0.1 The polynomials f_1 and f_2

0.1.1 Polynomial f_1

The polynomial f_1 is obtained by straigntenning monomial x_1x_2

$$x_1x_2 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3&4\\\hline 2&2&3&4&3&4&5&5\\\hline 3&4&5&5\\\cline{1-4}\end{array}$$

Observe that the monomial is nonstandard and noncomparable pair is $[1,4,5],[2,3]$. We will use following Plúcker relation.

$$\begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline 5\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array} = 0$$

We get

$$\begin{aligned} x_1x_2 &= \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3&4\\\hline 2&2&3&4&3&4&5&5\\\hline 3&4&5&5\\\cline{1-4}\end{array} \\ &= \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3&4\\\hline 2&2&3&3&4&4&5&5\\\hline 3&4&5&5\\\cline{1-4}\end{array} - \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&3&3&4\\\hline 2&2&2&3&4&4&5&5\\\hline 3&4&5&5\\\cline{1-4}\end{array} - \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3&4\\\hline 2&2&3&3&4&5&5&5\\\hline 3&4&4&5\\\cline{1-4}\end{array} \\ &\qquad + \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&3&3&4\\\hline 2&2&2&3&4&5&5&5\\\hline 3&4&4&5\\\cline{1-4}\end{array} - \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&3&4&4\\\hline 2&2&2&3&4&5&5&5\\\hline 3&3&4&5\\\cline{1-4}\end{array} \\ &= t_1t_0t_3^2 - t_1t_0t_2t_3 - t_1t_0t_4t_3 + t_1t_0^2t_3 - t_1^2t_0t_3 \end{aligned}$$

Hence we define

$$\begin{aligned} f_1(t_0,t_1,t_2,t_3,t_4) &:= x_1x_2 \\ &= t_1t_0t_3^2 - t_1t_0t_2t_3 - t_1t_0t_4t_3 + t_1t_0^2t_3 - t_1^2t_0t_3 \end{aligned}$$

0.1.2 Polynomial f_2

To get f_2 we will straigntain following monomial

$$t_2t_4 = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&5&4\\\hline 5&4\\\cline{1-2}\end{array}$$

We will use following Plúcker relations

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 5&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline 3&5\\\hline\end{array} - \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 4&5\\\hline\end{array} = 0 \qquad \cdots (\mathrm{R1})$$

$$\begin{array}{|c|c|}\hline 2&3\\\hline 5&4\\\hline\end{array} - \begin{array}{|c|c|}\hline 2&3\\\hline 4&5\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&4\\\hline 3&5\\\hline\end{array} = 0 \qquad \cdots (\mathrm{R2})$$

$$\begin{ytableau} 1 & 2 \\ 4 & 3 \\ 5 \end{ytableau} - \begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 \end{ytableau} + \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \end{ytableau} + \begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 \end{ytableau} - \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau} + \begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau} = 0 \qquad \cdots (\text{R3})$$

We get

$$\begin{aligned}
t_2 t_4 &= \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 4 \\ 5 & 4 \end{ytableau} \\
&= -\begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 4 & 5 & 4 \\ 3 & 5 \end{ytableau} + \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 4 \\ 4 & 5 \end{ytableau} && \cdots (\text{After applying R1}) \\
&= \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 4 \\ 4 & 5 \end{ytableau} - \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 4 & 4 & 5 \\ 3 & 5 \end{ytableau} + \begin{ytableau} 1 & 1 & 2 & 4 \\ 2 & 4 & 3 & 5 \\ 3 & 5 \end{ytableau} && \cdots (\text{Term 1, Apply R2}) \\
&= -\begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 4 & 4 & 5 \\ 3 & 5 \end{ytableau} + \begin{ytableau} 1 & 1 & 2 & 4 \\ 2 & 4 & 3 & 5 \\ 3 & 5 \end{ytableau} + \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 4 & 5 \end{ytableau} - \begin{ytableau} 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & 5 \\ 4 & 5 \end{ytableau} && \cdots (\text{Term 1, Apply R2}) \\
&= -\begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 4 & 4 & 5 \\ 3 & 5 \end{ytableau} + \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 4 & 5 \end{ytableau} - \begin{ytableau} 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & 5 \\ 4 & 5 \end{ytableau} + \begin{ytableau} 1 & 1 & 2 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 5 \end{ytableau} \\
&\qquad - \begin{ytableau} 1 & 1 & 3 & 4 \\ 2 & 2 & 4 & 5 \\ 3 & 5 \end{ytableau} - \begin{ytableau} 1 & 1 & 2 & 4 \\ 2 & 3 & 5 & 5 \\ 3 & 4 \end{ytableau} + \begin{ytableau} 1 & 1 & 3 & 4 \\ 2 & 2 & 5 & 5 \\ 3 & 4 \end{ytableau} - \begin{ytableau} 1 & 1 & 4 & 4 \\ 2 & 2 & 5 & 5 \\ 3 & 3 \end{ytableau} \\
& && \cdots (\text{Term 2, Apply R3}) \\
&= -x_2 + t_0 t_3 - x_1 + t_1 t_3 - t_1 t_2 - t_1 t_4 + t_1 t_0 - t_1^2
\end{aligned}$$

Define f_2 as follows

$$\begin{aligned}
f_2(t_0, t_1, t_2, t_3, t_4) &:= x_1 + x_2 \\
&= -t_2 t_4 + t_0 t_3 + t_1 t_3 - t_1 t_2 - t_1 t_4 + t_1 t_0 - t_1^2
\end{aligned}$$