· scattering, absorption and extinction cross sections
· Vector harmonics
· scattering by a sphere.

1 Formulation of scattering problem:

Input: incident radiation, object: scatterer.

Solving: B. C. on scatterer's surface are imposed.
Require continuity of tangential components of $\vec{E}, \vec{B}$ fields on the object's surface.

Decompose total field into superposition of different contributions:

i) Incident field, defined at each point of space as if the obstacle was not there.
Usually taken as a plane wave, but we can take it as a cylindrical or gaussian beam.

ii) scattered field: value of the field external to the object.

iii) Internal field in the obstacle.

These fields are represented as series expansions of suitable vector harmonics

Mathematical tool: Vector harmonics (or wavefunctions)

Vector functions: solutions of the vector wave equatio
Any E, B field or potential can be expanded in series of these harmonics with suitable coefficient

Particular properties of the vector harmonics allow one to easily compute the different fields and easily impose boundary conditions.
Using vector harmonics, one can express EM fields in the same form independently of the scatterer's shape.
∴ It will be possible to couple the spherical wave with cylindrical one.

2. Scattering, Extinction and absorption cross sections

cross section: quantity with dimension of Area related with EM power scattered or absorbed by the object interacting with the incident wave.

Incident : $\vec{E}_i$, $\vec{H}_i$
Scattered : $\vec{E}_s$, $\vec{H}_s$

Total Field outside scatterer: $\vec{E} = \vec{E}_i + \vec{E}_s$
$\vec{H} = \vec{H}_i + \vec{H}_s$

$$\vec{E}_i = \vec{E}_0 \, e^{i(\vec{k}.\vec{r} - \omega t)}$$

$$\vec{H}_i = \vec{H}_0 \, e^{i(\vec{k}.\vec{r} - \omega t)}$$

; $\vec{k}$ is the wave vector appropriate to surrounding medium

E_0, H_0 are electric and magnetic polarization vectors.
Time dependence is omitted ~~throughout~~ the paper (quasi static)

Poynting vector of total field:

$$S = \frac{1}{2} \, Re \left[E \times H^* \right]$$

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$$S = \frac{1}{2} Re[E \times H^*]$$

$$= \frac{1}{2} Re[(E_i + E_s) \times (H_i^* + H_s^*)]$$

$$= \frac{1}{2} Re[E_i \times H_i^*] + \frac{1}{2} Re[E_s \times H_s^*] +$$

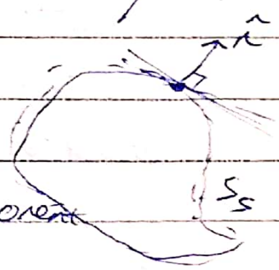
$$\frac{1}{2} Re[E_i \times H_s^* + E_s \times H_i^*]$$

$$S = S_i + S_s + S_e$$

S_i = poynting vector of incident wave.
S_s = " " scattered wave
S_e = " " interaction of incident and scattered fields

Inside the scatterer: $\vec{E}_p$, $\vec{H}_p$
Consider the scatterer's surface S_s, the amount of energy absorbed by the object can be computed as:

$$W_a = -\int_{S_s} \hat{n} \cdot S \, dS$$



(Equality due to continuity of tang. component of E and H field on the surface.

Applying Green's second vector thm, it can be proved that this integral is equal to the integral extended to a spherical surface S, centered on the scatterer with large radius

$$W_a = -\int_S \hat{n} \cdot S \, dS = W_i - W_s + W_e$$

where $W_i = -\int_S \hat{n} \cdot S_i \, dS$, $W_s = \int_S \hat{n} \cdot S_s \, dS$, $W_e = -\int_S \hat{n} \cdot S_e \, dS$

Power associated with incident wave is 0
(Incident power is both incoming and outgoing from surface (?)).

$W_e = W_a + W_s$

↳ sum of energy absorbed by scatterer and of the scattered energy

ie: Amount of energy subtracted from incident wave because of the interacⁿ with scatterer.

∴ called Extinction Power (W_e)

Define Cross sections:

$\sigma_s = \frac{W_s}{|S_i|}$ $\sigma_a = \frac{W_a}{|S_i|}$ $\sigma_e = \frac{W_e}{|S_i|}$ (dim = Ar

$\sigma_e = \sigma_a + \sigma_s$

Scattering cross secⁿ represents amount of power scattered by object over amount of power per unit area carried by incident wave.

