Spring 2019

Stanford
AA 203: Introduction to Optimal Control and Dynamic Optimization
Problem set 7, due on May 22

For this problem set you need to install the following software:

- `http://cvxr.com/cvx/download/` (referred to as CVX)
- `https://www.mpt3.org/` (referred to as MPT)

Please remember to attach your code to your problem set.

Problem 1: To familiarize yourself with the implementation of MPC controllers with CVX, read the example on slide 27 in Lecture 13, have a look at the CVX documentation: `http://web.cvxr.com/cvx/doc/`, and run the corresponding CVX/MATLAB script in `AA203-Examples/Lecture-13/`: `mpc_example.m`. Show and discuss the results of your computations.

Problem 2: Consider the second-order, discrete-time LTI system

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

We want to compute a receding horizon controller for the case where the cost is quadratic, i.e., $p(x_N) = x_N^T P x_N$, $q(x_k, u_k) = x_k^T Q x_k + u_k^T R u_k$. Assume (if not otherwise stated) that $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 0.01$, and that the system is subject to the input constraints

$$-\bar{u} \leq u(k) \leq \bar{u}, \quad k = 0, \ldots, N-1,$$

and to the state constraints

$$\begin{bmatrix} -\bar{x} \\ -\bar{x} \end{bmatrix} \leq x(k) \leq \begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix}, \quad k = 0, \ldots, N.$$

(Note: this setup is similar to the setup in Example 13.1 in the BBM book). Let P_∞ be the solution to the algebraic Riccati equation:

$$P_\infty = A^T P_\infty A + Q - A^T P_\infty B (B^T P_\infty B + R)^{-1} B^T P_\infty A.$$

a) Implement in CVX/MATLAB the receding horizon control strategy for this system.

b) Let $\bar{x} = 5$, $\bar{u} = 0.5$, $N = 3$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 10$, and $X_f = \mathbb{R}^2$. Simulate the closed-loop trajectories with initial states $x(0) = [-4.5,\, 2]$ and $x(0) = [-4.5,\, 3]$. Compare your results with those in Figure 13.2 in the BBM book.

c) Let $\bar{x} = 10$, $\bar{u} = 1$, $N = 2$, $P = P_\infty$, and $X_f = 0$. Discretize the state space (pick a reasonable discretization step) and find the set of initial points leading to feasible closed-loop trajectories converging to the origin (i.e., the domain of attraction for the RHC policy).

d) Let $\bar{x} = 10$, $\bar{u} = 1$, $N = 6$, $P = P_\infty$, and $X_f = 0$. Discretize the state space and find the domain of attraction for the RHC policy.

e) Let $\bar{x} = 10$, $\bar{u} = 1$, $N = 2$, $P = P_\infty$, and $X_f = \mathbb{R}^2$. Discretize the state space and find the domain of attraction for the RHC policy.

f) Let $\bar{x} = 10$, $\bar{u} = 1$, $N = 6$, $P = P_\infty$, and $X_f = \mathbb{R}^2$. Discretize the state space and find the domain of attraction for the RHC policy.

g) Discuss and compare the results in parts c), d), e), and f).

h) Consider the case $\bar{x} = 10$, $\bar{u} = 1$, $P = P_\infty$, and $X_f = 0$. Consider several different values of N and discuss, by presenting simulation experiments, how the trajectory and its cost are affected by the choice of N (you should use an appropriate initial condition).

Problem 3: Consider the discrete-time LTI system $x(t+1) = Ax(t) + Bu(t)$, with

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0.9 & 1 \\ 0 & 0.2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and state and control constraints

$$\begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, \qquad -0.5 \leq u(t) \leq 0.5.$$

Using the Matlab-based Multi Parametric Toolbox (MPT), compute and plot the control invariant set for the system. *Hint: look at the example at* *https://www.mpt3.org/UI/Invariance.*

Problem 4: Consider the discrete-time LTI system $x(t+1) = Ax(t) + Bu(t)$, with

$$A = \begin{bmatrix} 0.99 & 1 \\ 0 & 0.99 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and state and control constraints

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \qquad -0.5 \leq u(t) \leq 0.5.$$

We wish to synthesize a controller to stabilize the system to the origin while minimizing a quadratic cost function

$$J = x(t_f)^T P x(t_f) + \sum_{t=1}^{t_f-1} \left(x(t)^T Q x(t) + u(t)^T R u(t) \right).$$

1. Consider the weights

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R = 1.$$

   Propose a final condition X_f and a final cost P that, together, guarantee asymptotic stability and persistent feasibility of the closed-loop system. Plot the region X_f.
2. Implement an online MPC controller with MPT. (*Hint: have a look at the tutorial at* ***https://www.mpt3.org/UI/RegulationProblem****.*) Make sure to include the terminal penalty and the terminal set computed in Part 1 (*Hint: see* ***https://www.mpt3.org/UI/Filters***). Simulate the performance of the controller for $x_0 = [-4.7, 2]$ with an horizon $N = 4$. Plot the resulting trajectory.
3. Convert the online controller from Part 2 to an explicit (offline) MPC controller. Simulate the performance of the controller for the same initial condition $x_0 = [-4.7, 2]$ with an horizon $N = 4$. Plot the resulting trajectory (*Hint: use* ***expmpc.partition.plot()*** *to plot*).
4. Compare the execution time (both setup time and time required to compute the closed-loop trajectory) for the online controller and the explicit controller.
5. One can show that, for an LTI system with a quadratic cost function, the optimal policy is piecewise affine. Plot the partition of the state space for the controller computed in Part 3, i.e., the regions within the state space such that, within one region, the optimal control law is affine.

Learning goals for this problem set:

Problem 1: To familiarize with CVX and its application to MPC.

Problem 2: To gain experience with "tuning" MPC controllers.

Problem 3: To gain experience with control invariant sets.

Problem 4: To gain experience with MPT and implicit/explicit MPC.