Optimization
SC 607 IITB
by
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https://somphene.github.io/notes/

Notes
by
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A visitor remarked that the library at the newly constructed IHES was lacking. Grothendieck replied "We don't read books, we write them."

This is an **incomplete draft**. Please send corrections, comments, pictures of bad drawings, etc. to somphene1@gmail.com.

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Contents

1 Lecture Jan 10

§1.1 Introduction

Question 1.1.1. What is Optimization?

(Answer). Choosing the best alternative out of all the given alternatives for a specific job.

§1.1.i Principle

For finite or infinite lists of alternatives in which the alternatives can be run through in finite or reasonable time, optimization is easy: can be done by direct checking of the alternatives in the list. It is the other case where things get interesting. The problems we encounter in daily life are primarily of the second case. Hence formulating techniques better than just going through every alternative to solve such complex problems are needed. This is what we plan to cover in our course.

Basic elements of any Optimization problem are :

- Quantification of alternatives using a real valued function which we call f. Convention: We would like to find the least value of f.
- Set of alternatives is called **feasible region**, which we denote by S.

In this course we mainly deal with optimization problems for which S is a subset of Euclidean Space ($S \subseteq \mathbb{R}^n; f \colon \mathbb{R}^n \to \mathbb{R}$)

Question 1.1.2. (Optimization Problem) Find an $x^* \in S$ such that

$$f(x^*) \leq f(x) \quad \forall x \in S \tag{1.1}$$

Or sometimes we don't necessarily want x^* but the value of $f(x^*)$ such that eq. (1.1) holds, Such an x^* is called **Optimal Solution** and $f(x^*)$ is called **Optimal Value**.

Example 1.1.3 (Google Maps)

Optimization Problem of finding the optimal path to take from IIT Bombay main gate to IIT Delhi main gate. Here optimal may mean distance, time or some other cost function, denoted by f, associated with the path. Google maps solves for the time optimal path constrained to given traffic rules within seconds.

Example 1.1.4 (Classroom Assignment to Courses)

With hundreds of courses running in IIT Bombay, which of the hundreds of classrooms to assign to the course with constraints of size, computers, labs, preferences of instructor and students, so as to maximize closeness to the ideal scenario for everyone becomes an optimization problem.

§1.1.ii Classical Approach

> **Question 1.1.5.** **Queen Dido's problem**, also known as **Isoperimetric problem**: Given a rope of constant length l, find the shape of maximum area that can be enclosed by the rope.

Solution: We shall present a specific way of solving the problem so as to highlight some features that will be needed later, We outline the steps as follows:

1. The optimal shape can not have a dent protruding inwards (as shown in fig. 1.1), ie. it must be convex.

   Definition 1.1.6. **(Convex Set)** A set S is said to be **convex** if

   $$\forall\, x, y \in S; \quad \forall\, \lambda \in [0,1]; \quad \lambda x + (1-\lambda) y \in S$$

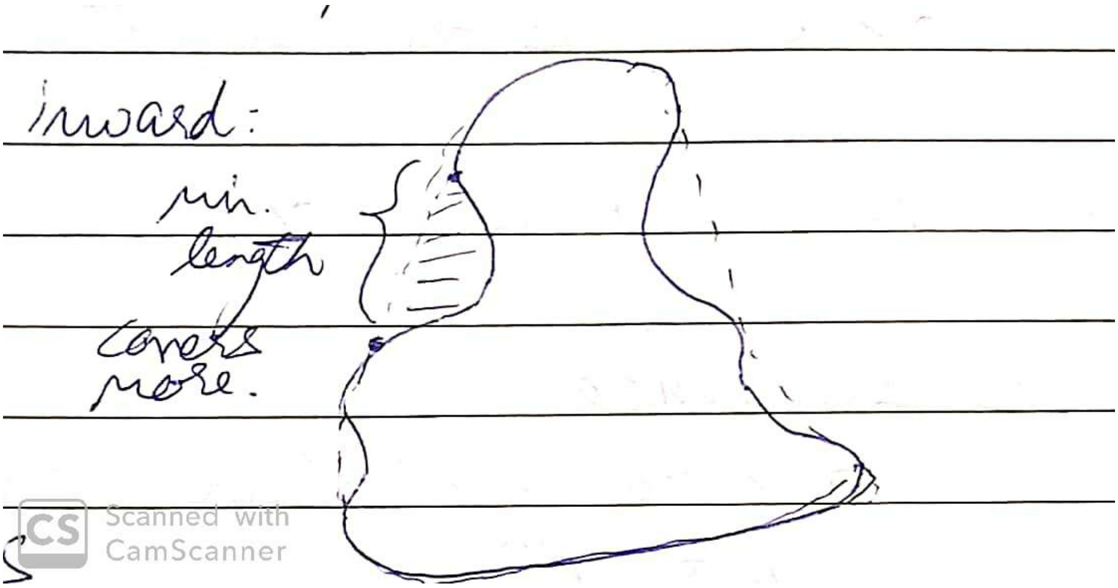

Figure 1.1: A non-convex set is not optimal.

