Exponential Lower-bounds via Exponential Sums

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Outline

NP Problems: Is Brute Force Optimal?

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- **ETH** says NO! (informally)

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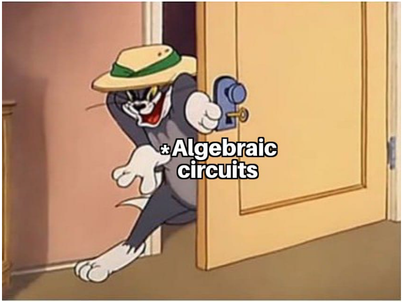
- What is even the *Computational Model* in that setting?
- Hence Algebraic Circuit enters the picture



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- Complexity Measure: $\#$ Edges in the circuit

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We believe these are $\geq \Omega(n)$

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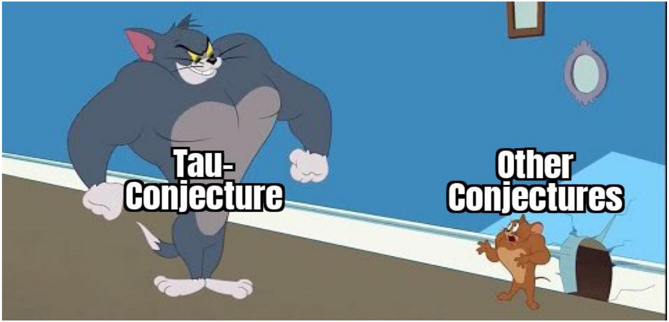

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- Does τ-conjecture imply some lower bound?
- [Bürgisser'07] showed super-polynomial lowerbound on $P_{m,n}$ assuming τ-conjecture

Main result

Conditional Optimal Lower Bound

Assuming τ-conjecture $\exists$ a polynomial family $P_{n,m}(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ of exponential sum which requires $2^{\Omega(n)} poly(m)$ size circuit.

Bürgisser's Proof analysis

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$$\prod_{i=1}^{n}(x+i)$$ has easy coefficients

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It has poly($\log n$) size circuit

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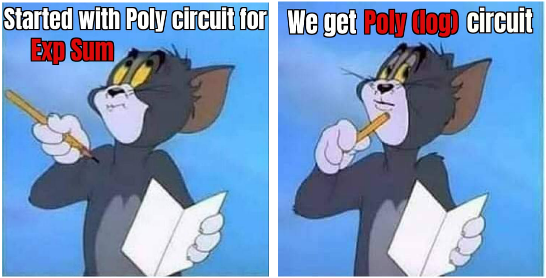

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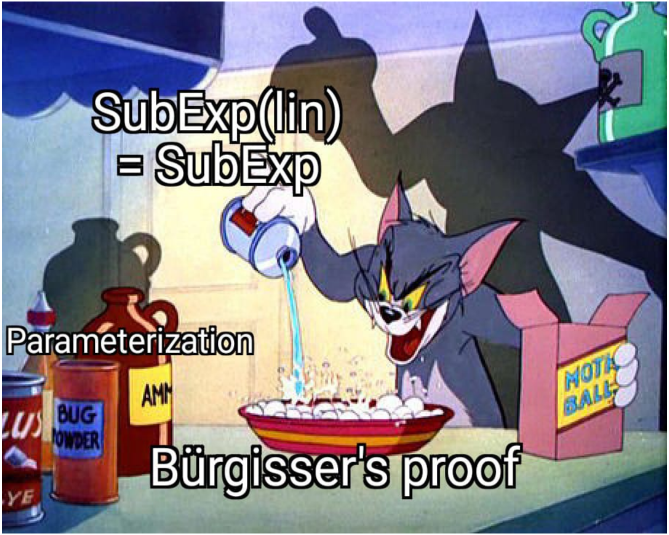
But we can improve it!



What's the Magic

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where

$$S_{n,k}(X_1,\ldots,X_n)=\sum_{S\subseteq[n],|S|=k}\prod_{i\in S}X_i$$

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3 level

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2 level

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1 level

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0 level

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such that,

$$x \in A \iff |\{y \in \{0,1\}^{\ell(|x|)} : \langle x, y \rangle \in B\}| > f(x).$$

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The *linear counting hierarchy* is $\text{CH}_{\text{lin}} K := \bigcup_{k \geq 0} \text{C-lin}_k K$

Linear Counting Hierarchy ($\mathsf{CH}_{\mathrm{lin}}$)

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Hence Exponential sum is EASY $\implies$ $\mathrm{CH}_{\mathrm{lin}}$ collapses

$$\implies \prod_{i=1}^{n}(x+i) \text{ is EASY}$$

Permanent

Given a variable matrix

$$\mathbf{X} := \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \vdots & & \ddots & \vdots \\ X_{n1} & X_{s2} & \dots & X_{nn} \end{bmatrix}_{n\times n}$$

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$$Per_n(\mathbf{X}) := \sum_{\sigma \in S_n} X_{1,\sigma_1} X_{2,\sigma_2} \dots X_{n,\sigma_n}$$

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Other Results

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2. We achieved completeness result for parameterized valiant classes.

Open Problems

1. Can we established conditional truly exponential (ie, $2^{\Omega(n)}poly(n)$) lower bound for Per_n? (Unconditional will be better :))

2. Can we get Lower Bounds for NP from tau-conjecture? (We don't know even super-polynomial bound)