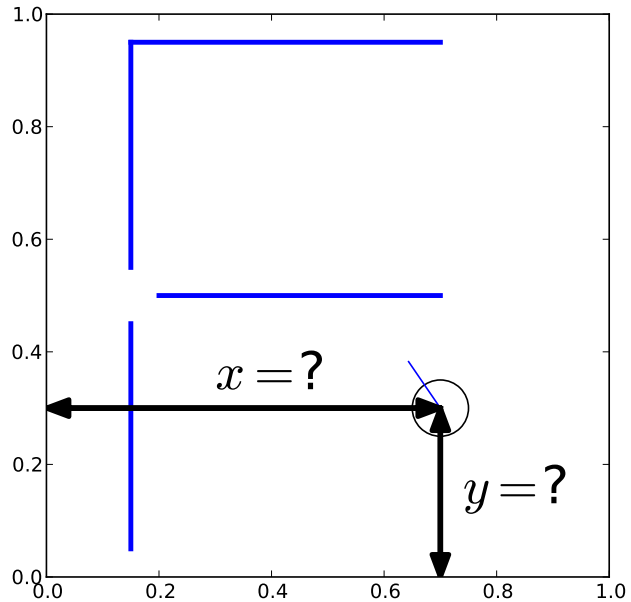
Let us begin with a motivating example

Given:

- the terrain map
- the history of moves a robot has made
- the history of robot's sensor readings

Find:

- Robot's position on the map



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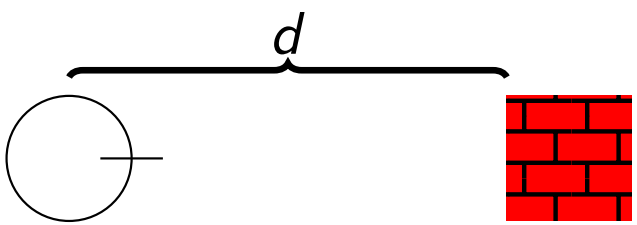
Example: robot localization problem

- Robot's position (internal state of the DBN):

$$\mathbf{X}^t = \begin{pmatrix} x^t \\ y^t \\ \alpha^t \end{pmatrix}$$

- There is only one sensor which measures the distance from obstacles

$$\mathbf{Y} = \frac{1}{d^2(\mathbf{X})} + N(0, \epsilon)$$



- $d(\mathbf{X})$ is the distance to the nearest obstacle in the direction the sensor is pointing
- it is a function of $\mathbf{X}$ and of the coordinates of walls on the map

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Example: robot localization problem

- We assume the uniform a-priori distribution

$$x^0, y^0 \sim U(0,1), \quad \alpha^0 \sim U(0, 2\pi)$$

- $P(\mathbf{X}^{t+1}|\mathbf{X}^t)$ depends on the move the robot made at time t
- Since the moves are not perfectly precise, the new position has random errors
- Possible moves:
  - rotate by angle β

$$\mathbf{X}^t = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix}, \qquad \mathbf{X}^{t+1} = \begin{pmatrix} x \\ y \\ \alpha + \beta + \epsilon \end{pmatrix}$$

  - move forward by p

$$\mathbf{X}^t = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix}, \qquad \mathbf{X}^{t+1} = \begin{pmatrix} x + (p + \epsilon)\cos\alpha \\ y + (p + \epsilon)\sin\alpha \\ \alpha \end{pmatrix}$$

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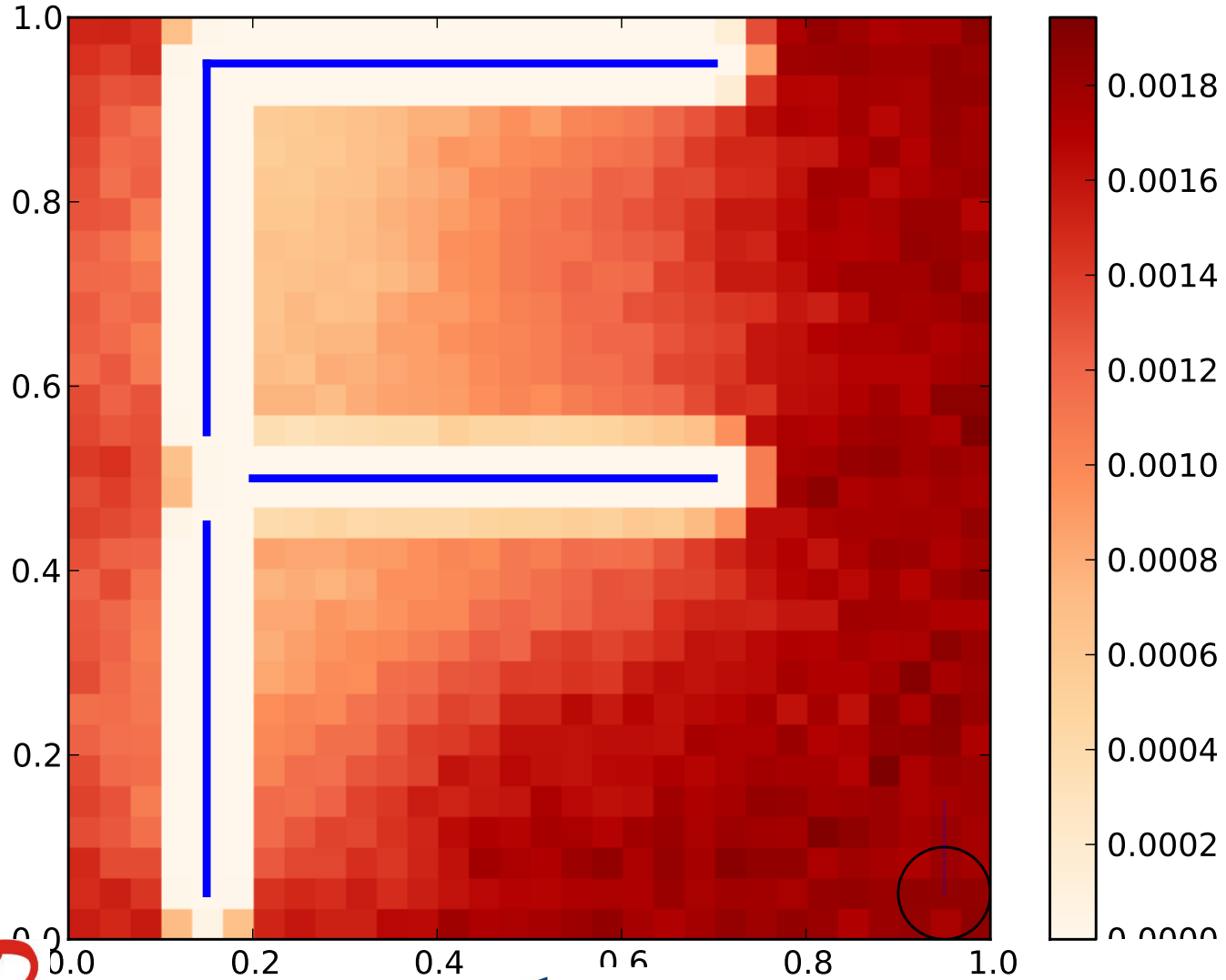

