1 Introduction

Throughout we will work over the field of complex numbers $\mathbb{C}$. Let $G = SL(n)$. There is a natural action of G on $\mathbb{C}^n$.

Let T be the subgroup of diagonal matrices in G. Consider the Grasmann variety $G_{r,n} \subseteq \mathbb{P}(\bigwedge^r \mathbb{C}^n)$ of r-dimensional subspaces of $\mathbb{C}^n$. We view $G_{r,n}$ as a projective subvariety of $\mathbb{P}(\bigwedge^r \mathbb{C}^n)$ via the Plücker embedding (see Section 2 for details).

Denote by $\mathcal{L}(\omega_r)$ the hyperplane bundle $\mathcal{O}(1)$ on $\mathbb{P}(\bigwedge^r \mathbb{C}^n)$ restricted to $G_{r,n}$. $\mathcal{L}(\omega_r)$ is a T-equivariant ample line bundle on $G_{r,n}$. So a natural question is to understand the T-quotient of $G_{r,n}$ with respect to this T-linearized bundle. To state the problem, we recall some definitions from GIT, [MFK94]. A point $p \in G_{r,n}$ is said to be $\mathcal{L}(\omega_r)$-semistable if there is a T-invariant section of a positive power of $\mathcal{L}(\omega_r)$ which does not vanish at p. We denote by $(G_{r,n})^{ss}_T\mathcal{L}(\omega_r)$ the $\mathcal{L}(\omega_r)$-semistable points of $G_{r,n}$. A $\mathcal{L}(\omega_r)$-semistable point is said to be stable if its orbit is closed in $(G_{r,n})^{ss}_T\mathcal{L}(\omega_r)$ and it has a finite stabilizer. Let $(G_{r,n})^{s}_T\mathcal{L}(\omega_r)$ denote the $\mathcal{L}(\omega_r)$-stable points. A long standing open question is to understand the geometry of the T-quotient, $T\backslash\backslash(G_{r,n})^{ss}_T\mathcal{L}(\omega_r)$,[GGMS87]. This problem has been studied in greater generality for flag varieties other than the Grassmannian and even in the case when G is a classical group of type different from A. However the problem remains far from being understood. For a comprehensive account of what is known about this problem see the introduction in [BSS20] and [Kan14]. In this paper we only deal with the type A case.

In [Sko93] Skorobogatov showed that when r and n are coprime then semistability is the same as stability. This was proved independently by Kannan [Kan98]. We will assume this is the case throughout the paper since it makes the question easier. In [HMSV05] the authors studied the problem in the case $G_{2,n}$, n even and gave a description of the generators and relations for the torus quotient. In [BSS20] the authors studied the same problem when n is odd and showed the projective normality of the torus quotient of $T\backslash\backslash(G_{2,n})^{ss}_T\mathcal{L}(\omega_2)$. Since an explicit description of the torus quotient of the Grassmannian is a difficult problem another fruitful approach has been studying the GIT quotient for Schubert varieties in $G_{r,n}$, initiated in [KP09b] and [KP09a].

In the type A case when $G = SL(n, \mathbb{C})$, and Q is any parabolic subgroup of G, Howard [How05] considered the problem of determining which line bundle on G/Q descends to an ample line bundle of the GIT quotientof G/Q by T. In [KP09b] Kannan and Sardar studied torus quotients of Schubert varieties in $G_{r,n}$. They showed in the case r, n coprime that $G_{r,n}$ has a unique minimal Schubert variety $X(w_{r,n})$ admitting semistable points with respect to the T-linearized bundle $\mathcal{L}(\omega_r)$.

1.1 Our results

In this paper we show that $T\backslash\backslash(G_{3,7})^{ss}_T\mathcal{L}(\omega_r)$ is projectively normal.To prove the theorem we show that the ring of T-invariants $\oplus_m H^0(G_{3,7,,}\mathcal{L}(m\omega_3))^T$ is generated in degree 7. The line bundle $\mathcal{L}(7\omega_3)$ descends to the GIT quotient and gives an embedding of the quotient, for which the coordinate ring is generated in degree 1 proving projective normality. Our proof technique to show that the ring of invariants is generated in degree $m = 7$ is computational. We show that the T invariants are in bijection with lattice points of a polyhedral cone $P = \{x|Ax \geq 0\} \subset \mathbb{R}^d$, for an appropriately defined rational matrix A and d. We compute a Hilbert basis of the monoid $\mathbb{Z}^d_+ \cap P$ (see Section 4 for definitions). We observe that the Hilbert basis has points which are in bijection with some T-invariants of degree 7, 14 and 21.We call these invariants in the Hilbert basis. We then show that the invariants of degree 14 and 21 in the Hilbert basis are in fact in the polynomial span of invariants of degree 7 in the Hilbert basis modulo Plücker relations. This shows that the coordinate ring of the GIT quotient for the polarized bundle $\tilde{\mathcal{L}}(7\omega_3)$ (the descent bundle) is generated in degree

1.

The same proof techniques shows that when (r, n) are coprime, the Hilbert basis of T-invariants has elements of degree $n, 2n, \ldots, (n-r)n$. Whether the invariants of higher degree in the Hilbert basis of invariants are in the polynomial span of invariants of degree n modulo Plücker relations remains an intriguing question.

We consider the question of explicitly describing the T-quotient of the Schubert variety indexed by the Weyl group element [367] in $G_{3,7}$ (see Section 2 for the definition and notation) and make some progress.

