Governing equations, weak forms, and numerical computations for seismic fluid-structure interaction

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Contents

1. Acoustic fluid domain

The following assumptions are made:

- The fluid is inviscid.
- The fluid is irrotational.
- The fluid is subjected to small displacements.
- There is no addition or subtraction of fluid.

The momentum equation is given by:

$$\rho_o \frac{\partial^2 \mathbf{u}_f}{\partial t^2} + \nabla p = \rho_o \mathbf{g} \tag{1}$$

where ρ_o is the static fluid density, $\mathbf{u}_f$ is the fluid displacement vector, p is the scalar pressure, and $\mathbf{g}$ is the gravity vector with one non-zero component which is the acceleration due to gravity in the vertical direction. The continuity equation assuming there is no net mass inflow or outflow is given by:

$$\frac{\partial \rho_f}{\partial t} + \rho_o \nabla \cdot \frac{\partial \mathbf{u}_f}{\partial t} = 0 \tag{2}$$

where ρ_f is the variable fluid density that is a function of time. The constitutive law is given by:

$$p = c_o^2 \rho_f \tag{3}$$

where c_o is the speed of sound. Substituting the constitutive law in equation (2) and taking a derivative with time:

$$\frac{1}{c_o^2} \frac{\partial^2 p}{\partial t^2} + \rho_o \nabla \cdot \frac{\partial^2 \mathbf{u}_f}{\partial t^2} = 0 \tag{4}$$

Substituting equation (1) in equation (4):

$$\frac{1}{c_o^2} \frac{\partial^2 p}{\partial t^2} + \rho_o \nabla \cdot \Big(\mathbf{g} - \frac{1}{\rho_o} \nabla p \Big) = 0 \tag{5}$$

The resulting governing equation is given by:

$$\frac{\partial^2 p}{\partial t^2} - c_o^2 \nabla^2 p + \rho_o c_o^2 \nabla \cdot \mathbf{g} = 0 \tag{6}$$

1.1. Weak form and finite element approximation

Multiplying equation (6) with a test function ν_f and integrating over the fluid domain:

$$\int_{\Omega_f} \Big(\nu_f \ \frac{\partial^2 p}{\partial t^2} - \nu_f \ c_o^2 \ \nabla^2 p + \nu_f \ \rho_o c_o^2 \ \nabla \cdot \mathbf{g} \Big) dV = 0 \tag{7}$$

Using the Green's theorem, the weak form is given by:

$$\int_{\Omega_f} \nu_f \ \frac{\partial^2 p}{\partial t^2} \ dV + c_o^2 \int_{\Omega_f} \nabla \nu_f \cdot \nabla p \ dV = c_o^2 \int_{\Gamma_f} \nu_f \nabla p \cdot \mathbf{n}_f \ dA - \rho_o c_o^2 \int_{\Omega_f} \nu_f \ \nabla \cdot \mathbf{g} \ dV \tag{8}$$

The finite element approximations are given by:

$$p = \mathbf{N}_f \mathbf{p}_f; \quad \nu_f = \mathbf{N}_f \mathbf{c}_f \tag{9}$$

where $\mathbf{N}_f$ is the shape function vector as a function of space, $\mathbf{p}_f$ is the pressure vector at the nodal points, and $\mathbf{c}_f$ is the nodal weights. Introducing the finite element approximation into equation (8):

$$\int_{\Omega_f} \mathbf{N}_f^T \; \mathbf{N}_f \; dV \; \ddot{\mathbf{p}}_f + c_o^2 \int_{\Omega_f} (\nabla \mathbf{N}_f)^T \nabla \mathbf{N}_f \; dV \; \mathbf{p}_f = c_o^2 \int_{\Gamma_f} \mathbf{N}_f^T \nabla p \cdot \mathbf{n}_f \; dA - \rho_o c_o^2 \int_{\Omega_f} \mathbf{N}_f^T \; \nabla \cdot \mathbf{g} \; dV \tag{10}$$