2. **Symmetry argument**: Now that we have reduced the admissible set to convex shapes, choose a starting point on the rope. Travel a distance of $l/2$ along the rope, stop and mark the point. Draw a line segment L joining the starting point and the final point. This line L divides the enclosed shape into two parts. If the area enclosed by the rope on either of the two sides of the segment is not equal then simply reflect the side with larger area about L to increase the total area while maintaining the total length of the rope to be l (as shown in fig. 1.2). Hence the shape must be symmetric about L.

3. **Geometry of semicircle**: By the previous symmetry argument, it suffices to find the optimal shape for one side of the shape enclosed by the rope of length $l/2$ and the line segment L. Suppose that the optimal curve is traced by the rope of length $l/2$. Then fix a point on the rope and imagine it to be a hinge. The lengths of this point from the starting and final points on the rope are fixed, say a and b. Keeping the rope's $l/2$ length constant, area enclosed between the rope and line segment L can be increased by increasing the area of the inscribed triangle as shown in

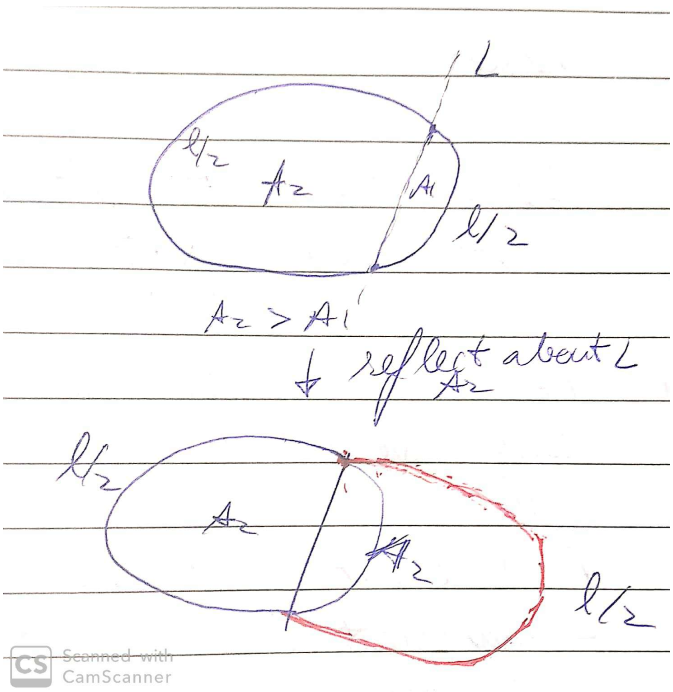

Figure 1.2: An asymmetric shape about L is not optimal.

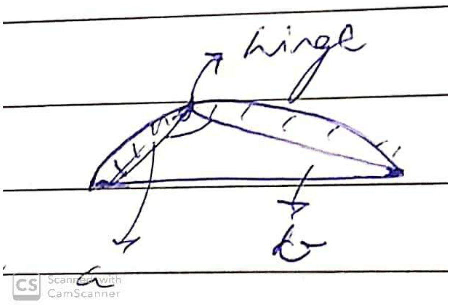

Figure 1.3: Any shape that is not the semicircle on L is not optimal.

fig. 1.3 (the shaded region has fixed area, only the inscribed triangle are can be varied). Fixing any one of a or b as the base, one may obtain the greatest area for the inscribed triangle by making the other side perpendicular to the base so as to get the maximum height (Area of $\Delta = 1/2 \times$ base $\times$ height). This achieves the maximum area of $\frac{ab}{2}$ for the inscribed triangle. Note that the angle subtended by the line segment L at the fixed hinge point is $\pi/2$ for the optimal solution. However the hinge point was arbitrarily chosen to be on the rope. Hence the line segment subtends an angle of $\pi/2$ at every point on the rope hat traces the optimal solution. This geometric property uniquely identifies that the optimal solution cannot be anything that is not a semicircle on L.