1.2 Organization

In the next section we set up the notation and recall important definitions and known results which we will need. In Section **??** we show the bijection between T invariant polynomials in the coordinate ring of the Grassmannian and lattice points of a pointed cone. In Section4 we recall the definition of the Hilbert basis of a lattice points in a pointed cone. We prove that the invariants corresponding to the Hilbert basis generate the ring of T-invariant polynomials in the coordinate ring of the Grassmannian. In Section 6 we restrict to the case of $G_{3,7}$. We describe the Hilbert basis of T-invariants in this particular example. We then give computational evidence of the fact that T-invariant polynomials of degree 14 and 21 in the Hilbert basis are in the algebraic span of T-invariant polynomials of degree 7 in the Hilbert basis. For the Schubert variety corresponding to the Weyl group element [367] in $G_{3,7}$ we describe the Hilbert basis of invariants and show by explicit calculations that the ring of non zero T-invariant polynomials on this Schubert variety is generated in degree 7. In Section**??** we prove that the GIT quotient $T\backslash\backslash(G_{3,7})^{ss}_{T}\mathcal{L}(\omega_3)$ is projectively normal.

2 Background and notations

We first recall some well known results about the Grassmannian. The proofs of these statements can be found in [LR07] and [Ses]. Let $G = SL(n)$ and consider the natural action of G in $\mathbb{C}^n$. We fix a standard basis of $\mathbb{C}^n$, $e_1, \ldots, e_n$. We write the elements of G with respect to this basis.

Let T be the maximal torus of diagonal matrices in G and B be the Borel subgroup of upper triangular matrices. B^- is the group of lower triangular matrices and $T = B \cap B^-$. The Weyl group of G, W, is the group of permutations of $[n]$ and we have the Bruhat decomposition $G = \bigcup_w BwB$, where w runs over all permutations of W.

Let $I_{r,n} := \{\underline{\mathbf{i}} = (i_1, i_2, \ldots, i_r) | 1 \leq i_1 < i_2 \cdots < i_r \leq n\}$ be the set of all strictly increasing sequences of legth r with entries in $[n]$. A canonical basis of $\bigwedge^r \mathbb{C}^n$ is given by $\{e_{\underline{\mathbf{i}}} = e_{i_1} \wedge \ldots \wedge e_{i_r}, \underline{\mathbf{i}} \in I_{r,n}\}$. We view the Grassmannian as a subvariety of $\mathbb{P}(\bigwedge^r \mathbb{C}^n)$, given by sending an r-dimensional subspace of $\mathbb{C}^n$ spanned by vectors $v_1, v_2, \ldots, v_r$ to the class $[v_1 \wedge v_2 \wedge \cdots v_r] \in \mathbb{P}(\wedge^r \mathbb{C}^n)$. The Grassmannian can be identified with the orbit of G/P_r, where P_r is the maximal parabolic subgroup of G fixing the subspace spanned by $[e_1, e_2, \ldots, e_r]$. The Weyl group of P_r, W_{P_r}, is a subgroup of W. The coset representatives of W/W_{P_r} of minimal length, W^{P_r}, can be identified with $I_{r,n}$, with the coset representative w corresponding to the subspace spanned by $[e_{w(1)}, \ldots, e_{w(r)}]$. Via this identification the point $e_w = e_{w(1)} \wedge e_{w(2)} \cdots \wedge e_{w(r)}$ in the Grassmannian is a T fixed point of $G_{r,n}$. The B orbit closure of e_w is called a Schubert variety in $G_{r,n}$ and we denote it by $X(w)$.

Denote by $\{p_{\underline{\mathbf{i}}} | \underline{\mathbf{i}} \in I_{r,n}\}$ the dual basis of the canonical basis in $(\bigwedge^r \mathbb{C}^n)^*$. The $p_{\underline{\mathbf{i}}}$ are called Plücker coordinates on the Grassmannian. The homogeneous coordinate ring of the Grassmannian $\mathbb{C}[G_{r,n}]$ is the quotient of $\mathbb{C}[p_{\underline{\mathbf{i}}} | \underline{\mathbf{i}} \in I_{r,n}]$ by the ideal of polynomials $I(G_{r,n})$ vanishing on the Grassmannian for its embedding in $\mathbb{P}(\bigwedge^r \mathbb{C}^n)$.

We define a partial order "$\leq$" on $I_{r,n}$ by defining $\underline{\mathbf{i}} \leq \underline{\mathbf{j}} \Leftrightarrow i_t \leq j_t$ for all $t = 1, 2, \cdots, r$. The set $I_{r,n}$ with this partial order is called the Brühat poset. It will be useful to refine the partial order to a total order which we will also denote by the same symbol. Then there is a natural lexicographic order on $\mathbb{C}[\bigwedge^r \mathbb{C}^n]$ which is a monomial order.

A monomial $p_{\underline{\mathbf{i}}^1} p_{\underline{\mathbf{i}}^2} \cdots p_{\underline{\mathbf{i}}^s}$ of degree s in the Plčker coordinates is said to be standard in $\mathbb{C}[\bigwedge^r \mathbb{C}^n]$ iff $\underline{\mathbf{i}}^1 \leq \underline{\mathbf{i}}^2 \cdots \leq \underline{\mathbf{i}}^s$. If a product of monomials $p_{\underline{\mathbf{i}}} p_{\underline{\mathbf{j}}}$ is not standard then it is known that

$$p_{\underline{\mathbf{i}}} p_{\underline{\mathbf{j}}} = p_{\underline{\mathbf{i}} \cup \underline{\mathbf{j}}} p_{\underline{\mathbf{i}} \cap \underline{\mathbf{j}}} + \text{other quadratic terms} \quad \text{modulo} \quad I(G_{r,n})$$

Here $(\underline{\mathbf{i}} \cup \underline{\mathbf{j}})_t = max(i_t, j_t), t = 1, \ldots, r$ and $(\underline{\mathbf{i}} \cap \underline{\mathbf{j}})_t = min(i_t, j_t), t = 1, \ldots, r$. Furthermore the other quadratic terms in the above expression are products of monomials which are strictly bigger than $p_{\underline{\mathbf{i}}} p_{\underline{\mathbf{j}}}$ in the monomial order on $\mathbb{C}[\bigwedge^r \mathbb{C}^n]$. Quadratic terms in the above expression which are nonstandard can be refined again using the same rule and, since the terms are increasing in lex order, we see that any non standard monomial has an expression in terms of standard monomials. The difference between the two expressions goes by the name "straightening law" and this is an element of the ideal $I(G_{r,n})$. As a result we have:

Theorem 1. *i Standard monomials span the coordinate ring of the Grassmannian.*

ii The ideal $I(G_{r,n})$ is generated by straightening laws.

