Optimization
SC 607 IITB
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https://somphene.github.io/notes/

Notes
by
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A visitor remarked that the library at the newly constructed IHES was lacking. Grothendieck replied "We don't read books, we write them."

This is an **incomplete draft**. Please send corrections, comments, pictures of bad drawings, etc. to somphene1@gmail.com.

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Contents

3 Lecture January 17

§3.1 Weierstrass theorem

Theorem 3.1.1 (Weierstrass)
Let $S \subseteq \mathbb{R}^n$ be closed and bounded. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then $\exists\ x^* \in S$ such that

$$f(x^*) = \inf\{f(x)\,|x \in S\}$$

Infimum of f is attained in S.
$\exists\, \hat{x} \in S$ such that

$$f(\hat{x}) = \sup\{f(x)\,|x \in S\}$$

Problems demonstrated by examples given previously in **??** are treated by avoiding the following conditions in Weierstrass's theorem:

1. discontiuity in f
2. S is unbounded
3. S is not closed

Definition 3.1.2 (Open Set)**.** Set S is said to be **Open** if $\forall x \in S$, $\exists r > 0$ such that $B(x, r) \subseteq S$.

Definition 3.1.3 (Interior)**.** Let $C \subseteq \mathbb{R}^n$ be any set.

$$\dot{C} = \text{interior of C} = \bigcup_{S \text{ is open } \& S \subseteq C} S$$

Definition 3.1.4 (Closure)**.** Let $C \subseteq \mathbb{R}^n$ be any set.

$$\overline{C} = \text{closure of C} = \bigcap_{S \text{ is closed } \& S \supseteq C} S$$

Definition 3.1.5 (Boundary)**.** $\partial C = \overline{C} \setminus \dot{C}$.

Remark 3.1.6 — **Interior** of a set is an open set since it is an arbitrary union of open sets. In fact it is the largest open set contained in C. **Closure** of a set is a closed set since it is an arbitrary intersection of closed sets (**??**). In fact it is the smallest closed set containing C. It follows that if C is open, $\dot{C} = C$ and if C is closed, then $\overline{C} = C$. This means that if C is open, ∂C contains no point of C and if C is closed, all points of ∂C are points of C.

Example 3.1.7 (NaN error in Matlab)
For our algorithms to find the optimal solution, we must work with compact sets so that the domain is bounded and the boundary points can be reached. In open sets, the algorithm will keep searching and never reach the boundary. Hence taking closure will provide a handy tool.

Definition 3.1.8 (Feasible point)**.** Denote by S, the feasible region, ie. $S =$ feasible region. Then any point in S is called **feasible point**

Definition 3.1.9 (Infeasible points)**.** Points not in S are called **infeasible points**.

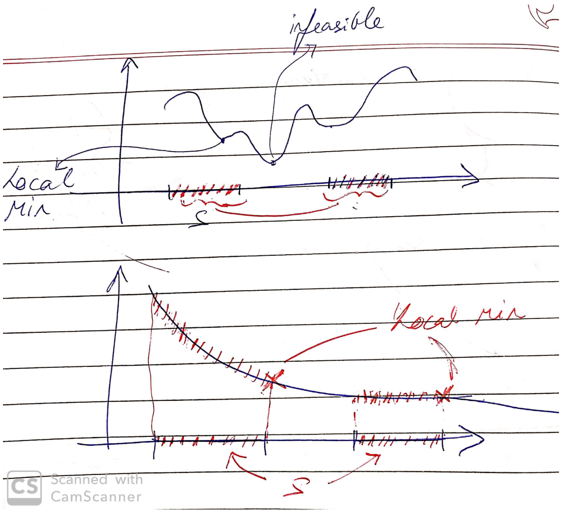

Figure 3.1: Local minimum of f in S

Definition 3.1.10 (Local Minimum)**.** $x^* \in S$ is said to be a local minimum if $\exists\, r > 0$ s.t.

$$f(x^*) \leq f(x) \quad \forall x \in B(x^*, r) \cap S$$

For an example consider Figure 3.1.