Combining the above arguments, we see that any shape that is not a **circle** cannot be the optimal solution. This approach is due to Jakob Steiner. The **logic** underlying all arguments was that there was a method by which area could be increased while keeping the perimeter lesser than or equal to the given length l.

Claimed Theorem: Of all positive integers, 1 is the largest.

Claimed Proof: Based on the same logic that was stated above, given any positive integer that is not 1, we can increase its value by some method (say by squaring). Hence proved.

Flaw in logic: We may not have an optimal solution in the first place. The above example shows that we cannot take the existence of optimal solution for granted. We must prove the existence of an optimal solution. This naturally needs a review of concepts from Analysis, here in particular that of Real Analysis, which forms the topic of our next section 1.2.

§1.1.iii Another Approach

Here, I independently give an alternative proof to the isoperimetric problem, which is only based on reflection symmetrization and hence simpler to follow.

Proof. I shall use two lemmas.

Lemma 1.1.7 (Reflection symmetrization for non-convex to convex sets)
Any S that is not convex, can be transformed to a convex set with increase in the area enclosed while keeping its perimeter constant.

Proof of Lemma by Contradiction. Denote the region enclosed in the rope by S. Suppose that S is not convex. Then there must exist at least one pair of points $x, y \in S$ such that the line segment L in not entirely inside S, ie. $\lambda x + (1-\lambda)y \notin S$, $\forall\, \lambda \in [0,1]$. In case the set of points on the segment L not in S are isolated, they have measure zero and the area enclosed is unaffected by their addition, hence these points can be added to make S convex. In case the set of points on $L \notin S$ are not isolated, the measure (one-dimensional) may be positive and hence we cannot simply add them as the perimeter could increase. In this case, take the minimal union of intervals on L which cover these points to give subsegments of L. For each such subsegment, reflect the points on the rope about that subsegment as shown in fig. 1.4. Reflection in the plane being an isometry, keeps the length of the points on the rope and hence that of the entire rope fixed, but the reflected part of the rope encloses more area.

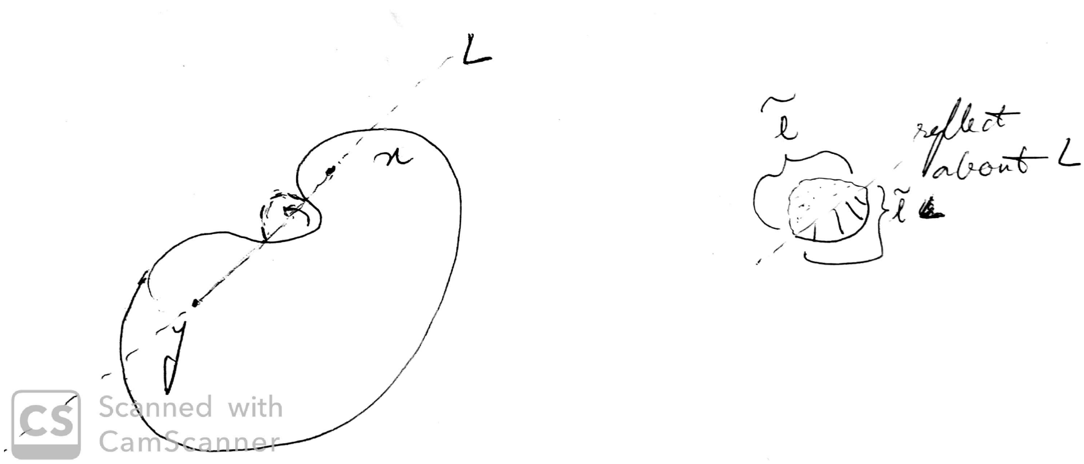

Figure 1.4: Any shape that is not the semicircle on L is not optimal.

□

Lemma 1.1.8 (Reflection Symmetrization of Area)
Given any two points on the rope, dividing it into arcs of equal length, the line segment joining them must also divide the optimal S into two parts of equal area.

Proof. Suppose not, then reflect part of greater area in the line segment to increase the bounded area. This keeps the perimeter constant as reflection in the plane is an isometry. Clearly then, the S we started with was not optimal. □

Suppose we assume that the optimal solution exists, lemma 1.1.7 gives us that the optimal S must be convex, in particular it is connected and lemma 1.1.8 gives us that the optimal S must have a reflection axis of symmetry in every direction. This uniquely identifies the optimal S as the region in the plane enclosed by a circle. □

Exercise 1.1.9. In the above proof, a subtle result was used: Show that the segments found by joining two points on the rope that divide it into arcs of equal length, are in bijective correspondence with directions in the plane.