If $w \in W^{P_r}$ corresponds to the element $\underline{\mathbf{i}} \in I_{r,n}$, the Schubert variety $X(w) \subset G_{r,n}$ is given by the vanishing of the $p_{\underline{\mathbf{j}}}, \underline{\mathbf{j}} \not\leq \underline{\mathbf{i}}$.

2.1 T-action on Plücker coordinates

Since we are interested in the T quotient of the Grassmannian, it is natural to consider the action of T on the coordinate ring $\mathbb{C}[G_{r,n}]$. A diagonal matrix t with entries $(t_1, t_2, \ldots, t_n)$ acts on $p_{\underline{\mathbf{i}}}$, $\underline{\mathbf{i}} = (i_1, \ldots, i_r)$, by scaling it by $(t_{i_1} t_{i_2} \ldots t_{i_r})^{-1}$. It is clear that the action of t on a monomial in the Plücker coordinates $p_{\underline{\mathbf{i}}^1} p_{\underline{\mathbf{i}}^2} \cdots p_{\underline{\mathbf{i}}^s}$ is $(t_1^{\#1} t_2^{\#2} \ldots t_n^{\#n})^{-1}$ where $\#t$ is the number of k such that t occurs in $\underline{\mathbf{i}}^k$. It is clear that the monomial is invariant under T if for each $t \in [n]$, the number of k for which t occurs in $\underline{\mathbf{i}}^k$ is the same. Now assume that $(r, n) = 1$. It is clear from the discussion so far that if $p_{\underline{\mathbf{i}}^1} p_{\underline{\mathbf{i}}^2} \cdots p_{\underline{\mathbf{i}}^s}$ is invariant, s must be a multiple of n. It is easy to see that there are monomials of degree n in the Plücker coordinates which are T-invariant, see [BSS20] for example. It follows:

Theorem 2. *Let $R(m) = H^0(G_{r,n}, \mathcal{L}(nm))^T$. Then $T\backslash\backslash(G_{r,n})^{ss}_T \mathcal{L}(\omega_r) = Proj(\oplus_m R(m))$.*

Since $H^0(G_{r,n}, \mathcal{L}(nm))$ is spanned by standard monomials, the homogeneous coordinate ring of the T-quotient, $R = \oplus_m R(m)$, is spanned by T-invariants which are standard. We call these standard T-invariants. Now standardness is defined by the "$\leq$" relation on the Brühat poset. So a set of T-invariant monomials which are products of Plücker coordinates with support in a maximal chain in the Bruhat poset is naturally standard. Taking the union of such T-invariant monomials over all maximal chains in the Bruhat poset gives us a generating set for the ring R. This follows since every standard monomial which is T-invariant Plücker coordinates with are pairwise comparable and the Bruhat poset has a minimal element and a maximal element. This is the approach we take.

We first formulate the question of obtaining T-invariant monomials with support in a fixed maximal chain of the Bruhat poset.

3 T invariants and Hilbert basis of lattices in pointed cones

We start with few notations and observations.

Let $\lambda = (\lambda_1^{r_1}, ..., \lambda_k^{r_k})$ be column shape of first graded component ring of T-invariants of Flag variety. Let $I(\lambda, n) = \bigcup_{i=1}^{k} I(\lambda_i, n)$. Bruhat order on set $I(\lambda, n)$ is defined as follows: For $\underline{p} \in I(\lambda_i, n), \underline{q} \in I(\lambda_j, n)$ define $\underline{p} \leq \underline{q}$ if $\lambda_i \geq \lambda_j$ and for $1 \leq l \leq \lambda_j$ we have $\underline{p}_l \leq \underline{q}_l$. Poset $(I(\lambda, n), \leq)$ is called Bruhat poset.

Let $\mathcal{C} = \underline{p}_1 \leq \cdots \leq \underline{p}_l$ be a chain in Bruhat subposet. We associate with every standard monomial, $x_{\underline{p}_1}^{a_1} x_{\underline{p}_2}^{a_2} \cdots x_{\underline{p}_l}^{a_l}$, a semistandard Young tableau $S$with column $\underline{p}_i$ appearing a_i times, note that some of the a_i's may be 0. Conversely, to every semistandard tableau with columns in $\mathcal{C}$ one can associate a standard monomial. Denote by $\boldsymbol{v}'_S$ the degree vector $(a_1, \cdots, a_l)$. Observe that if T be another tableau whose columns are in $\mathcal{C}$ then $\boldsymbol{v}'_{ST} = \boldsymbol{v}'_S + \boldsymbol{v}'_T$.

