PS 4 Problem 2

Monday, April 29, 2019 10:40 AM

$$J = \int_0^{\pi/2} \underbrace{\left[\dot{x}_1^2(t) + \dot{x}_2^2(t) + 2x_1(t)\,x_2(t)\right]}_{g} dt$$

Euler's eq^n : $\frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} = 0$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \partial g/\partial x_1 \\ \partial g/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 2x_1 \end{bmatrix}$$

$$\frac{\partial g}{\partial \dot{x}} = \begin{bmatrix} \partial g/\partial \dot{x}_1 \\ \partial g/\partial \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2\dot{x}_1 \\ 2\dot{x}_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_2 \\ 2x_1 \end{bmatrix} - \begin{bmatrix} 2\ddot{x}_1 \\ 2\ddot{x}_2 \end{bmatrix} = 0$$

$$\Rightarrow x_2 = \ddot{x}_1$$
$$x_1 = \ddot{x}_2$$

$$x_1 = \ddot{x}_2 \Rightarrow \ddot{x}_1 = \frac{d^4 x_2}{dt^4} = x_2$$

$$\frac{d^4 x_2}{dt^4} = x_2 \qquad \text{Let } x_2 = e^{rt}$$

$$\Rightarrow r^4 = 1 \quad \Rightarrow r^2 = 1 \text{ or } -1$$
$$\Rightarrow r = 1 \text{ or } -1 \text{ or } i \text{ or } -i$$

$$x_2 = c_1 e^{t} + c_2 e^{-t} + c_3 e^{it} + c_4 e^{-it} + \cancel{c_5}^{\,0}$$

equivalently, $x_2 = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$

$$x_1 = \ddot{x}_2 = c_1 e^{t} + c_2 e^{-t} - c_3 \cos t - c_4 \sin t$$

Boundary conditions * 4 boundary cond^{n}s

$x_1(0) = 0 \quad \Rightarrow c_1 + c_2 - c_3 = 0$ *

$x_1(\pi/2)$ free

$x_2(0) = 0 \quad \Rightarrow c_1 + c_2 + c_3 = 0$ *

$\Rightarrow c_3 = 0$

$c_1 + c_2 = 0$

$\Rightarrow c_2 = -c_1$

$x_2(\pi/2) = 1 \quad \Rightarrow c_1 e^{\pi/2} + c_2 e^{-\pi/2} + \cancel{c_3}^{\,0} + c_4 = 1$ *

$$\Rightarrow c_1 e^{\pi/2} - c_1 e^{-\pi/2} + c_4 = 1$$

$$\Rightarrow c_4 = 1 - c_1\left(e^{\pi/2} - e^{-\pi/2}\right)$$

$\delta t_f = 0$
$\delta x_2 = 0$
δx_1 arbitrary

$$\Rightarrow \left.\frac{\partial g}{\partial \dot{x}_1}\right|_{t_f} = 0 \quad \Rightarrow 2\dot{x}_1(t_f) = 0$$

$$\Rightarrow \dot{x}_1(\pi/2) = 0$$

$$\Rightarrow \left[c_1 e^{t} - c_2 e^{-t} + \underbrace{\cancel{c_3}}_{0} \sin t - c_4 \cos t\right]_{\pi/2} = 0$$

$$\Rightarrow c_1 e^{\pi/2} - (-c_1)\, e^{-\pi/2} = 0$$

$$\Rightarrow c_1 = 0$$

$$\Rightarrow c_2 = -c_1 = 0$$

$$\Rightarrow c_4 = 1 - c_1\left(e^{\pi/2} - e^{-\pi/2}\right) = 1$$

$$\Rightarrow \boxed{x_1 = -\sin t \quad \& \quad x_2 = \sin t}$$