Optimization
SC 607 IITB
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https://somphene.github.io/notes/

Notes
by
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A visitor remarked that the library at the newly constructed IHES was lacking.
Grothendieck replied "We don't read books, we write them."

This is an **incomplete draft**. Please send corrections, comments, pictures of bad drawings, etc. to somphene1@gmail.com.

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Contents

3 Lecture January 17

§3.1 Weierstrass theorem

Theorem 3.1.1 (Weierstrass)
Let $S \subseteq \mathbb{R}^n$ be closed and bounded. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then $\exists x^* \in S$ such that

$$f(x^*) = \inf \{f(x) \,|x \in S\}$$

Infimum of f is attained in S.
$\exists\, \hat{x} \in S$ such that

$$f(\hat{x}) = \sup \{f(x) \,|x \in S\}$$

Problems demonstrated by examples given previously are treated by avoiding the following conditions in Weierstrass's theorem:

1. discontiuity in f
2. S is unbounded
3. S is not closed

Definition 3.1.2 (Open Set)**.** Set S is said to be Open if $\forall x \in S \quad \exists r > 0$ such that $B(x, r) \subseteq S$.

Definition 3.1.3 (Interior)**.** Let $C \subseteq \mathbb{R}^n$ be any set.

$$\dot{C} = \text{interior of C} = \bigcup_{S \text{ is open} \,\& S \subseteq C} S$$

Interior of a set is an open set.

Definition 3.1.4 (Feasible point)**.** Denote by S, the feasible region, ie. $S =$ feasible region. Then any point in S is called **feasible point**

Definition 3.1.5 (Infeasible points)**.** Points not in S are called **infeasible points**.

Definition 3.1.6 (Local Minimum)**.** $x^* \in S$ is said to be a local minimum if $\exists\, r > 0$ s.t.

$$f(x^*) \leq f(x) \quad \forall x \in B(x^*, r) \cap S$$

For an example consider Figure 3.1.

Definition 3.1.7 (Global minimum)**.** $x^* \in S$ is said to be a **global minimum** if

$$f(x^*) \leq f(x) \quad \forall x \in S$$

Definition 3.1.8 (Unconstrained minimum)**.** $x^* \in \mathbb{R}^n$ s.t.

$$f(x^*) \leq f(x) \quad \forall x \in \mathbb{R}^n$$

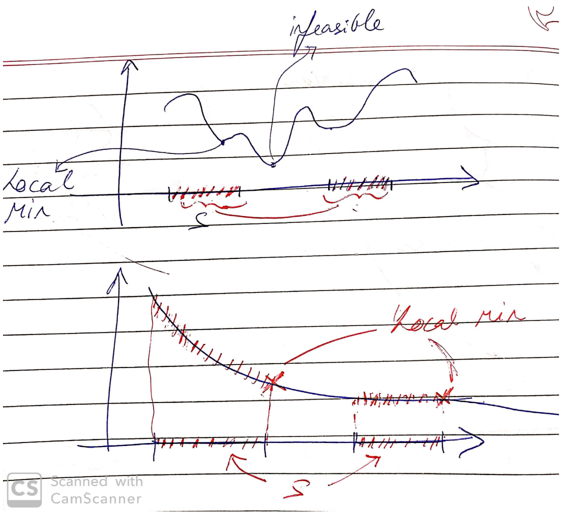

Figure 3.1: Local minimum of f in S

Definition 3.1.9 (Strict local minimum)**.** $x^* \in S$ is said to be a **strict local minimum** if $\exists\, r > 0$ s.t.

$$f(x^*) < f(x) \quad \forall x \in B(x^*, r) \cap S,\, x \neq X^*$$

Definition 3.1.10 (Isolated local Minimum)**.** $x^* \in S$ is said to be a **isolated local minimum** if $\exists\, r > 0$ s.t. x^* is the only local minimum in $B(x^*, r) \cap S$

Remark 3.1.11 (Isolated $\implies$ strict) — Every isolated point is a strict local minimum.

Proof by contradiction. Let x^* be an isolated point. Suppose x^* is not a strict local minimum point then

$$\forall r > 0, \exists x \in B(x^*, r) \cap S, x \neq x^* \quad \text{s.t.} \quad f(x^*) = f(x)$$

$\implies$ for small enough r, x is a local minima. $\implies$ x^* is not isolated. □

However, the converse is not true. **Every strict local minimum need not be isolated**. Proof by example: Consider the following function plotted in Figure 3.1

$$f(x) = \left\{ x^4 \cos\left(\frac{1}{x}\right) + 2x^4 \quad x \in [-1, 1] \setminus 0; \quad 0 \text{ at } x = 0 \right.$$

§3.2 Optimization with Constraints

Goal: min $f(x)$ ie. the objective function, s.t. $x \in S$, ie. the feasible region.

Question 3.2.1. What if there are additional constraints?

Goal: min $f(x)$ s.t. $g_i(x) \leq 0 \quad \forall i = 1, \ldots, m; \quad h_j(x) = 0 \quad \forall j = 1, \ldots, p$ are satisfied.

Here $g_i : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ are called **constraints**. $\leq$ are called Inequality constraints, $=$ are called Equality constraints. Geometry of the problem is very different

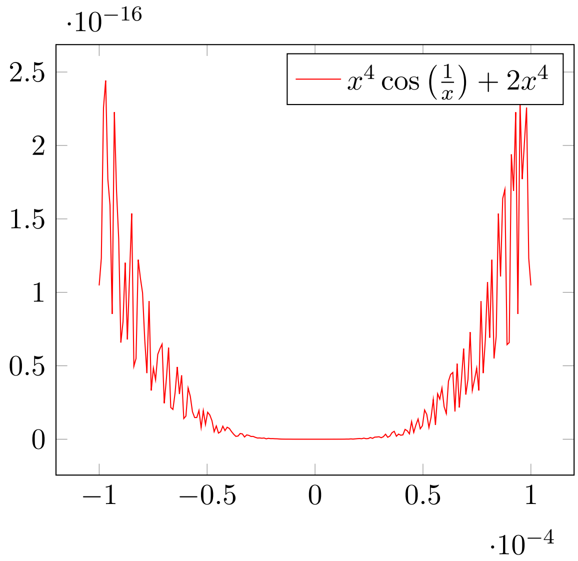

Figure 3.2: Example showing strict local minima does not $\implies$ isolated local minima

in the two different kinds of constraints. Algorithms are allowed to search in all directions if interior is within the feasible regions. Equalities can always be replaced by two inequalities in the opposite direction. This can be used to take advantage of the geometry.

§3.3 Other type of constraints

1. **Bound constraints**
2. **Either-or** constraints: Need not satisfy all constraints but just one out of them.
3. **if-then-else** constraint: choice followed by taking decisions.

We shall deal with only inequality and equality constraints.