1st February, 2022

Definition A Lie algebra L is said to be Abelian if $[x, y] = 0$ for all $x, y \in L$.

Example

$\delta(n, F)$ = the set of all $n \times n$ diagonal matrices with entries in F is abelian.

Derivations of an algebra

Let A be an F-algebra need not be associative and need not be finite dimensional.

For example, it may be a Lie algebra.

<u>Definition</u> By a derivation of A, we mean an F-linear map $\delta : A \longrightarrow A$ such that

$$\delta(ab) = a \cdot \delta(b) + (\delta(a)) \cdot b$$

for all $a, b \in A$.

For example, if $A = L$, a Lie algebra, then a derivation of L is an F-linear map $\delta : L \longrightarrow L$ such that

$\delta([x,y]) = [x, \delta y] + [\delta x, y]$
for all $x, y \in L$.

For an F-algebra A, let $\mathrm{Der}\, A$ denote the set of all derivations of A.

__Lemma__ $\mathrm{Der}\, A$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

__Proof:__ Clearly, $\mathrm{Der}(A)$ is a vector subspace of $\mathfrak{gl}(A)$.

To prove that $\mathrm{Der}\, A$ is a Lie subalgebra of of $\ell(A)$, it suffices to prove that

$[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1 \in Der(A)$

for all $\delta_1, \delta_2 \in Der(A)$.

So, let $\delta_1, \delta_2 \in Der(A)$, and let $a, b \in A$.

$$[\delta_1, \delta_2](ab) = (\delta_1\delta_2 - \delta_2\delta_1)(ab)$$

$$= \delta_1(a\delta_2(b) + \delta_2(a)\cdot b)$$

$$- \delta_2(a\delta_1(b) + \delta_1 a \cdot (b))$$

$$= a\delta_1\delta_2(b) + \underline{\delta_1(a)\delta_2(b)} + \underline{\delta_2(a)\delta_1(b)}$$

$$+ \delta_1\delta_2(a)\cdot b - a\delta_2\delta_1(b) - \underline{\delta_2(a)\delta_1(b)}$$

$$- \underline{\delta_1(a)\cdot\delta_2(b)} - \delta_2\delta_1(a)\cdot b$$

$$= a(\delta_1\delta_2 - \delta_2\delta_1)(b)$$

$+ (\delta_1\delta_2 - \delta_2\delta_1)(a)\cdot b$

$= a\,[\delta_1, \delta_2](b) + [\delta_1, \delta_2](a)\cdot b$

$\therefore\ [\delta_1, \delta_2] \in \mathrm{Der}(A).$

Let L be a Lie algebra.

Now, we give an example of a derivation of L.

Let $x \in L$.

Consider the map $\mathrm{ad}_L x : L \to L$ defined by $\mathrm{ad}_L(x)(y) = [x, y]$, $y \in L$.

Since $[\ ,\ ]$ is bilinear

$ad_L x$ is linear.

$\therefore ad_L(x) \in gl(L)$.

Lemma $ad_L(x) \in Der(L)$.

That is $ad_L(x)$ is a derivation of L.

Proof:- Let $y, z \in L$.

Now, $ad_L(x)([y,z]) = [x,[y,z]]$

$= -[y,[z,x]] - [z,[x,y]]$, by Jacobi identity.

Further $-[y,[z,x]] = [y,[x,z]]$ since $[z,x] = -[x,z]$.

Further, we have $-[z,[x,y]] = [[x,y],z]$.

$$\therefore \; ad_L(x)([y,z]) = [y,[x,z]] + [[x,y],z]$$
$$= [y, ad_L(x)(z)] + [ad_L(x)(y), z]$$

$\therefore$ $ad_L\, x$ is a derivation of L.

Actually, it is important to write $ad_L\, x$, $ad_K\, x$ in general to avoid ambiguity.

However, we will simply denote $ad_L\, x$ by $ad\, x$ when the content is clear.

Ideal of L

Definition Let L be a Lie algebra over F. A vector subspace I of L is said to be an ideal of L if

$[x, y] \in I$ for all $x \in L$ and for all $y \in I$.

Since $[x, y] = -[y, x]$, I is an ideal of L iff $[y, x] \in I$ for all $x \in L$ and $y \in I$.

Clearly 0 and L are

ideals of L.

Center of L

Let L be a Lie algebra.

