PS 1 Problem 5

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$x_{t+1} = Ax_t + Bu_t$ LTI

$$\min_{u \in \mathbb{R}^{mT}} J(u) := x_T^T Q_T x_T + \sum_{t=0}^{T-1} x_t^T Q x_t + u_t^T R u_t$$

equivalent to

$$\min_{u \in \mathbb{R}^{mT}} \frac{1}{2} u^T \tilde{Q} u - \tilde{b}^T u$$

$x_1 = Ax_0 + Bu_0$

$x_2 = A(Ax_0 + Bu_0) + Bu_1$

$= A^2 x_0 + ABu_0 + Bu_1$

$x_3 = A^3 x_0 + A^2 Bu_0 + ABu_1 + Bu_2$

$$\Rightarrow x_T = A^T x_0 + \sum_{i=0}^{T-1} A^{T-1-i} B u_i$$

$$x_t = A^t x_0 + \sum_{i=0}^{t-1} A^{t-1-i} B u_i$$

$$x_t^T Q x_t + u_t^T R u_t = \left(A^t x_0 + \sum_{i=0}^{t-1} A^{t-1-i} B u_i\right)^T Q \left(A^t x_0 + \sum_{i=0}^{t-1} A^{t-1-i} B u_i\right) + u_t^T R u_t$$

$\because \left(\sum_{i=0}^{t-1} A^{t-1-i} B u_i\right)' Q A^t x_0$ is a scalar

$$= (A^t x_0)' Q \left(\sum_{i=0}^{t-1} A^{t-1-i} B u_i\right)$$

$$\therefore x_t' Q x_t + u_t' R u_t = x_0' A^{t'} Q A^t x_0 + \left(\sum_{i=0}^{t-1} A^{t-1-i} B u_i\right)' Q \left(\sum_{i=0}^{t-1} A^{t-1-i} B u_i\right) + 2 x_0' A^{t'} Q \left(\sum_{i=0}^{t-1} A^{t-1-i} B u_i\right) + u_t' R u_t$$

We can write
$$\left(\sum_{i=0}^{t-1} A^{t-1-i} B u_i\right) = \begin{bmatrix} A^{t-1}B & & & 0 \\ & A^{t-2}B & & \\ & & \ddots & \\ 0 & & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{t-1} \end{bmatrix}$$

$$= C(t)\, u_{0:t-1}$$

$C(t) \in \mathbb{R}^{mt \times mt}$ $u_{0:t-1} \in \mathbb{R}^{mt}$

$$\therefore x_t' Q x_t + u_t' R u_t = x_0' A^t Q A^t x_0 + u_{0:t-1}' C(t)' Q C(t) u_{0:t-1} + 2 x_0' A^{t'} Q C(t) u_{0:t-1} + u_t' R u_t$$

$$\sum_{t=0}^{T-1} x_t' Q x_t + u_t' R u_t = \underbrace{\sum_{t=0}^{T-1} x_0' A^{t'} Q A^t x_0}_{1} + \underbrace{\sum_{t=0}^{T-1} u_{0:t-1}' C(t)' Q C(t) u_{0:t-1}}_{2} + \underbrace{\sum_{t=0}^{T-1} 2 x_0' A^{t'} Q C(t) u_{0:t-1}}_{3} + \underbrace{\sum_{t=0}^{T-1} u_t' R u_t}_{4}$$

power T

power T

$$\because x_T = A^T x_0 + \sum_{i=0}^{T-1} A^{T-1-i} B u_i = A^T x_0 + C(T) u_{0:T-1}$$

$$J(u) = x_T' Q_T x_T + ① + ② + ③ + ④$$

$$= \underbrace{x_0'(A^T)'Q_T A^T x_0}_{⑤} + \underbrace{u_{0:T-1}' C(T)' Q_T C(T) u_{0:T-1}}_{⑥}$$

$$+ \underbrace{2x_0'(A^T)' Q_T C(T) u_{0:T-1}}_{⑦} + ① + ② + ③ + ④$$

① and ⑤ are constant wrt u and are only functions of x_0 so we can leave them out of $\min_{u \in \mathbb{R}^{mT}} J(u)$

③ and ⑦
$$\sum_{t=0}^{T-1} 2x_0' A^{t\prime} Q C(t) u_{0:t-1} + 2x_0'(A^T)' Q_T C(T) u_{0:T-1}$$

$$= 2x_0' \Big[\left(A^{0\prime} Q A^0 B + A^{1\prime} Q A^1 B + \dots + A^{T-1\prime} Q A^{T-1} B\right) u_0$$
$$+ \left(A^{1\prime} Q A^0 B + A^{2\prime} Q A^1 B + \dots + A^{T-1\prime} Q A^{T-2} B\right) u_1$$
$$+ \left(A^{2\prime} Q A^0 B + A^{3\prime} Q A^1 B + \dots + A^{T-1\prime} Q A^{T-3} B\right) u_2$$
$$+ \dots$$
$$+ \left(A^{T-1\prime} Q A^0 B\right) u_{T-1} \Big]$$

$$+ 2x_0'(A^T)' Q_T \left[A^{T-1} B u_0 + A^{T-2} B u_1 + \dots A^0 B u_{T-1}\right]$$