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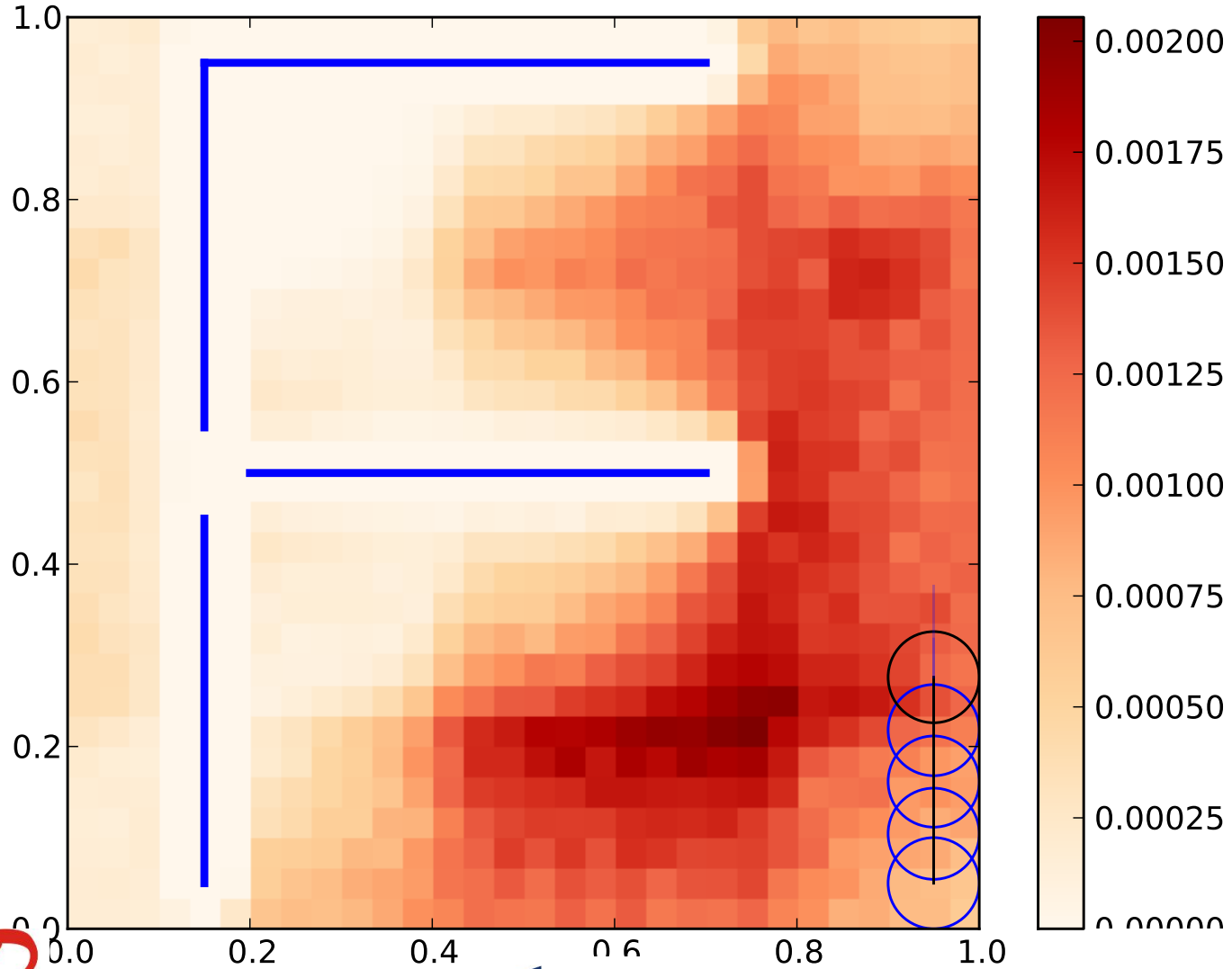

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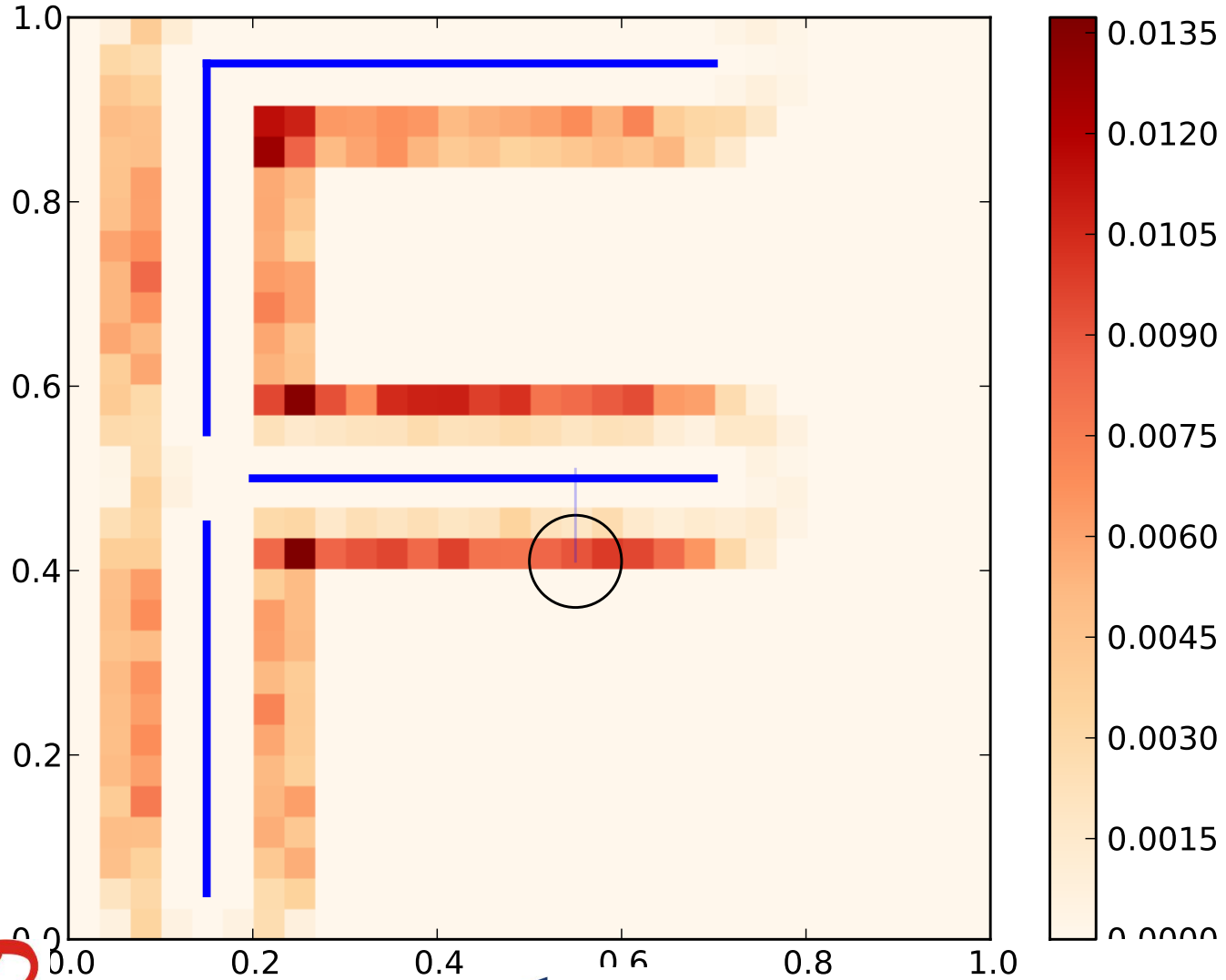

