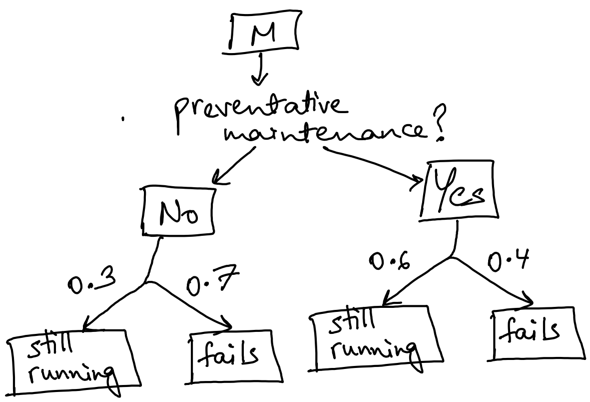
PS 3 Problem 1

Sunday, April 21, 2019 6:24 PM



Consider $J_4(x_4) :=$ profit at end of week 4 when machine is in state x_4 at end of wk 4

$x_4 = 0 \longrightarrow$ broken

$x_4 = 1 \longrightarrow$ running

$J_4(0) = \$0$

$J_4(1) = \$100$

machine running at end of wk 3

$J_3(1) = \$100 + \left(p(x_4=1)\, J_4(1) + p(x_4=0)\, J_4(0)\right)$

Let $k=1$ if preventative maintenance done

$k=0$ if " " not done

$= \$100 + \max\limits_{k=0,1} \left(E(J_4)\right)$

$= \$100 + \max\Big(\underbrace{0}_{\text{no maintenance}} + (0.7)(0) + (0.3)(100),$

maintenance cost $\rightarrow$ $-20 + (0.4)(0) + (0.6)(100)\Big)$

$= 100 + \max(\underbrace{30}_{k=0}, \underbrace{40}_{k=1})$

$\Rightarrow J_3(1) = \$140$ $\qquad u^*(x_3=1) : k=1$ (do maintenance)

no profit for wk 3

$J_3(0) = \$0 + \max\limits_{f=0,1} \left(E(J_4)\right)$

repair ($f=0$), replace ($f=1$)

$= \max\Big(-40 + 0.4(J_4(0)) + 0.6(J_4(1)),$

$-150 + 1.0 (J_4(1))\Big)$

$= \max(-40 + 0 + 60, -150 + 100)$

$\Rightarrow J_3(0) = \$20$ $\qquad u^*(x_3=0) : f=0$ (repair)

$$J_2(1) = \$100 + \max\Big(0 + 0.7(J_3(0)) + 0.3(J_3(1)),$$
$$-20 + 0.4(J_3(0)) + 0.6(J_3(1))\Big)$$
$$= 100 + \max(0 + 14 + 42,\ -20 + 8 + 84)$$
$$= 100 + \max(56, 72)$$

$J_2(1) = \$172$ $\quad u^*(x_2 = 1) : k = 1$ (do mainten)

$$J_2(0) = \$0 + \max\Big(-40 + 0.4(J_3(0)) + 0.6(J_3(1)),$$
$$-150 + 1.0(J_3(1))\Big)$$
$$= \max(-40 + 8 + 84,\ -150 + 140)$$
$$= \max(52, -10)$$

$J_2(0) = \$52$ $\quad u^*(x_2 = 0) : f = 0$ (repair)

We start w/ new machine so it is guaranteed to not fail in 1st week.
$\Rightarrow x_1 = 0$ is impossible

$$J_1(1) = \$100 + \max\Big(0 + 0.7(J_2(0)) + 0.3(J_2(1)),$$
$$-20 + 0.4(J_2(0)) + 0.6(J_2(1))\Big)$$
$$= 100 + \max(0 + 36.4 + 51.6,\ -20 + 20.8 + 103.2)$$
$$= 100 + \max(88, 104)$$

$J_1(1) = \$204$ $\quad u^*(x_1 = 1) : k = 1$ (do maintenance)

Optimal policy $\rightarrow$ $u^*(x_1=1) = u^*(x_2=1) = u^*(x_3=1) : k = 1$
i.e. Always do preventative maintenance if machine is running

($x_2 = 0$ impossible) $\quad u^*(x_2=0) = u^*(x_3=0) : f = 0$
i.e. Always repair, don't replace a failed machine.

Maximized expected profit = \$204

PS 3 Problem 2

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$$\mathcal{H} = (u(t))^2 + J_x^{*T} u(t)$$

transpose unnecessary $\because$ scalars

$$0 = J_t^* + \min_u \underbrace{\left[(u(t))^2 + J_x^{*T} u(t)\right]}_{\mathcal{H}}$$

NOC $\nabla_u \mathcal{H} = 0 \Rightarrow 2u(t) + J_x^* = 0$

$$\Rightarrow u(t) = -\frac{1}{2} J_x^*$$

$\because \nabla_{uu}^2 \mathcal{H} = 2 > 0 \Rightarrow$ this is global minimizer

$$u^*(t) = -\frac{1}{2} J_x^* \quad \text{if } \left|\frac{1}{2} J_x^*\right| \leq 1$$

Note that $|u(t)| \leq 1$ has to be satisfied as well.

