PS 1 Problem 1

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$$f_1(x,y) = -\log(10-2x^2-y^2)$$
$$f_2(x,y) = x^2(1+2y-x^2)$$

a) NOC $\Rightarrow \nabla f(x^*) = 0$

$x^* = (0,0) \quad x=0,\ y=0$

$$\nabla f_1(0,0) = \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_1}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1(-4x)}{10-2x^2-y^2} \\ \frac{-1(-2y)}{10-2x^2-y^2} \end{bmatrix} = \begin{bmatrix} \frac{4x}{10-2x^2-y^2} \\ \frac{2y}{10-2x^2-y^2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\nabla^2 f_1(0,0) = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x^2} & \frac{\partial^2 f_1}{\partial x \partial y} \\ \frac{\partial^2 f_1}{\partial x \partial y} & \frac{\partial^2 f_1}{\partial y^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4(10-2x^2-y^2)-(4x)(-4x)}{(10-2x^2-y^2)^2} & \frac{0-(4x)(-2y)}{(10-2x^2-y^2)^2} \\ \frac{0-(2y)(-4x)}{(10-2x^2-y^2)^2} & \frac{2(10-2x^2-y^2)-(2y)(-4y)}{(10-2x^2-y^2)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4(10)}{10^2} & 0 \\ 0 & \frac{2(10)}{10^2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$\therefore$ eig $= 0.4, 0.2 \quad > 0$

$\therefore \nabla^2 f_1(0,0)$ is pos def

$\Rightarrow$ also pos semi-def

f_1: $(0,0)$ satisfies NOC to 2^{nd} order ($\nabla f_1 = 0$, $\nabla^2 f_1$ pos semi-def) for local min

$(0,0)$ also satisfies SOC ($\nabla f_1 = 0$, $\nabla^2 f_1$ pos def) for local min

$$\nabla f_2(0,0) = \begin{bmatrix} \frac{\partial f_2}{\partial x} \\ \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 2x + 4xy - 4x^3 \\ 2x^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\nabla^2 f_2(0,0) = \begin{bmatrix} \frac{\partial^2 f_2}{\partial x^2} & \frac{\partial^2 f_2}{\partial x \partial y} \\ \frac{\partial^2 f_2}{\partial x \partial y} & \frac{\partial^2 f_2}{\partial y^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 4y - 12x^2 & 4x \\ 4x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

eig are 2 & 0

$\Rightarrow \nabla^2 f_2$ is pos semi-def

NOT pos def

$\boxed{\begin{array}{l} f_2: (0,0) \text{ satisfies NOC to } 2^{nd} \text{ order} \\ \quad (\nabla f_2 = 0 \quad \nabla^2 f_2 \text{ pos semi-def}) \text{ for local min} \\ \quad (0,0) \text{ does NOT satisfy SOC} \\ \quad (\nabla f_1 = 0 \quad \nabla^2 f_1 \text{ not pos def}) \text{ for local min} \end{array}}$

b) f_1 : $(0,0)$ is a local $\min^m$ $\because$ SOC satisfied

$(0,0)$ is also global $\min^m$ $\because f_1$ convex

f_2 : $(0,0)$ is NOT a local $\min^m$

@ $(0,0)$ $f_2 = 0$

@ $(10,0)$ $f_2 = 100\,(1+0-100) = -9900$

@ $(\varepsilon, -\gamma)$ $f_2 = \varepsilon^2(1-\gamma-\varepsilon^2)$

$f_2(\varepsilon,-\gamma) < f_2(\varepsilon,0)$

$\therefore$ NOT a local $\min^m$

$\therefore$ Also not a global $\min^m$