Coefficients: $Q_s = \frac{\sigma_s}{G}$ $Q_a = \frac{\sigma_a}{G}$ $Q_e = \frac{\sigma_e}{G}$

G is particle cross sectional area projected onto plane ⊥ to incident wave.

3. Wave vector (Vector wave) Formalism

Hansen first proposed this sol^n to vector wave eq^n.
Maxwell's Eq^n:

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla \times H + \frac{\partial D}{\partial t} = J$$

Can describe any EM field or potential through Helmholtz Equation. Consider any generic field $\vec{C}$.
$C = \{E, H, D, B, F, A, \Pi e, \Pi m\}$ then such a field in the absence of sources and in a linear, isotropic, homogeneous medium, must satisfy the general homogeneous wave eq^n:

$$\nabla^2 C - \mu\varepsilon \frac{\partial^2 C}{\partial t^2} - \mu\sigma \frac{\partial C}{\partial t} = 0$$

In frequency domain: Helmholtz Eq^n

$$\nabla^2 \vec{C} + k^2 \vec{C} = 0$$

$$k^2 = \omega^2 \varepsilon_c \mu + i\sigma\omega \quad (\text{omitting } e^{-i\omega t})$$

This vector Eq^n can be projected along unit vectors of a generic ref system along 3 scalar differential Eq^{n}s. However such a system is not easy to solve in most coordinate systems.

For scalar is: $\nabla^2 \psi + k^2 \psi = 0$ sol^n known in all coord. systems

Define vector harmonics:

$$\vec{L} = \nabla\psi \qquad M = \nabla \times (\hat{a}\psi) \qquad N = \frac{1}{k}\nabla \times M$$

$\hat{a}$ is typically a const. unit vector called pilot vector ($\nabla \times \hat{a} = 0$ in gen.)

Check that these satisfy vector Helmholtz Eqn:
div of curl = 0 and $M = \nabla \times (\hat{a}\psi)$
$\therefore \nabla \cdot M = 0$, using vector identities:

$$\nabla^2 M + k^2 M = \nabla \times [\hat{a}(\underbrace{\nabla^2\psi + k^2\psi}_{0})] = 0$$

$\therefore$ M satisfies vector wave equation if ψ is solⁿ to scalar wave eqⁿ $\nabla^2\psi + k^2\psi = 0$
Similarly for N and L.

Properties: $M = \frac{1}{k}\nabla \times N$ and $\nabla \cdot L = \nabla^2\psi = -k^2\psi$

$$\nabla \times L = 0 \qquad \nabla \cdot M = 0 \qquad \nabla \cdot N = 0$$

L is irrotational, M and N are solenoidal.
$\therefore$ EM fields in the absence of sources can be expressed only by a superposition of M and N. while L must necessarily be taken into account only when EM sources.

generic soln ψ's to Helmholtz Eqn is a set of solutions ψ_n (eigenvectors) that form a vector space basis L^2 of square integrable funcⁿs (Hilbert Space)

Set of vectors L_n, M_n and N_n are associated with this set of scalar solutions.

Example: Consider Magnetic Potential $\vec{A}$ to be solⁿ of vector Helmholtz then:

$$A = \frac{1}{i\omega}\sum_{n=0}^{\infty}(a_n M_n + b_n N_n + c_n L_n)$$

Applying Maxwell's eqⁿs $E = \sum_{n=0}^{\infty}(a_n M_n + b_n N_n)$

$$H = \frac{k}{i\omega\mu}\sum_{n=0}^{\infty}(a_n N_n + b_n M_n)$$

In cartesian coordinates, sol^n of scalar Helmholtz Eq^n: (unitary amplitudes):

$$\psi(r) = e^{i\vec{k}\cdot\vec{r}}$$

Then

(24) $L = i\psi\vec{k}$ $\quad M = i\psi\vec{k}\times\hat{a}$ $\quad N = \frac{1}{k}\psi(k\times a_0)\times k$

These allow plane-wave expansions of a generic field

$$\psi(r) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(k_x, k_y)\, e^{ik\cdot r}\, dk_x\, dk_y$$

(25) $\psi(r) = \iint g(\alpha,\beta)\, e^{ik\cdot r}\, d\alpha\, d\beta$ $\quad$ | α, β angles of $\vec{k}$ with axes.

$g(\alpha,\beta)$: plane wave angular spectrum.

