Optimization
SC 607 IITB
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https://somphene.github.io/notes/

Notes
by
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Spring 2020

A visitor remarked that the library at the newly constructed IHES was lacking.
Grothendieck replied "We don't read books, we write them."

This is an **incomplete draft**. Please send corrections, comments, pictures of bad drawings, etc. to somphene1@gmail.com.

Last updated January 25, 2020.

Contents

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§2.1 Bolzano-Weierstrass Theorem

Last time we gave an introduction to optimization and reviewed basic concepts from Real Analysis (**??**). We defined **sequence (??), bounded sequence, convergence, not convergent, limit and subsequence**.

Proposition 2.1.1

If a sequence $\{x_k\}_{k\in\mathbb{N}}$ converges to a limit L, then every subsequence $\{x_{k_i}\}_{i\in\mathbb{N}}$ converges, further it converges to the same limit L. Conversely, if all subsequences converge, then the sequence itself converges.

Proof of converse($\Longleftarrow$). The sequence itself is also a subsequence of itself. □

Proof of direct ($\Longrightarrow$). Convergence of subsequence is trivial, follows straight from the definition. Uniqueness of limit proof by use of triangle inequality is similar to proof of **??**. □

Theorem 2.1.2 (Bolzano Weierstrass)

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

We will state a more general version later in **??**.

Remark 2.1.3 — This theorem is very useful to show convergence of algorithms. Challenge in analysis of algorithm. Bolzano-Weierstrass theorem gives at least a subsequence that convergence.

Include diagram (later)

Example 2.1.4 (Rational Numbers)

S = Set of positive Rational Numbers.

$$S = \left\{\frac{m}{n} | m, n > 0 \quad \text{for} \quad m, n \in \mathbb{N}\right.$$

Choose rationals whose squares are bounded by 2, ie, $\{x \in S : x^2 \leq 2\}$. By Bolzano-Weierstrass, there exists a subsequence of rationals converging to $\sqrt{2}$

§2.2 GLB and LUB

Definition 2.2.1 (**Lower Bound**)**.** $L \in \mathbb{R}$ is a lower bound of a set S if
$\forall x \in S;\ x \geq L$

Definition 2.2.2 (**Upper Bound**)**.** $U \in \mathbb{R}$ is an upper bound of a set S if $\forall x \in S;\ x \leq U$

If a set S is **bounded**, for some $M > 0$, $\quad |x| \leq M; \quad \forall x \in S$
$\implies -M \leq x \leq M \quad \forall x \in S$
$\implies M$ is an upper bound and $-M$ is a lower bound on S

Definition 2.2.3 (**Least Upper bound (lub)**)**.** Least of all upper bounds. M is said to be the least upper bound in $S \subseteq \mathbb{R}$ if $\forall \epsilon > 0$; $\exists x \in S$ such that

$$x > M - \epsilon$$

Definition 2.2.4 (**Greatest Lower bound (glb)**)**.** Greatest of all lower bounds. m is said to be the greatest lower bound in $S \subseteq \mathbb{R}$ if $\forall \epsilon > 0$; $\exists x \in S$

$$x < m + \epsilon$$

Completeness Axiom : Chicken and egg problem: to show existence of glb, we need lub to exist and vice versa. Assume that glb and lub of any subset of $\mathbb{R}$ exist. (Comment: Based on what I had seen in the Real Analysis course, Real numbers are complete by construction (Dedekind cuts or Cauchy's construction). Well-ordering, Zorn's Lemma and Axiom of choice all are equivalent and give the existence of lub or glb so maybe the Prof. didn't want to go into this discussion and replaced it with the so-called completeness axiom. I asked this in class but due to video recording (CDEEP) constraints, the Prof. called it a digression.)

§2.3 Open sets, Closed sets and Compact Sets

Let $S \subseteq \mathbb{R}^n$.
Define for $r > 0$, $\quad B(x, r) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$

Definition 2.3.1 (Open Set)**.** A set S is an **Open Set** if $\forall x \in S$, $\exists r > 0$ such that $B(x, r) \subseteq S$.

Example 2.3.2 (Discrete points)

$S = \{x_1, x_2, \cdots, x_n\}$. Then S is not open. Any open ball contains infinitely many points. Hence no ball can be contained in S.

Definition 2.3.3 (Closed Set)**.** A set S is a **Closed Set** if its complement is open.

Definition 2.3.4 (Compact Set)**.** A set $S \subseteq \mathbb{R}^n$ is a **Compact Set** if its closed and bounded.

Remark 2.3.5 (Heine-Borel Theorem) — The above definition of compact sets is actually a characterization of compact sets in $\mathbb{R}^n$ made possible by the Heine-Borel Theorem which states that in Euclidean spaces compact sets are equivalent to those that are closed and bounded. Compact $\implies$ closed and bounded is true in any Metric space but the converse may not hold. It holds in the case of Euclidean metric.

Remark 2.3.6 (Dependence on Ambient Space) — The notions of open and closed depend on the ambient space. However the notion of compactness does not.

Example 2.3.7 (Open intervals are not open sets in $\mathbb{R}^2$)

Open intervals are open sets when viewed as subsets of $\mathbb{R}$, as open balls in $\mathbb{R}$ are open intervals itself. However open sets in $\mathbb{R}^2$ are open discs and they are not contained in any open interval. Hence Open intervals are not open sets in $\mathbb{R}^2$. In fact they are neither open nor closed (check boundary points).

Exercise 2.3.8. Give examples for the statement on compact sets in remark 2.3.6.