Exercise 1.1.10. Show that there exists an optimal solution to the isoperimetric problem.

Remark 1.1.11 (Method of Moving Planes) — The technique I used in the above proof can be generalized to give what is called the **Method of Moving planes** [Ca].

§1.2 Review of Real Analysis

Definition 1.2.1. (**Sequences in $\mathbb{R}^n$**) Function $f\colon \mathbb{N} \to \mathbb{R}^n$ defines a collection of points in $\mathbb{R}^n$ which is called a sequence and denoted by $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$.

Definition 1.2.2. (**Bounded Sequences**) We say $\{x_k\}_{k\in\mathbb{N}}$ is bounded if

$$\exists\, M > 0 \quad \text{such that} \quad \|x_k\| \leq M \quad \forall\, k \in \mathbb{N}$$

Example 1.2.3 (Unbounded sequence)
$x_k = k^2$ is not bounded.

Example 1.2.4 (Bounded but not convergent sequence)
$x_k = (-1)^k$ is bounded. Note that even though it is bounded, it is oscillating. There is no single value which the sequence eventually attains or settles towards.

Definition 1.2.5 (Convergence)**.** We say that $\{x_k\}_{k\in\mathbb{N}}$ **converges to** x^* if

$$\forall\, \epsilon > 0; \quad \exists\, n_0 \in \mathbb{N}; \quad \text{such that } \|x_k - x*\| < \epsilon \quad \forall\, k > n_0$$

x^* is called the **limit** of the sequence.

Proposition 1.2.6

$x_k = \frac{1}{k}$ converges to 0.

Proof. Given any $\epsilon > 0$, take $n_0 = \lceil \frac{1}{\epsilon} \rceil + 1$ so that $x_{n_0} - 0 < \epsilon$. $\{x_k\}_{k \in \mathbb{N}}$ being a monotone decreasing sequence, satisfies the criterion for convergence to 0. □

Proposition 1.2.7

If the limit of a sequence exists, it is unique.

Proof. Suppose $\{x_k\}_{k \in \mathbb{N}}$ has two distinct limits l_1 and l_2. Straightforward application of Triangle Inequality gives

$$\|l_1 - l_2\| \leq \|l_1 - x_k\| + \|x_k - l_2\|$$

Since $\{x_k\}_{k \in \mathbb{N}}$ converges to both l_1 and l_2, both the terms on the right hand side are arbitrarily close to 0 for sufficiently large k, ie. For any small $\epsilon > 0$

$$\|l_1 - l_2\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for} \quad k > n_0$$

$$\implies \|l_1 - l_2\| = 0$$

By definition of metric, $l_1 = l_2$ □

Definition 1.2.8. A sequence $\{x_k\}_{k \in \mathbb{N}}$ **does not converge** if

$$\forall\, x^* \in \mathbb{R}^n,\, \exists\, \epsilon > 0; \quad \forall\, n_0 \in \mathbb{N}; \quad \|x_k - x*\| \geq \epsilon \quad \text{for some} \quad k > n_0$$

The above definition can be used to generate a point $x_{k_{n_0}}$ by taking for each n_0, the smallest k for which the condition in the definition holds. This is itself a sequence $\{x_{k_{n_0}}\}_{n_0 \in \mathbb{N}}$. This gives rise to the notion of a subsequence.

Definition 1.2.9. A **subsequence** $\{x_{k_i}\}_{i \in \mathbb{N}}$ of $\{x_k\}_{k \in \mathbb{N}}$ is a sub-collection of points x_{k_i} from the original sequence such that $k_{i+1} > k_i$ is satisfied.

For more details of the notions defined in the above review of Real Analysis, there are many standard references such as [Ru+64] or [Pu02].

2 Lecture Jan 14

§2.1 Bolzano-Weierstrass Theorem

Last time we gave an introduction to optimization and reviewed basic concepts from Real Analysis (section 1.2). We defined **sequence (definition 1.2.1), bounded sequence, convergence, not convergent, limit and subsequence**.

Proposition 2.1.1

If a sequence $\{x_k\}_{k\in\mathbb{N}}$ converges to a limit L, then every subsequence $\{x_{k_i}\}_{i\in\mathbb{N}}$ converges, further it converges to the same limit L. Conversely, if all subsequences converge, then the sequence itself converges.

