Linear Algebra and Vector Calculus
Notes and Exercises

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January 26, 2020

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Chapter 1

Linear Algebra Basics

1.1 Linear Functions

A function $L\colon \mathbf{R}^m \to \mathbf{R}^n$ is a linear function if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^m$ and for all $a, b \in \mathbf{R}$

$$L(a\boldsymbol{x} + b\boldsymbol{y}) = aL(\boldsymbol{x}) + bL(\boldsymbol{y}).$$

It follows (by induction) that for all $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r \in \mathbf{R}^m$ and all $a_1, \ldots, a_r \in \mathbf{R}$

$$L(a_1\boldsymbol{x}_1 + \cdots + a_r\boldsymbol{x}_r) = a_1L(\boldsymbol{x}_1) + \cdots + a_rL(\boldsymbol{x}_r).$$

Theorem 1.1. *A linear function $L\colon \mathbf{R}^m \to \mathbf{R}^n$ is completely determined by its effect on the standard basis vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ of $\mathbf{R}^m$. An arbitrary choice of vectors $L(\boldsymbol{e}_1), \ldots, L(\boldsymbol{e}_m)$ of $\mathbf{R}^n$ determines a linear function from $\mathbf{R}^m$ to $\mathbf{R}^n$.*

Proof. Given any vector $\boldsymbol{x} \in \mathbf{R}^m$, we can express it as a unique linear combination $\sum_{i=1}^m \alpha_i \boldsymbol{e}_i$ of the basis vectors. By the linearity of L, $L(\boldsymbol{x}) = \sum_i \alpha_i L(\boldsymbol{e}_i)$ which is completely specified by $L(\boldsymbol{e}_1), \ldots, L(\boldsymbol{e}_m)$.

Let $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m$ be any vectors in $\mathbf{R}^n$. Define a map L from $\mathbf{R}^m$ to $\mathbf{R}^n$ as follows: for $\boldsymbol{x} = \sum_{i=1}^m \alpha_i \boldsymbol{e}_i \in \mathbf{R}^m$, $L(\boldsymbol{x}) = \sum_{i=1}^m \alpha_i \boldsymbol{b}_i$. Then $L(\boldsymbol{e}_i) = \boldsymbol{b}_i$ for all $1 \leq i \leq m$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^m$ and all $a, b \in \mathbf{R}$:

$$L(a\boldsymbol{x} + b\boldsymbol{y}) = \sum_{i=1}^m (ax_i + by_i)\boldsymbol{b}_i = a\sum_i x_i\boldsymbol{b}_i + b\sum_i y_i\boldsymbol{b}_i = aL(\boldsymbol{x}) + bL(\boldsymbol{y})$$

$\square$

Note that the domain of definition of a linear function must be a vector space. An non-linear function can be defined on a subset of a vector space.

1.2 Image and Kernel of a Linear Function

The image $\mathrm{Im}(L)$ of a linear function $L\colon \mathbf{R}^m \to \mathbf{R}^n$ is the set of vectors in $\mathbf{R}^n$ that L maps $\mathbf{R}^m$ to. In symbols, $\mathrm{Im}(L) := \{L(\boldsymbol{x}) \in \mathbf{R}^n \colon \boldsymbol{x} \in \mathbf{R}^m\}$. The kernel $\mathrm{Ker}(L)$ of L is the set of vectors in $\mathbf{R}^m$ that L maps to the zero vector in $\mathbf{R}^n$: $\mathrm{Ker}(L) := \{\boldsymbol{x} \in \mathbf{R}^m \colon L(\boldsymbol{x}) = \mathbf{0}_n\}$.