AA 203: *Introduction to Optimal and Learning-based Control* M. Pavone
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Problem Set Solutions #4

Comments: Compiled by J. Lacotte.

Problem 1.

Solution

We need to solve the Euler equation $\frac{\partial g}{\partial x} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right) = 0$.

$$\frac{\partial g}{\partial x} = 2x(t) + 5 + 5\dot{x}(t) \qquad \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right) = \ddot{x}(t) + 5\dot{x}(t)$$

So the equation is:

$$\begin{aligned} &2x(t) + 5 + 5\dot{x}(t) - (\ddot{x}(t) + 5\dot{x}(t)) = 0 \\ \implies & -\ddot{x}(t) + 2x(t) + 5 = 0. \end{aligned}$$

Making the substitution $y = x + 5/2$, we have the simple harmonic oscillator system:

$$\ddot{y}(t) - 2y(t) = 0.$$

The solution is

$$y(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}$$

which corresponds to:

$$x(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} - \frac{5}{2}.$$

The boundary conditions $x(0) = 1$ and $x(1) = 3$ give:

$$c_1 = \frac{-7 + 11e^{\sqrt{2}}}{2\left(-1 + e^{2\sqrt{2}}\right)} \approx 1.201 \qquad c_2 = \frac{7}{2} - \frac{-7 + 11e^{\sqrt{2}}}{2\left(-1 + e^{2\sqrt{2}}\right)} \approx 2.299.$$

Problem 2.

Solution

The Euler equation is:

$$\begin{bmatrix} 2x_2 \\ 2x_1 \end{bmatrix} - \begin{bmatrix} 2\ddot{x}_1 \\ 2\ddot{x}_2 \end{bmatrix} = 0$$

i.e. $\ddot{x}_1 = x_2$ and $\ddot{x}_2 = x_1$. Then the additional condition we need for fixed final time, free final state is:

$$\left.\frac{\partial g}{\partial \dot{x}_1}\right|_{t=t_f} = 2\dot{x}_1(t_f) = 0$$

So we need to solve the differential equations given by the Euler equations with boundary conditions:

$$x_1(0) = 0 \qquad \dot{x}_1(\pi/2) = 0 \qquad x_2(0) = 0 \qquad x_2(\pi/2) = 1$$

Noting that the general solution of $\frac{d^4 x_1}{dt^4} = \ddot{x}_2 = x_1$ is $x_1(t) = c_1 \sin(t) + c_2 \cos(t) + c_3 e^t + c_4 e^{-t}$, we may solve for all constants to determine that:

$$x_1(t) = -\sin(t) \qquad x_2(t) = \sin(t).$$

Problem 3.

Solution

Since $\frac{\partial g}{\partial t} = 0$, we may apply the Beltrami identity special case of the Euler equation (essentially, the lack of any explicit time dependence in g allows us to "integrate out" one of the time derivatives and transform the system to be solved from a second order ODE into a first order ODE):

$$\begin{aligned}
C &= g(x(t), \dot{x}(t)) - \dot{x}(t)\frac{\partial g}{\partial \dot{x}} g(x(t), \dot{x}(t)) \\
&= x(t)\sqrt{1 + \dot{x}^2(t)} - \frac{x(t)\dot{x}^2(t)}{\sqrt{1 + \dot{x}^2(t)}} \\
\implies C\sqrt{1 + \dot{x}^2(t)} &= x(t) + x(t)\dot{x}^2(t) - x(t)\dot{x}^2(t) \\
\implies C\sqrt{1 + \dot{x}^2(t)} &= x(t)
\end{aligned}$$

