Spring 2019

Stanford
AA203: Optimal and Learning-based Control
Problem set 4, due on May 1

Problem 1: Find the curve $x^*(t)$ that minimizes the functional

$$J = \int_0^1 \left[\frac{1}{2}\dot{x}^2(t) + 5x(t)\dot{x}(t) + x^2(t) + 5x(t)\right] dt$$

and passes through the points $x(0) = 1$ and $x(1) = 3$.

Problem 2: Find extremals for the functional

$$J = \int_0^{\pi/2} \left[\dot{x}_1^2(t) + \dot{x}_2^2(t) + 2x_1(t)x_2(t)\right] dt,$$

with boundary conditions $x_1(0) = 0$, $x_1(\pi/2)$ free, $x_2(0) = 0$, $x_2(\pi/2) = 1$.

Problem 3 (optional, no credits given for this question): Find the curve joining points $(-1, 5)$ and $(1, 5)$ and that generates the surface of minimum area when rotated about the t-axis, i.e., that minimizes the functional

$$J = 2\pi \int_{-1}^{1} x(t)\,\sqrt{1 + \dot{x}^2(t)}dt.$$

Plot the solution. Remarkably, the resulting surface is the shape that a thin soap film assumes when suspended between two concentric wire rings. (*Hint: use the Beltrami identity.*)

Problem 4: A ship must travel through a region of strong currents, which depend on position. The ship has a constant speed V, and its heading $\theta(t)$ can be controlled. The current is directed in the x direction with a speed

$$u = \frac{Vy(t)}{h}$$

for a given h. It is desired to find the ship's heading $\theta(t)$ required to move from a given initial position $(x(t_0), y(t_0))$ to the origin in minimum time. The equations of motion are

$$\dot{x}(t) = V\,\cos\theta(t) + \frac{Vy(t)}{h}$$
$$\dot{y}(t) = V\sin\theta(t)$$

and the performance index is

$$J = \int_{t_0}^{T} 1\,dt.$$

(a) Show that the optimal control law takes the form of

$$\tan\theta(t) = \alpha + \frac{V(T-t)}{h},$$

where α is a constant. This law is referred to as linear tangent law.

(b) Compute the optimal transfer time, i.e., $T - t_0$, for the case where the current's speed is equal to a constant, i.e., $u = \beta > 0$.

Problem 5: Find the Hamiltonian and then solve the necessary conditions to compute the optimal control and state trajectory that minimize

$$J = \int_0^1 u^2(t)dt$$

for the system $\dot{x}(t) = -2x(t) + u(t)$ with initial state $x(0) = 2$ and terminal state $x(1) = 0$. Plot the optimal control and state response using MATLAB.

Learning goals for this problem set:

Problem 1: To familiarize with the process of solving calculus of variations problems.

Problem 2: To learn how to solve calculus of variations problems involving multiple *independent* functions, and how to make the appropriate substitutions in the boundary conditions for the Euler equation.

Problem 3 To learn how calculus of variations can be useful for physics problems.

Problem 4 & 5 To familiarize with the Hamiltonian equations for optimal control.