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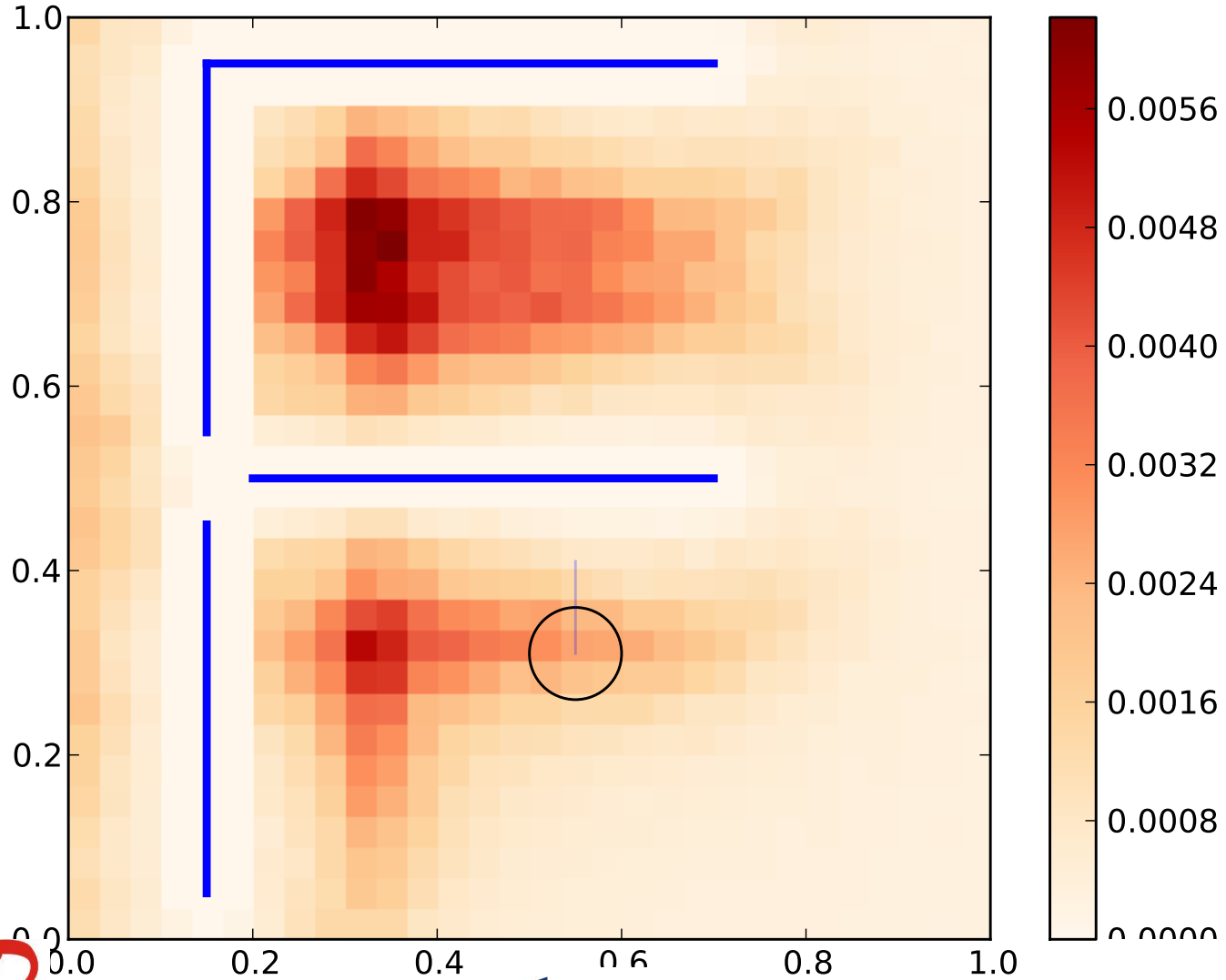

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Particle filters

- particle = sample + weight

$$(\mathbf{x}_i^t, w_i^t)$$

- A set of particles $\{(\mathbf{x}_i^t, w_i^t) : i = 1, \ldots, N\}$ approximates the distribution $P(\mathbf{X}^t)$ of states at time t
- Samples and weights are modified appropriately as we move from t to $t + 1$

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Particle filters – the basic algorithm

1. Initialization
   - $t \leftarrow 0$
   - draw N samples $\mathbf{x}_i$ from $P(\mathbf{X}^0)$
   - $w_i \leftarrow P(\mathbf{y}^0|\mathbf{x}_i^0)$ for $i = 1 \ldots N$
2. Draw new samples from $P(\mathbf{X}^{t+1}|\mathbf{x}_i^t)$

$$\mathbf{x}_i^{t+1} \sim P(\mathbf{X}^{t+1}|\mathbf{x}_i^t)$$

3. Update particle weights based on $\mathbf{y}^{t+1}$, the sensor readings at time $t+1$

$$w_i \leftarrow w_i \cdot P(\mathbf{y}^{t+1}|\mathbf{x}_i^{t+1})$$

4. Normalize weights

$$w_i \leftarrow \frac{w_i}{\sum_{i=1}^{N} w_i}$$

5. $t \leftarrow t+1$; **Goto** 2

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Particle filters – proof

- How do we know that a PF gives us the correct distribution?
- Let's rewrite

$$P(\mathbf{X}^{1..T}|\mathbf{y}^{1..T}) \propto P(\mathbf{X}^0)P(\mathbf{y}^0|\mathbf{X}^0)\prod_{t=1}^{T} P(\mathbf{X}^t|\mathbf{X}^{t-1})P(\mathbf{y}^t|\mathbf{X}^t)$$

$$= \underbrace{P(\mathbf{X}^0)\prod_{t=1}^{T} P(\mathbf{X}^t|\mathbf{X}^{t-1})}_{\text{samples}}\underbrace{P(\mathbf{y}^0|\mathbf{X}^0)\prod_{t=1}^{T} P(\mathbf{y}^t|\mathbf{X}^t)}_{\text{weights}}$$

- Note that samples are taken from a pure Markov Chain independent of outputs
- Weights are applied independently from sampling
- We've seen such a decomposition before...

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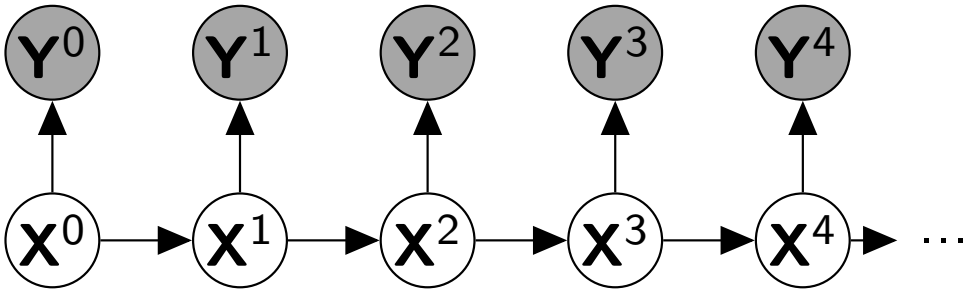
Particle filters – proof



$$P(\mathbf{X}^{1..T}|\mathbf{y}^{1..T}) \propto \underbrace{P(\mathbf{X}^0)\prod_{t=1}^{T} P(\mathbf{X}^t|\mathbf{X}^{t-1})}_{\text{samples}}\ \underbrace{P(\mathbf{y}^0|\mathbf{X}^0)\prod_{t=1}^{T} P(\mathbf{y}^t|\mathbf{X}^t)}_{\text{weights}}$$

- This is an example of ????
- We have already proved it works

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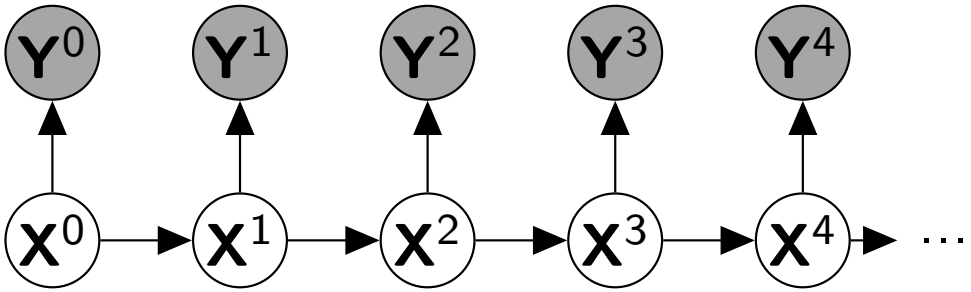
Particle filters – proof



$$P(\mathbf{X}^{1..T}|\mathbf{y}^{1..T}) \propto \underbrace{P(\mathbf{X}^0)\prod_{t=1}^{T} P(\mathbf{X}^t|\mathbf{X}^{t-1})}_{\text{samples}}\ \underbrace{P(\mathbf{y}^0|\mathbf{X}^0)\prod_{t=1}^{T} P(\mathbf{y}^t|\mathbf{X}^t)}_{\text{weights}}$$

- This is an example of likelihood weighting
- We have already proved it works

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Sample degeneracy

- Problem with the basic particle filter
- Sample degeneracy
  - one of the particles (not necessarily closest to the true location) initially gives sensor readings most similar to true readings
  - this particle gets the largest weight
  - this is likely to remain so for a few steps
  - after a few steps all other particles have weights practically equal to zero
- Result: estimates based on a single sample, hugely inaccurate