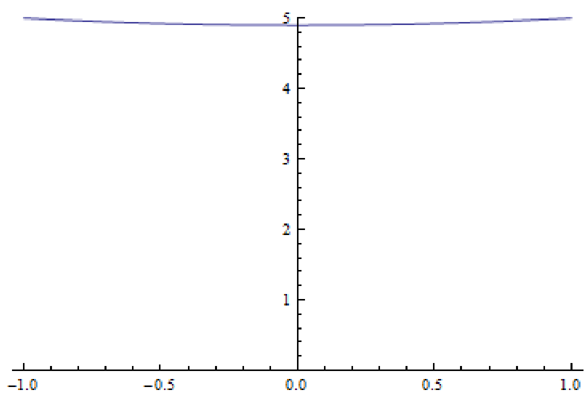
We can then use a separation of variables argument:

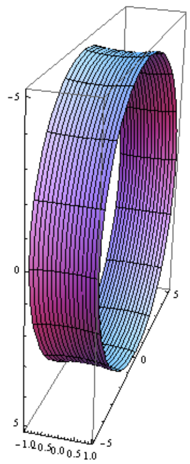
$$\begin{aligned}
&\dot{x}(t) = \sqrt{(x(t)/C)^2 - 1} \\
\implies &\int dt = \int \frac{dx}{\sqrt{(x/C)^2 - 1}} \\
\implies &t + K = \int \frac{dx}{\sqrt{(x/C)^2 - 1}} \\
\implies &t + K = C \cosh^{-1}(|x/C|) \\
\implies &x(t) = C \cosh(t/C + K/C)
\end{aligned}$$

Note that by symmetry of the initial and final conditions, we must have an even function, so $K = 0$. Then solving $5 = C \cosh(1/C)$ for $C = 4.898$, the solution is:

$$x(t) = 4.898 \cosh(.204t).$$

Note that the constant of integration actually assumes two admissible values, $C = 4.898$ and $\tilde{C} = 0.279$. As discussed in class, calculus of variations allows us to find extrema (Euler's equation is a necessary condition for optimality) but not to identify minima (sufficient conditions for optimality are generally beyond the scope of this class). However, in this exercise, one can manually verify that the solution given by $\tilde{C}$ yields higher cost ($J = 25.53$) than the solution given by C ($J = 9.93$).





Problem 4.

Solution

Part (a)

For this problem, the state is $[x(t), y(t)]^T$ and the control input is $\theta(t)$. The Hamiltonian is:

$$H = 1 + p_1\left(V\cos\theta + \frac{Vy}{h}\right) + p_2 V\sin\theta$$

The necessary conditions include

$$\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix} = -\nabla_{(x,y)}H = \begin{bmatrix}0\\ -p_1 V/h\end{bmatrix} \qquad\qquad 0 = \partial H/\partial\theta = -p_1 V\sin\theta + p_2 V\cos\theta.$$

Thus $p_1(t) = p_1$ is a constant, $p_2(t) = -p_1 Vt/h + p_2(0)$ is linear and

$$\begin{aligned}\tan\theta(t) &= \frac{p_2(t)}{p_1}\\ &= -\frac{V}{h}t + \frac{p_2(0)}{p_1}\\ &= \left(\frac{p_2(0)}{p_1} - \frac{VT}{h}\right) + \frac{V(T-t)}{h}\end{aligned}$$

Taking $\alpha = \frac{p_2(0)}{p_1} - \frac{VT}{h}$ (note that in the optimal solution, final time T is a constant, even though we haven't explicitly solved for it yet), we see that the optimal control is a linear tangent law.

Note that, since we do not require to solve for the initial conditions, showing that $\tan\theta(t) = -V/ht + C$ is sufficient to get full credit. The form $\tan\theta(t) = \alpha + V/h(T-t)$ merely highlights the fact that α is the final value assumed by the optimal steering angle θ.

