PS 1 Problem 5

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$x_{t+1} = Ax_t + Bu_t$ LTI

$$\min_{u \in \mathbb{R}^{mT}} J(u) := x_T^T Q_T x_T + \sum_{t=0}^{T-1} x_t^T Q x_t + u_t^T R u_t$$

equivalent to

$$\min_{u \in \mathbb{R}^{mT}} \frac{1}{2} u^T \tilde{Q} u - \tilde{b}^T u$$

$x_1 = Ax_0 + Bu_0$

$x_2 = A(Ax_0 + Bu_0) + Bu_1$

$= A^2 x_0 + ABu_0 + Bu_1$

$x_3 = A^3 x_0 + A^2 B u_0 + ABu_1 + Bu_2$

$$\Rightarrow x_T = A^T x_0 + \sum_{i=0}^{T-1} A^{T-1-i} B u_i$$

$$x_t = A^t x_0 + \sum_{i=0}^{t-1} A^{t-1-i} B u_i$$

$A \in \mathbb{R}^{n \times n}$ $B \in \mathbb{R}^{n \times m}$

$$x = \begin{bmatrix} I_{n \times n} \\ A \\ \vdots \\ A^T \end{bmatrix} x_0 + \begin{bmatrix} 0_{n \times m} & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & 0 & & \\ A^{T-1}B & A^{T-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{T-1} \end{bmatrix}$$

$$= \tilde{A} x_0 + \tilde{B} u$$

$x_0 \in \mathbb{R}^n$ $\tilde{A} \in \mathbb{R}^{(T+1)n \times n}$

$u \in \mathbb{R}^{Tm}$ $\tilde{B} \in \mathbb{R}^{(T+1)n \times mT}$

$x \in \mathbb{R}^{(T+1)n}$

$$J = x_T' Q_T x_T + \sum_{t=0}^{T-1} x_t' Q x_t + u_t' R u_t$$

$$= x_0' Q x_0 + x_1' Q x_1 + \cdots x_{T-1}' Q x_{T-1} + x_T' Q_T x_T$$
$$+ u_0' R u_0 + u_1' R u_1 + \cdots u_{T-1}' R u_{T-1}$$

$$= x' \underbrace{\begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & Q & \\ & & & Q_T \end{bmatrix}}_{\hat{Q} \in \mathbb{R}^{(T+1)n \times (T+1)n}} x + u' \underbrace{\begin{bmatrix} R & & \\ & R & \\ & & \ddots \\ & & & R \end{bmatrix}}_{\tilde{R} \in \mathbb{R}^{Tm \times Tm}} u$$

$$= (\tilde{A} x_0 + \tilde{B} u)' \hat{Q} (\tilde{A} x_0 + \tilde{B} u) + u' \tilde{R} u$$

$$= \underbrace{x_0' \tilde{A}' \hat{Q} \tilde{A} x_0}_{\text{const wrt } u} + u' \tilde{B}' \hat{Q} \tilde{B} u + 2 x_0' \tilde{A}' \hat{Q} \tilde{B} u$$
$$+ u' \tilde{R} u$$

$$\Rightarrow \min J = \min \frac{1}{2} u' \underbrace{(\tilde{B}' \hat{Q} \tilde{B} + \tilde{R})}_{\tilde{Q}} u + \underbrace{x_0' \tilde{A}' \hat{Q} \tilde{B}}_{-\tilde{b}'} u$$

$$\tilde{Q} = \tilde{B}' \hat{Q} \tilde{B} + \tilde{R}$$

$$\hat{Q} \in \mathbb{R}^{(T+1)n \times (T+1)n} = \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q & \\ & & & & Q_T \end{bmatrix}$$

$$\tilde{B} \in \mathbb{R}^{(T+1)n \times mT} = \begin{bmatrix} 0 & & & \\ B & & & \\ AB & B & & \\ \vdots & & & \\ A^{T-1}B & A^{T-2}B & \cdots & B \end{bmatrix}$$

$$\tilde{R} \in \mathbb{R}^{mT \times mT} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}$$

$$\Rightarrow \tilde{Q} \in \mathbb{R}^{mT \times mT}$$

$$\tilde{b} = -\left(x_0' \tilde{A}' \hat{Q} \tilde{B}\right)'$$

$$x_0 \in \mathbb{R}^n \quad \tilde{A} \in \mathbb{R}^{(T+1)n \times n} \quad \tilde{A} = \begin{bmatrix} I_{n \times n} \\ A \\ \vdots \\ A^T \end{bmatrix}$$

$$\hat{Q} \in \mathbb{R}^{(T+1)n \times (T+1)n}$$

$$\tilde{B} \in \mathbb{R}^{(T+1)n \times mT}$$

$$\tilde{b} \in \left(\mathbb{R}^{1 \times mT}\right)' \in \mathbb{R}^{mT}$$

$$u^* = \tilde{Q}^{-1} \tilde{b} \qquad J(u^*) = 2.9471$$

AA 203 HW 1 Question 5 again

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```
clc
clear all
close all
Q = eye(2);
QT = 10 * eye(2);
R = eye(1);
A=[1 1; 0 1];
B=[0;1];
x0=[1;0];
T=20;
btilde = zeros(T,1);
Qtilde = zeros(T,T);
Qhat = blkdiag(kron(eye(20),Q),QT);
Atilde=eye(2);
for i = 1:T
    Atilde=[Atilde;A^i];
end
```

```
end
Btilde = zeros((T+1)*2,T);
for i=1:T
    for j=1:i
        Btilde(2*i+1: 2*i+2,j)=(A^(i-j)) *B;
    end
end
Rtilde = kron(eye(20),R);
Qtilde = Btilde'*Qhat*Btilde + Rtilde;
btilde = -(x0'*Atilde'*Qhat*Btilde)';
uStar = Qtilde\btilde

u= uStar;
x = zeros(2, T+1);
x(:,1) = x0;
sumJ = 0;
for t = 0:T-1
    x(:,t+2) = A*x(:,t+1) + B*u(t+1);
    sumJ = sumJ + x(:,t+1)'*Q*x(:,t+1) + u(t+1)'*R*u(t+1);
end
J = x(:,T+1)'*QT*x(:,T+1) + sumJ
```

```
uStar =

   -0.4221
    0.1030
    0.1530
    0.0974
    0.0464
    0.0177
    0.0051
    0.0007
   -0.0004
   -0.0004
   -0.0002
   -0.0001
   -0.0000
   -0.0000
   -0.0000
    0.0000
    0.0000
    0.0000
    0.0000
    0.0000


J =

    2.9471
```