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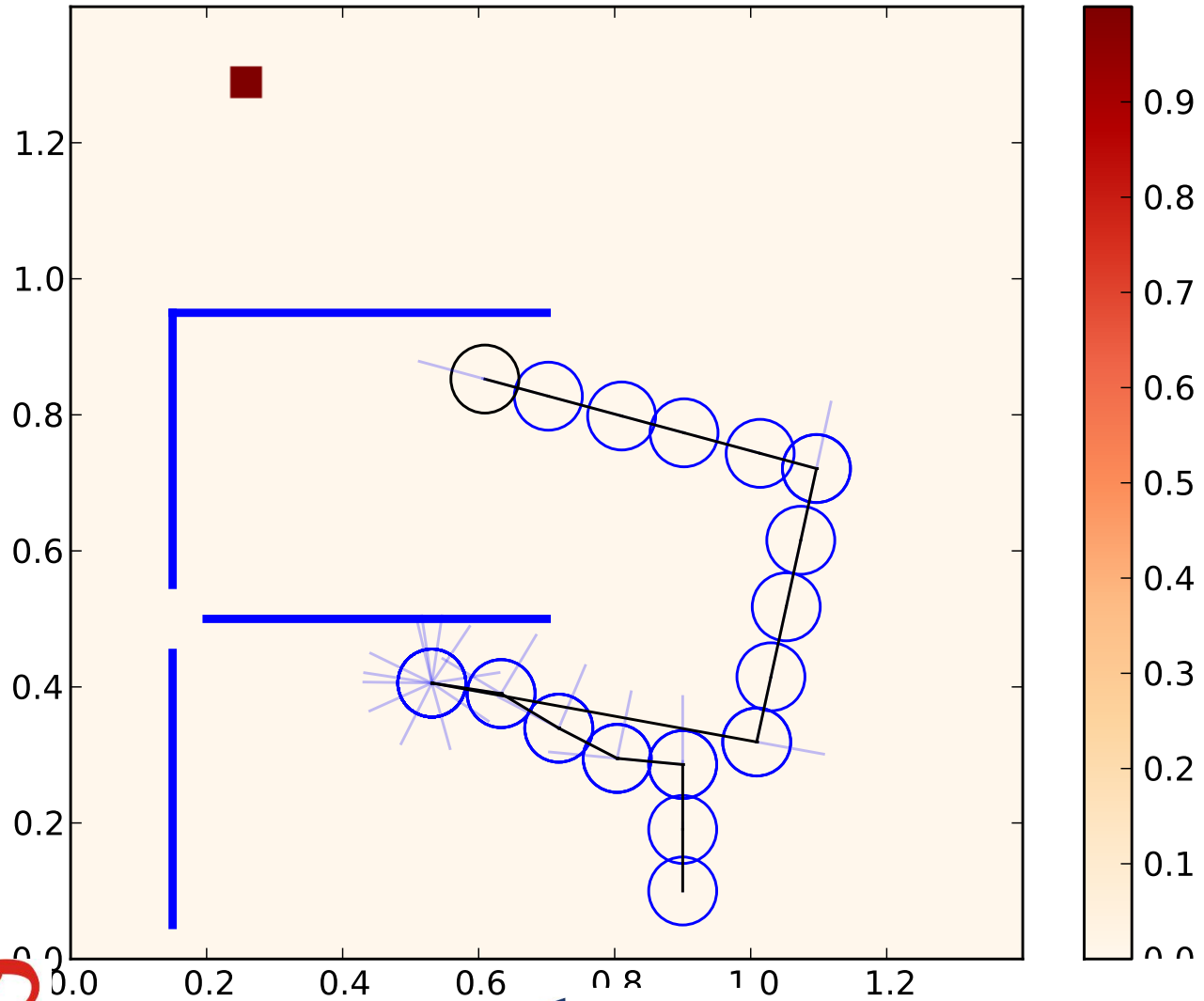
Sample degeneracy example (100000 samples)



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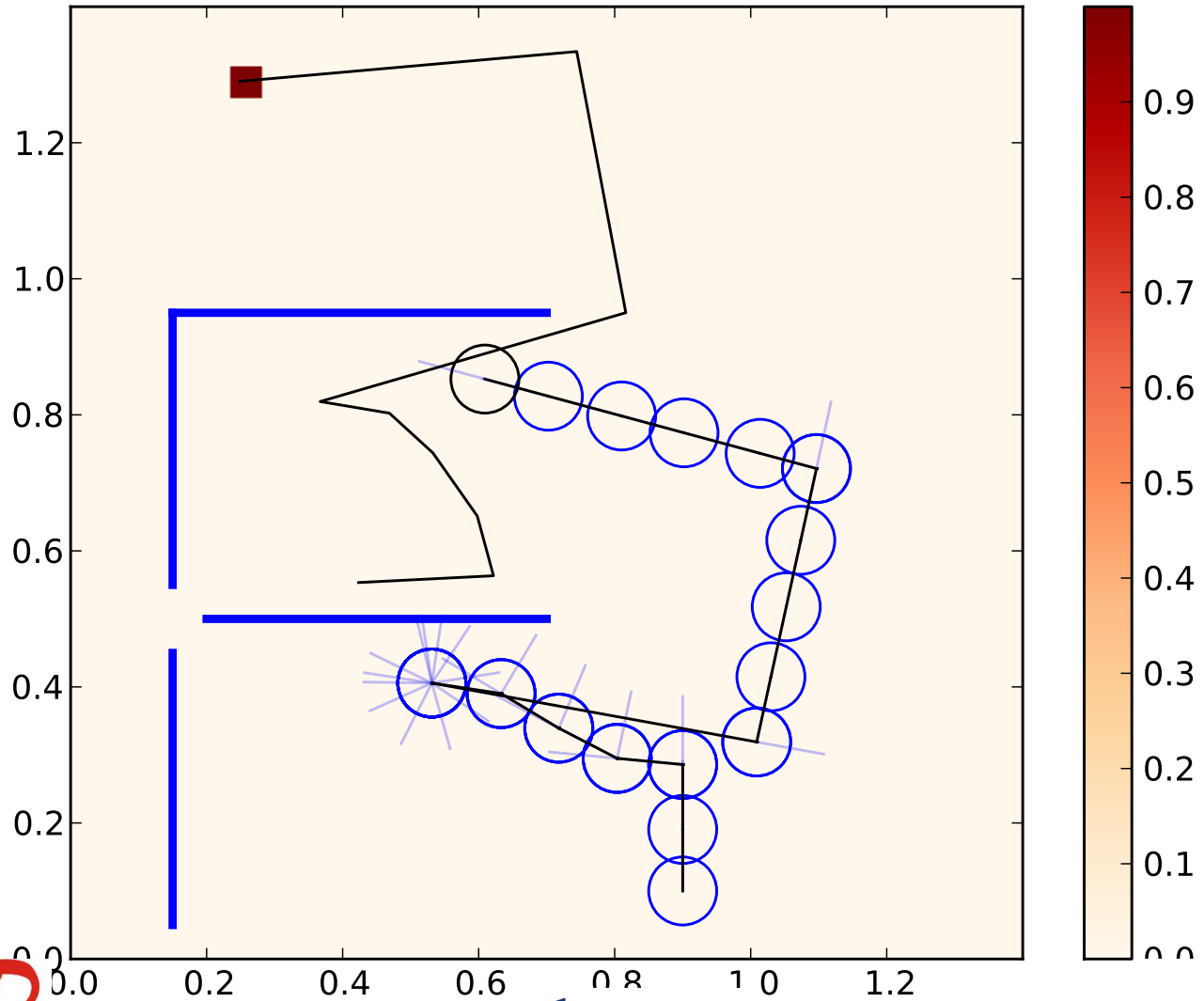
Sample degeneracy – trajectory of the best particle



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Sample degeneracy – some theory

Definition

Effective sample size is defined as

$$N_{eff} = \frac{1}{\sum {w_i}^2}$$

- The definition is intuitively correct:
  - When all samples have equal weights ($\frac{1}{N}$), $N_{eff} = N$
  - When one sample weight is equal to 1 and all other to 0, $N_{eff} = 1$
- Proper derivation of N_{eff}
  - use Taylor expansion of weighted sample variance
- Here we will use a simplified derivation assuming the weights are constant (not r.v.'s)

