PS 4 Problem 1

Sunday, April 28, 2019 11:40 PM

Find x^* to minimize

$$J = \int_0^1 \Big[\underbrace{\tfrac{1}{2}\dot{x}^2 + 5x\dot{x} + x^2 + 5x}_{g}\Big]\,dt$$

Using Euler eqn;

$$\frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} = 0$$

$$\Rightarrow 5\dot{x} + 2x + 5 - \frac{d}{dt}(\dot{x} + 5x) = 0$$

$$\Rightarrow \cancel{5\dot{x}} + 2x + 5 - \ddot{x} - \cancel{5\dot{x}} = 0$$

$$\Rightarrow \ddot{x} = 2x + 5$$

Let $x = e^{rt} \Rightarrow r^2 = 2 \Rightarrow r = \pm\sqrt{2}$

For $\ddot{x} = 2x + 5$ try $x = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + c_3$

$$2c_1 e^{\sqrt{2}t} + 2c_2 e^{-\sqrt{2}t} = 2c_1 e^{\sqrt{2}t} + 2c_2 e^{-\sqrt{2}t} + 2c_3 + 5$$

$$\Rightarrow c_3 = -5/2$$

$$x(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} - 5/2$$

$x(0) = 1 \Rightarrow c_1 + c_2 - 5/2 = 1 \Rightarrow c_1 + c_2 = 7/2$

$x(1) = 3 \Rightarrow c_1 e^{\sqrt{2}} + c_2 e^{-\sqrt{2}} - 5/2 = 3$

$$c_1 e^{2\sqrt{2}} + c_2 = \frac{11}{2} e^{\sqrt{2}}$$

$$c_1 \left(e^{2\sqrt{2}} - 1\right) = \frac{11}{2} e^{\sqrt{2}} - \frac{7}{2}$$

$$\Rightarrow c_1 \approx 1.2013$$

$$\Rightarrow c_2 = 7/2 - c_1$$

$$\Rightarrow c_2 \approx 2.2987$$

$$\boxed{x^*(t) = 1.2013\, e^{\sqrt{2}t} + 2.2987\, e^{-\sqrt{2}t} - 2.5}$$

PS 4 Problem 2

Monday, April 29, 2019 10:40 AM

$$J = \int_0^{\pi/2} \underbrace{\left[\dot{x}_1^2(t) + \dot{x}_2^2(t) + 2x_1(t)\,x_2(t)\right]}_{g} dt$$

Euler's eqⁿ : $\frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} = 0$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \partial g/\partial x_1 \\ \partial g/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 2x_1 \end{bmatrix}$$

$$\frac{\partial g}{\partial \dot{x}} = \begin{bmatrix} \partial g/\partial \dot{x}_1 \\ \partial g/\partial \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2\dot{x}_1 \\ 2\dot{x}_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_2 \\ 2x_1 \end{bmatrix} - \begin{bmatrix} 2\ddot{x}_1 \\ 2\ddot{x}_2 \end{bmatrix} = 0$$

$$\Rightarrow x_2 = \ddot{x}_1$$
$$x_1 = \ddot{x}_2$$

$$x_1 = \ddot{x}_2 \Rightarrow \ddot{x}_1 = \frac{d^4 x_2}{dt^4} = x_2$$

$$\frac{d^4 x_2}{dt^4} = x_2 \qquad \text{Let } x_2 = e^{rt}$$

$$\Rightarrow r^4 = 1 \quad \Rightarrow r^2 = 1 \text{ or } -1$$
$$\Rightarrow r = 1 \text{ or } -1 \text{ or } i \text{ or } -i$$

$$x_2 = c_1 e^{t} + c_2 e^{-t} + c_3 e^{it} + c_4 e^{-it} + \overset{0}{\cancel{c_5}}$$

equivalently, $x_2 = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$

$$x_1 = \ddot{x}_2 = c_1 e^{t} + c_2 e^{-t} - c_3 \cos t - c_4 \sin t$$

Boundary conditions * 4 boundary condⁿs

$x_1(0) = 0 \quad \Rightarrow c_1 + c_2 - c_3 = 0$ *

$x_1(\pi/2)$ free

$x_2(0) = 0 \quad \Rightarrow c_1 + c_2 + c_3 = 0$ *

$\Rightarrow c_3 = 0$

$c_1 + c_2 = 0$

$\Rightarrow c_2 = -c_1$

$x_2(\pi/2) = 1 \quad \Rightarrow c_1 e^{\pi/2} + c_2 e^{-\pi/2} + \overset{0}{\cancel{c_3}} + c_4 = 1$ *