Need to find $J^*(t,x)$ that satisfies this equ and boundary condition

$$J^*(T, x(T)) = (x(T))^2$$

Guess
$$J^*(t,x) = \begin{cases} (x-T+t)^2 + T-t & \text{if } x > 1+T-t \\ (x+T-t)^2 + T-t & \text{if } x < -(1+T-t) \\ x^2/(1+T-t) & \text{if } |x| \leq 1+T-t \end{cases}$$

This satisfies boundary condition @ $t=T$.

$$J_x^* = \begin{cases} 2(x-T+t) & \text{if } x > 1+T-t \\ 2(x+T-t) & \text{if } x < -(1+T-t) \\ \dfrac{2x}{1+T-t} & \text{if } |x| \leq 1+T-t \end{cases}$$

$$J_t^* = \begin{cases} 2(x-T+t)-1 & \text{if } x > 1+T-t \\ -2(x+T-t)-1 & \text{if } x < -(1+T-t) \\ \dfrac{-x^2}{(1+T-t)^2}(-1) & \text{if } |x| \leq 1+T-t \end{cases}$$

Case 1 $\quad x > 1+T-t$

HJB requires

$$0 = J_t^* + \min_u \underbrace{\left[(u(t))^2 + J_x^{*T} u(t)\right]}_{\mathcal{H}}$$

Minimizer is

$$u(t) = -\frac{1}{2} J_x^* = -(x-T+t)$$

But $\because x > 1+T-t \Rightarrow x-T+t > 1$

$\Rightarrow -(x-T+t) < -1$

$J^*_x = 2(x-T+t) > 0$

$\therefore$ minimizing $u(t) = -1$

$u^*(t) = -1$

HJB $\text{RHS} = J^*_t + (-1)^2 + J^*_x(-1)$

$= 2(x-T+t) - 1 + 1 - [2(x-T+t)]$

$= 0$

$\therefore$ This $J^*(t,x)$ works in this case.

Case 2 $\quad x < -(1+T-t)$

HJB requires

$$0 = J^*_t + \min_u [\underbrace{(u(t))^2 + J^{*T}_x u(t)}_{\mathcal{H}}]$$

Minimizer is

$u(t) = -\frac{1}{2} J^*_x = -(x+T-t)$

But $\because x < -(1+T-t) \Rightarrow -(x+T-t) > 1$

$J^*_x = 2(x+T-t) < 0$

$\therefore$ minimizing $u(t) = 1$

$u^*(t) = 1$

HJB $\text{RHS} = J^*_t + (1)^2 + J^*_x(1)$

$= -2(x+T-t) - 1 + 1 + [2(x+T-t)]$

$= 0$

$\therefore$ This $J^*(t,x)$ works in this case.

Case 3 $\quad |x| \leq 1+T-t$

HJB requires

$$0 = J^*_t + \min_u [\underbrace{(u(t))^2 + J^{*T}_x u(t)}_{\mathcal{H}}]$$

Minimizer is

$u(t) = -\frac{1}{2} J^*_x = -\frac{x}{1+T-t}$

$|x| \leq 1+T-t \Rightarrow |u| \leq 1$

$\therefore u^*(t) = -\frac{x}{1+T-t}$

HJB RHS $= J_t^* + u(t) + J_x u(t)$

$$= \frac{-x^2}{(1+T-t)^2}(-1) + \frac{x^2}{(1+T-t)^2} + \left(\frac{2x}{1+T-t}\right)\left(-\frac{x}{1+T-t}\right)$$

$$= \frac{x^2 + x^2 - 2x^2}{(1+T-t)^2} = 0$$

$\therefore$ This $J^*(t,x)$ works in this case.