Denote by $STab(\lambda, n, \mathcal{C})$ the set of all T-invariant semistandard tableaux with each column is in $\mathcal{C}$, filling from $[n]$ and column shape $(\lambda_1^{r_1 d}, \cdots, \lambda_k^{r_k d})$, for some positive integer d. Weight vector of tableau S with filling from $[n]$ is n-dimentional vector $wt(S) := (b_1, ..., b_n)$ such that i appear b_i times in S. Weight vector of any T-invariant tableau of shape μ is $\frac{|\mu|}{n}\mathbf{1}$. Denote $\boldsymbol{w} := \frac{|\lambda|}{n}\mathbf{1}$

Define the weight matrix $A'_{\mathcal{C}}$ for chain $\mathcal{C}$ whose entries are indexed by the set $[n] \times \mathcal{C}$. For $(i, \underline{p}) \in [n] \times \mathcal{C}$,

$$A'_{\mathcal{C}}(i, \underline{p}) = \begin{cases} 1 & i \in \underline{p} \\ 0 & i \notin \underline{p} \end{cases}$$

Note that the $\underline{p}^{th}$ column of $A'_{\mathcal{C}}$ is $wt(\underline{p})$. Denote by $A_{\mathcal{C}}$ an augmented weight matrix $A_{\mathcal{C}} := [A'_{\mathcal{C}}| - \boldsymbol{w}]$.

Define shape matrix $B'_{\mathcal{C}}$ for chain $\mathcal{C}$ whose entries are indexed by the set $[k] \times \mathcal{C}$. Recall k is number of distinct lengths of columns in $I(\lambda, n)$. $B'_{\mathcal{C}}$ is defined as follows:

$$B'_{\mathcal{C}}(i, \underline{p}) = \begin{cases} 1 & |\underline{p}| = \lambda_i \\ 0 & |\underline{p}| \neq \lambda_i \end{cases}$$

Where $|\underline{p}|$ is length of column $\underline{p}$. Let $\boldsymbol{r} = (r_1, \cdots, r_k)$. Denote by $B_{\mathcal{C}}$ the augmented shape matrix $B_{\mathcal{C}} := [B'_{\mathcal{C}}| - \boldsymbol{r}]$.

Following two observations justify names of matrices $A'_{\mathcal{C}}$ and $B'_{\mathcal{C}}$.

Observation 3. *Let S be a tableau whose columns are in $\mathcal{C}$. For $i \in [n]$ we have $A'_{\mathcal{C}}\boldsymbol{v}'_S(i)$ is number of columns in S which contain i. Hence we have $wt(S) = A'\boldsymbol{v}'_S$.*

Proof.

$$A'_{\mathcal{C}}\boldsymbol{v}'_S(i) = \sum_{\underline{p}_j \in \mathcal{C}} A'_{\mathcal{C}}(i, \underline{p}_j)\boldsymbol{v}'_S(j) = \sum_{i \in \underline{p}_j} \boldsymbol{v}'_S(j).$$

□

Observation 4. *Let S be a tableau whose columns are in $\mathcal{C}$. For $i \in [k]$ we have $B'_{\mathcal{C}}\boldsymbol{v}'_S(i)$ is number of columns in S of length λ_i. Hence the column shape of tableau S is $(\lambda_1^{B'_{\mathcal{C}}\boldsymbol{v}'_S(1)}, \cdots, \lambda_k^{B'_{\mathcal{C}}\boldsymbol{v}'_S(k)})$.*

Proof.

$$B'_{\mathcal{C}}\boldsymbol{v}'_S(i) = \sum_{\underline{p}_i \in \mathcal{C}} B'_{\mathcal{C}}(i, \underline{p}_i)\boldsymbol{v}'_S(i) = \sum_{|\underline{p}_i| = \lambda_i} \boldsymbol{v}'_S(i).$$

□

If $S \in STab(\lambda, n, \mathcal{C})$ then there is positive integer d such that $wt(S) = d\boldsymbol{w}$ and shape of S is $(\lambda_1^{r_1 d}, \cdots, \lambda_k^{r_k d})$. For such tableau S associate vector $\boldsymbol{v}_S := (\boldsymbol{v}'_S, d)$, the vector of length $l+1$. From above two observations we have $\boldsymbol{v}_S \in ker(A_{\mathcal{C}}) \cap ker(B_{\mathcal{C}})$. Conversly we have:

Observation 5. *Let $\boldsymbol{x} = (\boldsymbol{x}', d)$ be a nonzero non-negative integer vector in $ker(A_{\mathcal{C}}) \cap ker(B_{\mathcal{C}})$ then there exist $T_x \in STab(\lambda, n, \mathcal{C})$ such that $\boldsymbol{x} = \boldsymbol{v}_{T_{\boldsymbol{x}}}$.*

Proof. Define tableau S such that, $\underline{p}_i \in \mathcal{C}$ appears x_i times. Hence we have $\boldsymbol{x}' = \boldsymbol{v}'_S$. Note since columns for S are from chain, S semistandard. It remains to show that $wt(S) = d\boldsymbol{w}$ and column shape of S is $(\lambda_1^{r_1 d} \cdots \lambda_k^{r_k d})$. This follows from calculations below.

$$A_{\mathcal{C}}\boldsymbol{x} = 0 \implies [A'_{\mathcal{C}}| - \boldsymbol{w}](\boldsymbol{x}', d) = 0 \implies A'_{\mathcal{C}}\boldsymbol{x}' = d\boldsymbol{w} \implies wt(S) = d\boldsymbol{w}.$$

$$B_{\mathcal{C}}\boldsymbol{x} = 0 \implies [B'_{\mathcal{C}}| - \boldsymbol{r}](\boldsymbol{x}', d) = 0 \implies B'_{\mathcal{C}}\boldsymbol{x}' = d\boldsymbol{r}.$$

□

Let $P_{\mathcal{C}} = ker(A_{\mathcal{C}}) \cap ker(B_{\mathcal{C}}) \cap \mathbb{R}^{l+1}_{\geq 0}$. We have following corollary