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Effective sample size

- Let f be a function of $\mathbf{X}^t$. We want to estimate $\mathrm{E}_{P(\mathbf{X}^t)} f$
- If we had N true unweighted i.i.d. samples, the variance of the estimate would be $\frac{1}{N}\mathrm{Var}(f(\mathbf{X}^t))$
- Suppose we have a weighted estimate $\sum_{i=1}^{N} w_i f(\mathbf{x}_i^t)$
- Now

$$\mathrm{Var}\left[\sum_{i=1}^{N} w_i f(\mathbf{x}_i^t)\right] = \sum_{i=1}^{N} w_i^2 \mathrm{Var}(f(\mathbf{x}_i^t)) \approx \mathrm{Var}(f(\mathbf{X}^t)) \sum_{i=1}^{N} w_i^2$$

  the last assumption is necessary since $\mathbf{x}_i^t$ comes from the distribution ignoring output symbols
- How many unweighted i.i.d. samples N_{eff} do we need to get the same variance?

$$\frac{1}{N_{eff}}\mathrm{Var}(f(\mathbf{X}^t)) = \mathrm{Var}(f(\mathbf{X}^t)) \sum_{i=1}^{N} w_i^2 \quad \Rightarrow \quad N_{eff} = \frac{1}{\sum w_i^2}$$

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Effective sample size

Theorem

Effective sample size N_{eff}

- is never greater than the number of particles N
- can only decrease with each time step

- For details and proofs see
  - Kong, Liu, Wong 1994
  - Doucet, Godsill, Andrieu, 2000

- Avoiding sample degeneracy:
  - resampling
  - importance sampling

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Resampling

Main idea:

- From the current set of particles pick N samples with replacement
- Sampling probability is proportional to particle weight
- Set weight of each sample to $\frac{1}{N}$
- Replace the current set of particles with the new sample

Advantages:

- This way we will get a more varied sample
- If we resample early enough, new sample is based on several important particles which are all assigned high weights

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Particle filter with resampling

1. Initialization
   - $t \leftarrow 0$
   - draw N samples $\mathbf{x}_i$ from $P(\mathbf{X}^0)$
   - $w_i \leftarrow P(\mathbf{y}^0|\mathbf{x}_i^0)$ for $i = 1 \ldots N$
2. Draw new samples from $P(\mathbf{X}^{t+1}|\mathbf{x}_i^t)$
3. Update particle weights based on $\mathbf{y}^{t+1}$
4. Normalize particle weights
5. Compute effective sample size N_{eff}
6. **If** $N_{eff} < \epsilon N$
7. Resample
8. $t \leftarrow t + 1$; **Goto** 2

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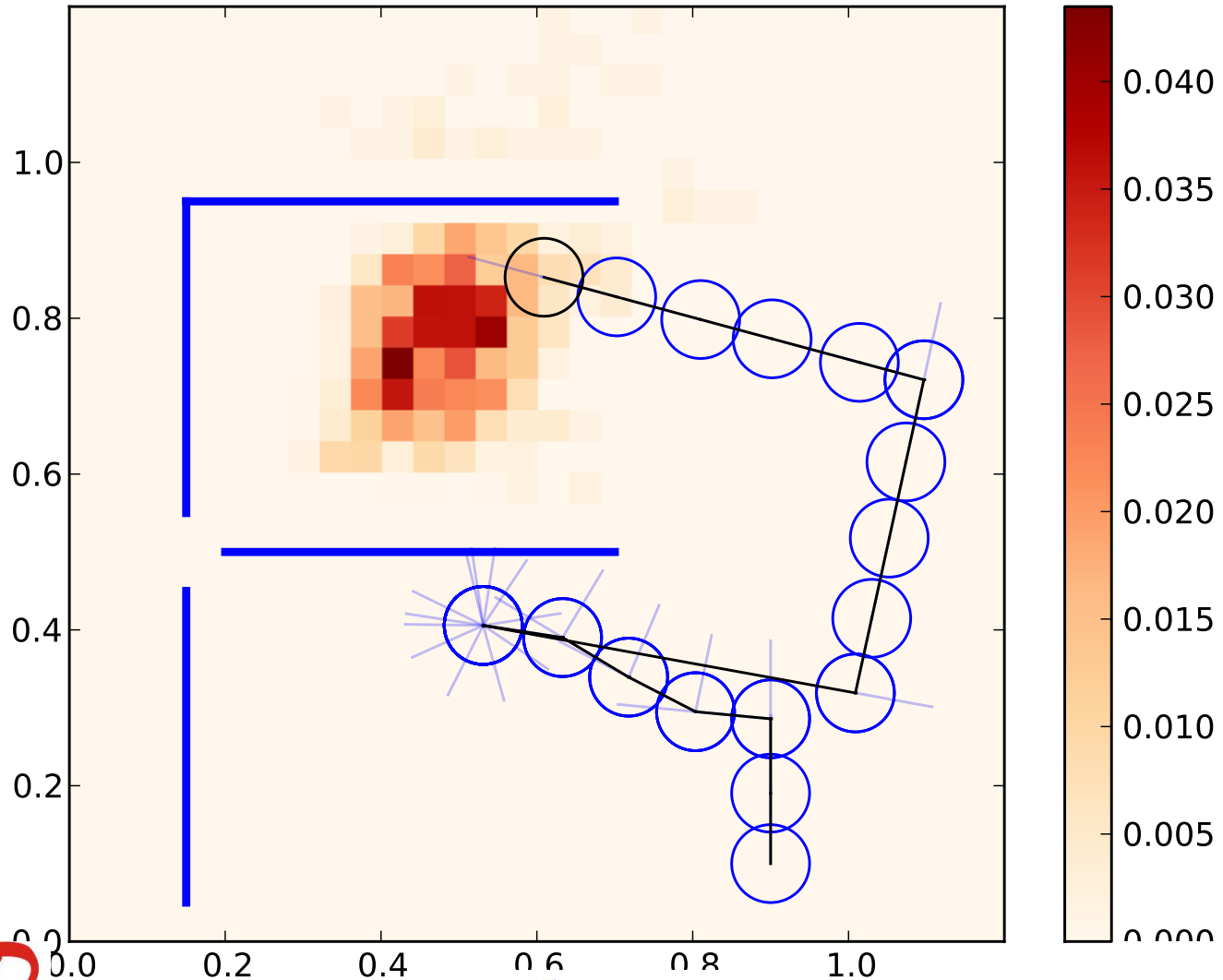
Example: resampling with 1000 samples