Proof of converse($\Longleftarrow$). The sequence itself is also a subsequence of itself. □

Proof of direct ($\Longrightarrow$). Convergence of subsequence is trivial, follows straight from the definition. Uniqueness of limit proof by use of triangle inequality is similar to proof of proposition 1.2.7. □

Theorem 2.1.2 (Bolzano Weierstrass)

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

We will state a more general version later in section 3.1.

Remark 2.1.3 — This theorem is very useful to show convergence of algorithms. Challenge in analysis of algorithm. Bolzano-Weierstrass theorem gives at least a subsequence that convergence.

Include diagram (later)

Example 2.1.4 (Rational Numbers)

S = Set of positive Rational Numbers.

$$S = \left\{\frac{m}{n} | m, n > 0 \quad \text{for} \quad m, n \in \mathbb{N}\right.$$

Choose rationals whose squares are bounded by 2, ie, $\{x \in S : x^2 \leq 2\}$. By Bolzano-Weierstrass, there exists a subsequence of rationals converging to $\sqrt{2}$

§2.2 GLB and LUB

Definition 2.2.1 (**Lower Bound**). $L \in \mathbb{R}$ is a lower bound of a set S if
$\forall x \in S;\ x \geq L$

Definition 2.2.2 (**Upper Bound**). $U \in \mathbb{R}$ is an upper bound of a set S if $\forall x \in S;\ x \leq U$

If a set S is **bounded**, for some $M > 0$, $\quad |x| \leq M; \quad \forall x \in S$
$\implies -M \leq x \leq M \quad \forall x \in S$
$\implies M$ is an upper bound and $-M$ is a lower bound on S

Definition 2.2.3 (**Least Upper bound (lub)**)**.** Least of all upper bounds. M is said to be the least upper bound in $S \subseteq \mathbb{R}$ if $\forall \epsilon > 0$; $\exists x \in S$ such that

$$x > M - \epsilon$$

Definition 2.2.4 (**Greatest Lower bound (glb)**)**.** Greatest of all lower bounds. m is said to be the greatest lower bound in $S \subseteq \mathbb{R}$ if $\forall \epsilon > 0$; $\exists x \in S$

$$x < m + \epsilon$$

Completeness Axiom : Chicken and egg problem: to show existence of glb, we need lub to exist and vice versa. Assume that glb and lub of any subset of $\mathbb{R}$ exist. (Comment: Based on what I had seen in the Real Analysis course, Real numbers are complete by construction (Dedekind cuts or Cauchy's construction). Well-ordering, Zorn's Lemma and Axiom of choice all are equivalent and give the existence of lub or glb so maybe the Prof. didn't want to go into this discussion and replaced it with the so-called completeness axiom. I asked this in class but due to video recording (CDEEP) constraints, the Prof. called it a digression.)

§2.3 Open sets, Closed sets and Compact Sets

Let $S \subseteq \mathbb{R}^n$.
Define for $r > 0$, $\quad B(x, r) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$

Definition 2.3.1 (Open Set)**.** A set S is an **Open Set** if $\forall x \in S$, $\exists r > 0$ such that $B(x, r) \subseteq S$.

Example 2.3.2 (Discrete points)

$S = \{x_1, x_2, \cdots, x_n\}$. Then S is not open. Any open ball contains infinitely many points. Hence no ball can be contained in S.

Definition 2.3.3 (Closed Set)**.** A set S is a **Closed Set** if its complement is open.

Definition 2.3.4 (Compact Set)**.** A set $S \subseteq \mathbb{R}^n$ is a **Compact Set** if its closed and bounded.

Remark 2.3.5 (Heine-Borel Theorem) — The above definition of compact sets is actually a characterization of compact sets in $\mathbb{R}^n$ made possible by the Heine-Borel Theorem which states that in Euclidean spaces compact sets are equivalent to those that are closed and bounded. Compact $\implies$ closed and bounded is true in any Metric space but the converse may not hold. It holds in the case of Euclidean metric.

Remark 2.3.6 (Dependence on Ambient Space) — The notions of open and closed depend on the ambient space. However the notion of compactness does not.

Example 2.3.7 (Open intervals are not open sets in $\mathbb{R}^2$)

Open intervals are open sets when viewed as subsets of $\mathbb{R}$, as open balls in $\mathbb{R}$ are open intervals itself. However open sets in $\mathbb{R}^2$ are open discs and they are not contained in any open interval. Hence Open intervals are not open sets in $\mathbb{R}^2$. In fact they are neither open nor closed (check boundary points).

Exercise 2.3.8. Give examples for the statement on compact sets in remark 2.3.6.

Proposition 2.3.9

The following operations preserve notions of openness, closedeness and compactness.

