0.1 History of Wildfire Modeling

- First model: Wallace Fons in 1946.
- First landmark development: Richard Rothermel (1972) developed an equation for forward fire spread
$$R = \underbrace{I_p(1+\phi_w+\phi_s)}_{\text{Heat Source}} \cdot \underbrace{(\rho_b \epsilon Q_{ig})^{-1}}_{\text{Heat Sink}}.$$
- Second landmark development: Sanderlin and Sunderson (1975), along with Anderson (1982), applied Huygen's Principle to fireline propagation.
- Third landmark development: Computers.

0.2 Our Problem

We wish to track the **fireline** of a wildfire with respect to time. The **fireline** of a wildfire is the boundary of the region that is on fire. All atmospheric coupling will be ignored.

0.3 Our Approaches

Two approaches to fireline propagation:

1. Utilizing Huygen's Principle (Anderson, 1982)
2. Utilizing the Level Set Method (Mandel, 2011)

1 Rothermel Model

1.1 Setup

As a theoretical base, we are going to assume that the spread rate is given by the following, which was determined by Frandsen (1971).

$$R = \frac{I_{xig} + \int_{-\infty}^{0} \left(\frac{\partial I_z}{\partial z}\right)_{z_C} dx}{\rho_{be} Q_{ig}}.$$

Where

- x is the horizontal coordinate and z is the vertical coordinate.
- R = rate of spread.
- I_{xig} = horizontal heat flux absorbed by a unit volume of fuel at the time of ignition. Essentially, the effective amount of heat flowing from the fire.
- $\left(\frac{\partial I_z}{\partial z}\right)_{z_C}$ = the gradient of the vertical intensity evaluated at a plane at a constant depth, z_C, of the fuel bed. Essentially, this is greater if there is a high wind or a steep slope since then the fire is angled on top of the fuel, instead of being just in front.
- ρ_{be} = effective bulk density, which is the amount of fuel involved in the ignition process.
- Q_{ig} = the heat of preignition, which is the heat required to bring a unit weight of fuel to ignition.

The numerator describes the heat source and the denominator describes the heat sink.

1.2 Derivation

We want to simplify this theoretical base. We do this by:

1. Adding terms that are easy to determine (for example, by a mapping software or a data table)
2. Removing terms that are hard to determine

1.2.1 Heat Sink

First, we simplify the heat sink term.

- Q_{ig}, easy to store, does not need to be simplified.
- ρ_{be} can be simplified to $\epsilon\rho_b$ where $\epsilon = \rho_{be}/\rho_b$ and ρ_b denotes bulk density, $\rho_b = M/V$.

1.2.2 Heat Source

Next, we simplify the heat source term. This is given by

$$I_p = \underbrace{I_{xig}}_{\text{horizontal flux}} + \underbrace{\int_{-\infty}^{0} \left(\frac{\partial I_z}{\partial z}\right)_{z_C} dx}_{\text{vertical component}}.$$

First, we will assume that there is no wind, which you will come to see is a very common approach. Additionally, we'll assume that the vertical flux is very small for no wind fires, which essentially eliminates the vertical component. Thus, we can let $I_p = (I_p)_0$.

1.2.3 Adding Wind and Slope

Then, all together, we get that in a no wind situation the spread rate is given by

$$R = \frac{(I_p)_0}{\rho_b \epsilon Q_{ig}}.$$

Then, we account for wind and slope by writing

$$I_p = \underbrace{(I_p)_0}_{\text{Fuel Factor}} (1 + \underbrace{\phi_w}_{\text{Wind Factor}} + \underbrace{\phi_s}_{\text{Slope Factor}}).$$

Note that each both the wind factor and slope factor may depend on wind, slope, and fuel properties, and are experimentally determined. This gives us our final spread rate, with wind and slope included, of

$$R_{sw} = R(1 + \phi_w + \phi_s).$$

The benefit of this form is that everything can be stored as region specific variables and experimental data.

