PS 2 Problem 2

Wednesday, April 17, 2019 12:31 AM

$\dot{x}_1(t) = u_1(t)$

$\dot{x}_2(t) = u_2(t)$

$\|u(t)\| = 1 \Rightarrow u_1^2(t) + u_2^2(t) = 1$

Starts at $x(0)$

Ends at $x(T)$

$\min \int_0^T r(x(t))\,dt$

$r(\cdot) \geq 0$ and continuous

From $\bar{x} = (\bar{x}_1, \bar{x}_2)$

we can go to $(\bar{x}_1 + \Delta, \bar{x}_2)$

$(\bar{x}_1 - \Delta, \bar{x}_2)$

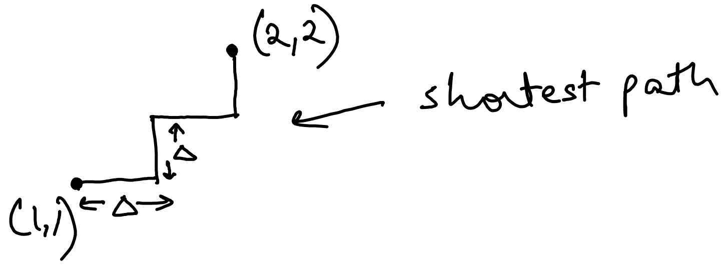
$(\bar{x}_1, \bar{x}_2 + \Delta)$

$(\bar{x}_1, \bar{x}_2 - \Delta)$

cost $r(\bar{x})\Delta$

Consider going from $(1,1)$ to $(2,2)$

With this discretization, let $\Delta = 0.5$ at first



Here, we need $2\ \Delta$ steps in (+)ve x_1 dir^n

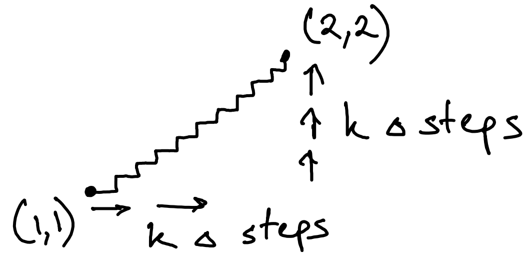
and $2\ \Delta$ steps in (+)ve x_2 dir^n

Assume $r(\cdot)$ uniform everywhere $= r^*$

Cost for this path $= 2\Delta r^* + 2\Delta r^*$

$= 4(0.5)r^* = 2r^*$

Let's make Δ smaller $= 1/k$ where k large



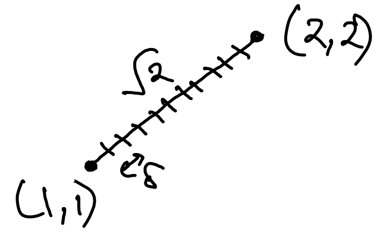
Again, we need $k\ \Delta$ steps in (+) ve x_1 &
$k\ \Delta$ steps in (+) ve x_2

$$\text{Cost} = 2k(\Delta)\, r^{\alpha} = 2r^{\alpha}$$

Even in the limit $\Delta \to 0$ $\quad k \to \infty$

$$\text{Cost} = 2r^{*}$$

However, optimal cost of original problem is a straight line of length $\sqrt{2}$.



The optimal path involves small steps δ in the $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ direction such that $\|u(t)\| = 1$.

$$\text{Cost of this path} = \underbrace{\delta}_{\text{step length}} \underbrace{\left(\frac{\sqrt{2}}{\delta}\right)}_{\#\text{ of steps}} r^{\alpha} = \sqrt{2}\, r^{\alpha}$$

$$\text{True optimal} = \sqrt{2}\, r^{*} < \text{Cost of discretized} = 2r^{\alpha}$$

$\therefore$ This is a bad discretization of the original problem.