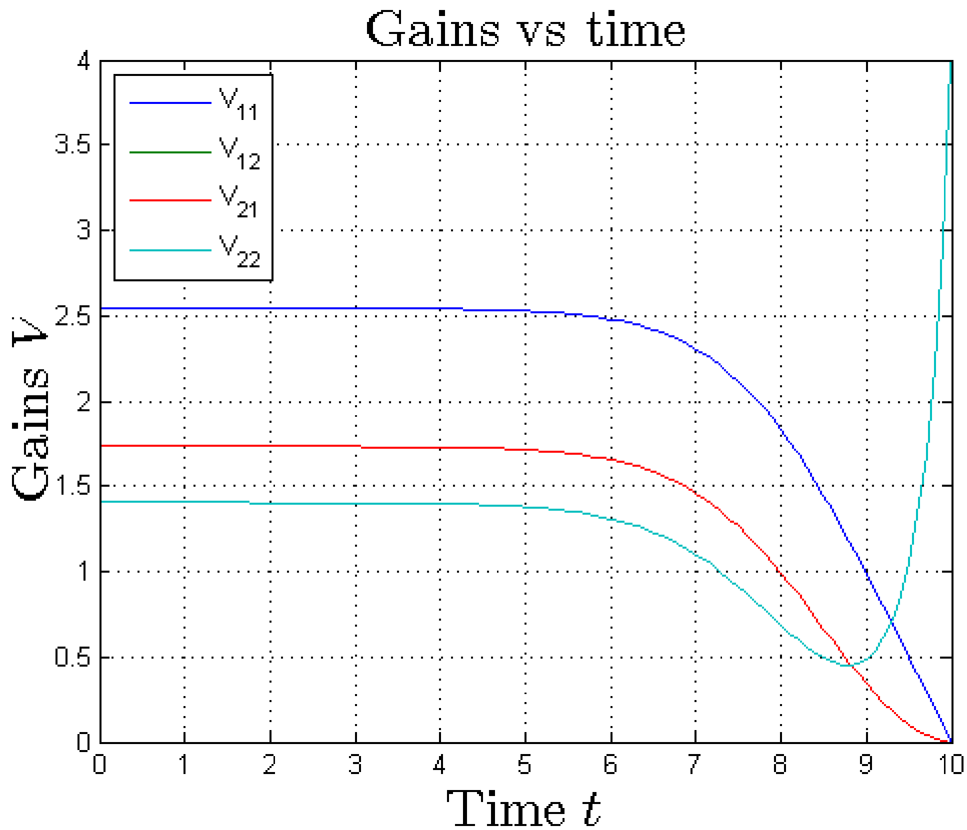
$$\boxed{u^*(t) = \begin{cases} -1 & \text{if } x > 1+T-t \\ 1 & \text{if } x < -(1+T-t) \\ -\dfrac{x}{1+T-t} & \text{if } |x| \le 1+T-t \end{cases}}$$

Problem Set 3 Question 3

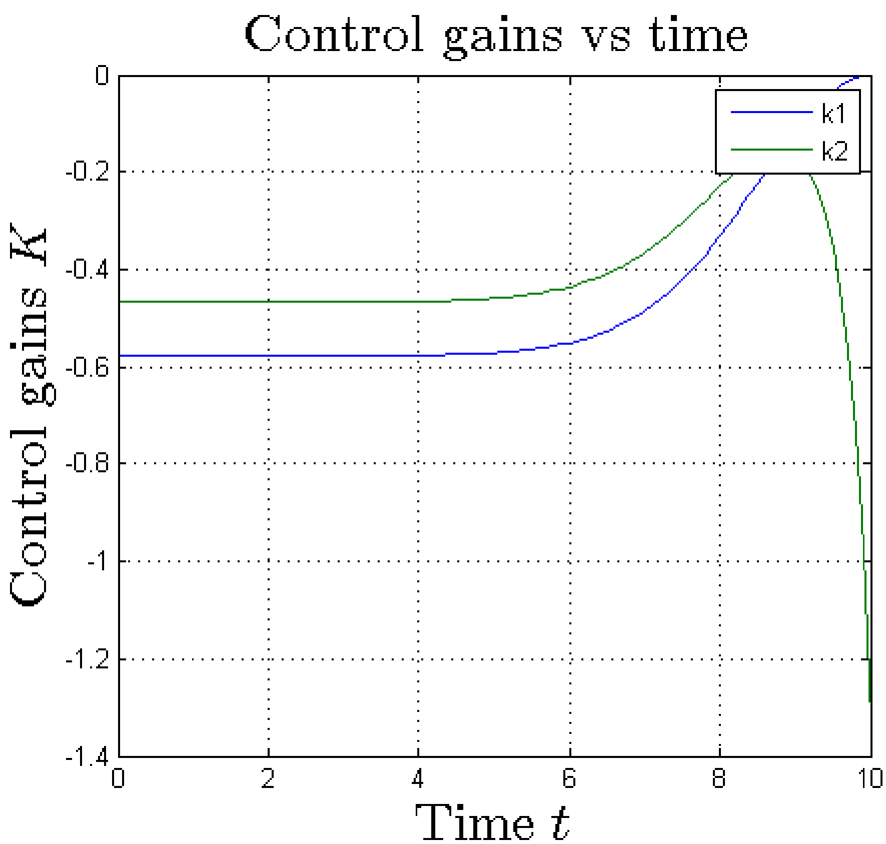
Somrita Banerjee

```
clc
clear all
close all

q = 1;
r = 3;
h = 4;
tf = 10;

A = [0 1; 0 -1];
B = [0;1];
Qf = [0 0; 0 h];
Q = [q 0; 0 0];
R = [r];

% initialize V[tf] as Qf
V_final = Qf(:);

for tf =[10,100]
    % Time in reverse
    dt = tf/100;
    rt = tf:-dt:0;

    [T, V] = ode45(@(t,V)mRiccati(t, V, A, B, Q, R), rt, V_final);

    [m, n] = size(V);
    VV = mat2cell(V, ones(m,1), n);
    fh_reshape = @(V)reshape(V,size(A));
    VV = cellfun(fh_reshape,VV,'UniformOutput',false);
    % Method inspired by https://www.mathworks.com/matlabcentral/answers/94722
    % -how-can-i-solve-the-matrix-riccati-differential-equation-within-matlab
    revV = flip(VV);

    uStar = zeros(size(revV,1),1);
    x = zeros(2,size(revV,1));
    K = zeros(2,size(revV,1));
    x(:,1) = [1;1];
    for i = 1: size(revV,1)
        K(:,i) = -(R\(B.'))*revV{i};
        uStar(i) = K(:,i)'*x(:,i);
        if i ~= size(revV,1)
            dxdt = A*x(:,i) + B*uStar(i);
            x(:,i+1) = x(:,i) + dxdt * dt;
        end
    end

    forwardTime = fliplr(rt);

    matVV=cat(3, revV{:});

    V1 = matVV(1,1,:);
    V1 = V1(:);
    V2 = matVV(1,2,:);
    V2 = V2(:);
```