Proposition 2.3.9

The following operations preserve notions of openness, closedeness and compactness.

- Union of any collection open sets is open.
- Intersection of any collection closed sets is closed.
- Union of finite collection closed sets is closed.
- Intersection of finite collection open sets is open.
- Intersection of any collection of compact sets is compact.
- Union of any collection of finitely many compact sets is compact.

Proof. Each compact set is closed and bounded. Take the maximum of of lub's of the radius of Balls centred at the origin containing the sets. This maximum is finite as the maximum of finitely many numbers is finite. Hence the union is bounded. Finite union of closed sets is closed. By characterization of compact sets in $\mathbb{R}^n$, the union is a compact set. □

Question 2.3.10 (Answer)**.** Why is the intersection of infinitely many open sets not necessarily open?

Demonstrate by example: Take a sequence of open sets where the n^{th} term is $(-1/n, 1/n)$. Then the intersection is singleton set 0 which is not open.

§2.4 Continuity

Definition 2.4.1. $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be **continuous at a point** $x \in \mathbb{R}^n$ if $\forall \epsilon > 0, \quad \exists \delta > 0, \quad$ such that

$$|f(y) - f(x)| < \epsilon, \quad \forall y \in B(x, \delta)$$

We say f is **continuous** if it is continuous at all points in the domain.

Definition 2.4.2. f is **sequentially continuous** at x if $\forall \{x_k\}_{k\in\mathbb{N}} \to x$ then

$$\lim_{k\to\infty} f(x_k) = f(x) = f(\lim_{k\to\infty} x_k)$$

Proposition 2.4.3

Sequentially continuous is equivalent to continuous in metric spaces (Assuming Axiom of choice or Axiom of Countability).

Proof. Out of scope for this course. If interested, refer to proofwiki. □

§2.5 Weierstrass theorem

We restate the optimization problem

Question 2.5.1. Find an $x^* \in S$ such that

$$f(x^*) \leq f(x) \quad \forall x \in S$$

Or sometimes we don't necessarily want x^* but the value of $f(x^*)$ such that

$$f(x^*) \leq f(x) \quad \forall x \in S$$

Question 2.5.2. Given that $\inf_{x\in S} f(x)$ exists, does there exist $x^* \in S$ such that $f(x^*) = \inf_{x\in S} f(x)$

(Answer) **NO**, Demonstrate by example. $\exp(-x)$ has inf 0 but no real x attains it, as seen in fig. 2.1

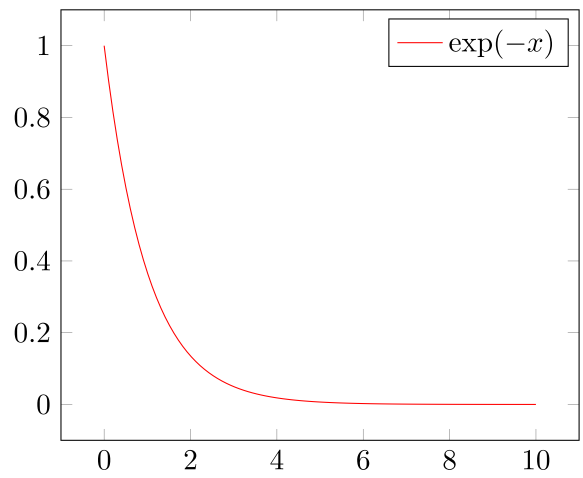

Figure 2.1: Example showing infimum exists but is not attained for any $x \in \mathbb{R}$

As seen in fig. 2.2, for the discontinuous function, infimum is the left hand limit of f at a however no $x \in \mathbb{R}$ attains it.

Theorem 2.5.3 (Weierstrass)

Let $S \in \mathbb{R}^n$ be closed and bounded (compact). Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Then f attains its infimum on S, ie. $\exists x^* \in S$ that $f(x*) = \inf\{f(x)| \, x \in S\}$. In this case $\min\{f(x)| \, x \in S\}$ is used to denote that the infimum is attained.

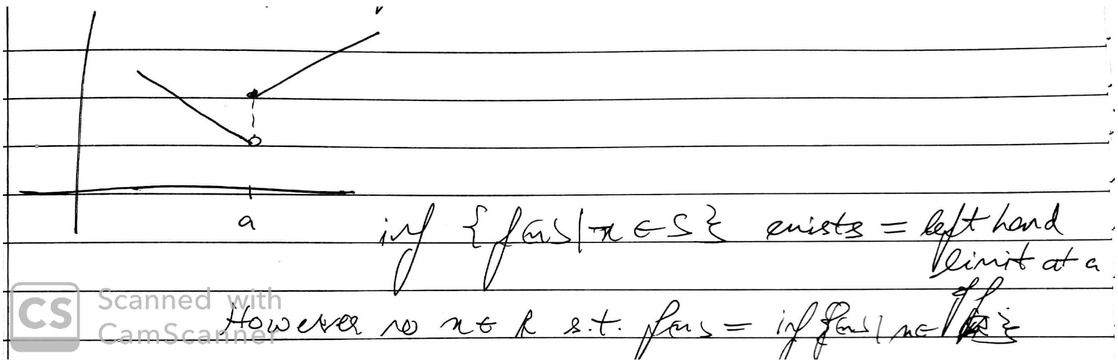

Figure 2.2: Infimum exists and is the left hand limit of f at $x = a$ but not attained in $S = \mathbb{R}$

Remark 2.5.4 (Convention to find min is justified) — take g to be negative of f then sup becomes inf.