This equation can be re-written in matrix form as:

$$\mathbf{M}_f \; \ddot{\mathbf{p}}_f + \mathbf{K}_f \; \mathbf{p}_f = \mathbf{f}_s + \mathbf{f}_g \tag{11}$$

where, $\mathbf{M}_f$ and $\mathbf{K}_f$ are similar to mass and stiffness matrices of a fluid, $\mathbf{f}_s$ is the force vector coming from the structure, and $\mathbf{f}_g$ is the gravity force vector. These quantities are further expanded as:

$$\begin{aligned} \mathbf{M}_f &= \int_{\Omega_f} \mathbf{N}_f^T \; \mathbf{N}_f \; dV \\ \mathbf{K}_f &= c_o^2 \int_{\Omega_f} (\nabla \mathbf{N}_f)^T \nabla \mathbf{N}_f \; dV \\ \mathbf{f}_s &= c_o^2 \int_{\Gamma_f} \mathbf{N}_f^T \nabla p \cdot \mathbf{n}_f \; dA \\ \mathbf{f}_g &= -\rho_o c_o^2 \int_{\Omega_f} \mathbf{N}_f^T \; \nabla \cdot \mathbf{g} \; dV \end{aligned} \tag{12}$$

1.2. Homogeneous gravity vector

In equation (6), there is a divergence of the gravity vector term (i.e., $\nabla \cdot \mathbf{g}$). If the gravity vector has constant values that are not a function of space, then its divergence becomes zero. Is such a case, the governing equation reduces to:

$$\frac{\partial^2 p}{\partial t^2} - c_o^2 \nabla^2 p = 0 \tag{13}$$

The finite element approximation in matrix form then becomes:

$$\mathbf{M}_f \; \ddot{\mathbf{p}}_f + \mathbf{K}_f \; \mathbf{p}_f = \mathbf{f}_s \tag{14}$$

1.3. Re-writing the gravity term as a body force term

The term $\rho_o \mathbf{g}$ in equation (1) can be expressed as a generic body force vector $\mathbf{f}_{bf}$. Each term in this vector has the dimension force per unit volume. The governing equation can now be expressed as:

$$\frac{\partial^2 p}{\partial t^2} - c_o^2 \nabla^2 p + c_o^2 \nabla \cdot \mathbf{f}_{bf} = 0 \tag{15}$$

The finite element approximation in matrix form is now given by:

$$\mathbf{M}_f \; \ddot{\mathbf{p}}_f + \mathbf{K}_f \; \mathbf{p}_f = \mathbf{f}_s + \mathbf{F}_{bf} \tag{16}$$

where $\mathbf{F}_{bf}$ is given by:

$$\mathbf{F}_{bf} = -c_o^2 \int_{\Omega_f} \mathbf{N}_f^T \; \nabla \cdot \mathbf{f}_{bf} \; dV \tag{17}$$

1.4. Assuming an incompressible fluid

If the density of the fluid is unchanging with time, then the time derivative of ρ_f vanishes in equation (2). Then the following holds:

$$\nabla \cdot \frac{\partial \mathbf{u}_f}{\partial t} = 0 \iff \nabla \cdot \frac{\partial^2 \mathbf{u}_f}{\partial t^2} = 0 \tag{18}$$

The above equation implies that the fluid is incompressible. Substituting equation (1) into the above equation when gravity vector is expressed as a body force vector:

$$\begin{aligned} &\nabla \cdot \Big(\mathbf{f}_{bf} - \nabla p \Big) = 0 \\ &\nabla^2 p = \nabla \cdot \mathbf{f}_{bf} \iff c_o^2 \nabla^2 p = c_o^2 \; \nabla \cdot \mathbf{f}_{bf} \end{aligned} \tag{19}$$

It is evident from the above equation that if the body force vector does not change with the spatial dimension, then its divergence equals zero. Consequently, $\nabla^2 p = 0$, which is a diffusion equation. The weak form for equation (19) is given by:

$$c_o^2 \int_{\Omega_f} \nabla \nu_f \cdot \nabla p \; dV = c_o^2 \int_{\Gamma_f} \nu_f \nabla p \cdot \mathbf{n}_f \; dA - c_o^2 \int_{\Omega_f} \nu_f \; \nabla \cdot \mathbf{f}_{bf} \; dV \tag{20}$$

Introducing the finite element approximation:

$$c_o^2 \int_{\Omega_f} (\nabla \mathbf{N}_f)^T \nabla \mathbf{N}_f \; dV \; \mathbf{p}_f = c_o^2 \int_{\Gamma_f} \mathbf{N}_f^T \nabla p \cdot \mathbf{n}_f \; dA - c_o^2 \int_{\Omega_f} \mathbf{N}_f^T \; \nabla \cdot \mathbf{f}_{bf} \; dV \tag{21}$$

The finite element approximation in matrix form is given by:

$$\mathbf{K}_f \; \mathbf{p}_f = \mathbf{f}_s + \mathbf{F}_{bf} \tag{22}$$

2. Linear elastic structural domain

The governing equation is given by:

$$\widetilde{\nabla}^T \boldsymbol{\sigma}_s + \mathbf{f}_{bs} = \rho_s \; \frac{\partial^2 \mathbf{u}_s}{\partial t^2} \tag{23}$$

