0.1 Notations

We will start with few notations required along the way. Let $\mathcal{C} = p_1 < \cdots < p_l$ be a chain in Bruhat poset. Denote by $STab(r, n, \mathcal{C})$ the set of all torus-invariant tableaux in chain $\mathcal{C}$.

1. For $p \in \mathcal{C}$ denote by u_p, the n-dimentional indicator vector defined as follows: For $1 \leq i \leq n$, $u_p(i)$ is defined as follows:
$$u_p(i) = \begin{cases} 1 & i \in p \\ 0 & i \notin p \end{cases}$$
Here tuple p is treated as set.

2. We will construct matrix A' of order $n \times l$. Entries of A' are indexed by tuple (i, p) where $1 \leq i \leq n$ and $p \in \mathcal{C}$. A' is constructed as follows:
$$A'_{(i,p)} = \quad u_p(i)$$
We call A' the indicator matrix of chain $\mathcal{C}$. Note that p^{th} column of A' is u_p, the indicator vector of $p \in \mathcal{C}$.

3. Let $w \in \mathbb{R}^n$ be a vector defined as follows:
$$w = \begin{bmatrix} -r \\ -r \\ \vdots \\ -r \end{bmatrix}$$
We construct one more matrix denoted by A, defined by
$$A = [A'|w].$$
Call this matrix the augmented indicator matrix of chain $\mathcal{C}$. Let $P_{\mathcal{C}} = ker(A)_{\geq 0}$.

4. For column-standard tableau S whose columns are in $\mathcal{C}$, define a column vector $v'_{_S}$ of length l, indexed by set $[l]$ (or $\mathcal{C}$), as follows:
$$v'_{_S}(i) = \quad \text{number of times } p_i \text{ appear in } S \text{ as column}$$
Observe that:
   (a) For $p \in \mathcal{C}$, $v'_p(q) = \delta_{pq}$.
   (b) If S and T are two tableaux and ST is tableau obtained by appending T to S, then
$$v'_{_{ST}} = v'_{_S} + v'_{_T}.$$

5. For torus-invarient semi-standard tableau S in $\mathcal{C}$, define a column vector $v_{_S}$ of length $l+1$, indexed by set $[l+1]$, as follows:
$$v_{_S}(i) = \begin{cases} v'_{_S}(i) & i \leq l \\ \text{degree of } S & i = l+1 \end{cases}$$
   (a) If S and T are two tableaux and ST is tableau obtained by appending T to S, then
$$v_{_{ST}} = v_{_S} + v_{_T}.$$

6. For non-negative integer vector $v = [u|d]^t \in \mathbb{R}^{l+1}$ where $u \in \mathbb{R}^l, d \in \mathbb{R}$, define tableau T^v such that
$$v'_{T^v} = u$$

7. Weight of column-standard tableau S, denoted by $wt(S)$, is n-tuple $(a_1, ..., a_n)$ such that i appear a_i times in S. We will think of $wt(S)$ as vector in $\mathbb{R}^n$. Observe that:
   (a) If S contain only one column p then
   $$wt(S) = u_p.$$
   (b) If S and T are two tableaux and ST is tableau obtained by appending T to S, then
   $$wt(ST) = wt(S) + wt(T).$$
   (c) S is torus-invarient of degree d iff weight of S is $-dw$.

0.2 Relation between Indicator Matrix and Weight

Lemma 1. *For any column-standard tableau S, $A'v'_S$ is weight of S.*

Proof. We prove by induction on number of columns m in tableau S. For $m = 1$, let p be the column of S, then $v'_S(q) = \delta_{pq}$. Thus we have, $A'v'_S = u_p$, which is exactly the weight of S.
Let S contain $m + 1$ columns, we think of S as tableau obtained by appending a column p to another tableau S', such S' and p exist. We have

$$v'_S = v'_{S'} + v'_p \implies A'v'_S = A'v'_{S'} + A'v'_p = wt(S') + wt(p) = wt(S).$$

□

Corollary 2. *If a semi-standard tableau S is torus-invarient then v_S is in the kernal of A.*

Proof. Let S is torus-invarient of degree d then $wt(S) = -dw$, thus we have,

$$Av_S = [A'|w][v'^t_S|d]^t = A'v'_S + dw = wt(S) + dw = 0.$$

Where notation x^t for vector x denotes transpose. □

Corollary 3. *Let $x \neq 0$ be a non-negative integer vector in the kernal of A then there exist a torus-invarient semistandard tableau S of degree x_{l+1} such that $x = v_S$.*

