PS 4 Problem 4

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Part a

$\dot{x}(t) = V\cos\theta(t) + \frac{Vy(t)}{h}$

$\dot{y}(t) = V\sin\theta(t)$

$x(t_0), y(t_0)$ given $\quad x(T) = 0 = y(T)$

δt_f arbitrary

$J = \int_{t_0}^{T} 1\, dt \quad \Rightarrow g = 1$

$\mathcal{H} = g + p'f = 1 + \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} V\cos\theta + \frac{Vy}{h} \\ V\sin\theta \end{bmatrix}$

$\mathcal{H} = 1 + p_1\left(V\cos\theta + \frac{Vy}{h}\right) + p_2\left(V\sin\theta\right)$

Unconstrained control $\theta(t)$

$\therefore$ NOC: 1 $\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{x} = V\cos\theta + \frac{Vy}{h}$

$\dot{y} = V\sin\theta$

2 $\dot{p}_1 = -\frac{\partial H}{\partial x} \quad \Rightarrow \dot{p}_1 = 0 \quad \Rightarrow p_1 = c_1$

$\dot{p}_2 = -\frac{\partial H}{\partial y} \quad \Rightarrow \dot{p}_2 = -\frac{p_1 V}{h} = -\frac{c_1 V}{h}$

$\Rightarrow p_2 = -\frac{c_1 V}{h} t + c_2$

3 $0 = \frac{\partial H}{\partial \theta} \Rightarrow -p_1 V\sin\theta + p_2 V\cos\theta = 0$

$\Rightarrow \tan\theta^* = p_2/p_1$

$\Rightarrow \tan\theta^*(t) = \dfrac{-\frac{c_1 V}{h} t + c_2}{c_1}$

$= -\frac{V}{h} t + \frac{c_2}{c_1}$

Let $\tan\theta^*(T) = \alpha$ (const)

$\Rightarrow \alpha = -\frac{V}{h} T + \frac{c_2}{c_1}$

$\Rightarrow \frac{c_2}{c_1} = \frac{V}{h} T + \alpha$

$\Rightarrow \boxed{\tan(\theta^*(t)) = \frac{V}{h}(T-t) + \alpha}$

where $\alpha = \tan(\theta^*(T))$

Part b

$\alpha = \beta > 0$

$$\mathcal{H} = 1 + p_1\left(V\cos\theta(t) + \beta\right) + p_2\left(V\sin\theta(t)\right)$$

$$\dot{p}_1 = -\frac{\partial H}{\partial x} = 0 \Rightarrow p_1 = c_1$$

$$\dot{p}_2 = -\frac{\partial H}{\partial y} = 0 \Rightarrow p_2 = c_2$$

$$\frac{\partial H}{\partial \theta} = 0 \Rightarrow -p_1 V\sin\theta + p_2 V\cos\theta = 0$$

$$\Rightarrow \tan(\theta(t)) = p_2/p_1 = c_2/c_1$$

So, optimal $\theta^*(t) = \text{const} = \tan^{-1}(c_2/c_1)$

$$\dot{x} = V\cos\theta^*(t) + \beta = V\cos\theta^\alpha + \beta = \text{const}$$

$$\Rightarrow x(t) = (V\cos\theta^\alpha + \beta)t + k_1$$

$$x(t_0) = x(t_0) \Rightarrow (V\cos\theta^* + \beta)\, t_0 + k_1 = x(t_0)$$

$$x^*(t) = (V\cos\theta^* + \beta)t + x(t_0) - (V\cos\theta^\alpha + \beta)t_0$$

$$x(T) = 0 \Rightarrow (V\cos\theta^\alpha + \beta)(T - t_0) + x(t_0) = 0$$

$$\Rightarrow V\cos\theta^* = -\frac{x(t_0)}{T - t_0} - \beta \quad \text{— (1)}$$

$$\dot{y} = V\sin\theta(t) \Rightarrow \dot{y}^* = V\sin\theta^\alpha$$

$$\Rightarrow y(t) = V\sin\theta^\alpha t + k_2$$

$$y(t_0) = y(t_0) \Rightarrow V\sin\theta^\alpha\, t_0 + k_2 = y(t_0)$$

$$y(t) = V\sin\theta^\alpha t + y(t_0) - V\sin\theta^\alpha t_0$$

$$y(T) = 0 \Rightarrow V\sin\theta^\alpha (T - t_0) + y(t_0) = 0$$

$$\Rightarrow V\sin\theta^\alpha = \frac{-y(t_0)}{T - t_0} \quad \text{— (2)}$$

Square and add (1) & (2)

$$V^2 = \left(\frac{-x(t_0)}{T - t_0} - \beta\right)^2 + \left(\frac{-y(t_0)}{T - t_0}\right)^2$$

$$\Rightarrow V^2 = \frac{x^2(t_0)}{(T - t_0)^2} + \beta^2 + \frac{2x(t_0)\beta}{T - t_0} + \frac{y^2(t_0)}{(T - t_0)^2}$$

Let $T - t_0 = \Delta t$

$$\Rightarrow V^2\Delta t^2 = x^2(t_0) + y^2(t_0) + 2x(t_0)\beta\Delta t + \beta^2\Delta t^2$$

$$\Rightarrow \Delta t^2(\beta^2 - V^2) + \Delta t\,(2x(t_0)\beta) + x^2(t_0) + y^2(t_0) = 0$$

$$\Rightarrow \Delta t = \frac{-2x(t_0)\beta \pm \sqrt{4x^2(t_0)\beta^2 - 4(\beta^2 - V^2)(x^2(t_0) + y^2(t_0))}}{2(\beta^2 - V^2)}$$

$$\Rightarrow \boxed{T - t_0 = \frac{-2x(t_0)\beta \pm \sqrt{4x^2(t_0)\beta^2 - 4(\beta^2 - V^2)(x^2(t_0) + y^2(t_0))}}{2(\beta^2 - V^2)}}$$