Corollary 6. *There is bijection between nonzero integral points in $P_{\mathcal{C}}$ and $STab(\lambda, n, \mathcal{C})$, given by $S \mapsto \boldsymbol{v}_S$, $\boldsymbol{x} \mapsto T_x$. We also have $\boldsymbol{v}_{ST} = \boldsymbol{v}_S + \boldsymbol{v}_T$*

3.1 Examples

3.1.1 The Grassmannian $G_{3,7}$

For the grassmannian $G_{3,7}$ the shape of first graded component of ring of T-invariants is $\lambda = (3^7)$ then $I(\lambda, n) = (I(3, 7))$ of size 35. There are [465] chains in Bruhat poset $I(3, 7)$ with lowest element $[1, 2, 3]$ and top element $[5, 6, 7]$. Also any maximal chain contain 13 elements. For fix maximal chain $\mathcal{C}$ below, the marix $A'_{\mathcal{C}}$ is of order 7×13 and $\boldsymbol{w} = \frac{|\lambda|}{n}\boldsymbol{1} = 3.\boldsymbol{1}$. The matrix $B'_{\mathcal{C}}$ is of order 1×13. We fix following maximal chain:

$$\mathcal{C} = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 4\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 5\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 6\\ \hline\end{array} \leq \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline 6\\ \hline\end{array} \leq \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 6\\ \hline\end{array} \leq \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 2\\ \hline 4\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 3\\ \hline 4\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 3\\ \hline 5\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 4\\ \hline 5\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 4\\ \hline 6\\ \hline 7\\ \hline\end{array} \leq \begin{array}{|c|}\hline 5\\ \hline 6\\ \hline 7\\ \hline\end{array}$$

$$A_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \end{pmatrix}$$

Tableau Example in above chain is

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4\\ \hline 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 5 & 5 & 5 & 5 & 5\\ \hline 4 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7\\ \hline\end{array} \qquad \boldsymbol{v}_S = [0, 1, 1, 1, 3, 2, 1, 0, 0, 0, 5, 0, 0, 2]$$

3.1.2 Flag Varieties for $n = 5$

For the flag variety with the shape of first graded component of ring of T-invariants is $\lambda = (3, 2)$ we have $I(\lambda, 5) = I(3, 5) \cup I(2, 5)$ of size [20]. There are [42] chains in Bruhat poset $I(\lambda, 5)$ with lowest element $[1, 2, 3]$ and top element $[4, 5]$. Also any maximal chain contain 10? elements. For fix maximal chain $\mathcal{C}$ below, the marix $A'_{\mathcal{C}}$ is of order 5×10 and $\boldsymbol{w} = \frac{|\lambda|}{n}\mathbf{1} = \mathbf{1}$. The matrix $B'_{\mathcal{C}}$ is of order 2×10. $A_{\mathcal{C}}$ and $B_{\mathcal{C}}$ are of order 5×11 and 2×11 respectively. We fix following maximal chain:

$$\mathcal{C} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \leq \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} \leq \begin{array}{|c|}\hline 1\\\hline 2\\\hline 5\\\hline\end{array} \leq \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \leq \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \leq \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} \leq \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array} \leq \begin{array}{|c|}\hline 2\\\hline 5\\\hline\end{array} \leq \begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} \leq \begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$$

$$A_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

$$B_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

$$S = \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & 5\\\hline 4 & \multicolumn{1}{c}{}\\\cline{1-1}\end{array} \qquad \boldsymbol{v}_S = [0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1]$$

4 Hilbert Basis

In this section we use ideas from polyhedral combinatorics to compute the generating set for $STab(\lambda, n, \mathcal{C})$. We begin with some definitions. A standard and wonderful reference is Schrijver's book [Sch98]. To compute the Hilbert basis we used the algorithm given in [Hem02]. We do not go into details of that algorithm, except mention how we used it. Details of that algorithm for the specific problem of computing a generating set of $R = \oplus_m$ are described in a companion paper [DS21]. We continue to use the notation $\mathcal{C}, A_{\mathcal{C}}, B_{\mathcal{C}}$ from the previous section. To simolify notation we use A instead of $A_{\mathcal{C}}$.

Definition 7 (Cone)**.** *A set $P \in \mathbb{R}^n$ is cone if for any $\boldsymbol{x}, \boldsymbol{y} \in P$ and non-negative λ_1, λ_2, we have $\lambda_1\boldsymbol{x} + \lambda_2\boldsymbol{y} \in P$. A cone P is pointed if $P \cap -P = \{0\}$. A cone P is polyhedral if $P = \{\boldsymbol{x} \in \mathbb{R}^n | B\boldsymbol{x} \geq 0\}$ for some matrix B. A set $\{\boldsymbol{g}_1, ...\boldsymbol{g}_k\} \subset P$ is called a (conical) generating set of P if for all $\boldsymbol{x} \in P$ there are non-negative $\lambda_1, ..., \lambda_k$ such that $\boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{g}_i$.*

Definition 8 (Hilbert basis)**.** *Let $P \in \mathbb{R}^n$ be a polyhedral cone with rational generators. We call a finite set $H = \{\boldsymbol{h}_1, ..., \boldsymbol{h}_t\} \subset \mathbb{Z}^n \cap P$ a Hilbert basis of P if for every integral vector $v \in P$ there are non-negative integers $\lambda_1, ..., \lambda_t$ such that $v = \sum_{i=1}^{t} \lambda_i h_i$.*

Every rational polyhedral cone P has a Hilbert basis and if P is pointed cone then there is unique inclusion minimal Hilbert basis [Sch98, Theorem 16.4]. The following corollary is immediate.