$$\Rightarrow c_1 e^{\pi/2} - c_1 e^{-\pi/2} + c_4 = 1$$

$$\Rightarrow c_4 = 1 - c_1\left(e^{\pi/2} - e^{-\pi/2}\right)$$

$\delta t_f = 0$

$\delta x_2 = 0$

δx_1 arbitrary

$$\Rightarrow \left.\frac{\partial g}{\partial \dot{x}_1}\right|_{t_f} = 0 \qquad \Rightarrow 2\dot{x}_1(t_f) = 0$$

$$\Rightarrow \dot{x}_1(\pi/2) = 0$$

$$\Rightarrow \left[c_1 e^{t} - c_2 e^{-t} + \underbrace{\cancel{c_3}}_{0} \sin t - c_4 \cos t\right]_{\pi/2} = 0$$

$$\Rightarrow c_1 e^{\pi/2} - (-c_1)\, e^{-\pi/2} = 0$$

$$\Rightarrow c_1 = 0$$

$$\Rightarrow c_2 = -c_1 = 0$$

$$\Rightarrow c_4 = 1 - c_1\left(e^{\pi/2} - e^{-\pi/2}\right) = 1$$

$$\Rightarrow \boxed{x_1 = -\sin t \quad \& \quad x_2 = \sin t}$$

PS 4 Problem 3

Wednesday, May 1, 2019 11:11 AM

$$J = 2\pi \int_{-1}^{1} \underbrace{x(t)\sqrt{1+\dot{x}(t)^2}}_{g}\, dt$$

$(-1,5)$ to $(1,5)$

$\Rightarrow x(-1) = 5 \qquad x(1) = 5$

Could use Euler's eqn

$$\frac{\partial g}{\partial x} - \frac{d}{dt}\frac{dg}{d\dot{x}} = 0$$

but since $g(x,\dot{x})$ and not $g(x,\dot{x},t)$, it might be easier to use Beltrami's eqn

$$g - \dot{x}\frac{dg}{\partial \dot{x}} = c$$

$$\Rightarrow x\sqrt{1+\dot{x}^2} - \dot{x}\,\frac{x\,2\dot{x}}{2\sqrt{1+\dot{x}^2}} = c$$

$$\Rightarrow x(1+\dot{x}^2) - x\dot{x}^2 = c\sqrt{1+\dot{x}^2}$$

$$\Rightarrow x = c\sqrt{1+\dot{x}^2}$$

$$\Rightarrow \frac{x^2}{c} = 1+\dot{x}^2$$

$$\Rightarrow \dot{x}^2 = Kx^2 - 1$$

$$\Rightarrow \dot{x} = \sqrt{kx^2-1}$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{kx^2-1}$$

$$\Rightarrow x(t) = \frac{e^{-c_1\sqrt{k}-\sqrt{k}\,t}\left(e^{2\sqrt{k}\,(c_1+t)} + k\right)}{2k}$$

Now we need to plug in $t=-1,\ x=5$ and $t=1,\ x=5$ to solve for k and c_1

PS 4 Problem 4

Tuesday, April 30, 2019 12:20 AM

Part a

$\dot{x}(t) = V\cos\theta(t) + \frac{Vy(t)}{h}$

$\dot{y}(t) = V\sin\theta(t)$

$x(t_0), y(t_0)$ given $\quad x(T) = 0 = y(T)$

δt_f arbitrary

$J = \int_{t_0}^{T} 1 \, dt \quad \Rightarrow g = 1$

$$\mathcal{H} = g + p'f = 1 + \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} V\cos\theta + \frac{Vy}{h} \\ V\sin\theta \end{bmatrix}$$

$$\mathcal{H} = 1 + p_1\left(V\cos\theta + \frac{Vy}{h}\right) + p_2\left(V\sin\theta\right)$$

Unconstrained control $\theta(t)$

$\therefore$ NOC: ① $\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{x} = V\cos\theta + \frac{Vy}{h}$

$\qquad \dot{y} = V\sin\theta$

② $\dot{p}_1 = -\frac{\partial H}{\partial x} \quad \Rightarrow \dot{p}_1 = 0 \quad \Rightarrow p_1 = c_1$

$\dot{p}_2 = -\frac{\partial H}{\partial y} \quad \Rightarrow \dot{p}_2 = -\frac{p_1 V}{h} = -\frac{c_1 V}{h}$

$\Rightarrow p_2 = -\frac{c_1 V}{h} t + c_2$

③ $0 = \frac{\partial H}{\partial \theta} \quad \Rightarrow -p_1 V\sin\theta + p_2 V\cos\theta = 0$