Let $Z(L) := \{ z \in L : [x, z] = 0 \text{ for all } x \in L \}$

Since $[\,,\,]$ is bilinear, $Z(L)$ is a vector subspace of L.

Clearly, $[x, z] = 0 \in Z(L)$ for all $x \in L$, and $z \in Z(L)$

$\therefore$ $Z(L)$ is an ideal of L.

Note that $Z(L) = L$ iff L is abelian.

For $L = gl(n, F)$,
$Z(L) =$ the set of all $n \times n$ scalar matrices with entries in F.

Another example of an ideal of L:-

Let $D'L := [L, L]$.
Recall that $[L, L]$ is the vector subspace of L generated by vectors of the form $[x, y]$, $x, y \in L$.

That is, $[L, L] := \left\{ \sum_{i=1}^{n} [x_i, y_i], \; x_i, y_i \in L \; (1 \le i \le n), \; n \in \mathbb{N} \right\}$

By definition, $[L, L]$ is a vector subspace of L.

We now prove that $[L, L]$ is an ideal of L.

To do this, it suffices to prove that $[x, [y, z]] \in [L, L]$ for all $x, y, z \in L$.

(This is because, an element of $[L, L]$ is a sum of elements of the form $[y, z]$, $y, z \in L$ and $[\cdot, \cdot]$ is bilinear.)

So, let $x, y, z \in L$.

$$[x,[y,z]] = [y,[x,z]] + [[x,y],z] \quad \text{(1)}$$

Since $ad_L(x)$ is a derivation of L.

Clearly, both $[y,[x,z]]$ and $[[x,y],z]$ belongs to $[L,L]$.

$\therefore$ By (1), $[x,[y,z]] \in [L,L]$.

Note that $[L,L]=0$ iff L is abelian.

Another example of an ideal: $sl(n,F)$ is an ideal of $gl(n,F)$.

This is because,

$Tr([x,y]) = 0$ for all $x, y \in gl(n,F)$.

In particular, $Tr([x,y]) = 0$ $\forall x \in gl(n,F)$ and $y \in sl(n,F)$

__Exercise__

Assume that char $F = 0$.

Prove the following.

(i) $[gl(n,F), gl(n,F)] = sl(n,F)$.

(ii) $[sl(n,F), sl(n,F)] = sl(n,F)$.

You can use $(*)$ for computational purpose.

Now, let I and J be two ideals of L.

Then the sum $I+J$ defined by $I+J := \{x+y : x \in I, y \in J\}$ is an ideal of L.

Clearly $I+J$ is a vector subspace of L.

To prove that $I+J$ is an ideal of L, let $x \in L$, $y \in I$, $z \in J$

$$[x, y+z] = \underbrace{[x, y]}_{\in I} + \underbrace{[x, z]}_{\in J}$$

$\therefore [x, y+z] \in I+J$.

Further for any two ideals I and J of L $[I, J]$ is also an ideal of L.

Recall $[I, J]$ = the vector subspace of L generated by elements of the form $[x, y]$, $x \in I$, $y \in J$.

So, let $x \in L$, $y \in I$, $z \in J$.

$$[x, [y, z]] = [y, [x, z]] + [[x, y], z],$$

since $ad_L x$ is a derivation of L.

Further, since J is an ideal of L, and $z \in J$, we have $[x, z] \in J$.

$$\therefore \; [\underset{\substack{\uparrow \\ I}}{y}, [x, \underset{\substack{\uparrow \\ J}}{z}]] \in [I, J]$$

Similarly, we have

$[[\underset{\substack{\uparrow \\ I}}{x}, y], \underset{\substack{\uparrow \\ J}}{z}] \in [I, J]$, since $y \in I$.

$\therefore$ Since $[I, J]$ is the vector subspace of L generated by elements of the form $[x_i, y_i]$ $x_i \in I$, $y_i \in J$, and $ad_L(x)$ is linear for any $x \in L$, we have $ad_L(x)(w) \in [I, J]$ $\forall$ $w \in [I, J]$. $\therefore$ $[I, J]$ is an ideal of L.

Simple Lie algebras

A Lie algebra L is said to be simple if $[L, L] \neq 0$ (that is, L is not abelian) and 0 and L are the only ideals of L.

For example, one dimensional Lie algebras are not simple.

Note that if L is simple then we have

$$Z(L) = 0 \quad \text{and} \quad [L, L] = L.$$