Corollary 9. *$P_{\mathcal{C}} = ker(A_{\mathcal{C}}) \cap ker(B_{\mathcal{C}}) \cap \mathbb{R}_+^{r(n-r)}$ is a pointed cone. So $P_{\mathcal{C}}$ has a unique inclusion minimal Hilbert basis.*

Definition 10 (Direct spliting)**.** *We say $T \in STab(\lambda, n, \mathcal{C})$ splits directly if there are two tableaux $X, Y \in STab(\lambda, n, \mathcal{C})$ such that $T = XY$ up to rearranging columns.*

For a maximal chain $\mathcal{C}$, let $H_{\mathcal{C}}$ denote the unique Hilbert basis. The following lemma is now immediate.

Lemma 11. *$T \in STab(\lambda, n, \mathcal{C})$ splits directly iff $\boldsymbol{v}_T \notin H_{\mathcal{C}}$.*

Proof. ($\Longrightarrow$) Let T splits directly then there exist $X, Y \in STab(\lambda, n, \mathcal{C})$ such that $T = XY$ up to rearranging columns. This implies $\boldsymbol{v}_T = \boldsymbol{v}_X + \boldsymbol{v}_Y$ and thus $\boldsymbol{v}_T$ is decomposible hence $\boldsymbol{v}_T \notin H$.

($\Longleftarrow$) If $\boldsymbol{v}_T \notin H$ then there are non-negative integers $\lambda_1, ..., \lambda_t$ such that $\boldsymbol{v}_T = \sum_{i=1}^{t} \lambda_i \boldsymbol{h}_i$. Note that $\sum_{i=1}^{t} \lambda_i \geq 2$ otherwise $\boldsymbol{v}_T \in H$. This implies there is nonzero integer $\boldsymbol{u} \in P_{\mathcal{C}}$ and $\boldsymbol{h} \in H$ such that $\boldsymbol{v}_T = \boldsymbol{h} + \boldsymbol{u}$. Hence we get $T = T_{\boldsymbol{h}} T_{\boldsymbol{u}}$. □

Here onwards, by *Hilbert basis of variety* we mean $\cup_{\mathcal{C}} \mathcal{H}_{\mathcal{C}}$ where union is over all maximal chains in bruhat poset B.

Corollary 12 (Direct Factoring Algorithm)**.** *There is algorithm which takes T-invariant tableau S and produces tableaux X, Y such that $S = XY$ and following holds:*

1. *If S splits directly then X, Y are proper T-invariant factors of S*
2. *If S do not splits directly then $X = S$ and $Y = 1$.*

Proof. Let $\mathcal{C}$ be the chain of all columns in S and $H_{\mathcal{C}}$ be the Hilbert basis of cone $P_{\mathcal{C}}$ which can be computed using alhorithm given in []. If $S \notin H_{\mathcal{C}}$ then tableau do not factor directly, In this case our algorithm return $X = S$ and $Y = 1$. Otherwise we find hilbert basis elements which are factor of S. □

5 Computation of Hilbert basis using `4ti2`.

We computed the Hilbert basis for each maximal chain in $G_{3,7}$. This data has is available at github.

Observation 13. *For $Gr(3, 7)$ the following holds*

1. *There are 462 maximal chains of which 131 have no T-invariants.*
2. *The maximum degree of a T-invariant corresponding to an element in the Hilbert basis is 3x7=21.*
3. *We observe that each element of the Hilbert basis of degree 14 and 21 is in the polynomial span of T-invariants of degree 7, using straightening. We show some of these calculations for the Schubert variety $X[367]$ in Section [**?**].*

Theorem 14. *$\mathbb{C}[G_{3,7}]^T$ is projectively normal.*

6 Flag varieties in $n = 5$

6.1 Shape $(r, 1^s)$

In this subsection we will assume that $\lambda = (r, 1^s)$ and $r + s = n$. We have following lemma

Lemma 15. *Let $S \in STab(\lambda, n, \mathcal{C})$ of degree d where $\mathcal{C}$ is maximal chain in Bruhat poset $(I(\lambda, n), \leq)$. For any column $\underline{p}$ of length r in S, there is factor of S of degree 1 which contain $\underline{p}$.*

Proof. Shape and weight of S are $(r^d, 1^{sd})$ and $d\mathbf{1}$ respectively. Let $S = X_1X_2$ such that shape of X_1 is (r^d) and shape of X_2 is (1^{sd}). Let $\underline{p}$ be a column in X_1 and $i \in [n]$ such that i do not appear in $\underline{p}$. We have that weight of i in X_1 is at most $d-1$ (There are exactly d columns in X_1 and entries in colums are distinct). This implies weight of i in X_2 is at least 1 (weight of i in $S = X_1X_2$ is d).

Factor of S of degree 1 is constructed by taking any column p in X_1 and all i's not appearing in $\underline{p}$ from X_2. $\square$

Corollary 16. *Flag variety $\mathcal{FL}_\lambda$ is projectively normal.* $\square$

We define linear map $\phi_d : \mathbb{C}[\mathcal{FL}_\lambda]_d^T \to \mathbb{C}[G_{n-1,r-1}]_d$ as follows

$$S = \begin{array}{|c|c|c|c|c|c|}\hline S_{11} & \cdots & S_{1d} & S_{1(d+1)} & \cdots & S_{1(d+sd)} \\ \hline S_{21} & \cdots & S_{2d} \\ \cline{1-3} \vdots & \vdots & \vdots \\ \cline{1-3} S_{r1} & \cdots & S_{rd} \\ \cline{1-3}\end{array} \mapsto S' = \begin{array}{|c|c|c|}\hline S_{21} & \cdots & S_{2d} \\ \hline \vdots & \vdots & \vdots \\ \hline S_{r1} & \cdots & S_{rd} \\ \hline\end{array}$$

Tableau S' is obtained by removing first row in tableau S. Weight of 1 in tableau S is d hence weight of 1 in S' is 0. We get ϕ_d is isomorphism of vector spaces.