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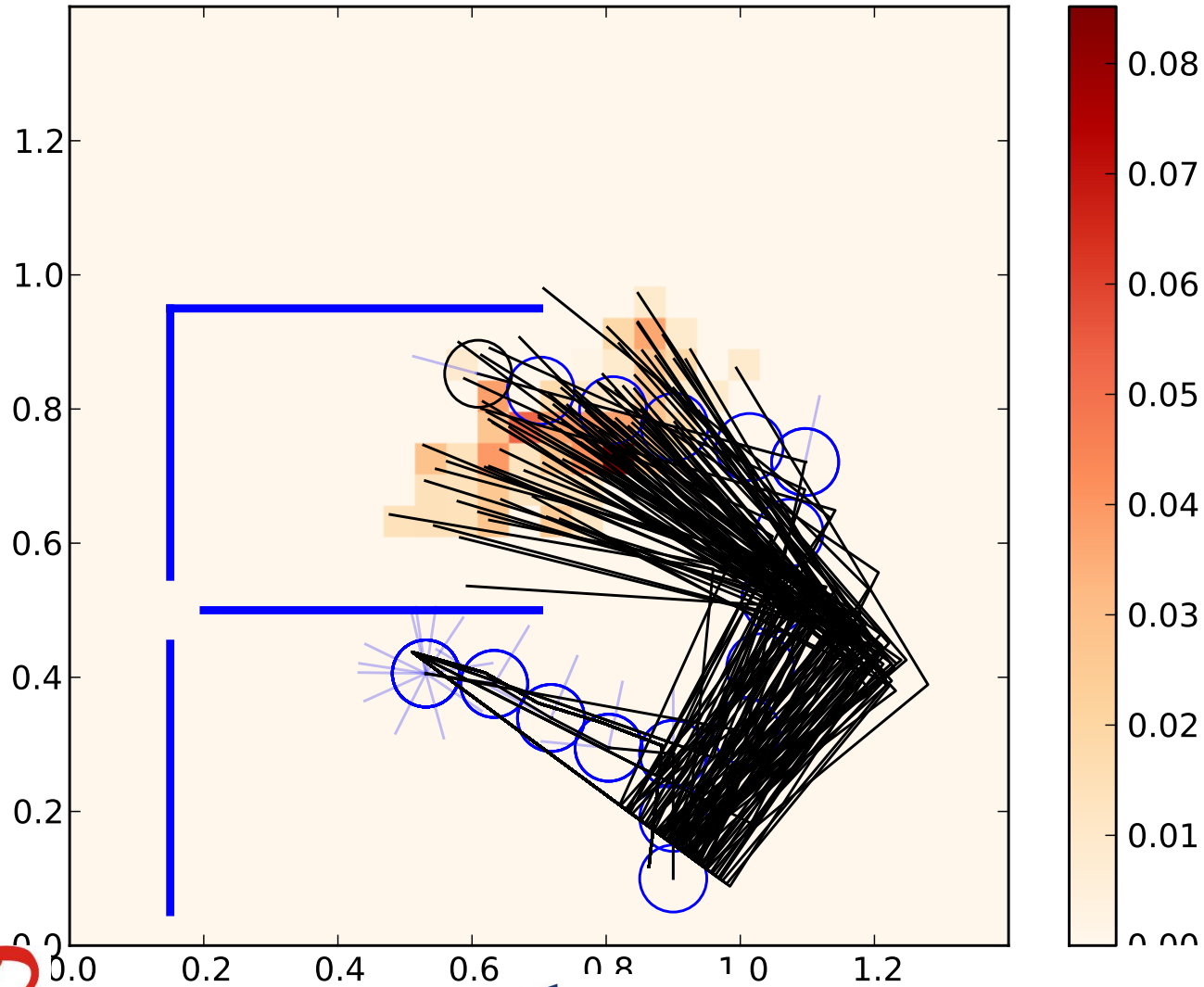
Example: resampling with 100 samples



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Importance sampling

- The standard particle filter draws samples from

$$P(\mathbf{X}^{t+1}|\mathbf{x}_i^t)$$

- However, we also know $\mathbf{y}^{t+1}$ which could be used to get better estimates

- Idea: use importance sampling
  - instead of sampling from $P(\mathbf{X}^{t+1}|\mathbf{x}_i^t)$
  - sample $\mathbf{X}^{t+1}$ from some other proposal distribution

$$\pi(\mathbf{X}^{t+1}|\mathbf{x}^t, \mathbf{y}^{t+1})$$

  - use appropriate weights to accommodate the difference

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Particle filters – importance sampling

Only small changes to the main loop are necessary:

- Draw new samples from $\pi(\mathbf{X}^{t+1}|\mathbf{x}_i^t, \mathbf{y}^{t+1})$

$$\mathbf{x}_i^{t+1} \sim \pi(\mathbf{X}^{t+1}\ |\mathbf{x}_i^t, \mathbf{y}^{t+1})$$

- Update particle weights

$$w_i \leftarrow w_i \cdot P(\mathbf{y}^{t+1}|\mathbf{x}_i^{t+1}) \frac{P(\mathbf{x}_i^{t+1}|\mathbf{x}_i^t)}{\pi(\mathbf{x}_i^{t+1}|\mathbf{x}_i^t, \mathbf{y}^{t+1})}$$

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Importance sampling

- The optimal proposal distribution for sampling $\mathbf{X}^{t+1}$ is

$$P(\mathbf{X}^{t+1}|\mathbf{x}_i^t, \mathbf{y}^{t+1})$$

- Unfortunately
  - this distribution is hard to sample from
  - updating weights (see previous slide) requires computing

$$P(\mathbf{y}^{t+1}|\mathbf{x}^t) = \int P(\mathbf{y}^{t+1}|\mathbf{X}^{t+1})P(\mathbf{X}^{t+1}|\mathbf{x}^t)\, d\mathbf{X}^{t+1}$$

- Typically we have to use other approximations

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Importance sampling: robot localization problem

- We will only discuss the initial position
- Instead of sampling the initial position from $U([0,1] \times [0,1])$ we will take $\mathbf{y}^0$ into account
- High value of $\mathbf{y}^0$ means we are close to a wall
- Our importance distribution:
  - a mixture of uniform distributions on rectangles around walls
  - the higher $\mathbf{y}^0$, the narrower the rectangles
- Other proposals are of course possible e.g. Gaussians 'stretched' along the walls

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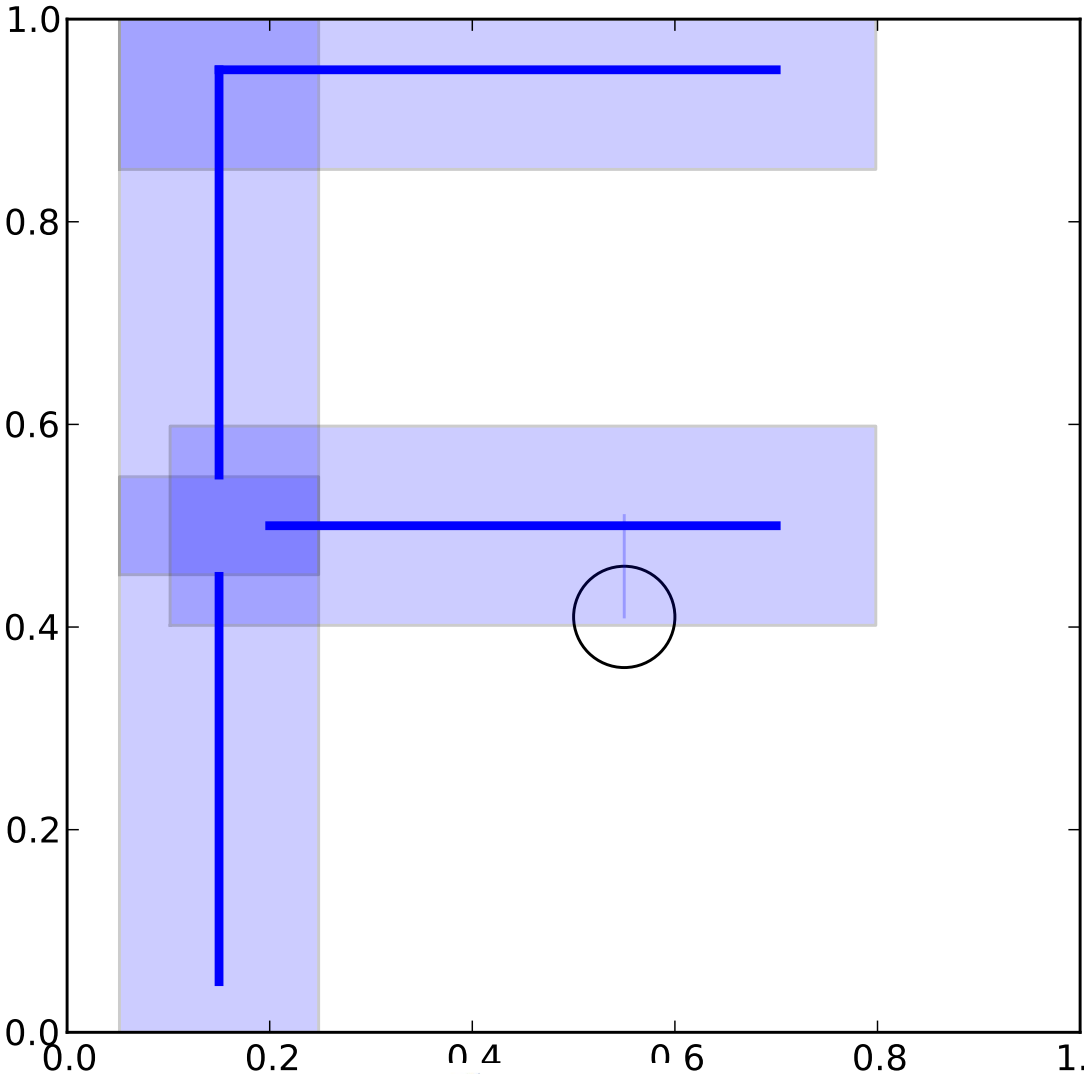
Importance sampling: proposal distribution