- Union of any collection open sets is open.
- Intersection of any collection closed sets is closed.
- Union of finite collection closed sets is closed.
- Intersection of finite collection open sets is open.
- Intersection of any collection of compact sets is compact.
- Union of any collection of finitely many compact sets is compact.

Proof. Each compact set is closed and bounded. Take the maximum of of lub's of the radius of Balls centred at the origin containing the sets. This maximum is finite as the maximum of finitely many numbers is finite. Hence the union is bounded. Finite union of closed sets is closed. By characterization of compact sets in $\mathbb{R}^n$, the union is a compact set. $\square$

Question 2.3.10 (Answer)**.** Why is the intersection of infinitely many open sets not necessarily open?

Demonstrate by example: Take a sequence of open sets where the n^{th} term is $(-1/n, 1/n)$. Then the intersection is singleton set 0 which is not open.

§2.4 Continuity

Definition 2.4.1. $f \colon \mathbb{R}^n \to \mathbb{R}$ is said to be **continuous at a point** $x \in \mathbb{R}^n$ if $\forall \epsilon > 0, \quad \exists \delta > 0, \quad$ such that

$$|f(y) - f(x)| < \epsilon, \quad \forall y \in B(x, \delta)$$

We say f is **continuous** if it is continuous at all points in the domain.

Definition 2.4.2. f is **sequentially continuous** at x if $\forall \{x_k\}_{k \in \mathbb{N}} \to x$ then

$$\lim_{k \to \infty} f(x_k) = f(x) = f(\lim_{k \to \infty} x_k)$$

Proposition 2.4.3

Sequentially continuous is equivalent to continuous in metric spaces (Assuming Axiom of choice or Axiom of Countability).

Proof. Out of scope for this course. If interested, refer to proofwiki. □

§2.5 Weierstrass theorem

We restate the optimization problem

Question 2.5.1. Find an $x^* \in S$ such that

$$f(x^*) \leq f(x) \quad \forall x \in S$$

Or sometimes we don't necessarily want x^* but the value of $f(x^*)$ such that

$$f(x^*) \leq f(x) \quad \forall x \in S$$

Question 2.5.2. Given that $\inf_{x \in S} f(x)$ exists, does there exist $x^* \in S$ such that $f(x^*) = \inf_{x \in S} f(x)$

(Answer) **NO**, Demonstrate by example. $\exp(-x)$ has inf 0 but no real x attains it, as seen in fig. 2.1

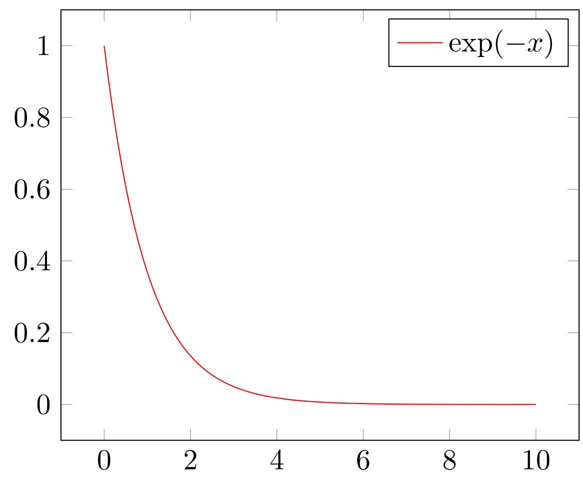

Figure 2.1: Example showing infimum exists but is not attained for any $x \in \mathbb{R}$

As seen in fig. 2.2, for the discontinuous function, infimum is the left hand limit of f at a however no $x \in \mathbb{R}$ attains it.

Theorem 2.5.3 (Weierstrass)

Let $S \in \mathbb{R}^n$ be closed and bounded (compact). Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous. Then f attains its infimum on S, ie. $\exists x^* \in S$ that $f(x*) = \inf\{f(x) | x \in S\}$. In this case $\min\{f(x) | x \in S\}$ is used to denote that the infimum is attained.