Observation 17. *Let $X, Y \in STab(\lambda, n, \mathcal{C})$ be two tableaux of degree $(d_1, d_2$ respectively. $S = XY$ is nonstandard iff there are two columns of length r in S which are noncomparable and both column contain 1 in first box.*

Proof. Reverse direction is trivial. To prove forward direction observe that there are d_1+d_2 columns of length r in S and first entry in all of these columns is 1. Hence each of the column of length r in S is comparable with any column of length 1. Given that S is nonstandard and any two columns of length 1 are comparable implies there must be two columns of length r which are not comparable. $\square$

Corollary 18. *There is bijection between ideal generators of T quotient of flag variety $\lambda = (r, 1^s)$ and $G_{r-1,n-1}$.*

Corollary 19. *T quotient of flag variety $\lambda = (r, 1^s)$ is isomorphic to $G_{r-1,n-1}$. In perticular this settles the question of T quotient shapes $(4, 1), (3, 1, 1), (2, 1, 1, 1)$ when $n = 5$.*

6.2 Shape $(3, 2)$

Hilbert basis of $\mathcal{FL}^T_{(3,2)}$ is

$$t_0 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \quad t_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \quad t_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \quad t_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \quad t_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

$$x_1 = \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\ \hline 2&3&3&5\\ \hline 4&5&\multicolumn{2}{c}{}\\ \cline{1-2}\end{array} \qquad x_2 = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&4&4&5\\ \hline 3&5&\multicolumn{2}{c}{}\\ \cline{1-2}\end{array}$$

Lemma 20. *$x_1 + x_2, x_1x_2 \in \mathbb{C}[t_0, t_1, t_2, t_3, t_4]$ using straightaining relations. Hence $z = x_1 - x_2$ is integral over $\mathbb{C}[t_0, t_1, t_2, t_3, t_4, z]$ and Krull dimension of $\mathbb{C}[t_0, t_1, t_2, t_3, t_4, z]$ is 5.*

Proof. We have following two straightainning relations (see apendix for details):

$$x_1x_2 = f_1(t_0, t_1, t_2, t_3, t_4,)$$

$$x_1 + x_2 = f_2(t_0, t_1, t_2, t_3, t_4,)$$

this implies $x_1 - x_2$ is integral over $\mathbb{C}[t_0, t_1, t_2, t_3, t_4]$. $\square$

6.3 Shape $(2, 2, 1)$

Hilbert basis of $\mathcal{FL}^T_{(2,2,1)}$ is

$$t_0 = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} \qquad t_1 = \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} \qquad t_2 = \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} \qquad t_3 = \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} \qquad t_4 = \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array}$$

$$x_1 = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&5\\ \hline 2&4&4&5&\multicolumn{2}{c}{}\\ \cline{1-4}\end{array} \qquad x_2 = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&4&4&5\\ \hline 2&3&3&5&\multicolumn{2}{c}{}\\ \cline{1-4}\end{array}$$

Lemma 21. *$x_1, x_2 \in \mathbb{C}[t_0, \cdots, t_4]$ using straightaining relations. Hence Krull dimension of $\mathbb{C}[t_0, \cdots, t_4, x_1, x_2]$ is 5*

Proof. See appendix for details. $\square$

7 The Schubert variety [367] in $G_{3,7}$

List of degree 2 tableaux which cannot be splitted directly is

$$t_0 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\ \hline 2&2&2&4&4&4&4&4&4&5&5&5&5&6\\ \hline 3&5&5&6&6&6&6&6&7&7&7&7&7&7\\ \hline\end{array}\ , \qquad t_1 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\ \hline 2&2&2&4&4&4&4&4&4&5&5&5&5&5\\ \hline 3&5&6&6&6&6&6&6&7&7&7&7&7&7\\ \hline\end{array},$$

$$t_2 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\ \hline 2&2&2&4&4&4&4&4&4&5&6&6&6&6\\ \hline 3&5&5&5&5&5&6&6&7&7&7&7&7&7\\ \hline\end{array}\ , \qquad t_3 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\ \hline 2&2&2&4&4&4&4&4&4&5&5&6&6&6\\ \hline 3&5&5&5&5&6&6&6&7&7&7&7&7&7\\ \hline\end{array},$$

$$t_4 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\ \hline 2&2&2&4&4&4&4&4&4&5&5&5&6&6\\ \hline 3&5&5&5&6&6&6&6&7&7&7&7&7&7\\ \hline\end{array}\ , \qquad t_5 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\ \hline 2&2&2&4&4&4&4&4&4&6&6&6&6&6\\ \hline 3&5&5&5&5&5&5&6&7&7&7&7&7&7\\ \hline\end{array},$$

$$t_6 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\ \hline 2&2&2&4&4&4&4&5&5&6&6&6&6&6\\ \hline 3&4&4&5&5&5&5&6&7&7&7&7&7&7\\ \hline\end{array}\ , \qquad t_7 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\ \hline 2&2&2&3&4&4&4&5&5&6&6&6&6&6\\ \hline 4&4&4&5&5&5&5&6&7&7&7&7&7&7\\ \hline\end{array}.$$

7.1 Factoring t_0, t_3, t_4

We will use following plucker relations:

$$R_1 : \quad 0 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 5 & 4 \\ \hline 7 & 7 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 7 & 7 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline 7 & 7 \\ \hline \end{array}$$

On $[3, 6, 7]$, last monomial in R_1 is 0.