$\Rightarrow \tan\theta^* = p_2/p_1$

$$\Rightarrow \tan\theta^*(t) = \frac{-\frac{c_1 V}{h} t + c_2}{c_1}$$

$$= -\frac{V}{h} t + \frac{c_2}{c_1}$$

Let $\tan\theta^*(T) = \alpha$ (const)

$\Rightarrow \alpha = -\frac{V}{h} T + \frac{c_2}{c_1}$

$\Rightarrow \frac{c_2}{c_1} = \frac{V}{h} T + \alpha$

$$\Rightarrow \boxed{\tan(\theta^*(t)) = \frac{V}{h}(T - t) + \alpha}$$

where $\alpha = \tan(\theta^*(T))$

Part b

$\alpha = \beta > 0$

$\mathcal{H} = 1 + p_1\left(V\cos\theta(t) + \beta\right) + p_2\left(V\sin\theta(t)\right)$

$\dot{p}_1 = -\frac{\partial H}{\partial x} = 0 \Rightarrow p_1 = c_1$

$\dot{p}_2 = -\frac{\partial H}{\partial y} = 0 \Rightarrow p_2 = c_2$

$\frac{\partial H}{\partial \theta} = 0 \Rightarrow -p_1 V\sin\theta + p_2 V\cos\theta = 0$

$\Rightarrow \tan(\theta(t)) = p_2/p_1 = c_2/c_1$

So, optimal $\theta^*(t) = \text{const} = \tan^{-1}(c_2/c_1)$

$\dot{x} = V\cos\theta^*(t) + \beta = V\cos\theta^\alpha + \beta = \text{const}$

$\Rightarrow x(t) = (V\cos\theta^\alpha + \beta)t + k_1$

$x(t_0) = x(t_0) \Rightarrow (V\cos\theta^* + \beta)\, t_0 + k_1 = x(t_0)$

$x^*(t) = (V\cos\theta^* + \beta)t + x(t_0) - (V\cos\theta^\alpha + \beta)t_0$

$x(T) = 0 \Rightarrow (V\cos\theta^* + \beta)(T - t_0) + x(t_0) = 0$

$\Rightarrow V\cos\theta^* = -\frac{x(t_0)}{T - t_0} - \beta$ — (1)

$\dot{y} = V\sin\theta(t) \Rightarrow \dot{y}^* = V\sin\theta^\alpha$

$\Rightarrow y(t) = V\sin\theta^\alpha t + k_2$

$y(t_0) = y(t_0) \Rightarrow V\sin\theta^\alpha\, t_0 + k_2 = y(t_0)$

$y(t) = V\sin\theta^\alpha t + y(t_0) - V\sin\theta^\alpha t_0$

$y(T) = 0 \Rightarrow V\sin\theta^\alpha (T - t_0) + y(t_0) = 0$

$\Rightarrow V\sin\theta^\alpha = \frac{-y(t_0)}{T - t_0}$ — (2)

Square and add (1) & (2)

$$V^2 = \left(\frac{-x(t_0)}{T - t_0} - \beta\right)^2 + \left(\frac{-y(t_0)}{T - t_0}\right)^2$$

$$\Rightarrow V^2 = \frac{x^2(t_0)}{(T - t_0)^2} + \beta^2 + \frac{2x(t_0)\beta}{T - t_0} + \frac{y^2(t_0)}{(T - t_0)^2}$$

Let $T - t_0 = \Delta t$

$\Rightarrow V^2\Delta t^2 = x^2(t_0) + y^2(t_0) + 2x(t_0)\beta\Delta t + \beta^2\Delta t^2$

$\Rightarrow \Delta t^2(\beta^2 - V^2) + \Delta t\,(2x(t_0)\beta) + x^2(t_0) + y^2(t_0) = 0$

$$\Rightarrow \Delta t = \frac{-2x(t_0)\beta \pm \sqrt{4x^2(t_0)\beta^2 - 4(\beta^2 - V^2)\left(x^2(t_0) + y^2(t_0)\right)}}{2(\beta^2 - V^2)}$$

$$\Rightarrow \boxed{T - t_0 = \frac{-2x(t_0)\beta \pm \sqrt{4x^2(t_0)\beta^2 - 4(\beta^2 - V^2)\left(x^2(t_0) + y^2(t_0)\right)}}{2(\beta^2 - V^2)}}$$

PS 4 Problem 5

Tuesday, April 30, 2019 1:44 AM

$J = \int_0^1 u^2(t)\,dt$ $\quad\quad$ $\dot{x}(t) = -2x(t) + u(t)$

$x(0) = 2$

$x(1) = 0$

$\mathcal{H} = g + p^T f$

$\Rightarrow \mathcal{H} = u^2 + p(-2x + u)$

$\dot{p} = -\frac{\partial H}{\partial x} = 2p$ $\quad\quad$ $p = e^{rt}$ $\quad$ $r = 2$