$$= 2x_0' \Big[\sum_{i=0}^{T-1} (A^i)' Q A^i B + (A^T)' Q_T A^{T-1} B, \ \sum_{i=1}^{T-1} (A^i)' Q A^{i-1} B + (A^T)' Q_T A^{T-2} B \ \dots \ \sum_{i=T-1}^{T-1} (A^i)' Q A^{i-(T-1)} B + (A^T)' Q$$

$$= -2\tilde{b}^T u$$

$$\boxed{\text{where } \tilde{b} = \begin{bmatrix} \tilde{b}_0 \\ \tilde{b}_1 \\ \vdots \\ \tilde{b}_{T-1} \end{bmatrix} \quad \tilde{b}_j = -x_0' \left(\sum_{i=j}^{T-1} (A^i)' Q A^i B + (A^T)' Q_T A^{T-1-j} B \right)}$$

Check $\tilde{b}_j$: $1 \times n \quad n \times n \quad n \times n \quad n \times n \quad n \times m \quad = 1 \times m$

$\Rightarrow \tilde{b} : 1 \times mT \quad \Rightarrow \tilde{b}^T : \mathbb{R}^{mT}$

② & ④ & ⑥ $= \sum_{t=0}^{T-1} u_{0:t-1}' C(t)' Q C(t) u_{0:t-1}$

$+ \sum_{t=0}^{T-1} u_t' R u_t + u_{0:T-1}' C(T)' Q_T C(T) u_{0:T-1}$

$= u' \tilde{Q} u$

$C(t) u_{0:t-1} = \sum_{i=0}^{t-1} A^{t-1-i} B u_i$

$C(t)$ is a diagonal matrix.

$$\boxed{\begin{aligned} \tilde{Q} &= \begin{bmatrix} \tilde{Q}_{00} & & & 0 \\ & \tilde{Q}_{11} & & \\ & & \ddots & \\ 0 & & & \tilde{Q}_{(T-1)(T-1)} \end{bmatrix} \\ \tilde{Q}_{jj} &= \sum_{i=j}^{T-1} B' A^{i'} Q A^{i} B + R + B' A^{T-1-j'} Q_T A^{T-1-j} B \end{aligned}}$$

Check $\tilde{Q}_{jj}$: $m \times n \quad n \times n \quad n \times n \quad n \times n \quad n \times m$

$= m \times m$

$\Rightarrow \tilde{Q} : mT \times mT$

b) $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $Q^T = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ $R = 1$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $x_0 = [1 \ 0]$ $T = 20$

$\Rightarrow n = 2 \quad m = 1$

$\tilde{b} \in \mathbb{R}^{mT} = \mathbb{R}^{20}$

$\tilde{b}_0 = -x_0' \left(\sum_{i=0}^{19} (A^i)' Q A^i B + (A^{20})' Q_T A^{19} B \right)$

$\tilde{Q} : mT \times mT = 20 \times 20$

$\tilde{Q}_{00} = \sum_{i=0}^{19} B' A^{19-i'} Q A^{19-i} B + R + B' A^{19'} Q_T A^{19} B$

$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \end{bmatrix}$

$A^2 = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix} = \begin{bmatrix}1&2\\0&1\end{bmatrix}$

$A^n = \begin{bmatrix}1&n\\0&1\end{bmatrix}$

Solving $u^* = \tilde{Q}^{-1}\tilde{b}$

```
uStar =

   -0.0622
   -0.0645
   -0.0666
   -0.0687
   -0.0706
   -0.0723
   -0.0738
   -0.0752
   -0.0763
   -0.0773
   -0.0779
   -0.0783
   -0.0784
   -0.0783
   -0.0777
   -0.0766
   -0.0749
   -0.0720
   -0.0664
   -0.0509
```

cost

```
J =

   2.0933e+03
```

AA 203 HW 1 Question 5

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```
clc
clear all
close all
Q = eye(2);
QT = 10 * eye(2);
R = eye(1);
A=[1 1; 0 1];
B=[0;1];
x0=[1;0];
T=20;
btilde = zeros(T,1);
Qtilde = zeros(T,T);
for j = 0: T-1
    sumb = 0;
    sumQ = 0;
    for i = j: T-1
        sumb = sumb + (-x0' * (A^i)' * Q * (A^i) * B);
        sumQ = sumQ + B' * (A^i)' * Q * (A^i) *B;
    end
    btilde(j+1,1) =  sumb - x0'*((A^T)'*QT*(A^(T-1-j))*B);
    Qtilde(j+1,j+1) = sumQ + R + B' * (A^(T-1-j))' * QT * (A^(T-1-j)) * B;
end
uStar = inv(Qtilde)*btilde;
x = zeros(2, 21);
x(:,1) = [1; 0];
i = 1;
u = uStar;

sumJ = 0;
for t = 0:T-1
    x(:,t+2) = A*x(:,t+1) + B*u(t+1);
    sumJ = sumJ + x(:,t+1)'*Q*x(:,t+1) + u(t+1)'*R*u(t+1);
end
J = x(:,T+1)'*QT*x(:,T+1) + sumJ

QuadCost = 0.5*u'*Qtilde*u - btilde'*u
```

```
J =

   2.0933e+03


QuadCost =

 -164.2338
```

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