PS 3 Problem 2

Sunday, April 21, 2019 7:13 PM

$$\mathcal{H} = (u(t))^2 + J_x^{*T} u(t)$$

transpose unnecessary ∵ scalars

$$0 = J_t^* + \min_u \underbrace{\left[(u(t))^2 + J_x^{*T} u(t)\right]}_{\mathcal{H}}$$

NOC $\nabla_u \mathcal{H} = 0 \Rightarrow 2u(t) + J_x^* = 0$

$$\Rightarrow u(t) = -\frac{1}{2} J_x^*$$

$\because \nabla_{uu}^2 \mathcal{H} = 2 > 0 \Rightarrow$ this is global minimizer

$$u^*(t) = -\frac{1}{2} J_x^* \quad \text{if } \left|\frac{1}{2} J_x^*\right| \le 1$$

Note that $|u(t)| \le 1$ has to be satisfied as well.

Need to find $J^*(t,x)$ that satisfies this equ and boundary condition

$$J^*(T, x(T)) = (x(T))^2$$

Guess
$$J^*(t,x) = \begin{cases} (x-T+t)^2 + T-t & \text{if } x > 1+T-t \\ (x+T-t)^2 + T-t & \text{if } x < -(1+T-t) \\ x^2/(1+T-t) & \text{if } |x| \le 1+T-t \end{cases}$$

This satisfies boundary condition @ $t=T$.

$$J_x^* = \begin{cases} 2(x-T+t) & \text{if } x > 1+T-t \\ 2(x+T-t) & \text{if } x < -(1+T-t) \\ \dfrac{2x}{1+T-t} & \text{if } |x| \le 1+T-t \end{cases}$$

$$J_t^* = \begin{cases} 2(x-T+t) - 1 & \text{if } x > 1+T-t \\ -2(x+T-t) - 1 & \text{if } x < -(1+T-t) \\ \dfrac{-x^2}{(1+T-t)^2}(-1) & \text{if } |x| \le 1+T-t \end{cases}$$

Case 1 $x > 1+T-t$

HJB requires

$$0 = J_t^* + \min_u \underbrace{\left[(u(t))^2 + J_x^{*T} u(t)\right]}_{\mathcal{H}}$$

Minimizer is

$$u(t) = -\frac{1}{2} J_x^* = -(x-T+t)$$

But $\because x > 1+T-t \Rightarrow x-T+t > 1$
$\Rightarrow -(x-T+t) < -1$

$J^*_x = 2(x-T+t) > 0$

$\therefore$ minimizing $u(t) = -1$

$u^*(t) = -1$

HJB $RHS = J^*_t + (-1)^2 + J^*_x(-1)$

$= 2(x-T+t) - 1 + 1 - [2(x-T+t)]$

$= 0$

$\therefore$ This $J^*(t,x)$ works in this case.

Case 2 $\quad x < -(1+T-t)$

HJB requires

$$0 = J^*_t + \min_u \underbrace{\left[(u(t))^2 + J_x^{*T} u(t)\right]}_{\mathcal{H}}$$

Minimizer is

$u(t) = -\frac{1}{2} J^*_x = -(x+T-t)$

But $\because x < -(1+T-t) \Rightarrow -(x+T-t) > 1$

$J^*_x = 2(x+T-t) < 0$

$\therefore$ minimizing $u(t) = 1$

$u^*(t) = 1$

HJB $RHS = J^*_t + (1)^2 + J^*_x(1)$

$= -2(x+T-t) - 1 + 1 + [2(x+T-t)]$

$= 0$

$\therefore$ This $J^*(t,x)$ works in this case.

Case 3 $\quad |x| \leq 1+T-t$

HJB requires

$$0 = J^*_t + \min_u \underbrace{\left[(u(t))^2 + J_x^{*T} u(t)\right]}_{\mathcal{H}}$$

Minimizer is

$u(t) = -\frac{1}{2} J^*_x = -\frac{x}{1+T-t}$

$|x| \leq 1+T-t \Rightarrow |u| \leq 1$

$\therefore u^*(t) = -\frac{x}{1+T-t}$

HJB RHS $= J^*_t + u(t) + J_x u(t)$

$$= \frac{-x^2}{(1+T-t)^2}(-1) + \frac{x^2}{(1+T-t)^2} + \left(\frac{2x}{1+T-t}\right)\left(-\frac{x}{1+T-t}\right)$$

$$= \frac{x^2 + x^2 - 2x^2}{(1+T-t)^2} = 0$$

$\therefore$ This $J^*(t,x)$ works in this case.

$$\boxed{u^*(t) = \begin{cases} -1 & \text{if } x > 1+T-t \\ 1 & \text{if } x < -(1+T-t) \\ -\dfrac{x}{1+T-t} & \text{if } |x| \le 1+T-t \end{cases}}$$