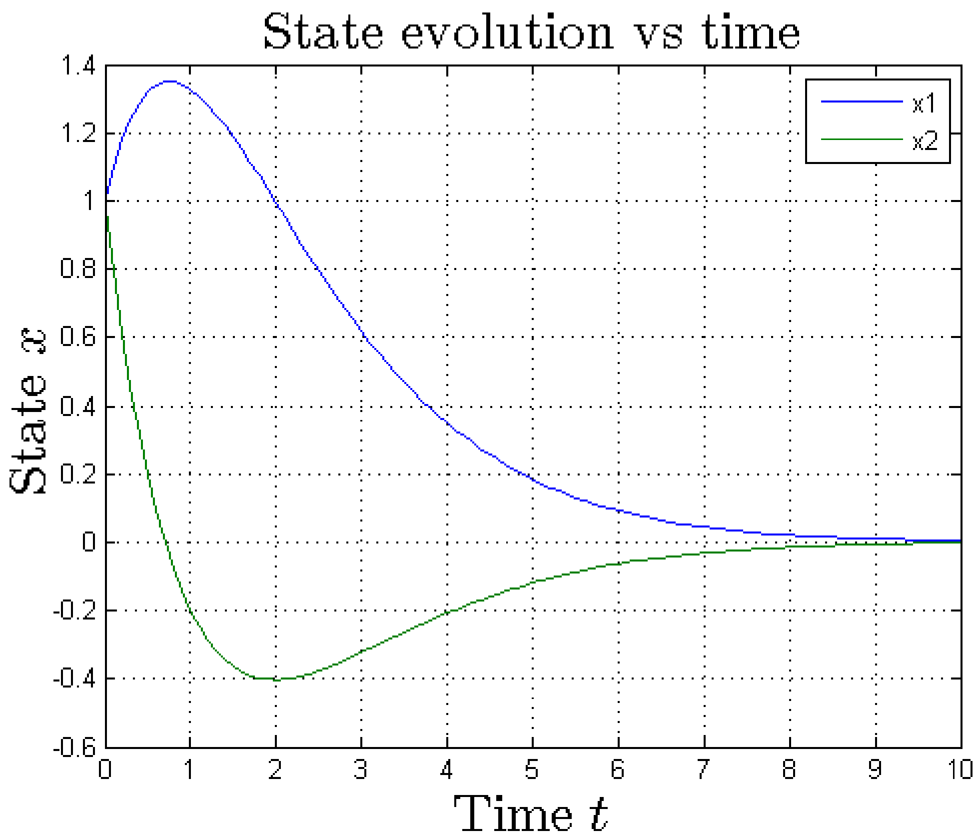
```
    V3 = matVV(2,1,:);
    V3 = V3(:);
    V4 = matVV(2,2,:);
    V4 = V4(:);
    figure
    plot(forwardTime,V1,forwardTime,V2,forwardTime,V3,forwardTime,V4)
    title('Gains vs time','Interpreter','latex','FontSize',20)
    xlabel('Time $$t$$','Interpreter','latex','FontSize',20)
    ylabel('Gains $$V$$','Interpreter','latex','FontSize',20)
    legend({'V_{11}','V_{12}','V_{21}','V_{22}'},'location','northwest');
    grid on

    figure
    plot(forwardTime,K)
    title('Control gains vs time','Interpreter','latex','FontSize',20)
    xlabel('Time $$t$$','Interpreter','latex','FontSize',20)
    ylabel('Control gains $$K$$','Interpreter','latex','FontSize',20)
    legend('k1','k2');
    grid on

    figure
    plot(forwardTime,x)
    title('State evolution vs time','Interpreter','latex','FontSize',20)
    xlabel('Time $$t$$','Interpreter','latex','FontSize',20)
    ylabel('State $$x$$','Interpreter','latex','FontSize',20)
    legend('x1','x2');
    grid on
end
```

