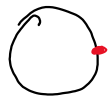
PS 2 Problem 3

Tuesday, April 16, 2019 4:32 PM

N side polygon w/ maximal perimeter

Placement of 1^{st} pt doesn't matter, say at 0°



2^{nd} pt placed at θ

"perimeter" = c

$c^2 = r^2 + r^2 - 2r^2\cos\theta$ by law of cosines

$c^2 = 2r^2(1-\cos\theta) = 4r^2\sin^2\theta/2$

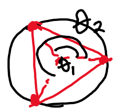
gradient $\Rightarrow \sin\theta = 0$ or $\sin\theta/2 = 1$

$\hookrightarrow$ c max for $\cos\theta = -1 \Rightarrow \theta = \pi$

$\Rightarrow c = 2r$

$\leftarrow$ optimal 2^{nd} pt

Given soln for 2 pts, let's find soln for 3 pts



$$\text{perimeter} = 2r\sin\theta_1/2 + 2r\sin\left(\frac{\theta_2-\theta_1}{2}\right) + 2r\sin\left(\frac{2\pi-\theta_2}{2}\right)$$

$\therefore$ For N pts,

$$\text{perimeter} = 2r\sin\theta_1/2 + 2r\sin\frac{\theta_2-\theta_1}{2} + 2r\sin\frac{\theta_3-\theta_2}{2} + \dots$$
$$\dots + 2r\sin\frac{\theta_N-\theta_{N-1}}{2} + 2r\sin\underbrace{\left(\frac{2\pi-\theta_N}{2}\right)}_{=\sin\frac{\theta_N}{2}}$$

Let $\theta_0 = 0$ by def^n

$$\Rightarrow \text{perimeter} = 2r\sin\left(\frac{\theta_N}{2}\right) + \sum_{i=1}^{N} 2r\sin\frac{\theta_i-\theta_{i-1}}{2}$$

$\therefore$ objective is to place $\theta_1, \dots \theta_N$ such that we maximize

$$\boxed{2r\sin\left(\frac{\theta_N}{2}\right) + \sum_{i=1}^{N} 2r\sin\frac{\theta_i-\theta_{i-1}}{2}}$$

b) Additive cost so apply Dynamic Programming

Cost at N^{th} step

$$J_N(\theta_N) = 2r\sin\left(\frac{\theta_N}{2}\right) = \overset{\text{optimal}}{J^*_N(\theta_N)}$$

Cost at $N-1^{th}$ step

$$J_{N-1}(\theta_{N-1}) = 2r\sin\left(\frac{\theta_N}{2}\right) + 2r\sin\left(\frac{\theta_N-\theta_{N-1}}{2}\right)$$

$$J^*_{N-1}(\theta_{N-1}) = \max_{\theta_N}\left(2r\sin\frac{\theta_N}{2} + 2r\sin\left(\frac{\theta_N-\theta_{N-1}}{2}\right)\right)$$

differentiate wrt θ_N & set to 0

$$\Rightarrow \frac{2r}{2}\cos\left(\frac{\theta_N}{2}\right) + \frac{2r}{2}\cos\left(\frac{\theta_N-\theta_{N-1}}{2}\right) = 0$$

$$\Rightarrow \cos\left(\frac{\theta_N}{2}\right) = -\cos\left(\frac{\theta_N-\theta_{N-1}}{2}\right)$$

satisfied if $\frac{\theta_N}{2} = \pi - \left(\frac{\theta_N-\theta_{N-1}}{2}\right)$

$$\Rightarrow \theta_N = \pi + \frac{\theta_{N-1}}{2}$$

$$\Rightarrow J^*_{N-1}(\theta_{N-1}) = 2r\sin\left(\frac{\pi}{2}+\frac{\theta_{N-1}}{4}\right) + 2r\sin\left(\frac{\pi}{2}-\frac{\theta_{N-1}}{4}\right)$$

$$\frac{\pi}{2}+\frac{\theta_{N-1}}{4} = \pi - \left(\frac{\pi}{2}-\frac{\theta_{N-1}}{4}\right)$$

