

ML Notes

Somnath Sikdar

December 20, 2020

Contents

1	Bias-Variance Decomposition	2
---	-----------------------------	---

Chapter 1

Bias-Variance Decomposition

The bias-variance decomposition is the decomposition of the generalization error of a learning algorithm into the sum of the bias, the variance and an irreducible error term.

Consider a setting where a response variable Y is related to a set of predictor variables $X \in \mathbf{R}^p$ as follows: $Y = f(X) + \varepsilon$, where $f: \mathbf{R}^p \rightarrow \mathbf{R}$ is some deterministic function and ε is white noise. That is, $\mathbf{E}[\varepsilon|X = x] = 0$ and $\text{Var}(\varepsilon|X = x) = \sigma^2$, for some variance σ^2 . Let us assume that there is an underlying data distribution $\mathcal{D}(X, Y)$ which is unknown to us. We have an algorithm that, given an n -sized training set $S = \{(x_1, y_1), \dots, (x_n, y_n)\}$ sampled iid from this distribution \mathcal{D} , yields \hat{f}_S , which is an approximation to the true function f . We assume that the algorithm itself is deterministic: that is, given the same n -sized sample S , it produces the same output \hat{f}_S .

What is the expected generalization error of this algorithm? This is defined as:

$$\mathbf{E}_{S \sim \mathcal{D}^n, (X, Y) \sim \mathcal{D}} \left[(Y - \hat{f}_S(X))^2 \right]. \quad (1.1)$$

This is the expected error when we input the random sample S to our algorithm to obtain \hat{f}_S and then evaluate it on the random point (X, Y) . We may write this expected error as:

$$\mathbf{E}_{S \sim \mathcal{D}^n, (X, Y) \sim \mathcal{D}} \left[(Y - \hat{f}_S(X))^2 \right] = \int \mathbf{E}_{(X, Y) \sim \mathcal{D}} \left[(Y - \hat{f}_S(X))^2 | S \right] \cdot \mathbf{Pr}_{\mathcal{D}^n} \{S\} dS. \quad (1.2)$$

We will first work with the expectation term on the right-hand side of the above equation. That is, we will assume that the sample S is fixed and then calculate the expected error wrt the approximation \hat{f}_S . In order to simplify our notation a bit, we will index the expectation operator with the random variable to indicate which distribution is being referred to and not use the symbol for the distribution anymore.

Let us write $(Y - \hat{f}_S(X))^2$ as $(Y - f(X) + f(X) - \hat{f}_S(X))^2$. Expanding, we get:

$$\begin{aligned} \mathbf{E}_{(X, Y)} \left[(Y - \hat{f}_S(X))^2 | S \right] &= \mathbf{E}_{(X, Y)} \left[(f(X) - \hat{f}_S(X))^2 | S \right] + \\ &\quad \mathbf{E}_{(X, Y)} \left[(Y - f(X))^2 | S \right] + \\ &\quad \mathbf{E}_{(X, Y)} \left[2(Y - f(X))(f(X) - \hat{f}_S(X)) | S \right] \end{aligned} \quad (1.3)$$

Consider the last term on the right-hand side. We claim that this is 0. Indeed, we may write $\mathbf{E}_{(X,Y)} \left[2(Y - f(X))(f(X) - \hat{f}_S(X)) | S \right]$ as

$$\begin{aligned} &= \iint 2 \cdot (y - f(x)) \cdot (f(x) - \hat{f}_S(x)) \cdot p_{X,Y}(x, y) dy dx \\ &= \iint 2 \cdot (y - f(x)) \cdot (f(x) - \hat{f}_S(x)) \cdot p_{Y|X}(y|x) \cdot p_X(x) dy dx \\ &= \int 2 \cdot (f(x) - \hat{f}_S(x)) \cdot \mathbf{E}_{Y|X} [Y - f(x) | X = x] \cdot p_X(x) dx. \end{aligned}$$

Now we know that $\mathbf{E}_{Y|X} [Y - f(x) | X = x] = \mathbf{E}_{Y|X} [\varepsilon | X = x] = 0$. Hence this whole expression evaluates to 0 as claimed.