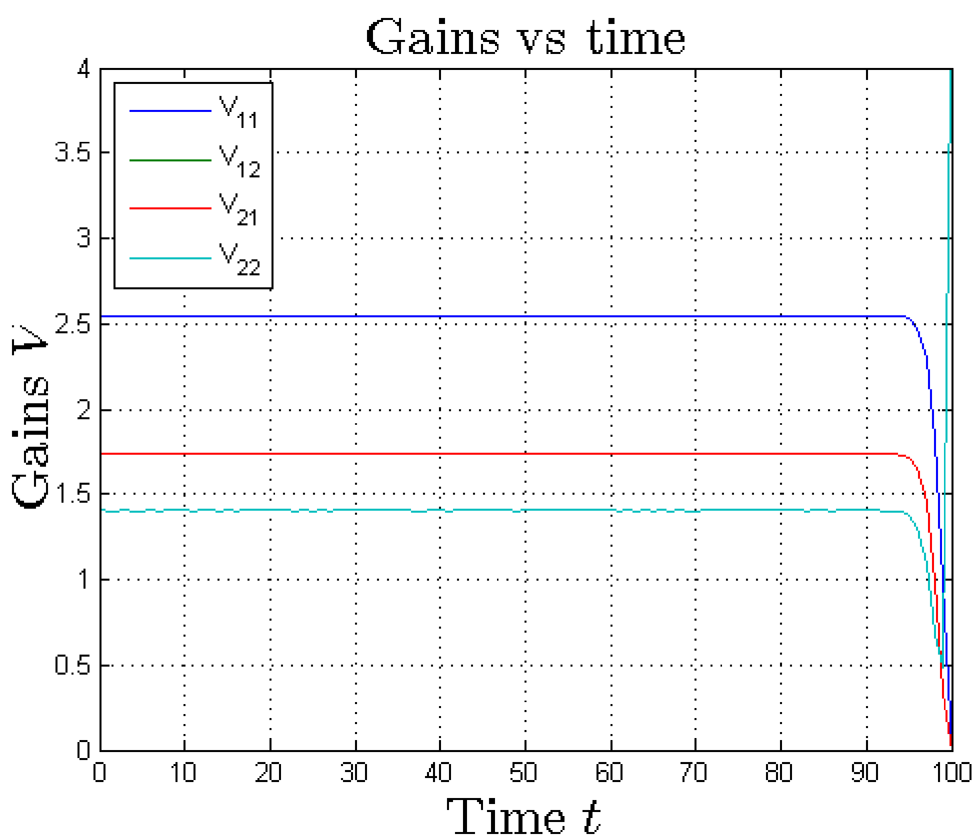


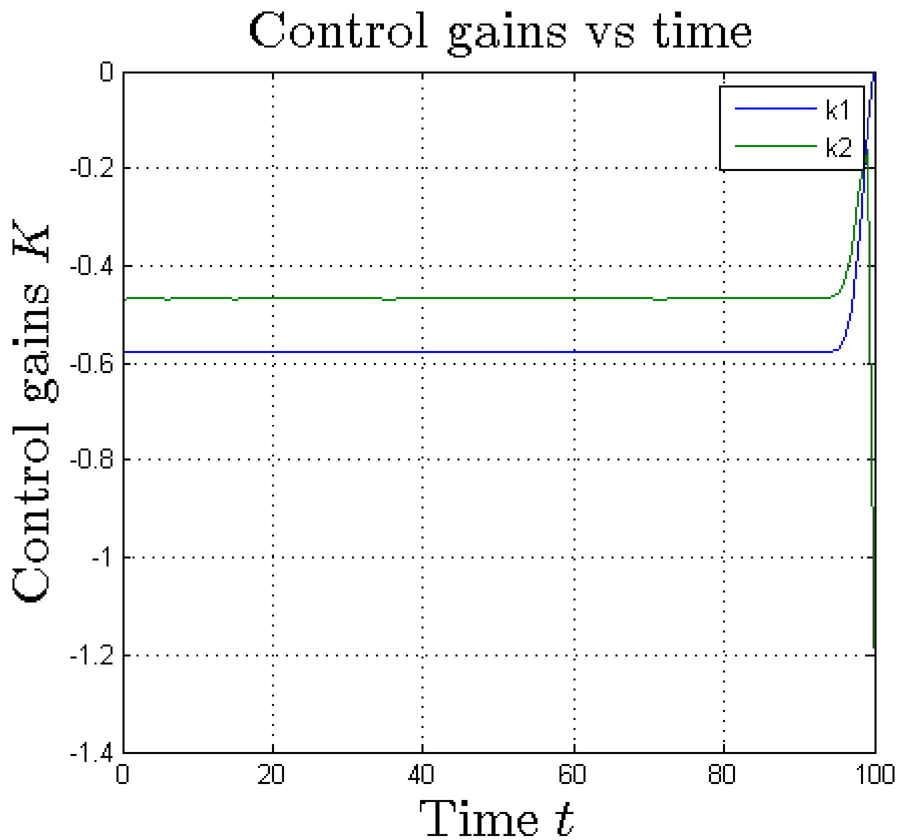
Gains vs time
Gains V
Time t
V_{11}
V_{12}
V_{21}
V_{22}
0
0.5
1
1.5
2
2.5
3
3.5
4
0
1
2
3
4
5
6
7
8
9
10