Proof. Define tableau S such that, $p_i \in \mathcal{C}$ appears x_i times. Note since columns for S are from chain, S semistandard. It remains to show that degree of S is x_{l+1}. Given that,

$$Ax = 0 \implies [A'|w][v'^t_S|x_{l+1}]^t = 0 \implies A'v'_S = -x_{l+1}w \implies wt(S) = -x_{l+1}w.$$

The notation x^t for vector x denotes transpose. □

Corollary 4. *The map in notation 5. is bijection:*

$$\begin{aligned} STab(r, n, \mathcal{C}) &\longrightarrow P_{\mathcal{C}} \cap \mathbb{Z}^{l+1} \setminus \{0\} \\ T &\longmapsto v_T \\ T^v &\longleftarrow v \end{aligned}$$

0.3 Hilbert basis

Definition 5 (Cone)**.** *A set $P \in \mathbb{R}^n$ is cone if for any $x, y \in P$ and non-negative λ_1, λ_2, we have $\lambda_1 x + \lambda_2 y \in P$. A cone P is pointed if $P \cap -P = \{0\}$. A cone P is polyhedral if $P = \{x \in \mathbb{R}^n | Bx \geq 0\}$ for some matrix B. Set $\{g_1, ...g_k\}$ is called generators of P if for all $x \in P$ there are non-negative $\lambda_1, ..., \lambda_k$ such that $x = \sum_{i=1}^{k} \lambda_i c_i$.*

Definition 6 (Lattice)**.** *A lattice in $\mathbb{R}^n$ is subgroup of $\mathbb{R}^n$ under addition.*

Definition 7 (From paper on Hilbert basis)**.** *Let $P \in \mathbb{R}^n$ be a polyhedral cone with rational generators and let $\Lambda \subset \mathbb{Z}^n$ be a lattice. We call finite set $H = \{h_1, ..., h_t\} \subset \Lambda \cap P$ generate monoid $(\Lambda \cap P, +)$ if for every $x \in \Lambda \cap P$ there are non-negative integers $\lambda_1, ..., \lambda_t$ such that $\sum_{i=1}^{t} \lambda_i h_i$. Following are known results:*

1. *If P is pointed cone then there is unique inclusion minimal generating set.*
2. *If $\Lambda = \mathbb{Z}^n$ and P is pointed cone then the inclusion minimal set H is called Hilbert basis of monoid P.*
3. *Every element of hilbert basis is indecomposible, i.e., it cannot be writen as sum of two elements of the monoid.*

Note that $P_{\mathcal{C}} = ker(A)_{\geq 0}$ is pointed cone: 1. Addition of two non-negative vectors in $ker(A)$ is non-negative and in $ker(A)$. 2. Non-negative scalar multiple of non-negative vector in $ker(A)$ is non-negative and in $ker(A)$. 3. $x, -x \in P_{\mathcal{C}} \implies x = 0$.

Definition 8. *We say $T \in STab(r, n, \mathcal{C})$ splits directly if there are two tableaux $X, Y \in STab(r, n, \mathcal{C})$ such that $T = XY$ up to rearranging columns.*

Lemma 9. *$T \in STab(r, n, \mathcal{C})$ splits directly iff $v_T \notin H$. Where H is hilbert basis of $P_{\mathcal{C}}$.*

Proof. ($\implies$) Let T splits directly then there exist $X, Y \in STab(r, n, \mathcal{C})$ such that $T = XY$ up to rearranging columns. This implies $v_T = v_X + v_Y$ and thus v_T is decomposible hence $v_T \notin H$.

($\impliedby$) If $v_T \notin H$ then there are non-negative integers $\lambda_1, ..., \lambda_t$ such that $v_T = \sum_{i=1}^{t} \lambda_i h_i$. Note that $\sum_{i=1}^{t} \lambda_i \geq 2$, if not then $v_T \in H$. This implies there is $u \in Z^{l+1} \cap P_{\mathcal{C}}$ and $h, h' \in H$ such that $v_T = h + h' + u$. Using property $v_{ST} = v_S + v_T$ and corollary 4, we get $T = T^h T^{h'} T^u$. ☐

0.4 Computation of Hilbert basis using `4ti2`.

The software package `4ti2` ('fourtytwo or for tea too') implements hilbert basis computation algorithm given in paper 'On the Computation of Hilbert Bases and Extreme Rays of Cones' (https://arxiv.org/abs/math/0203105). Using this implementation, we computed hilbert basis for chains in $Gr(3, 7)$. Following are the observations:

1. There are 462 maximal chains in bruhat poset and 131 out of them do not admit T-invariants.
2. Maximum degree of the element which cannot be directly splitted is 3.

Data will be uploaded soon on github.