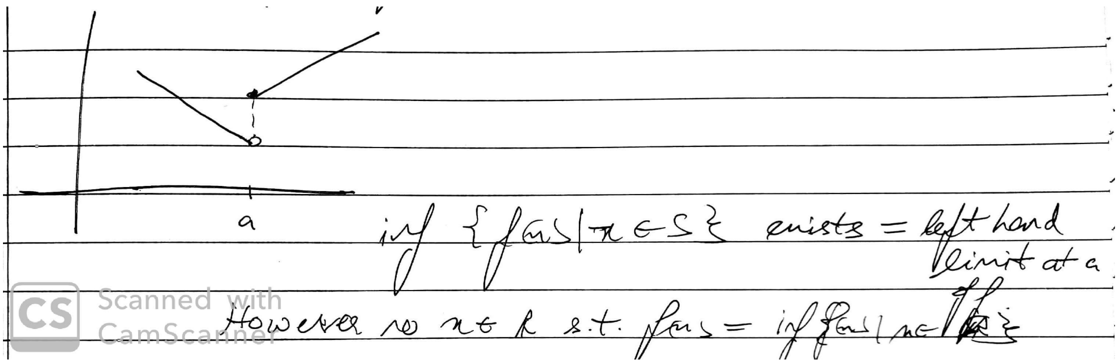

Figure 2.2: Infimum exists and is the left hand limit of f at $x = a$ but not attained in $S = \mathbb{R}$

Remark 2.5.4 (Convention to find min is justified) — take g to be negative of f then sup becomes inf.

3 Lecture January 17

§3.1 Weierstrass theorem

Theorem 3.1.1 (Weierstrass)
Let $S \subseteq \mathbb{R}^n$ be closed and bounded. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then $\exists\ x^* \in S$ such that

$$f(x^*) = \inf \{f(x)\,|x \in S\}$$

Infimum of f is attained in S.
$\exists\, \hat{x} \in S$ such that

$$f(\hat{x}) = \sup \{f(x)\,|x \in S\}$$

Problems demonstrated by examples given previously in chapter 2 are treated by avoiding the following conditions in Weierstrass's theorem:

1. discontiuity in f
2. S is unbounded
3. S is not closed

Definition 3.1.2 (Open Set)**.** Set S is said to be **Open** if $\forall x \in S$, $\exists\, r > 0$ such that $B(x, r) \subseteq S$.

Definition 3.1.3 (Interior)**.** Let $C \subseteq \mathbb{R}^n$ be any set.

$$\dot{C} = \text{interior of C} = \bigcup_{S \text{ is open} \,\& S \subseteq C} S$$

Definition 3.1.4 (Closure)**.** Let $C \subseteq \mathbb{R}^n$ be any set.

$$\overline{C} = \text{closure of C} = \bigcap_{S \text{ is closed} \,\& S \supseteq C} S$$

Definition 3.1.5 (Boundary)**.** $\partial C = \overline{C} \setminus \dot{C}$.

Remark 3.1.6 — **Interior** of a set is an open set since it is an arbitrary union of open sets. In fact it is the largest open set contained in C. **Closure** of a set is a closed set since it is an arbitrary intersection of closed sets (proposition 2.3.9). In fact it is the smallest closed set containing C. It follows that if C is open, $\dot{C} = C$ and if C is closed, then $\overline{C} = C$. This means that if C is open, ∂C contains no point of C and if C is closed, all points of ∂C are points of C.

Example 3.1.7 (NaN error in Matlab)
For our algorithms to find the optimal solution, we must work with compact sets so that the domain is bounded and the boundary points can be reached. In open sets, the algorithm will keep searching and never reach the boundary. Hence taking closure will provide a handy tool.

Definition 3.1.8 (Feasible point)**.** Denote by S, the feasible region, ie. $S =$ feasible region. Then any point in S is called **feasible point**

Definition 3.1.9 (Infeasible points)**.** Points not in S are called **infeasible points**.

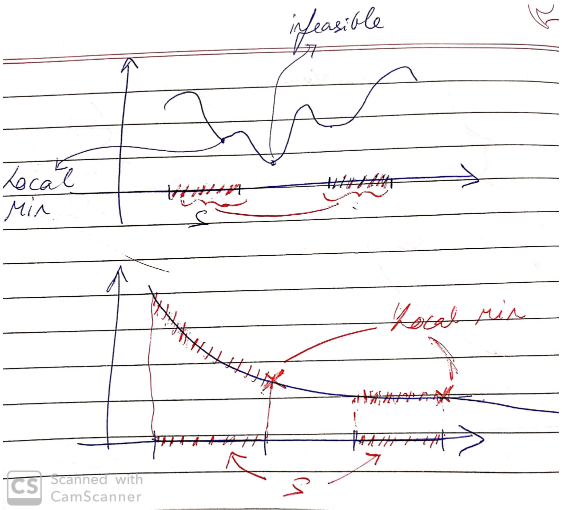

Figure 3.1: Local minimum of f in S

Definition 3.1.10 (Local Minimum)**.** $x^* \in S$ is said to be a local minimum if $\exists\, r > 0$ s.t.

$$f(x^*) \leq f(x) \quad \forall x \in B(x^*, r) \cap S$$

For an example consider Figure 3.1.