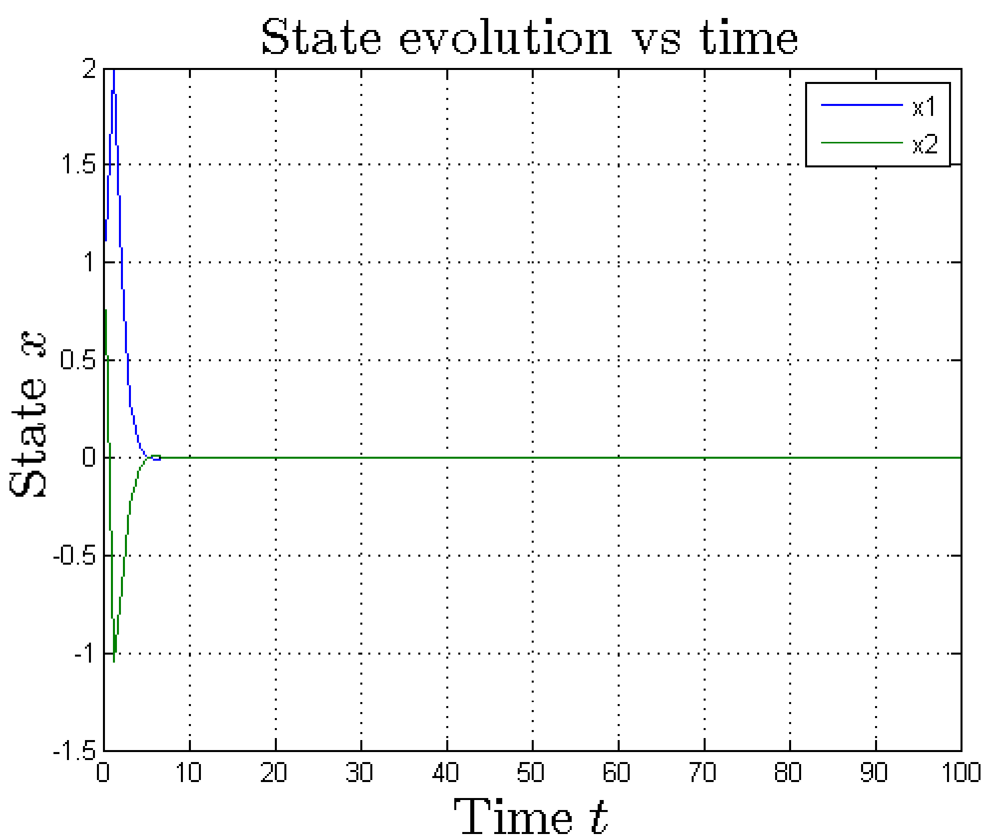
Control gains vs time
Control gains K
Time t
k1
k2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
0
2
4
6
8
10


State evolution vs time
x1
x2
State x
Time t
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
1
2
3
4
5
6
7
8
9
10


Gains vs time
V_{11}
V_{12}
V_{21}
V_{22}
Gains V
Time t
4
3.5
3
2.5
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70
80
90
100



Control gains vs time
Control gains K
Time t
k1
k2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
0
20
40
60
80
100



State evolution vs time
State x
Time t
x1
x2
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
10
20
30
40
50
60
70
80
90
100

```
function dVdt = mRiccati(t, V, A, B, Q, R)
V = reshape(V, size(A)); %Convert from "n^2"-by-1 to "n"-by-"n"
dVdt = -(Q - V*B*(R\(B.'))*V + V*A + A.'*V); %Determine derivative
dVdt = dVdt(:); %Convert from "n"-by-"n" to "n^2"-by-1

% Method inspired by https://www.mathworks.com/matlabcentral/answers/94722
% -how-can-i-solve-the-matrix-riccati-differential-equation-within-matlab
```

