PART 6

We know $\phi(t) < 0$ for $t = 0$

We also know $\dot{\phi}(t) = \frac{1}{m(t)} > 0$

$\because$ Final time is free $\Rightarrow \delta t_f$ arbitrary

$$\Rightarrow \left[\mathcal{H} + \frac{\partial h}{\partial t}\right]_{t_f} = 0$$

$$\Rightarrow \left. \underbrace{p_y}_{p_y(t_f) = -1} v + \underbrace{p_v}_{p_v(t_f) = 0}\left(\frac{u}{m} - g\right) + p_m(-bu) + \left(-\underbrace{\frac{dy}{dt}}_{\frac{dy}{dt} = v(t)}\right)\right|_{t_f} = 0$$

$$\Rightarrow \cancel{-v(t_f)} + 0 - \underbrace{b}_{b > 0} u(t_f) p_m(t_f) - \cancel{v(t_f)} = 0$$

$$\Rightarrow u(t_f) p_m(t_f) = 0$$

$\Rightarrow$ either $u(t_f) = 0 \Rightarrow \phi(t_f) > 0$

or $p_m(t_f) = 0$

$$\Rightarrow \phi(t_f) = \frac{p_v(t_f)}{m(t_f)} - p_m(t_f) b$$

$$= 0 - 0 = 0$$

A) if $p_m(t_f) = 0 \Rightarrow t_{sw} = t_f \Rightarrow \phi(t_f) = 0$

and $\phi(t) < 0$ for $t \in [0, t_{sw}]$

$\phi(t) > 0$ for $t \in (t_{sw}, t_f]$

$\because \phi(t)$ can't be 0 for a finite time interval

B) If $p_m(t_f) \neq 0$ and $u(t_f) = 0$

$\Rightarrow \phi(t_f) > 0$

$\Rightarrow \exists\, t_{sw} \in (0, t_f)$

$\phi(t) < 0$ for $t \in [0, t_{sw}]$

$\phi(t) > 0$ for $t \in (t_{sw}, t_f]$

$\because \phi(0) < 0$ and $\dot{\phi}(t) > 0$

$\therefore$ In both cases, $\exists\, t_{sw} \in (0, t_f]$ such that
$\phi(t) < 0$ for $t \in [0, t_{sw}]$
$\phi(t) > 0$ for $t \in (t_{sw}, t_f]$

Note: another way of showing same result

$\because$ Final time is free δt_f arbitrary

$$\Rightarrow \mathcal{H} + \frac{\partial h}{\partial t}\Big|_{t_f} = 0$$

$$\Rightarrow \left(p_y v + p_v\left(\frac{u}{m} - g\right) + p_m(-bu)\right) - \frac{\partial y(t_f)}{\partial t} = 0$$

$$\Rightarrow \left(-1\, v(t) + u(t)\left(\frac{p_v(t)}{m(t)} - b p_m(t)\right) - g(t - t_f)\right)\Big|_{t_f} - \dot{y}(t_f) = 0$$

$$\Rightarrow -v(t_f) + u(t_f)\,\phi(t_f) - \dot{y}(t_f) = 0$$

$\dot{y} = v$

$$\Rightarrow u(t_f)\,\phi(t_f) = 0$$

$\therefore$ Either $\phi(t_f) = 0$ momentaily or
$u(t_f) = 0 \Rightarrow \phi(t_f) > 0$

However, because $\mathcal{H}$ does not depend explicitly on time, we also have the NOC

$$\mathcal{H}(x^*, u^*, p^*) = 0 \quad \forall\ t \in [t_0, t_f]$$

$$\Rightarrow p_y(t)\, v_0 + p_v(t)\left(\frac{u(t)}{m(t)} - g\right) + p_m(t)\left(-b u(t)\right) = 0$$

$p_y(t) = -1$ $\quad$ $p_v(t) = t - t_f$

$$\Rightarrow -1\, v(t) + u(t)\left(\frac{p_v(t)}{m(t)} - b p_m(t)\right) - (t - t_f)\, g = 0$$

$$\Rightarrow -v(t) + u(t)\,\phi(t) - (t - t_f)\, g = 0$$

$$\Rightarrow u(t)\,\phi(t) = v(t) + (t - t_f)\, g$$

At $t = t_f$ $\quad \mathcal{H} + \frac{\partial h}{\partial t} = 0 \Rightarrow \mathcal{H}(t_f) + (-\dot{y}) = 0$

$$\Rightarrow v(t_f) = 0$$

$$\Rightarrow u(t_f)\,\phi(t_f) = 0$$

$\therefore$ Either $\phi(t_f) = 0$ momentaily or $u(t_f) = 0 \Rightarrow \phi(t_f) > 0$