Definition 3.1.11 (Global minimum)**.** $x^* \in S$ is said to be a **global minimum** if

$$f(x^*) \leq f(x) \quad \forall x \in S$$

Definition 3.1.12 (Unconstrained minimum)**.** $x^* \in \mathbb{R}^n$ is said to be a **global minimum** if

$$f(x^*) \leq f(x) \quad \forall x \in \mathbb{R}^n$$

Remark 3.1.13 — The unconstrained minimum need not be finite or may not exist. f may not even be defined outside S so all kinds of possibilities exist.

Remark 3.1.14 (Global Minimum) — Weierstrass Theorem gives existence of a global minimum.

Definition 3.1.15 (Strict local minimum)**.** $x^* \in S$ is said to be a **strict local minimum** if $\exists\, r > 0$ s.t.

$$f(x^*) < f(x) \quad \forall x \in B(x^*, r) \cap S,\ x \neq x^*$$

Definition 3.1.16 (Isolated local Minimum)**.** $x^* \in S$ is said to be a **isolated local minimum** if $\exists\, r > 0$ s.t. x^* is the only local minimum in $B(x^*, r) \cap S$

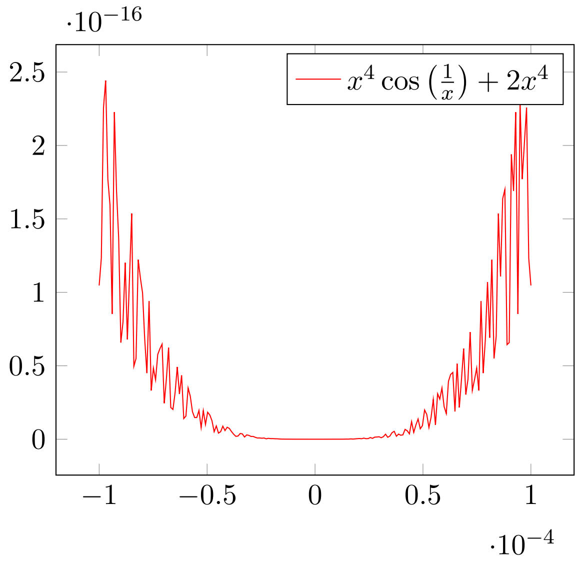

Figure 3.2: Example showing strict local minima does not $\implies$ isolated local minima

Proposition 3.1.17 (Isolated $\implies$ strict)
Every isolated point is a strict local minimum.

Proof by contradiction. Let x^* be an isolated point. Suppose x^* is not a strict local minimum point then

$$\forall r > 0, \exists x \in B(x^*, r) \cap S, x \neq x^* \quad \text{s.t.} \quad f(x^*) = f(x)$$

$\implies$ for small enough r, x is a local minima. $\implies$ x^* is not isolated. $\square$

However, the converse is not true. **Every strict local minimum need not be isolated**. Proof by example: Consider the following function plotted in Figure 3.1

$$f(x) = \left\{ x^4 \cos\left(\frac{1}{x}\right) + 2x^4 \quad x \in [-1, 1] \setminus 0; \quad 0 \text{ at } x = 0 \right.$$

§3.2 Optimization with Constraints

Goal: min $f(x)$ ie. the objective function, s.t. $x \in S$, ie. the feasible region.

> **Question 3.2.1.** What if there are additional constraints?

Goal: min $f(x)$ s.t. $g_i(x) \leq 0 \quad \forall i = 1, \ldots, m; \quad h_j(x) = 0 \quad \forall j = 1, \ldots, p$ are satisfied.

Here $g_i : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ are called **constraints**. Those invloving $\leq$ are called Inequality constraints, and those involving $=$ are called Equality constraints. Geometry of the problem is very different in the two different kinds of constraints. Algorithms are allowed to search in all directions if interior is within the feasible regions. Equality can always be replaced by two inequalities in the opposite direction, that is $h_j(x) = 0 \iff h_j(x) \leq 0 \;\&\; -h_j(x) \leq 0$.

§3.3 Other type of constraints

1. **Bound constraints**: $m \leq h_j(x) \leq M$
2. **Either-or** constraints: Need not satisfy all constraints but just one or more out of them.
3. **if-then-else** constraint: choice followed by taking decisions.

We shall deal with only inequality and equality constraints.

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