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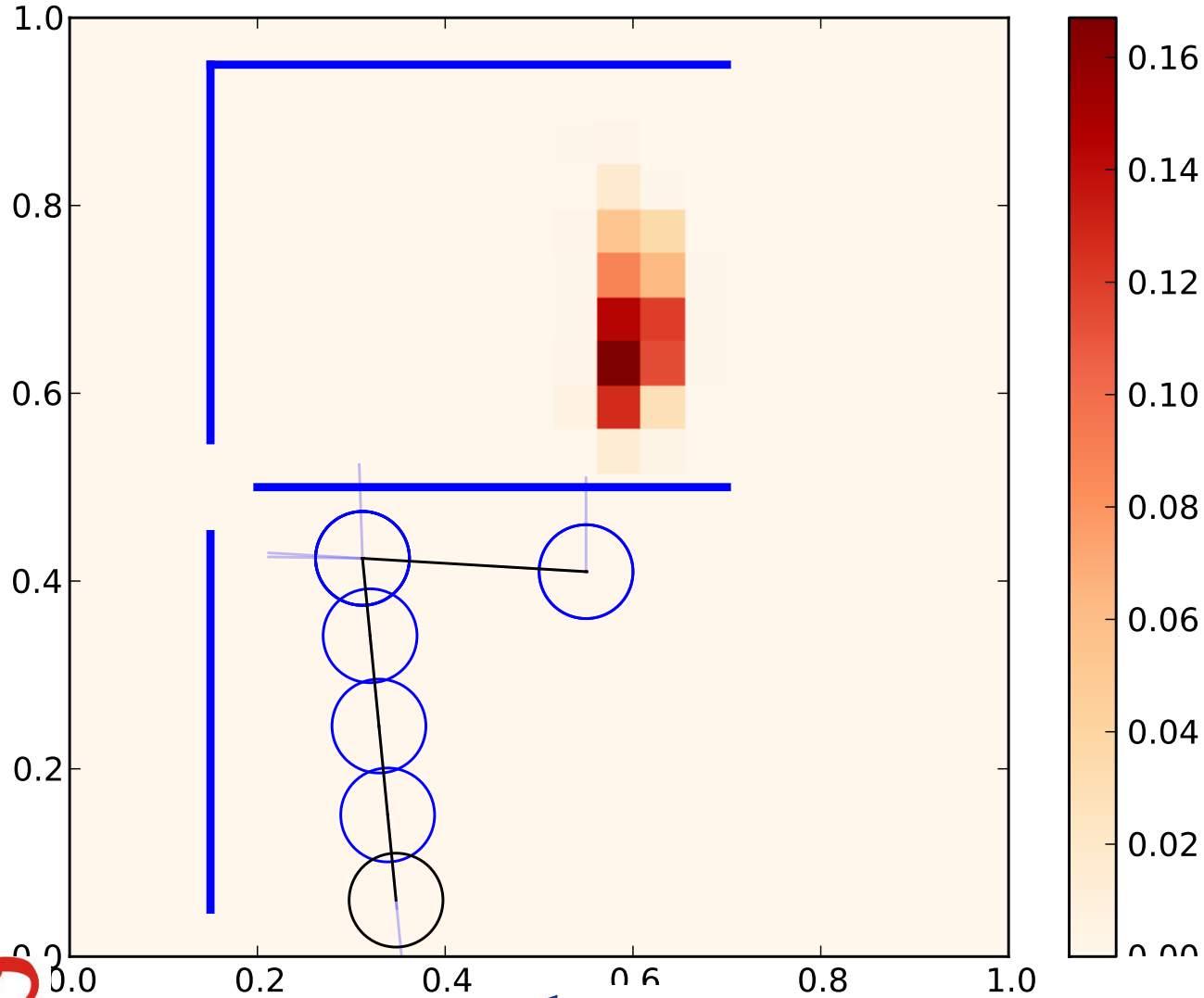
Example: no importance sampling



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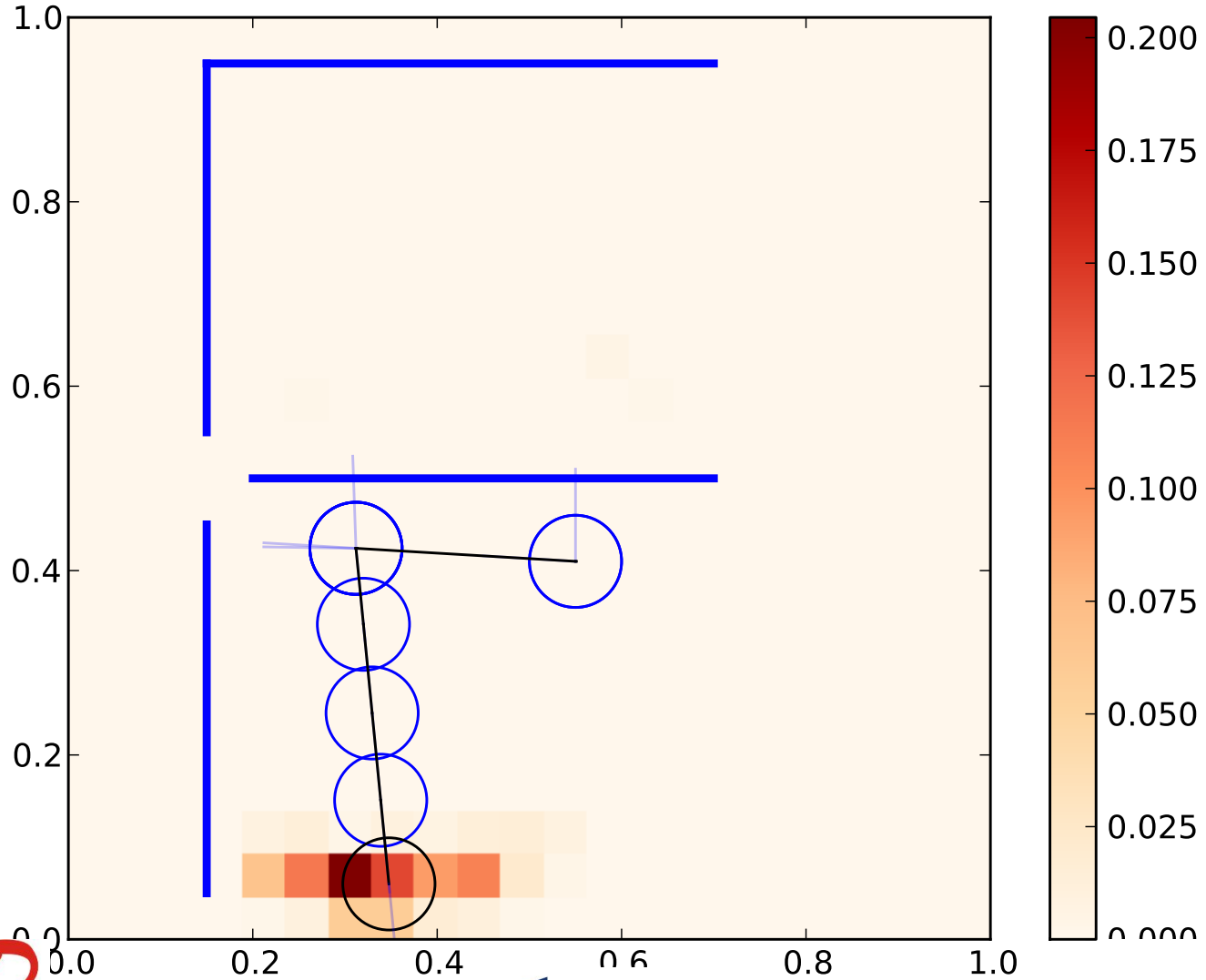
Example: with importance sampling



