We have the following:

Let $1 \le i, j, k, l \le n$.

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} \longrightarrow (*)$$

where $\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$

<u>Special linear Lie algebra</u>

Consider the trace map $Tr : gl(n, F) \longrightarrow F$ defined by $Tr(x) = \sum_{i=1}^{n} x_{ii}$ = the sum of all diagonal entries of x.

Clearly Tr is linear and surjective.

Note that for any $x, y \in gl(n, F)$ we have $Tr(xy) = Tr(yx)$.

Hence, we have

$$Tr([x, y]) = 0 \ \forall\, x, y \in gl(n, F).$$

Remember $[x, y] = xy - yx$ in $gl(n, F)$.

Now, let $sl(n, F) := Ker(Tr)$.

Since $Tr([x, y]) = 0$ for all $x, y \in gl(n, F)$, $sl(n, F)$ is a Lie subalgebra of $gl(n, F)$.

Since $Tr : gl(n, F) \to F$ is surjective, we have $\dim(sl(n, F)) = n^2 - 1$.

Now, let $V = F^n$.
For any $x \in \mathfrak{gl}(n,F)$ and $g \in GL(n,F)$, we have

$$\begin{aligned} Tr(gxg^{-1}) &= Tr(x) \\ \| \quad & \quad \| \\ Tr(g(xg^{-1})) &= Tr((xg^{-1})g) \end{aligned}$$

$\therefore$ Tr does not depend on a choice of a basis of V. Hence it is defined on $End(V)$.
Thus, $\mathfrak{sl}(V)$ makes sense.

For $1 \le i \le n-1$, let
$h_i = e_{ii} - e_{i+1,i+1}$.

So, $\{e_{ij} : 1 \le i \ne j \le n\}$ $\cup \{h_i : 1 \le i \le n-1\}$ is a basis of $sl(n, F)$. (char $F = 0$).

Special Orthogonal Lie algebra $n \geq 2$

Assume that char $F \neq 2$.

Consider the map

$\varphi : gl(n, F) \longrightarrow gl(n, F)$ defined by

$\varphi(x) = -x^t$, where x^t = transpose of x.

clearly φ is F-linear and φ^2 = identity map on $gl(n, F)$. $(-(-x^t))^t = x)$

Thus, φ is a linear

automorphism of $gl(n,F)$.

We now show that φ is a homomorphism of Lie algebras.

Proof:-

Let $x, y \in gl(n,F)$.

$$\varphi([x,y]) = -([x,y])^t$$
$$= -(xy - yx)^t$$
$$= -(xy)^t + (yx)^t$$
$$= -y^t x^t + x^t y^t$$
$$= [x^t, y^t]$$
$$= [-x^t, -y^t]$$
$$= [\varphi(x), \varphi(y)]$$

Thus, φ is a homomorphism of Lie algebras. $\therefore$ φ is a Lie algebra automorphism of $gl(n,F)$.

Let $so(n,F) := gl(n,F)^{\varphi}$

$$= \{x \in gl(n,F) : \varphi(x) = x\}.$$

Since φ is a homomorphism of Lie algebras, $so(n,F)$ is a Lie subalgebra of $gl(n,F)$.

For computational purpose, $so(n,F)$ can be studied by defining as follows:

Let $E_n = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}_{n \times n}$.

example $n=3$, $n=4$

$$E_n := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Now, consider the map
$\sigma : gl(n, F) \longrightarrow gl(n, F)$
defined by

$$\sigma(x) = E_n(-x^t) E_n^{-1}.$$

Since φ is a Lie algebra automorphism of $gl(n, F)$, σ is also a Lie algebra automorphism of $gl(n, F)$

$\therefore L := gl(n, F)^{\sigma}$ is

a Lie Subalgebra of $\mathfrak{gl}(n, F)$. We call this Lie algebra also special orthogonal Lie algebra.

Now, let $n \in \mathbb{N}_{\geq 2}$.

Let $J_n = \begin{pmatrix} O_n & E_n \\ -E_n & O_n \end{pmatrix}_{2n \times 2n}$

Consider the map $I: \mathfrak{gl}(2n, F) \longrightarrow \mathfrak{gl}(2n, F)$ defined by

$$I(x) = J_n(-x^t)J_n^{-1}.$$

Let $sp(2n, F) := \mathfrak{gl}(2n, F)^I$.

Since τ is a Lie algebra automorphism of $gl(2n,F)$, $sp(2n,F)$ is a Lie subalgebra of $gl(2n,F)$. We call this Lie algebra symplectic Lie algebra.

Upper triangular matrices

An element $A=(a_{ij})_{1\leq i,j\leq n} \in gl(n,F)$ is said to be upper triangular if $a_{ij}=0$ whenever $i>j$.

Example: $\overset{n=2}{\Rightarrow} A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Let $t(n,F)$ be the set of all $n\times n$ upper triangular matrices with entries in F.
By $(*)$, we have
$$[x,y] = xy - yx \in t(n,F)$$
for all $x, y \in t(n,F)$.
So, $t(n,F)$ is a Lie subalgebra of $\mathfrak{gl}(n,F)$.

<u>Lie algebra of strictly upper triangular matrices:</u>

An element $A = (a_{ij})_{1\le i,j\le n} \in \mathfrak{gl}(n,F)$ is said to be strictly upper triangular if $a_{ij} = 0$ whenever $i \geq j$.

Example $\underline{\underline{n=2}}$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $\eta(n, F)$ be the set of all $n \times n$ strictly upper triangular matrices with entries in F.

By $(*)$, we have

$$[x, y] = xy - yx \in \eta(n, F)$$
$$\forall \; x, y \in \eta(n, F).$$

$\therefore \eta(n, F)$ is a Lie subalgebra of $gl(n, F)$.

Diagonal matrices.

An element $A = (a_{ij})_{1 \le i,j \le n} \in \mathcal{S}(n,F)$ is said to be a diagonal matrix if $a_{ij} = 0$ whenever $i \neq j$.

Example $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Let $\mathcal{S}(n,F)$ be the set of all $n \times n$ diagonal matrices with entries in F.

Clearly, $[x,y] = 0$ $\forall\, x,y \in \mathcal{S}(n,F)$.

Therefore, $\mathcal{S}(n,F)$ is a

Lie subalgebra of $gl(n,F)$.

Notation Let L be a Lie algebra.
Let J, K be Lie subalgebras of L.
Set $[J,K] :=$ The vector subspace of L generated by elements of the form $[x,y]$, $x \in J$, $y \in K$.
$\left(\sum_{i=1}^{r} [x_i, y_i],\ x_i \in J,\ y_i \in K\right)$

By using $(*)$, we have
(i) $[t(n,F), t(n,F)] = \eta(n,F)$
(ii) $[t(n,F), \eta(n,F)] = \eta(n,F)$.
(iii) $[\delta(n,F), \eta(n,F)] = \eta(n,F)$.

Further, we also have

$$t(n, F) = \delta(n, F) \oplus \eta(n, F),$$

as direct sum of vector subspaces.

$$\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix} + \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

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