Lie Algebras

Book: J. E. Humphreys, Introduction to Lie algebras and Representation Theory.

Definition:

Let F be a field. A vector space L together with an operation $[\cdot,\cdot] : L \times L \longrightarrow L$
$(x, y) \mapsto [x, y]$ called the bracket or commutator of x and y is

said to be a Lie algebra over F if the following axioms are satisfied.

(i) $[\cdot,\cdot]$ is bilinear.

That is,

$[a_1 x_1 + a_2 x_2, y] = a_1 [x_1, y] + a_2 [x_2, y]$

$\forall\ a_1, a_2 \in F,\ x_1, x_2, y \in L,$

and

$[x, a_1 y_1 + a_2 y_2] = a_1 [x, y_1] + a_2 [x, y_2]$

$\forall\ a_1, a_2 \in F,\ x, y_1, y_2 \in L.$

(ii) $[x, x] = 0\ \ \forall\ x \in L.$

(iii) Jacobi identity:

$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

$\forall\ x, y, z \in L.$

Let L and L' be two Lie algebras over F.
An F-linear map $\varphi: L \longrightarrow L'$ is said to be a homomorphism of Lie algebras if

$$[\varphi(x), \varphi(y)] = \varphi([x, y])$$

for all $x, y \in L$.

<u>Definition</u> A homomorphism $\varphi: L \longrightarrow L'$ of Lie algebras is said to be an isomorphism of Lie algebras if φ is an isomorphism of vector spaces.

Two Lie algebras L and L' are said to be isomorphic if there is an isomorphism $\varphi : L \to L'$ of Lie algebras.

<u>Definition</u> Let L be a Lie algebra over F. A vector subspace K of L is said to be a Lie subalgebra of L if $[x, y] \in K$ for all $x, y \in K$.

$$K \times K \longrightarrow K$$
$$\cap \qquad\qquad \cap$$
$$L \times L \longrightarrow L$$

Let L be a non zero Lie algebra over F.

Let x be a non zero element of L.

Then the one dimensional vector subspace $K = F \cdot x$ generated by x is a Lie subalgebra of L.

$$[ax, bx] = ab[x, x] = ab \cdot 0 = 0$$

Throughout the course, we assume that all Lie algebras under consideration are finite dimensional as vector spaces, unless otherwise specified.

Let V be a finite dimensional vector space over F.

An F-linear map $T: V \longrightarrow V$ is called an endomorphism of V.

Let $End(V)$ denote the set of all endomorphisms of V.

$End(V)$ is an F-algebra with following operations

(i) $(T+S)(v) = Tv + Sv$ $\forall\, T, S \in End(V)$ and for all $v \in V$

(ii) $TS = T \circ S =$ the composition of mappings.

That is, the multiplication in $End(V)$ is given by composition of mappings

(iii) $(aT)v = a\cdot(Tv)$ $\forall a \in F$, $T \in End(V)$, $v \in V$.

In particular, the multiplication in $End(V)$ is associative.

<u>Lie algebra structure on $End(V)$</u> :-

$[\,,\,] : End(V) \times End(V) \to End(V)$ defined by

$[x, y] = xy - yx$, where the multiplication in $End(V)$ is as above.

Clearly, this bracket $[\cdot,\cdot]$ on $End(V)$ satisfies axioms (i) and (ii) of a Lie algebra.

We now prove the Jacobi identity.

Let $x, y, z \in End(V)$

$[x,[y,z]] = x(yz) - x(zy) - (yz)x + (zy)x \rightarrow ①$

$[y,[z,x]] = y(zx) - y(xz) - (zx)y + (xz)y \rightarrow ②$

$[z,[x,y]] = z(xy) - z(yx) - (xy)z + (yx)z \rightarrow ③$

① + ② + ③ $= 0$.

$\therefore$ $End(V)$ becomes a Lie algebra over F.

We denote this Lie algebra End(V) by $\mathfrak{gl}(V)$. $\mathfrak{gl}(V)$ is called general linear Lie algebra

$GL(V) :=$ The group of all invertible endomorphism of V = The units of End(V).

For computation purpose

Let V be finite dimensional vector space over F. Let $n = \dim_F V$.

Fix a basis $\{v_1, \ldots, v_n\}$ of V.

Let $M(n, F)$ denote the set of all $n \times n$ matrices with entries in F.
$M(n, F)$ is an F-algebra with the following operations.

(i) Addition is addition of matrices.

(ii) Multiplication = Multiplication of Matrices.

(iii) Scalar multiplication

$$a \cdot (x_{ij})_{1 \le i,j \le n} = (a x_{ij})_{1 \le i,j \le n}$$

In particular, matrix multiplication is associative.

Therefore $M(n,F)$ has a Lie algebra structure given by

$$[A,B] = AB - BA.$$

We denote this Lie algebra $M(n,F)$ by $\mathfrak{gl}(n,F)$.

Moreover, there is a natural map

$$\varphi : \mathrm{End}(V) \longrightarrow M(n,F)$$

defined by
$$\varphi(A) = (a_{ij})_{1 \leq i,j \leq n},$$
where $A v_j = \sum_{i=1}^{n} a_{ij} v_i$.

From results of Linear algebra
$\varphi : \mathrm{End}(V) \longrightarrow M(n, F)$
is an isomorphism of F-algebras.

In particular
$\varphi : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(n, F)$
is an isomorphism of Lie algebras.

So, we can study $\mathfrak{gl}(V)$ by studying $\mathfrak{gl}(n, F)$.

Fix a pair k, l of integers such that $1 \leq k, l \leq n$.

Let $X_{kl} : M(n, F) \to F$ be the linear form defined by
$$X_{kl}\left((a_{ij})_{1 \leq i, j \leq n} \right) = a_{kl}.$$

For $1 \leq k, l \leq n$, let $e_{kl} \in M(n, F)$ be the unique element given by

$$X_{kl}(e_{kl}) = 1, \text{ and}$$

$$X_{ij}(e_{kl}) = 0 \text{ if } (i,j) \neq (k,l).$$

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