$$\Rightarrow J^*_{N-1}(\theta_{N-1}) = 4r\sin\left(\frac{\pi}{2}-\frac{\theta_{N-1}}{4}\right)$$

$$J^*_{N-2}(\theta_{N-2}) = \max_{\theta_{N-1}}\left[2r\sin\left(\frac{\theta_{N-1}-\theta_{N-2}}{2}\right) + J^*_{N-1}(\theta_{N-1})\right]$$

$$\Rightarrow 2r\frac{1}{2}\cos\left(\frac{\theta_{N-1}-\theta_{N-2}}{2}\right) + 4r\left(-\frac{1}{4}\right)\cos\left(\frac{\pi}{2}-\frac{\theta_{N-1}}{4}\right) = 0$$

$$\Rightarrow \cos\left(\frac{\theta_{N-1}-\theta_{N-2}}{2}\right) = \cos\left(\frac{\pi}{2}-\frac{\theta_{N-1}}{4}\right)$$

satisfied if $\frac{\theta_{N-1}}{2} - \frac{\theta_{N-2}}{2} = \frac{\pi}{2} - \frac{\theta_{N-1}}{4}$

$$\Rightarrow 3\frac{\theta_{N-1}}{4} = \frac{\pi}{2} + \frac{\theta_{N-2}}{2}$$

$$\Rightarrow \theta_{N-1} = \frac{2\pi}{3} + \frac{2\theta_{N-2}}{3}$$

$$\Rightarrow J^*_{N-2}(\theta_{N-2}) = 2r\sin\left(\frac{\pi}{3}-\frac{\theta_{N-2}}{6}\right) + 4r\sin\left(\frac{\pi}{2}-\frac{\pi}{6}-\frac{\theta_{N-2}}{6}\right)$$

$$= 2r\sin\left(\frac{\pi}{3}-\frac{\theta_{N-2}}{6}\right) + 4r\sin\left(\frac{\pi}{3}-\frac{\theta_{N-2}}{6}\right)$$

$$= 6r\sin\left(\frac{\pi}{3}-\frac{\theta_{N-2}}{6}\right)$$

$$J^*_{N-3}(\theta_{N-3}) = \max_{\theta_{N-2}}\left[2r\sin\left(\frac{\theta_{N-2}-\theta_{N-3}}{2}\right) + J^*_{N-2}(\theta_{N-2})\right]$$

Diff wrt $\theta_{N-2} \rightarrow r\cos\left(\frac{\theta_{N-2}-\theta_{N-3}}{2}\right) + 6r\left(-\frac{1}{6}\right)\cos\left(\frac{\pi}{3}-\frac{\theta_{N-2}}{6}\right) = 0$

$$\Rightarrow \frac{\theta_{N-2}}{2} - \frac{\theta_{N-3}}{2} = \frac{\pi}{3} - \frac{\theta_{N-2}}{6}$$

$$\Rightarrow \theta_{N-2}\frac{(2)}{3} = \frac{\pi}{3} + \frac{\theta_{N-3}}{2}$$

$$\Rightarrow \theta_{N-2} = \frac{\pi}{2} + \frac{3\theta_{N-3}}{4}$$

So far, we have

$$\theta_N = \pi + \frac{\theta_{N-1}}{2} \qquad J^*_{N-1}(\theta_{N-1}) = 4r\sin\left(\frac{\pi}{2} - \frac{\theta_{N-1}}{4}\right)$$

$$\theta_{N-1} = \frac{2\pi}{3} + \frac{2\theta_{N-2}}{3} \qquad J^*_{N-2}(\theta_{N-2}) = 6r\sin\left(\frac{\pi}{3} - \frac{\theta_{N-2}}{6}\right)$$

$$\theta_{N-2} = \frac{2\pi}{4} + \frac{3\theta_{N-3}}{4}$$

Pattern emerging.