$\Rightarrow p = c_1 e^{2t}$

$\frac{\partial H}{\partial u} = 0$ (unbounded control)

$\Rightarrow 2u + p = 0 \quad \Rightarrow u = -P/2$

$\Rightarrow u = -\frac{c_1}{2} e^{2t}$

$\dot{x}(t) = -2x(t) - \frac{c_1}{2} e^{2t}$

$\dot{x}(t) + 2x(t) = -\frac{c_1}{2} e^{2t}$

$\mu = e^{\int 2\,dt} = e^{2t}$

$e^{2t}\dot{x} + 2x e^{2t} = -\frac{c_1}{2} e^{4t}$

$\Rightarrow (x e^{2t})' = -\frac{c_1}{2} e^{4t}$

$\Rightarrow x e^{2t} = \int -\frac{c_1}{2} e^{4t}\,dt$

$\Rightarrow x e^{2t} = -c_1 \frac{e^{4t}}{8} + c_2$

$\Rightarrow x = -\frac{c_1 e^{2t}}{8} + c_2 e^{-2t}$

$x(0) = 2 \Rightarrow -\frac{c_1}{8} + c_2 = 2 \quad\quad \Rightarrow c_2 = 2 + \frac{c_1}{8}$

$x(1) = 0 \Rightarrow -c_1 \frac{e^2}{8} + c_2 e^{-2} = 0$

$\Rightarrow c_2 = c_1 \frac{e^4}{8}$

$c_1 \frac{e^4}{8} = \frac{16 + c_1}{8}$

$\Rightarrow c_1 (e^4 - 1) = 16$

$\Rightarrow c_1 = \frac{16}{e^4 - 1}$

$c_2 = \frac{2e^4}{e^4 - 1}$

$u^* = -\frac{c_1}{2} e^{2t}$

$x^* = -\frac{c_1 e^{2t}}{8} + c_2 e^{-2t}$

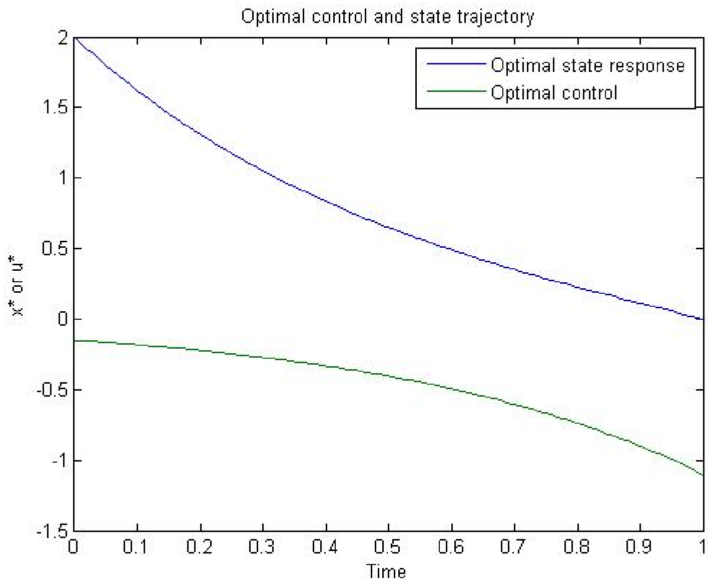
$$\boxed{u^*(t) = \frac{-8}{e^4 - 1} e^{2t} \qquad x^*(t) = \frac{-2}{e^4 - 1} e^{2t} + \frac{2e^4}{e^4 - 1} e^{-2t}}$$

Check: $\dot{x}^*(t) = \frac{-4}{e^4-1}e^{2t} - 4\frac{e^4}{e^4-1}e^{-2t}$

$-2x^* + u^* = \frac{4}{e^4-1}e^{2t} - \frac{4e^4}{e^4-1}e^{-2t} - \frac{8}{e^4-1}e^{2t}$

equal! ✓

$x^*(0) = 2 \qquad x^*(1) = 0$




```
%% AA 203 Homework 4
% Somrita Banerjee
clc
clear all
close all
t = linspace(0,1,100);
uStar = (-8/(exp(4)-1)).*exp(2.*t);
a = -2/(exp(4)-1);
b = 2*exp(4)/(exp(4)-1);
xStar = a.*exp(2.*t) + b.*exp(-2.*t);
figure
plot(t, xStar, t, uStar)
legend('Optimal state response','Optimal control')
xlabel('Time')
ylabel('x* or u*')
title('Optimal control and state trajectory');
```