Spring 2019

Stanford
AA203: Optimal and Learning-based Control
Problem set 5, due on May 9

Problem 1: In this problem we will investigate optimal trajectories for a rocket, by considering the classic, one-dimensional Goddard model widely used in the early days of rocket science. Specifically, consider the following Optimal Control Problem (**OCP**) which entails maximizing the height reached by the rocket given some limited amount of propellant:

$$
(\mathbf{OCP}) \begin{cases} \underset{u(\cdot)}{\text{maximize}} & y(t_f) \\ \text{subject to} & \dot{y}(t) = v(t)\ ,\ \dot{v}(t) = \dfrac{u(t)}{m(t)} - g\ ,\ \dot{m}(t) = -b\,u(t) \\ & y(0) = y_0, v(0) = v_0, m(0) = m_0 \\ & y(t_f) \in \mathbb{R}, v(t_f) \in \mathbb{R}, m(t_f) = m_f \\ & 0 \le u(t) \le u_{\max},\ t \in [0, t_f] \end{cases}
$$

where $y(t)$ represents the height reached by the rocket at time t, $v(t)$ is the velocity of the rocket at time t, $m(t)$ is the mass of the rocket at time t, $g > 0$ is the gravitational acceleration (assumed constant), $b > 0$ is the fuel consumption ratio. The final time t_f is free, and the final desired mass m_f accounts for the requirement to have some residual amount of propellant left for further maneuvers. We make the following (very mild) assumption:

(A) *The initial mass m_0 is such that $u_{\max} > g m_0$, and the initial acceleration is positive, i.e., $\dot{v}(0) > 0$.*

The condition $u_{\max} > g m_0$ enforces that enough thrust is available to escape Earth's gravity pull, while the second one requires that, at the very beginning, the rocket is heading up.

By leveraging the Pontryagin's Minimum Principle (PMP), the objective of this problem is to show that the optimal control sequence $u^*(t)$ is given by

$$
u^*(t) = \begin{cases} u_{\max},\ t \in [0, t_{sw}] \\ 0,\ t \in (t_{sw}, t_f] \end{cases} \tag{1}
$$

where $t_{sw} \in (0, t_f]$ is a *switching time* (possibly equal to t_f, in which case there is no switching). We will show this in steps:

1. Denote by $(p_y(t), p_v(t), p_m(t))$ the co-state variables for problem **OCP**. Write down the Hamiltonian and derive the co-state equations.
2. Recalling that the final boundary conditions in this case read as

$$(p_y(t_f), p_v(t_f), p_m(t_f)) - \nabla h(y(t_f), v(t_f), m(t_f)) \perp \ker \nabla F(y(t_f), v(t_f), m(t_f)) \quad (2)$$

where $F(y, v, m) := m - m_f$; show that

$$p_y(t_f) = -1 \ , \ p_v(t_f) = 0. \quad (3)$$

3. Define $\phi(t) := \dfrac{p_v(t)}{m(t)} - p_m(t)\, b$. Show that

$$\dot{\phi}(t) = -\frac{p_y(t)}{m(t)}, \quad (4)$$

and argue that $\phi(t)$ can not be zero on any non-zero time interval (i.e., this problem is not singular).

4. Exploiting the fact that $\phi(t) \neq 0$ on any non-zero time interval, show that the optimal control sequence for (**OCP**) is given by

$$u^*(t) = \begin{cases} u_{\max}, \text{ if } \phi(t) < 0, \\ 0, \text{ if } \phi(t) > 0. \end{cases} \quad (5)$$

5. Use Assumption **(A)** to show that the optimal control sequence satisfies $u^*(0) = u_{\max}$ and $\phi(0) < 0$.
6. Show that there exists a switching time $t_{sw} \in (0, t_f]$ such that $\phi(t) < 0$ for $t \in [0, t_{sw}]$ and $\phi(t) > 0$ for $t \in (t_{sw}, t_f]$.
7. Leveraging the dynamical equation of the mass and the structure of the optimal control sequence (1), prove that $t_{sw} = \dfrac{m_0 - m_f}{b\, u_{\max}}$.

Problem 2: Now that we have characterized the *structure* of an optimal control sequence for problem **OCP**, we want to compute it numerically by using numerical indirect methods. Specifically, we will consider two methods:. The first method solves **OCP** by performing a bisection search on the optimal final time t_f. The second method entails implementing in Matlab a shooting algorithm.

1. *Method # 1*: Leveraging the structure of the optimal control sequence $u^*(t)$ found in Problem 2, the task is to determine the optimal final time t_f that maximizes the final height $y(t_f)$ (i.e., the objective of problem **OCP**). The reasoning is as follows. The previous theoretical analysis shows that the rocket will keep accelerating until the switching time t_{sw}. From that time, the velocity

will start decreasing. The rocket will gain height until the velocity is positive and it will start losing height once the velocity switches sign: that time is the desired final time, for which the height is the highest possible. Use this insight to compute the optimal final time t_f via bisection search. To do so, use as a starting point the script skeletons `Xdyn.m`, `dichotomyFunc.m` and `exactGoddard.m` provided in the folder `GoddardExact`.

2. *Method # 2*: To implement a shooting method, one needs a good guess for $(t_f, p_y(0), p_v(0), p_m(0))$ in order to ensure convergence. Such a guess can be achieved by leveraging the theoretical analysis from Problem 2, along with a few additional steps.

   (a) Using the analysis from Problem 2, show that $p_v(t) = t - t_f$ for $t \in [0, t_f]$, and thus $p_y(0) = -1$ and $p_v(0) = -t_f$.

   (b) As seen in class, an additional necessary condition for this problem is that the Hamiltonian is identically equal to zero, thus $H(0) = H(t_f) = 0$. Use this fact to show that

   $$p_m(0) = -\frac{1}{b u_{\max}} \left(t_f \left(\frac{u_{\max}}{m_0} - g \right) + v_0 \right).$$

   (c) We now have explicit expressions for $(p_y(0),\ p_v(0),\ p_m(0))$ – the only value that needs to be guessed is t_f. Implement a shooting algorithm to find $u^*(t)$, by using as a starting point the script skeletons `Zdyn.m`, `hamiltonianFunc.m`, `shootingFunc.m` and `shootingGoddard.m` provided in the folder `GoddardShooting`, and by using as an initial guess for t_f the value obtained above via bisection search. What are the optimal switching time t_{sw} and the optimal final time t_f? Attach the plots of your results.

   (d) Let us play with the values for the guess of t_f to test the sensitivity of the shooting algorithm. Do you achieve convergence for $t_f = 255$ and, in case, how many iterations are needed? And for $t_f = 270$?

Learning goals for this problem set:

Problem 1: To familiarize with the process of using PMP to derive the "structure" of an optimal control sequence.

Problem 2: To learn how to apply numerical indirect methods to solve optimal control problems.