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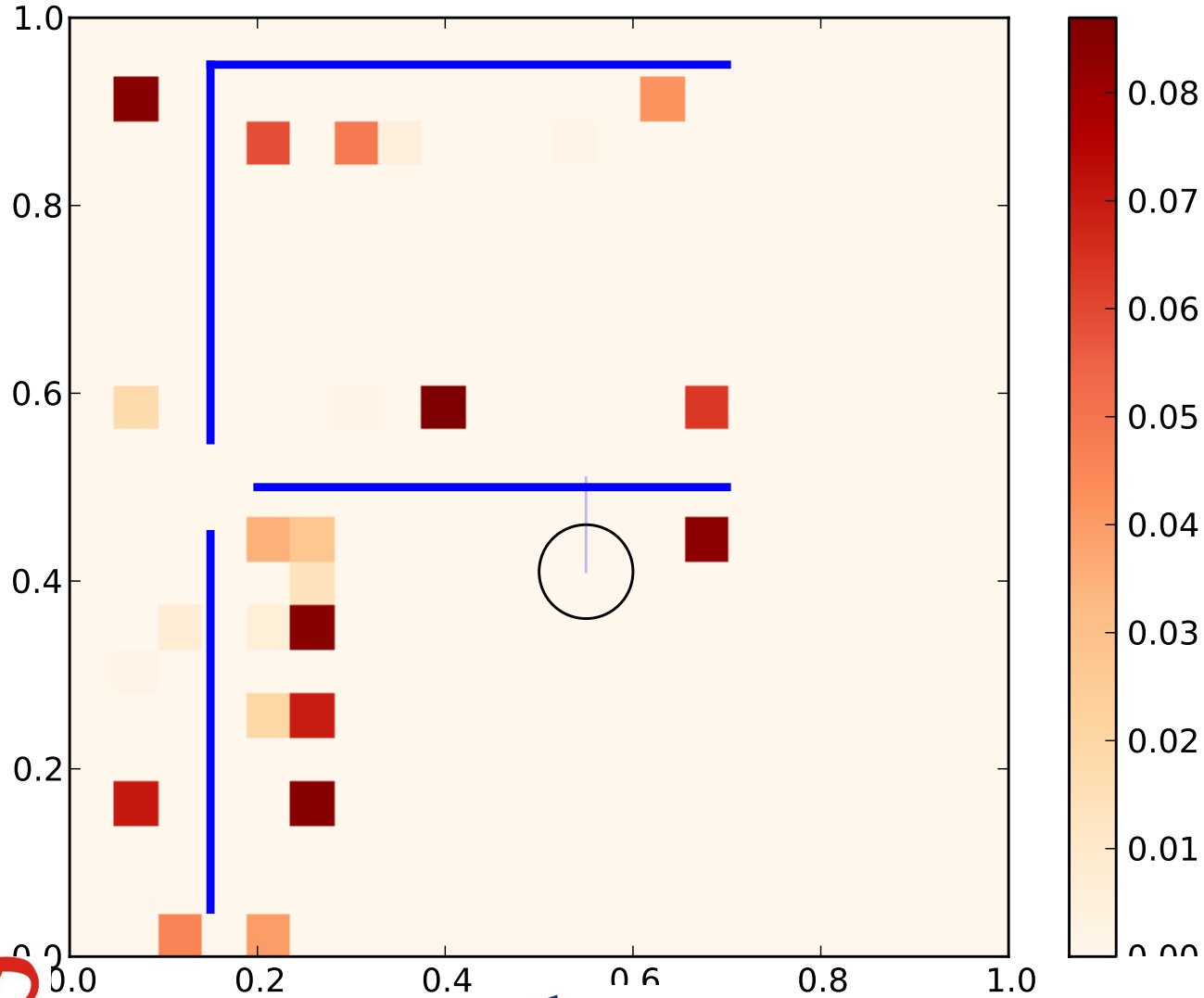
Example: initial position, no importance sampling



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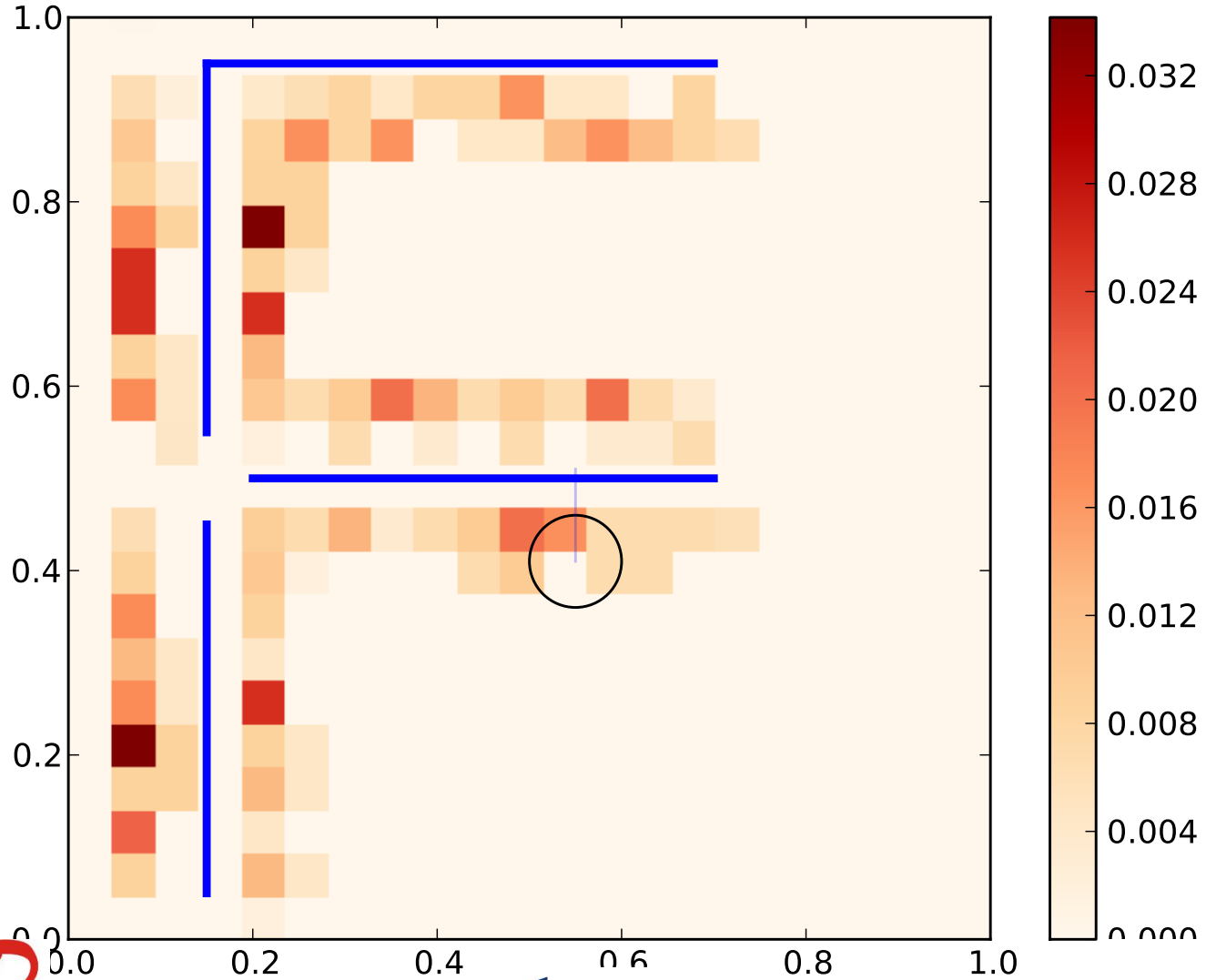
Example: initial position, with importance sampling



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Particle filters – most popular variations

SIS (Sequential Importance Sampling) – importance sampling based version

SIR (Sequential Importance Resampling) – importance sampling based version with resampling

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Extentsions

- Problem specific methods of choosing proposal distributions
- 'Backward' estimation – correcting samples at previous time steps based on future observations
  - Auxiliary Particle Filter – we sample from the previous time step
  - resampling whole fragments of trajectories
- SLAM – Simultaneous Localization and Mapping
- Particle Swarm Optimization (PSO) – find the most probable trajectory by simulating a swarm of particles

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