Insert (25) in (24):

$$\vec{L} = i\iint g(\alpha,\beta)\,\vec{k}(\alpha,\beta)\, e^{ik\cdot r}\, d\alpha\, d\beta$$

$$\vec{M} = i\iint g(\alpha,\beta)\,\vec{k}(\alpha,\beta)\times\hat{a}\, e^{ik\cdot r}\, d\beta\, d\alpha$$

$$\vec{N} = \frac{1}{k}\iint g(\alpha,\beta)\left[k(\alpha,\beta)\times\hat{a}\right]\times k(\alpha,\beta)\, e^{ik\cdot r}\, d\beta\, d\alpha$$

5. ## Scattering by a spherical object : 3D

Step 1. Determine the expressions of spherical harmonics (spherical vector wave functions) $M_{mn}(r)$ and $N_{mn}(r)$

Scalar Helmholtz Eq^n in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + k^2\psi = 0$$

Well known solution:

$$\psi(r) = A\, z_n(kr)\, P_n^m(\cos\theta)\, e^{im\varphi}$$

$$\psi(r) = A z_n(kr) P_n^m(\cos\theta) e^{im\varphi}$$

A is complex constant. $z_n(kr)$ represents a spherical Bessel function of 1st, 2nd, third or fourth kind

$$z_n(k_p\rho) = \{j_n(k_p\rho), y_n(k_p\rho), h_n^{(1)}(k_p\rho), h_n^{(2)}(k_p\rho)\}$$

$P_n^m(\cos\theta)$ is the associated Legendre function.

It is not easy to recognize a pilot vector $\hat{a}$ to insert ($\hat{r}, \hat{\theta}, \hat{\varphi}$) are not constant. We need $\hat{a}$ to be $\nabla \times \vec{a} = 0$ $\therefore$ $r\hat{r}$ is radial $\therefore$ curl is 0.

The vector harmonic : $M_{mn}(r) = \nabla \times [r\hat{r}\, \psi_{mn}(r)]$ is a solution of vector Helmholtz Equation.

$$M_{mn}(r) = \frac{im}{\sin\theta} z_n(kr) P_n^m(\cos\theta) e^{im\varphi}\, \hat{\theta}$$
$$- z_n(kr) \frac{\partial P_n^m(\cos\theta)}{\partial\theta} e^{im\varphi}\, \hat{\varphi}$$

$$N_{mn}(r) = \frac{z_n(kr)}{kr} n(n+1) P_n^m(\cos\theta) e^{im\varphi}\, \hat{r}$$
$$+ \frac{1}{kr} \frac{\partial[r z_n(kr)]}{\partial r} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} e^{im\varphi}\, \hat{\theta}$$
$$+ \frac{1}{kr} \frac{\partial[r z_n(kr)]}{\partial r} \frac{im}{\sin\theta} P_n^m(\cos\theta) e^{im\varphi}\, \hat{\varphi}$$

These can be simplified by introducing two scalar func^{n}s: Scalar Tesseral functions, related to associated Legendre:

$$\pi_{mn}(\theta) = \frac{m P_n^m(\cos\theta)}{\sin\theta}$$

&

$$\tau_{mn}(\theta) = \frac{dP_n^m(\cos\theta)}{d\theta}$$

Then $M_{mn} = z_n(kr)\,[\,i\pi_{mn}(\cos\theta)\,\hat{\theta} - \tau_{mn}(\cos\theta)\,\hat{\varphi}\,]\,e^{im\varphi}$

$$N_{mn} = \Big\{ n(n+1)\,\frac{z_n(kr)}{r}\, P_n^m(\cos\theta)\,\hat{r} +$$

$$\frac{1}{kr}\frac{\partial[r z_n(kr)]}{\partial r}\,[\,\tau_{mn}(\cos\theta)\,\hat{\theta} + i\pi_{mn}(\cos\theta)\,\hat{\varphi}\,\Big\}\, e^{im\varphi}$$

Introduce three vector functions

vector Tesseral functions

$$\begin{cases} m_{mn}(\theta,\varphi) = e^{im\varphi}\,[\,i\pi_{mn}(\cos\theta)\,\hat{\theta} - \tau_{mn}(\cos\theta)\,\hat{\varphi}\,] \\ n_{mn}(\theta,\varphi) = e^{im\varphi}\,[\,\tau_{mn}(\cos\theta)\,\hat{\theta} + i\pi_{mn}(\cos\theta)\,\hat{\varphi}\,] \\ p_{mn}(\theta,\varphi) = e^{im\varphi}\, n(n+1)\, P_n^m(\cos\theta)\,\hat{r} \end{cases}$$

These functions obey orthogonality relations.
They are independent of the particular spherical Bessel function involved in vector harmonics

