Question 2

Monday, June 3, 2019 7:30 PM

$\dot{y}(t) + a y(t) = b u(t)$ $\quad a, b$ unknown

$\dot{y}_m(t) + a_m y_m(t) = b_m r(t)$

Part a

$u(t) = k_r(t) r(t) + k_y(t)\, y(t)$

Dropping dep on t for clarity

$\Rightarrow \dot{y} + a y = b u = b(k_r r + k_y y)$

$\Rightarrow \dot{y} + (a - b k_y) y = b k_r r$

To match reference model, we choose

$a - b k_y = a_m$ and $b k_r = b_m$

$$\Rightarrow \boxed{k_y^* = \frac{a - a_m}{b} \quad \text{and} \quad k_r^* = \frac{b_m}{b}}$$

Part b

$e(t) := y(t) - y_m(t)$

$\delta_r(t) := k_r(t) - k_r^*$

$\delta_y(t) := k_y(t) - k_y^*$

$\dot{e}(t) = \dot{y}(t) - \dot{y}_m(t)$

$= b u(t) - a y(t) - b_m r(t) + a_m y_m(t)$

$= b\big(\underbrace{k_r(t)}_{\delta_r + k_r^*} r(t) + \underbrace{k_y(t)}_{\delta_y + k_y^*} y(t)\big) - a y(t)$

$+ a_m \underbrace{y_m(t)}_{y_m = y - e} - b_m r(t)$

(dropping dep on t for clarity)

$\Rightarrow \dot{e} = b\left(\left(\delta_r + \frac{b_m}{b}\right) r + \left(\delta_y + \frac{a - a_m}{b}\right) y\right) - a y$

$+ a_m y - a_m e - b_m r$

$= b \delta_r r + \cancel{b_m r} + b \delta_y y + \cancel{(a - a_m) y}$

$\cancel{- a y} + \cancel{a_m y} - a_m e - \cancel{b_m r}$

$$\Rightarrow \boxed{\dot{e}(t) + a_m e(t) = b \delta_r(t) r(t) + b \delta_y(t) y(t)}$$

Part a

Let $x := (e, \delta_r, \delta_y)$

$$V(x) = \frac{1}{2}e^2 + \frac{|b|}{2\gamma}(\delta_r^2 + \delta_y^2)$$

$$\Rightarrow \dot{V} = e\dot{e} + \frac{|b|}{2\gamma}(2\delta_r\dot{\delta}_r + 2\delta_y\dot{\delta}_y)$$

$$\delta_r = k_r(t) - k_r^* \Rightarrow \dot{\delta}_r = \dot{k}_r = -\operatorname{sgn}(b)\gamma e(t) r(t)$$

const

$$\delta_y = k_y(t) - k_y^* \Rightarrow \dot{\delta}_y = \dot{k}_y = -\operatorname{sgn}(b)\gamma e(t) y(t)$$

$$\Rightarrow \dot{V} = e(-a_m e + b\delta_r r + b\delta_y y) + \frac{|b|}{\gamma}\left(\delta_r(-\operatorname{sgn}(b)\gamma e r) + \delta_y(-\operatorname{sgn}(b)\gamma e y)\right)$$

$$= -a_m e^2 + eb(\delta_r r + \delta_y y) - \frac{|b|}{\cancel{\gamma}}\cancel{\gamma}\operatorname{sgn}(b) e(\delta_r r + \delta_y y)$$

$$|b|\operatorname{sgn}(b) = b \quad \forall b$$

$$\Rightarrow \dot{V} = -a_m e^2 + \cancel{eb(\delta_r r + \delta_y y)} - \cancel{eb(\delta_r r + \delta_y y)}$$

$$\Rightarrow \boxed{\dot{V} = -a_m e^2}$$

Consider Lyapunov Thm

1. if $x := (e, \delta_r, \delta_y)$

   then if $x = 0$ $\quad (e=0, \delta_r = 0, \delta_y = 0)$

   $$\dot{e} = -a_m \cancel{e}^{\,0} + b\cancel{\delta_r}^{\,0} r + b\cancel{\delta_y}^{\,0} y = 0$$
   $$\dot{\delta}_r = \dot{k}_r = -\operatorname{sgn}(b)\gamma \cancel{e}^{\,0} r = 0$$
   $$\dot{\delta}_y = \dot{k}_y = -\operatorname{sgn}(b)\gamma \cancel{e}^{\,0} y = 0$$