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PS 3 Problem 4

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Part 1

$$\mathcal{T} = \left\{ (h,v,m) : \left| h - \frac{150{,}000}{c} \right| \leq \frac{500}{c}, \left| v - \frac{28}{c} \right| \leq \frac{2.8}{c}, 250 \leq m \leq 500 \right\}$$

$$l(h,v,m) = \max\left(h - \frac{149500}{c}, v - \frac{25.2}{c}, m - 250, \frac{150{,}500}{c} - h, \frac{30.8}{c} - v, 500 - m \right)$$

$$l(h,v,m) \leq 0 \iff (h,v,m) \in \mathcal{T}$$

Part 2

$V(h,v,m,t)$ satisfies HJI PDE

$$\frac{\partial V}{\partial t}(h,v,m,t) + \min_{u \in U} \max_{d \in D} \nabla V(h,v,m,t)' f(h,v,m,u,d) = 0$$

$$\nabla V(h,v,m,t)' f(h,v,m,u,d)$$

$$= \begin{bmatrix} \frac{\partial V}{\partial h} & \frac{\partial V}{\partial v} & \frac{\partial V}{\partial m} \end{bmatrix} \begin{bmatrix} v \\ \frac{u}{m} - g + d \\ -bu \end{bmatrix}$$

$$= \frac{\partial V}{\partial h} v + \frac{\partial V}{\partial v}\left(\frac{u}{m} - g + d\right) + \frac{\partial V}{\partial m}(-bu)$$

$$= u\left(\frac{1}{m}\frac{\partial V}{\partial v} - b\frac{\partial V}{\partial m}\right) + \frac{\partial V}{\partial v} d + \frac{\partial V}{\partial h} v - \frac{\partial V}{\partial v} g$$

$\because$ u & d are linear terms here

$$\Rightarrow u^* = \arg\min_{u \in U} \left(\frac{1}{m}\frac{\partial V}{\partial v} - b\frac{\partial V}{\partial m}\right) u$$

$$\boxed{u^* = \begin{cases} 0 & \text{if } \frac{1}{m}\frac{\partial V}{\partial v} - b\frac{\partial V}{\partial m} \geq 0 \\ \frac{10000}{c} & \text{if } \frac{1}{m}\frac{\partial V}{\partial v} - b\frac{\partial V}{\partial m} < 0 \end{cases}}$$

$$d^* = \arg\max_{d \in D} \frac{\partial V}{\partial v} d$$

$$\boxed{d^* = \begin{cases} 1/c & \text{if } \frac{\partial V}{\partial v} \geq 0 \\ -1/c & \text{if } \frac{\partial V}{\partial v} < 0 \end{cases}}$$

Question 4 Part 3 Code GoddardAA203.m

Contents


```
% Goddard Space Launcher Model - AA203
```

Grid: generate a box-type grid of lower corner 'grid_min' and upper corner 'grid_max'

```
C = 100000;

grid_min = [0; 0; 250]; % Lower corner of computation domain
grid_max = [150500/C; 30/C; 500]; % Upper corner of computation domain
N = [20; 20; 50]; % Number of grid points per dimension
% pdDims = 3; % 3rd dimension is periodic
g = createGrid(grid_min, grid_max, N); % Generate the grid
```

Target set

```
toler = [500/C; 2.8/C; 125];
goal = [150000/C; 28/C; 375];
lower = goal - toler;
upper = goal + toler;
data0 = shapeRectangleByCorners(g, lower, upper);
```

Time vector

```
t0 = 0;
tMax = 6;
dt = 0.05;
tau = t0:dt:tMax;
```

Problem parameters

```
gValue = 9.81/C;
b = C*10^(-4);
uMax = 10000/C;
dMax = 1/C;
uMode = 'min'; % Minimize on controls
dMode = 'max'; % Maximize on disturbances
```

Pack problem parameters

```
% Define dynamic system
x0 = [0;0;500]; % Starting point
goddardLauncher = Goddard(x0, gValue, b, uMax, dMax);

% Put grid and dynamic systems into schemeData
schemeData.grid = g;
schemeData.dynSys = goddardLauncher;
schemeData.accuracy = 'veryHigh'; %set accuracy
schemeData.uMode = uMode;
schemeData.dMode = dMode;
```

Compute value function

```
HJIextraArgs.visualize = true; %show plot
HJIextraArgs.fig_num = 1; %set figure number
HJIextraArgs.deleteLastPlot = true; %delete previous plot as you update
data = HJIPDE_solve(data0, tau, schemeData, 'minVWithTarget', HJIextraArgs);
save('goddardAA203.mat', 'tau', 'g', 'data')
```