2 Huygen Fire Model

2.1 Huygen's Principle

Huygen's Principle originates from the wave equation,

$$\begin{cases} u_{tt} - c^2 \Delta u = 0,\ \Delta u = \sum_{k=1}^{n} u_{x_k x_k} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}.$$

Green's Function for the 2D wave equation is

$$G(x,y,t) = \frac{1}{2\pi c} \frac{1}{\sqrt{t^2 - x^2 - y^2}} H(t - |x|/c),\ H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases}.$$

In two dimensions (which is the dimension we care about), the solution to the wave equation is

$$u(x_0, y_0, t_0) = \iint_D \psi(x,y)G(x-x_0, y-y_0, t_0)dxdy + \frac{\partial}{\partial t_0}\iint_D \phi(x,y)G(x-x_0, y-y_0, t_0)dxdy$$
$$D = \left\{(x-x_0)^2 + (y-y_0)^2 \le c^2 t_0^2\right\}.$$

Which can be simplified to

$$u(x_0, y_0, t_0) = \iint \psi(x,y)G(x-x_0, y-y_0, t_0)dxdy + \frac{\partial}{\partial t_0}\iint \phi(x,y)G(x-x_0, y-y_0, t_0)dxdy$$

Huygen's Principle heuristic:

Each point on a wavefront can be seen as the source of new waves.
These new waves generate the new wavefront.

2.2 Setup

Assumptions

- Flat land.
- Constant fuel properties. Amount of fuel, moisture content, etc.
- Constant atmospheric conditions, except for wind.

Thus, we have a constant burning rate, a.

2.3 Empirical Expectations

2.3.1 No Wind

When there is no wind, we expect the fire to burn uniformly in a circle.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} at\cos\theta \\ at\sin\theta \end{pmatrix}.$$

2.3.2 Constant Wind

Experimentally, a constant wind in the positive x-direction shows a fireline taking the form of an ellipse, which is parameterized as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} at(f\cos\theta + g) \\ ath\sin\theta \end{pmatrix}.$$

Where f, g, h are parameters that depend on wind, and are determined experimentally.

2.4 Applying Huygen's Principle

We first apply Huygen's Principle to the situation with no wind change to verify that it does, indeed, work with our model.

At a time t_0 with a constant wind in the positive x-direction our fireline will be parameterized as

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} at_0(f\cos\theta + g) \\ at_0 h\sin\theta \end{pmatrix}.$$

Fix a point on the wave front, and keep the wind the same. Then, in local coordinates at this point, after a time t the local fireline will look like

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} at(f\cos\phi + g) \\ ath\sin\phi \end{pmatrix}.$$

Then, the claim is that the **envelope** of this fireline will be of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 + X \\ y_0 + Y \end{pmatrix} = \begin{pmatrix} a(t_0 + t)(f\cos\theta + g) \\ a(t_0 + t)h\sin\theta \end{pmatrix}.$$

This means Huygen's Principle does accurately predict the experimental results.

2.5 Single Wind Change

Situation:

1. Wind blowing in the positive x-direction at speed V for time t_0
2. Wind changes and is now blowing at an angle β to the positive x-axis at speed W for time t.

The fireline is given by the outer envelope to the following 2D surface:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 + X\cos\beta - Y\sin\beta \\ y_0 + X\sin\beta + Y\cos\beta \end{pmatrix}.$$

Some brutal computation shows that this envelope is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 + X \\ y_0 + Y \end{pmatrix} = \begin{pmatrix} x_0 + at(g(W)\cos\beta + r[pf^2(W)\cos\beta + qh^2(W)\sin\beta]) \\ y_0 + at(g(W)\sin\beta + r[pf^2(W)\sin\beta - qh^2(W)\cos\beta]) \end{pmatrix}$$

Where

$$\begin{aligned} p &= p(\theta,\beta,V) := h(V)\cos\beta\cos\theta + f(V)\sin\beta\sin\theta \\ q &= q(\theta,\beta,V) := h(V)\sin\beta\cos\theta - f(V)\cos\beta\sin\theta \\ r &= r(\theta,\beta,V,W) := \left[p^2 f^2(W) + q^2 h^2(W)\right]^{-1/2} \end{aligned}$$

2.6 Multiple Wind Changes

This is just a repeated application of the "Single Wind Change".

2.7 Non-Uniform Conditions

The idea is that we can adapt to non-uniform changes by changing a to the R in Rothermel's paper.

3 Level Set Model

The fundamental heuristic of this approach is that we only need to track the level set of the PDE.

3.1 Setup

Assumptions

- Speed of burning is independent of wind speed and fuel moisture.
- No atmospheric coupling.