Final form: $M_{mn}(r,\theta,\varphi) = z_n(r)\, m_{mn}(\theta,\varphi)$

$$N_{mn}(r,\theta,\varphi) = \frac{z_n(r)}{r}\, p_{mn}(\theta,\varphi) + \frac{1}{r}\frac{\partial[r z_n(r)]}{\partial r}\, n_{mn}(\theta,\varphi)$$

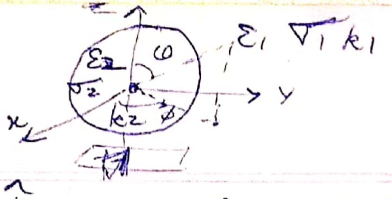
Start with PEC sphere radius a, in a linear homogen. isotropic medium with dielectric permittivity ε_1, wavenum k_1. Consider elliptically polarized plane wave incident on the sphere:

$$E_i(r) = e_{pol}\, e^{ik\cdot r} = (E_{\theta i}\,\hat{\theta}_i + E_{\varphi i}\,\hat{\varphi}_i)\, e^{ik_i\cdot r}$$

θ_i, φ_i are the unit vectors of the local spherical coordinate frame w.r.t. wave vector of the plane wave

$$k_i = k_1\hat{k} = k_1(\sin\theta_i\cos\varphi_i\,\hat{x} + \sin\theta_i\sin\varphi_i\,\hat{y} + \cos\theta_i\,\hat{z})$$

$$\hat{\varphi}_i = \frac{\hat{z}\times\hat{k}_i}{|\hat{z}\times\hat{k}_i|} = -\sin\varphi_i\,\hat{x} + \cos\varphi_i\,\hat{y}$$



$\hat{\theta}_i = \hat{\varphi} \times \hat{k}_i = \cos\theta_i \cos\varphi_i \, \hat{x} + \cos\theta_i \sin\varphi_i \, \hat{y} - \sin\theta_i \, \hat{z}$

Incident plane wave can be expanded in spherical harmonics as follows:

$$E_i(r) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[a_{mn} M^{(1)}_{mn}(r) + b_{mn} N^{(1)}_{mn}(r) \right]$$

(45) Boundary condition : tangential component continuous

$$a_{mn} = (-1)^m i^n \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!} \, e_{pol} \cdot m^*_{mn}(\theta_i, \varphi_i)$$

$$b_{mn} = (-1)^m i^{n-1} \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!} \, e_{pol} \cdot n^*_{mn}(\theta_i, \varphi_i)$$

$M^{(1)}$ indicates spherical Bessel func^n of 1st kind (typical for stationary waves)

Scattered field can be expanded in spherical vector wave functions as:

$$E_s(r) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[c_{mn} M^{(3)}_{mn}(r) + d_{mn} N^{(3)}_{mn}(r) \right]$$

(3) in vector harmonics : Bessel of 3rd kind i.e. spherical Hankel func^n of 1st type: typical for progressive waves.

c, d are unknowns to be obtained by B.C.

$(E_i + E_s) \times \hat{r} = 0$ for $r = a$

make explicit:

$$E_i = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[a_{mn} m_{mn}(\theta, \varphi) j_n(k_1 r) + b_{mn} n_{mn}(\theta, \varphi) j_n'(k_1 r) + b_{mn} p_{mn}(\theta, \varphi) \frac{j_n(k_1 r)}{k_1 r} \right]$$

$$E_s = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[c_{mn} m_{mn}(\theta, \varphi) h_n^{(1)}(k_1 r) + d_{mn} n_{mn}(\theta, \varphi) h_n'^{(1)}(k_1 r) + d_{mn} p_{mn}(\theta, \varphi) \frac{h_n^{(1)}(k_1 r)}{k_1 r} \right]$$

Then using $\mathbf{m}_{mn}(\theta,\varphi)\times\hat{r} = -\mathbf{n}_{mn}(\theta,\varphi)$

$$\mathbf{n}_{mn}(\theta,\varphi)\times\hat{r} = \mathbf{m}_{mn}(\theta,\varphi)$$

$$\mathbf{p}_{mn}(\theta,\varphi)\times\hat{r} = 0$$

substitute in B.C. :

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\{\mathbf{m}_{mn}(\theta,\varphi)[b_{mn}\, j_n(k_1a)+d_{mn}\, h_n^{(1)}(k_1a)] - \mathbf{n}_{mn}(\theta,\varphi)[a_{mn}\, j_n(k_1a) + c_{mn}\, h_n^{(1)}(k_1a)]\} = 0$$

Using orthogonality of vector spherical harmonics two scalar equations can be obtained :
i) take dot product with $\mathbf{m}^*_{m'n'}\sin\theta$ and integrating in θ,φ between $[0,\pi]$ and $[0,2\pi]$ resp.
and latter by $\mathbf{n}^*_{m'n'}\sin\theta$ with same integration.