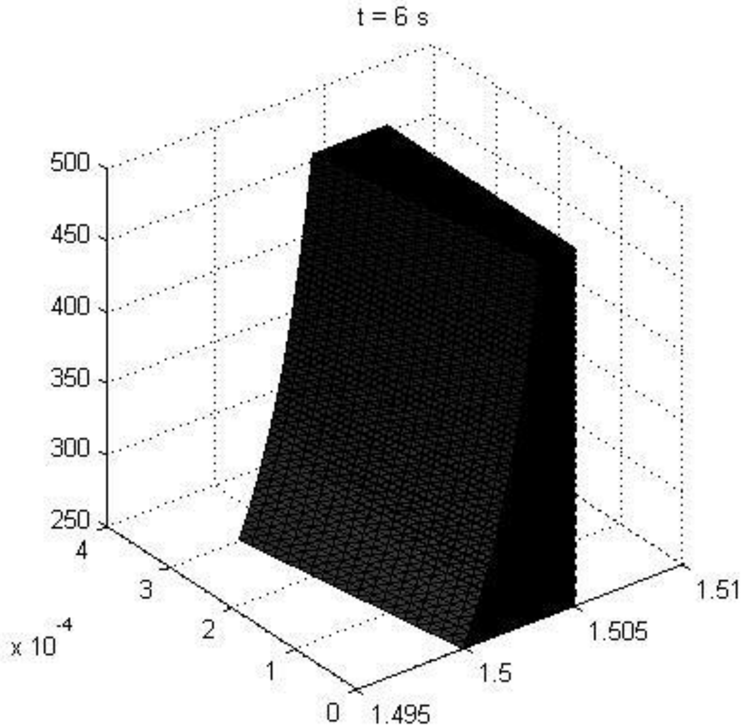
Visualize slices

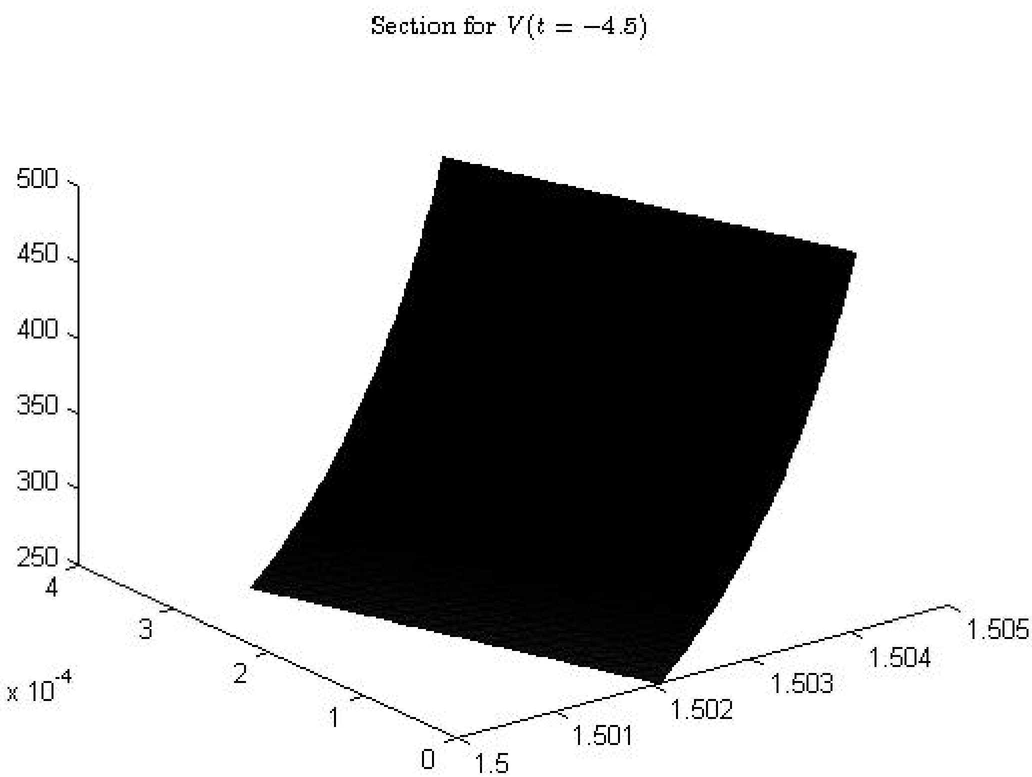
```
load('goddardAA203.mat')

% Section for J(t=-4.5)
figure;
ind = find(tau==4.5);
visSetIm(g, data(:,:,:,ind));
title('Section for $$V(t=-4.5)$$','Interpreter','latex');

% Section for J(t=-5.5)
figure;
ind = find(tau==5.5);
visSetIm(g, data(:,:,:,ind));
title('Section for $$V(t=-5.5)$$','Interpreter','latex');
```

Question 4 Part 4 Plots





Section for $V(t = -5.5)$