Next consider the second term $\mathbf{E}_{(X,Y)} [(Y - f(X))^2 | S]$ of Equation (1.3). Since the term $(Y - f(X))^2$ does not depend on the sample S that is chosen, this further simplifies to $\mathbf{E}_{(X,Y)} [(Y - f(X))^2]$. This is simply the variance of the error term and is equal to σ^2 . Finally, consider the first term of Equation (1.3) which is $\mathbf{E}_{(X,Y)} [(f(X) - \hat{f}_S(X))^2 | S]$. We use the trick of adding and subtracting as before. This time around, we add and subtract the term $g(X) := \mathbf{E}_S [\hat{f}_S(X)]$ to obtain:

$$\begin{aligned} (f(X) - g(X) + g(X) - \hat{f}_S(X))^2 &= (f(X) - g(X))^2 + (g(X) - \hat{f}_S(X))^2 + \\ &\quad 2 \cdot (f(X) - g(X)) \cdot (g(X) - \hat{f}_S(X)). \end{aligned} \tag{1.4}$$

We will evaluate each of these terms by directly plugging them in the expression in Equation (1.2). Let's examine the first term $(f(X) - g(X))^2$. Plugging this in the said expression, we see that we have to evaluate:

$$\begin{aligned} \int \mathbf{E}_{(X,Y)} [(f(X) - g(X))^2 | S] \mathbf{Pr}_{S \sim \mathcal{D}^n} \{S\} dS &= \mathbf{E}_{(X,Y)} [(f(X) - g(X))^2] \int \mathbf{Pr}_{S \sim \mathcal{D}^n} \{S\} dS \\ &= \mathbf{E}_{(X,Y)} [(f(X) - g(X))^2]. \end{aligned} \tag{1.5}$$

The first equality holds because neither $f(X)$ nor $g(X)$ depends on the sample S chosen. The resulting expression is the expected squared bias of the estimator \hat{f}_S .

Forging ahead, we evaluate the next term which is $(g(X) - \hat{f}_S(X))^2$.

$$\int \mathbf{E}_{(X,Y)} [(g(X) - \hat{f}_S(X))^2 | S] \mathbf{Pr}_{S \sim \mathcal{D}^n} \{S\} dS = \mathbf{E}_{S, (X,Y)} [(g(X) - \hat{f}_S(X))^2]. \tag{1.6}$$

This term is the variance of the estimator \hat{f}_S obtained by our algorithm.

Finally, we evaluate the term $2 \cdot (f(X) - g(X)) \cdot (g(X) - \hat{f}_S(X))$. We now have to evaluate this integral:

$$\int \mathbf{E}_{(X,Y)} [2 \cdot (f(X) - g(X)) \cdot (g(X) - \hat{f}_S(X)) | S] \mathbf{Pr}_{S \sim \mathcal{D}^n} \{S\} dS \tag{1.7}$$

Recall that this is just the expectation wrt the distribution of the random sample S and the random data point (X, Y) at which we want to evaluate the predictor \hat{f}_S output by our

algorithm. Instead of first conditioning on S , we can condition on the random data point (X, Y) and write:

$$\iint \mathbf{E}_S \left[2 \cdot (f(x) - g(x)) \cdot (g(x) - \hat{f}_S(x)) | X = x, Y = y \right] p_{X,Y}(x, y) dy dx. \quad (1.8)$$

Now the term $f(x) - g(x)$ does not depend on S and hence the expression inside the integral evaluates to $2 \cdot (f(x) - g(x)) \cdot \mathbf{E}_S [g(x) - \hat{f}_S(x)]$. Since $\mathbf{E}_S [g(x) - \hat{f}_S(x)] = 0$, the above integral evaluates to 0.

To summarize, we may write the expected generalization error of our algorithm as:

$$\begin{aligned} \mathbf{E}_{S \sim \mathcal{D}^n, (X, Y) \sim \mathcal{D}} \left[(Y - \hat{f}_S(X))^2 \right] &= \underbrace{\mathbf{E}_{(X, Y)} \left[(\mathbf{E}_S [\hat{f}_S(X)] - f(X))^2 \right]}_{\text{expected squared bias}} \\ &\quad + \underbrace{\mathbf{E}_{S, (X, Y)} \left[(\hat{f}_S(X) - \mathbf{E}_S [\hat{f}_S(X)])^2 \right]}_{\text{variance}} \\ &\quad + \underbrace{\sigma^2}_{\text{irreducible error}}. \end{aligned}$$