Applying R_1 on t_0, t_3, t_4 we get

$$t_0 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 5 & 4 & 5 & 5 & 5 & 6 \\ \hline 3 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 \\ \hline \end{array}$$

$$t_3 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 5 & 4 & 5 & 6 & 6 & 6 \\ \hline 3 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 \\ \hline \end{array}$$

$$t_4 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 5 & 4 & 5 & 5 & 6 & 6 \\ \hline 3 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 \\ \hline \end{array}$$

Note following tableau is factor of all of above tableaux

$$f_{034} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 2 & 4 & 4 & 4 & 5 & 6 \\ \hline 5 & 5 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}$$

7.2 Factoring t_2

We will use following plucker relations:

$$R_2 : \quad 0 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 6 & 4 \\ \hline 7 & 7 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 6 \\ \hline 7 & 7 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 6 \\ \hline 7 & 7 \\ \hline \end{array}$$

On $[3, 6, 7]$, last monomial in R_2 is 0.

Applying R_2 on t_2 we get

$$t_2 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 6 & 5 & 4 & 6 & 6 & 6 \\ \hline 3 & 5 & 5 & 5 & 5 & 5 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 \\ \hline \end{array}$$

Note following tableau is factor of above tableau

$$f_{034} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 2 & 4 & 4 & 4 & 6 & 6 \\ \hline 5 & 5 & 5 & 6 & 7 & 7 & 7 \\ \hline \end{array}$$

7.3 Factoring t_5

We define one more pluckre relation

$$R_3: \quad 0 = \begin{array}{|c|c|}\hline 2&3\\\hline 5&4\\\hline 6&7\\\hline\end{array} - \begin{array}{|c|c|}\hline 2&3\\\hline 4&5\\\hline 6&7\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&4\\\hline 3&5\\\hline 6&7\\\hline\end{array} - \begin{array}{|c|c|}\hline 2&4\\\hline 3&6\\\hline 5&7\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&3\\\hline 4&6\\\hline 5&7\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&5\\\hline 3&6\\\hline 4&7\\\hline\end{array}$$

Observe that $3^{rd}, 4^{th}$ and 6^{th} monomials are 0 on [3,6,7] Hence after multiplying following tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&&2&2&&3&3&3&3\\\hline 2&2&2&4&4&4&&4&4&&6&6&6&6\\\hline 3&5&5&5&5&5&&6&7&&7&7&7&7\\\hline\end{array}$$

to above relation in empty columns, we get

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&5&4&4&4&6&6&6&6\\\hline 3&5&5&5&5&5&6&6&7&7&7&7&7&7\\\hline\end{array}$$

$$= \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&4&5&6&6&6&6\\\hline 3&5&5&5&5&5&6&6&7&7&7&7&7&7\\\hline\end{array} - \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&4&4&6&6&6&6&6\\\hline 3&5&5&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$

$$= t_2 - t_5$$

Observe that following is factor of left hand side tableau

$$f_5 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&3&3&3\\\hline 2&2&4&4&4&6&6\\\hline 5&5&5&6&7&7&7\\\hline\end{array}$$

Hence t_5 factored.

7.4 Factoring t_7

Next we factor t_7, we will use following plucker relation:

$$R_4: \quad 0 = \begin{array}{|c|c|}\hline 1&3\\\hline 3&5\\\hline 7&6\\\hline\end{array} - \begin{array}{|c|c|}\hline 1&3\\\hline 3&5\\\hline 6&7\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&3\\\hline 3&6\\\hline 5&7\\\hline\end{array}$$

We multiply following tableau to above relation (in empty columns)

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&&1&1&2&2&2&&3&3&3&3\\\hline 2&2&2&&4&4&4&5&5&&6&6&6&6\\\hline 4&4&4&&5&5&5&6&7&&7&7&7&7\\\hline\end{array}$$

then we get following

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&6&6&6&6&6\\\hline 4&4&4&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$

$$= -\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&5&6&6&6&6\\\hline 4&4&4&7&5&5&5&6&7&6&7&7&7&7\\\hline\end{array} + \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&5&6&6&6&6\\\hline 4&4&4&6&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$

Observe that tableau on left hand side is t_7, also observe that following two tableaux are factors of two tableaux on right side in above relation

$$f_{71} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3\\\hline 2&3&4&4&5&6&6\\\hline 4&7&5&5&6&7&7\\\hline\end{array} \qquad f_{72} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3\\\hline 2&3&4&4&5&6&6\\\hline 4&6&5&5&7&7&7\\\hline\end{array}$$

7.5 Factoring t_6

Next we will use following plucker relation to factor t_6

$$R_5: \quad 0 = \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 5&4\\\hline\end{array} - \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 4&5\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline 3&5\\\hline\end{array}$$

We multiply following tableau to above relation (in empty columns)

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline &1&1&&1&1&2&2&2&3&3&3&3&3\\\hline &2&2&&4&4&4&5&5&6&6&6&6&6\\\hline &4&4&&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$

We get

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&6&6&6&6&6\\\hline 5&4&4&4&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$

$$= \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&3&4&4&4&5&5&6&6&6&6&6\\\hline 4&4&4&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array} - \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&2&3&3&3&3&3\\\hline 2&2&2&4&4&4&4&5&5&6&6&6&6&6\\\hline 3&4&4&5&5&5&5&6&7&7&7&7&7&7\\\hline\end{array}$$

$$= t_7 - t_6$$

Observe that following tableau ia factor of left hand side tableau in above relation

$$f_6 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&3&3&3\\\hline 2&2&4&4&6&6&6\\\hline 4&5&5&5&7&7&7\\\hline\end{array}$$

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