Definition 3.1.11 (Global minimum)**.** $x^* \in S$ is said to be a **global minimum** if

$$f(x^*) \leq f(x) \quad \forall x \in S$$

Definition 3.1.12 (Unconstrained minimum)**.** $x^* \in \mathbb{R}^n$ is said to be a **global minimum** if

$$f(x^*) \leq f(x) \quad \forall x \in \mathbb{R}^n$$

Remark 3.1.13 — The unconstrained minimum need not be finite or may not exist. f may not even be defined outside S so all kinds of possibilities exist.

Remark 3.1.14 (Global Minimum) — Weierstrass Theorem gives existence of a global minimum.

Definition 3.1.15 (Strict local minimum)**.** $x^* \in S$ is said to be a **strict local minimum** if $\exists\, r > 0$ s.t.

$$f(x^*) < f(x) \quad \forall x \in B(x^*, r) \cap S,\ x \neq x^*$$

Definition 3.1.16 (Isolated local Minimum)**.** $x^* \in S$ is said to be a **isolated local minimum** if $\exists\, r > 0$ s.t. x^* is the only local minimum in $B(x^*, r) \cap S$

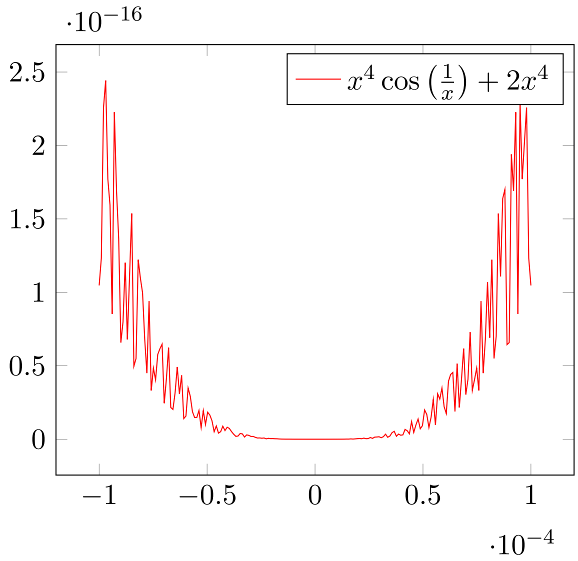

Figure 3.2: Example showing strict local minima does not $\implies$ isolated local minima

Proposition 3.1.17 (Isolated $\implies$ strict)
Every isolated point is a strict local minimum.

Proof by contradiction. Let x^* be an isolated point. Suppose x^* is not a strict local minimum point then

$$\forall r > 0, \exists x \in B(x^*, r) \cap S, x \neq x^* \quad \text{s.t.} \quad f(x^*) = f(x)$$

$\implies$ for small enough r, x is a local minima. $\implies$ x^* is not isolated. $\square$

However, the converse is not true. **Every strict local minimum need not be isolated**. Proof by example: Consider the following function plotted in Figure 3.1

$$f(x) = \begin{cases} x^4 \cos\left(\dfrac{1}{x}\right) + 2x^4 & x \in [-1, 1] \setminus 0; \quad 0 \text{ at } x = 0 \end{cases}$$

§3.2 Optimization with Constraints

Goal: min $f(x)$ ie. the objective function, s.t. $x \in S$, ie. the feasible region.

Question 3.2.1. What if there are additional constraints?

Goal: min $f(x)$ s.t. $g_i(x) \leq 0 \quad \forall i = 1, \ldots, m; \quad h_j(x) = 0 \quad \forall j = 1, \ldots, p$ are satisfied.

Here $g_i : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ are called **constraints**. Those invloving $\leq$ are called Inequality constraints, and those involving $=$ are called Equality constraints. Geometry of the problem is very different in the two different kinds of constraints. Algorithms are allowed to search in all directions if interior is within the feasible regions. Equality can always be replaced by two inequalities in the opposite direction, that is $h_j(x) = 0 \iff h_j(x) \leq 0 \;\&\; -h_j(x) \leq 0$.

§3.3 Other type of constraints

1. **Bound constraints**: $m \leq h_j(x) \leq M$
2. **Either-or** constraints: Need not satisfy all constraints but just one out of them.
3. **if-then-else** constraint: choice followed by taking decisions.

We shall deal with only inequality and equality constraints.