As a result:

$$a_{mn}\, j_n(k_1a) + c_{mn}\, h_n^{(1)}(k_1a) = 0$$

$$b_{mn}\, j_n(k_1a) + d_{mn}\, h_n^{(1)}(k_1a) = 0$$

obtaining two simple expressions for scattering coeffs:

$$\left.\begin{aligned} c_{mn} &= \frac{-a_{mn}\, j_n(k_1a)}{h_n^{(1)}(k_1a)} \\ d_{mn} &= \frac{-b_{mn}\, j_n(k_1a)}{h_n^{(1)}(k_1a)} \end{aligned}\right\}$$ Mie Scattering Coefficients for a PEC sphere

Scattering by Dielectric Sphere

ϵ_2, k_2 . consider internal field to the sphere:

$$E_p(\vec{r}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [e_{mn} M_{mn}^{(1)}(r) + f_{mn} N_{mn}^{(1)}(r)]$$

B.C.: $(E_i + E_s - E_p) \times \hat{r} = 0$ for $r = a$ ①

$[\nabla \times (E_i + E_s - E_p)] \times \hat{r} = 0$ for $r = a$ ②

substituting: two vector equations are:

$$\sum_{n=1}^{\infty} \sum_{m=-n}^{n} \{ m_{mn}(\theta,\varphi) [b_{mn} j_n(k_1 a) + d_{mn} h^{(1)}(k_1 a) - f_{mn} j_n(k_2 a)] - n_{mn}(\theta,\varphi) [a_{mn} j_n(k_1 a) + c_{mn} h_n^{(1)}(k_1 a) - e_{mn} j_n(k_2 a)] \} = 0$$

and

$$\sum_{n=1}^{\infty} \sum_{m=-n}^{n} \{ m_{mn}(\theta,\varphi) [k_1 a_{mn} j_n(k_1 a) + k_1 c_{mn} h_n^{(1)}(k_1 a) - k_2 e_{mn} j_n(k_2 a)] - n_{mn}(\theta,\varphi) [k_1 b_{mn} j_n(k_1 a) + k_1 d_{mn} h_n^{(1)}(k_1 a) - k_2 f_{mn} \tilde{j}_n(k_2 a)] \} = 0$$

Orthogonal: take dot product with $m^*_{m'n'} \sin\theta$, ... same

$$a_{mn} j_n(k_1 a) + c_{mn} h_n^{(1)}(k_1 a) - e_{mn} j_n(k_2 a) = 0$$
$$b_{mn} j_n(k_1 a) + d_{mn} h_n^{(1)}(k_1 a) - f_{mn} j_n(k_2 a) = 0$$
$$k_1 a_{mn} j_n(k_1 a) + k_1 c_{mn} h_n^{(1)}(k_1 a) - k_2 e_{mn} j_n(k_2 a) = 0$$
$$k_1 b_{mn} j_n(k_1 a) + k_1 d_{mn} h_n^{(1)}(k_1 a) - k_2 f_{mn} j_n(k_2 a) = 0$$

Introduce $x = \frac{k_2}{k_1}$ (dielectric contrast) and solving

$$c_{mn} = -a_{mn} \frac{j_n(k_1 a) j_n(k_2 a) - x j_n(k_1 a) j_n(k_2 a)}{h_n^{(1)}(k_1 a) j_n(k_2 a) - x h_n^{(1)}(k_1 a) j_n(k_2 a)}$$

and

$$d_{mn} = -b_{mn} \frac{j_n(k_1 a) j_n(k_2 a) - x j_n(k_1 a) j_n(k_2 a)}{h_n^{(1)}(k_1 a) j_n(k_2 a) - x h_n^{(1)}(k_1 a) j_n(k_2 a)}$$

- Mie scattering coefficients ~~for~~ by dielectric sphere

Compute cross sections as function of scattering coefficients:

$$C_s = \frac{2\pi}{k_i^2} \sum_{n=1}^{\infty} (2n+1) \left(\left|\frac{c_{in}}{a_{in}}\right|^2 + \left|\frac{d_{in}}{b_{in}}\right|^2 \right)$$