Part (b)

In the case that the current speed $u = Vy(t)/h = \beta$ is a constant, the Hamiltonian is

$$H = 1 + p_1(V\cos\theta + \beta) + p_2 V\sin\theta$$

and the necessary conditions become:

$$\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix} = -\nabla_{(x,y)}H = \begin{bmatrix}0\\0\end{bmatrix} \qquad 0 = \partial H/\partial\theta = -p_1 V\sin\theta + p_2 V\cos\theta.$$

This means that both costates are constant, and the control law,

$$\theta(t) = \tan^{-1}\left(\frac{p_2}{p_1}\right) = \theta$$

is also constant. Then the dynamics don't depend on time, so we can integrate them:

$$x(t) = (V\cos\theta + u)(T - t_0) + x(t_0) \qquad y(t) = (V\sin\theta)(T - t_0) + y(t_0)$$

Note that this is the equation for a line. At time $t = T$ we need to be at the origin, i.e.

$$0 = (V\cos\theta + u)(T - t_0) + x(t_0) \qquad 0 = (V\sin\theta)(T - t_0) + y(t_0)$$

This system of two equations in two variables (T and θ) may be solved as follows:

$$0 = (V\sin\theta)(T - t_0) + y(t_0) \implies \cos\theta = \frac{\sqrt{(T-t_0)^2V^2 - y(t_0)^2}}{(T-t_0)V}$$

so then:

$$\begin{aligned}0 &= \left(V\frac{\sqrt{(T-t_0)^2V^2 - y(t_0)^2}}{(T-t_0)V} + u\right)(T-t_0) + x(t_0)\\ 0 &= \sqrt{(T-t_0)^2V^2 - y(t_0)^2} + (T-t_0)u + x(t_0)\\ (T-t_0)^2V^2 - y(t_0)^2 &= ((T-t_0)u + x(t_0))^2\end{aligned}$$

which is a quadratic with solution

$$T - t_0 = \frac{ux(t_0) + \sqrt{V^2(x(t_0)^2 + y(t_0)^2) - u^2y(t_0)^2}}{V^2 - u^2}$$

To validate this answer, we note that with no current (i.e. $u = 0$) the optimal travel time is $\sqrt{x(t_0)^2 + y(t_0)^2}/V$, i.e. distance traveled divided by ship speed. If $u > V$, i.e. the current is faster than the ship can travel, then a necessary condition for a valid solution is that u and $x(t_0)$ have opposite signs.

Problem 5.

Solution

The Hamiltonian for this problem is:

$$H = u^2 - 2px + pu$$

Then the necessary conditions for an extremal are:

$$\dot{x} = -2x + u \qquad \dot{p} = -\frac{\partial H}{\partial x} = 2p \qquad 0 = \frac{\partial H}{\partial u} = 2u + p$$

$$\implies \begin{bmatrix}\dot{x}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}-2 & -\frac{1}{2}\\ 0 & 2\end{bmatrix}\begin{bmatrix}x\\ p\end{bmatrix}, \quad u = -p/2$$

This system of ODEs has the solution

$$\begin{aligned}\begin{bmatrix}x(t)\\p(t)\end{bmatrix} &= \exp\left(\begin{bmatrix}-2 & -\frac{1}{2}\\0 & 2\end{bmatrix}t\right)\begin{bmatrix}x(0)\\p(0)\end{bmatrix}\\ &= \begin{bmatrix}e^{-2t} & -\sinh(2t)/4\\0 & e^{2t}\end{bmatrix}\begin{bmatrix}x(0)\\p(0)\end{bmatrix}\end{aligned}$$

Using the initial and final conditions $x(0) = 2$ and $x(1) = 0$, we may solve for $p(0) = 16/(-1+e^4) = 0.299$. This gives:

$$\begin{aligned}x(t) &= 2.037e^{-2t} - 0.037e^{2t}\\ u(t) &= -p(t)/2 = -0.149e^{2t}\end{aligned}$$

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sys=ss(-2,1,1,0);
N=100;
T=linspace(0,2,N);
U=-.149*exp(2*T);
X=lsim(sys,U,T,2);
plot(T,U,T,X)
xlabel('t')
legend('U','X')
axis([0,1,-1.5,2])
```