3.2 Deriving the Level Set Equation

We define the level set function, ψ, as follows:

$$\psi(x,y,t) = \begin{cases} \psi(x,y,t) \le 0, & \text{if } (x,y) \text{ is on fire} \\ \psi(x,y,t) > 0, & \text{if } (x,y) \text{ is not on fire} \end{cases}.$$

Let $\vec{x}(t)$ on the fireline at all t. That is, $\psi(\vec{x}(t), t) = 0$ for all t. Then, differentiating with respect to t yields

$$\frac{d}{dt}\psi(\vec{x}(t), t) = \frac{\partial \psi}{\partial t} + \nabla \psi \cdot \vec{x}'(t)$$
$$0 = \frac{\partial \psi}{\partial t} + \vec{n} \cdot \vec{x}'(t)||\nabla \psi||, \ \vec{n} = \frac{\nabla \psi}{||\nabla \psi||} \text{ is the normal vector to the surface}$$

Now, notice that $\vec{n} \cdot \vec{x}'(t)$ is the **fire spread rate** at $\vec{x}$ in the direction of the normal to the fireline. We call this S. Note that we can get S from Rothermel's paper. Thus, we get the **level set equation**

$$\psi_t + S||\nabla \psi|| = 0.$$

So, whatever our solution is, it needs to satisfy this PDE on the fireline.

4 Basic Numerical Methods

There are three basic ways to numerically compute derivatives using finite differences. For this, our domain will be $x_1, ..., x_n \in \mathbb{R}$. Note that we can extend this to multivariable functions by just doing these as partial derivatives.

- Forward Difference:
$$f'(x_k) = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}.$$
- Backward Difference:
$$f'(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$
- Central Difference:
$$f'(x_k) = \frac{f(x_{k+1}) - f(x_{k-1})}{x_{k+1} - x_{k-1}}.$$

5 Conclusion

5.1 Review of Content

Okay, let's review what we've done so far. We

1. Looked at a model created by Rothermel in 1972 to determine how fast a fire spreads.
2. Next, we looked at a model created by Anderson in 1982 to determine how we can propagate firelines using Huygen's Principle.
3. Finally, we looked at a model created by Mandel et al. in 2011 that used the level set equation to model fireline propagation.

5.2 Further Interest

- If you're interested in applications of these models, you can look at the model by Finney in the resources for one based on Huygen's principle, and the one by Mandel et al. for one based on the level set equation.
- There are two big fire concepts that we have not addressed.
  1. Crown Fires: There is two levels of fire in wildfires. There is the normal fire on the forest floor, which is what we have looked at. But, there is also crown fires, which are fires at the top of trees. You can imagine that fire would spread faster up there because there is more leafy substances and there is more access to oxygen.

2. Spotting: Spotting occurs when large embers are blown in front of the fire and start their own little fires.

Both of these have shown to be significant, and have models for them.

- Now, we have not addressed a huge part of wildfire modeling, which is the atmosphere. Most modern day wildfire models have some degree of coupling with the atmosphere because there is an inherent interaction between fire and atmosphere.

Appendix A: Frandsen

Here is how Frandsen derived his equation. Conservation of energy states that:

$$\nabla \cdot \vec{I} = -\frac{\partial J}{\partial t}.$$

Our boundary conditions are

- At $x = -\infty$ we have $\vec{I} = 0$ and $J = 0$,
- At $x = 0$ we have $\vec{I} = I_{ig}$ and $J = J_{ig}$.

The left hand side of the above becomes,

$$\frac{\partial I_x}{\partial x} + \frac{\partial I_y}{\partial y} + \frac{\partial I_z}{\partial z}.$$

We assume that the fireline is long enough that the $\frac{\partial I_y}{\partial y}$ can be removed through symmetry.

Then, we note that by the mean value theorem for integrals that we can write,

$$\begin{aligned} \left(\frac{dI_x}{dx}\right)_{z_C} &= \frac{1}{\delta}\int_\delta \frac{\partial I_x}{\partial x} dz \\ \left(\frac{dJ}{dx}\right)_{z_C} &= \frac{1}{\delta}\int_\delta \frac{\partial J}{\partial x} dz. \end{aligned}$$

Additionally, using the chain rule on $-\frac{\partial J}{\partial t}$ we get

$$\frac{\partial J}{\partial t} = \frac{dx}{dt}\frac{\partial J}{\partial x}$$

where $R = \frac{dx}{dt}$ is our spread rate. Putting this all together, we get

$$-R\frac{dJ}{dx} = \frac{dI_x}{dx} + \left(\frac{\partial I_z}{\partial z}\right)_{z_C}$$

Integrating over our domain and applying boundary conditions yields

$$-RJ_{ig} = I_{xig} + \int_{-\infty}^{0} \left(\frac{\partial I_z}{\partial z}\right)_{z_C} dx.$$

Then, note that this is the speed of the volume going towards the fire, so the fire spread will just be the opposite, $-R \to R$.