   $\therefore \dot{x} = f(x,t) = 0 \quad @ \; x = 0$

2. $V(x) = \frac{1}{2}e^2 + \frac{|b|}{2\gamma}(\delta_r^2 + \delta_y^2)$ is clearly continuous & differentiable

3. $V(x)$ is a sum of squares so it is positive definite

   $$V(x) > 0 \quad \forall x \neq 0$$
   $$V(x) = 0 \quad \forall x = 0$$

④ $\dot{V}(x) = -a_m e^2 \qquad a_m > 0$

$\Rightarrow \dot{V}(x) < 0 \quad \forall\ e \neq 0$

and $\dot{V}(x) = 0 \quad \forall\ e = 0$

$\Rightarrow \dot{V}(x) \leq 0 \quad \forall\ x \neq 0$ [possible $e=0$, $\delta_r \neq 0$, $e_y \neq 0$]

and $\dot{V}(x) = 0 \quad \forall\ x = 0$

$\therefore \dot{V}(x)$ is negative semi-definite.

So, we can say $x = (e, \delta_r, \delta_y) = 0$ is a stable point in the sense of Lyapunov.
So $\|e(t)\|$, $\|\delta_r(t)\|$, $\|\delta_y(t)\|$ remain bounded $\forall\ t \in [0, \infty)$.

Consider Barbalat's Lemma.

Let $g(t) = \dot{V}(x) = -a_m e(t)^2$

① $g(t)$ is clearly differentiable $\dot{g} = -2a_m \dot{e}(t)$

② $g(t)$ has a finite limit as $t \to \infty$
because $e(t)$ is bounded $\forall t$
$\Rightarrow -a_m e(t)^2$ is bounded $\forall t$

③ $\dot{g}(t) = -2a_m \dot{e}(t)$ is uniformly continuous
because
$\ddot{g}(t) = -2a_m \ddot{e}(t)$ is bounded
because

$$\ddot{e}(t) = -a_m \dot{e} + b\dot{\delta}_r r + b\delta_r \dot{r} + b\dot{\delta}_y y + b\delta_y \dot{y}$$

$$= -a_m(-a_m e + b\delta_r r + b\delta_y y) + b(-\text{sgn}(b)\gamma e r) r + b\delta_r \dot{r} + b(-\text{sgn}(b)\gamma e y) y + b\delta_y \dot{y}$$

Using Lyapunov, we showed e, δ_y, δ_r are bounded
we know r & y are bounded (given to us)

$\therefore \ddot{e}(t)$ is bounded $\Rightarrow \ddot{g}(t)$ is bounded

Using Barbalat's, we can say

$\dot{g}(t) = -2a_m e(t) \to 0$ as $t \to \infty$

$a_m > 0 \quad \therefore \boxed{e(t) \to 0 \text{ as } t \to \infty}$

Part d

$\gamma = 2$

<u>Case 1</u>: $r(t) = 4$ <u>Case 2</u>: $r(t) = 4\sin(3t)$

y, y_m, k_r, k_y

$\dot{y}_m(t) + 4y_m(t) = 4r(t)$

$\dot{y}(t) - y(t) = 3u(t) = 3\left(k_r(t)r(t) + k_y(t)y(t)\right)$

We know $\text{sgn}(b) = 1$ (b is positive)

$\dot{k}_r(t) = -\text{sgn}(b)\gamma e(t) r(t)$

$= -\gamma\left(y(t) - y_m(t)\right) r(t)$

$\dot{k}_y(t) = -\text{sgn}(b)\gamma e(t) y(t)$

$= -\gamma\left(y(t) - y_m(t)\right) y(t)$

Now we can define

$$x = \begin{bmatrix} y(t) \\ y_m(t) \\ k_r(t) \\ k_y(t) \end{bmatrix} \text{ and } \dot{x} = \begin{bmatrix} \dot{y}(t) \\ \dot{y}_m(t) \\ \dot{k}_r(t) \\ \dot{k}_y(t) \end{bmatrix}$$