Let's guess $\theta_{N-l} = \frac{2\pi}{l+2} + \frac{(l+1)\theta_{N-l-1}}{(l+2)}$

$$J^*_{N-k}(\theta_{N-k}) = 2(k+1)r\sin\left(\frac{\pi}{k+1} - \frac{\theta_{N-k}}{2(k+1)}\right)$$

We know this holds for $l = 0, 1, 2$ & $k = 1, 2$

Let's show that if it holds for k, it also holds for $k+1$

$$J^*_{N-k-1}(\theta_{N-k-1}) = \max_{\theta_{N-k}}\left[2r\sin\left(\frac{\theta_{N-k} - \theta_{N-k-1}}{2}\right) + J^*_{N-k}(\theta_{N-k})\right]$$

diff wrt $\theta_{N-k} \Rightarrow r\cos\left(\frac{\theta_{N-k} - \theta_{N-k-1}}{2}\right) + \frac{2(k+1)r(-1)}{2(k+1)}\cos\left(\frac{\pi}{k+1} - \frac{\theta_{N-k}}{2(k+1)}\right) = 0$

$$\Rightarrow \frac{\theta_{N-k}}{2} - \frac{\theta_{N-k-1}}{2} = \frac{\pi}{k+1} - \frac{\theta_{N-k}}{2(k+1)}$$

$$\Rightarrow \frac{\theta_{N-k}(k+2)}{2(k+1)} = \frac{\pi}{k+1} + \frac{\theta_{N-k-1}}{2}$$

$$\Rightarrow \theta_{N-k} = \frac{2\pi}{k+2} + \frac{(k+1)\theta_{N-k-1}}{k+2}$$

$$\Rightarrow J^*_{N-k-1}(\theta_{N-k-1}) = 2r\sin\left(\frac{\pi}{k+2} - \frac{\theta_{N-k-1}}{2(k+2)}\right) + 2(k+1)r\sin\left(\frac{\pi}{k+1} - \frac{2\pi}{2(k+1)(k+2)} - \frac{(k+1)\theta_{N-k-1}}{2(k+1)(k+2)}\right)$$

$$= (2 + 2k+2)r\sin\left(\frac{\pi}{k+2} - \frac{\theta_{N-k-1}}{2(k+2)}\right)$$

$$J^*_{N-k-1}(\theta_{N-k-1}) = 2(k+2)\sin\left(\frac{\pi}{k+2} - \frac{\theta_{N-k-1}}{2(k+2)}\right)$$

$\therefore$ if it holds for k, also holds for $k+1$.
By induction, we've proven

$$\theta_{N-k} = \frac{2\pi}{k+2} + \frac{(k+1)\,\theta_{N-k-1}}{k+2}$$

Assume $\theta_1 = 0$ rad

$$\Rightarrow \theta_2 = \frac{2\pi}{(N-2)+2} + \frac{(N-2+1)\,\overset{\to 0}{\theta_1}}{(N-2+2)} = \frac{2\pi}{N}$$

$$\theta_3 = \frac{2\pi}{N-3+2} + \frac{N-3+1 \;\; \theta_2}{N-3+2}$$

$$= \frac{2\pi}{N-1} + \frac{N-2}{N-1}\,\frac{2\pi}{N}$$

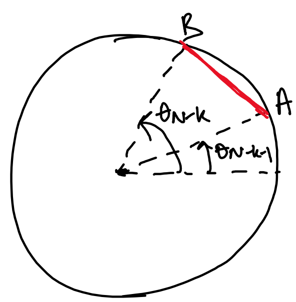
$$= \frac{2\pi}{N(N-1)}(N+N-2) = 2\cdot\frac{2\pi}{N}$$

Similarly, we can again show by induction

$$\theta_K = (k-1)\,\frac{2\pi}{N} \quad \text{for } k=2,3,\ldots N$$

$$\therefore \theta_{diff} = \theta_k - \theta_{k-1} = \frac{2\pi}{N} \quad \text{for } k=2,3,\ldots N$$

Since angles between any 2 adjacent points of polygon is const



Given two angles for points $N-k-1$ & $N-k$, side length

$$AB = l = 2r\sin\frac{\theta_{diff}}{2}$$

$$= 2r\sin\left(\frac{\pi}{N}\right)$$

$\therefore$ This is a regular polygon in order to maximize perimeter