$\boldsymbol{\sigma}_s$ is the Cauchy stress tensor in Voigt notation, $\mathbf{f}_{bs}$ is the body force vector, $\mathbf{u}_s$ is the displacement vector. The differential operator $\widetilde{\nabla}$ in the above equation is given by:

$$\widetilde{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \end{bmatrix} \tag{24}$$

The constitutive law is given by:

$$\boldsymbol{\sigma}_s = \mathbf{D}_s \; \boldsymbol{\varepsilon}_s \tag{25}$$

where $\mathbf{D}_s$ is the elasticity tensor and $\boldsymbol{\varepsilon}_s$ is the Cauchy strain tensor in Voigt notation. Additionally, the strain tensor can be expressed as:

$$\boldsymbol{\varepsilon}_s = \widetilde{\nabla} \mathbf{u}_s \tag{26}$$

Multiplying equation (23) with a test function vector $\boldsymbol{\nu}_s = [\nu_1\ \nu_2\ \nu_3]^T$ and integrating over the domain:

$$\int_{\Omega_s} \boldsymbol{\nu}_s^T\ \widetilde{\nabla}^T \boldsymbol{\sigma}_s\ dV + \int_{\Omega_s} \boldsymbol{\nu}_s^T\ \mathbf{f}_{bs}\ dV = \int_{\Omega_s} \boldsymbol{\nu}_s^T\ \rho_s\ \frac{\partial^2 \mathbf{u}_s}{\partial t^2}\ dV \tag{27}$$

Applying the Green's theorem:

$$\int_{\Omega_s} \boldsymbol{\nu}_s^T\ \rho_s\ \frac{\partial^2 \mathbf{u}_s}{\partial t^2}\ dV + \int_{\Omega_s} (\widetilde{\nabla} \boldsymbol{\nu}_s)^T\ \boldsymbol{\sigma}_s\ dV = \int_{\Gamma_s} \boldsymbol{\nu}_s^T\ \mathbf{t}_s\ dA + \int_{\Omega_s} \boldsymbol{\nu}_s^T\ \mathbf{f}_{bs}\ dV \tag{28}$$

where $\mathbf{t}_s$ is the traction vector given by:

$$\mathbf{t}_s = \mathbf{S}_s\ \mathbf{n}_s \tag{29}$$

where $\mathbf{S}_s$ is the Cauchy stress tensor and $\mathbf{n}_s$ is the normal vector. The finite element approximations are given by:

$$\mathbf{u}_s = \mathbf{N}_s \mathbf{d}_s;\ \boldsymbol{\nu}_s = \mathbf{N}_s \mathbf{c}_s \tag{30}$$

where $\mathbf{N}_s$ contain the finite element shape functions, $\mathbf{d}_s$ is the nodal displacement vector, and $\mathbf{c}_s$ is the nodal weights vector. The finite element approximation is given by:

$$\int_{\Omega_s} \mathbf{N}_s^T\ \rho_s\ \mathbf{N}_s\ dV\ \ddot{\mathbf{d}}_s + \int_{\Omega_s} (\widetilde{\nabla} \mathbf{N}_s)^T\ \mathbf{D}_s\ \widetilde{\nabla} \mathbf{N}_s\ dV\ \mathbf{d}_s = \int_{\Gamma_s} \mathbf{N}_s^T\ \mathbf{t}_s\ dA + \int_{\Omega_s} \mathbf{N}_s^T\ \mathbf{f}_{bs}\ dV \tag{31}$$

The finite element approximation can be expressed in matrix form as:

$$\mathbf{M}_s\ \ddot{\mathbf{d}}_s + \mathbf{K}_s\ \mathbf{d}_s = \mathbf{f}_f + \mathbf{F}_{bs} \tag{32}$$

where $\mathbf{M}_s$ and $\mathbf{K}_s$ are similar to the mass and stiffness matrices, respectively, $\mathbf{f}_f$ is the force vector coming from the fluid, and $\mathbf{F}_{bs}$ is the body force vector. These quantities are further expanded as:

$$\begin{aligned} \mathbf{M}_s &= \int_{\Omega_s} \mathbf{N}_s^T\ \rho_s\ \mathbf{N}_s\ dV \\ \mathbf{K}_s &= \int_{\Omega_s} (\widetilde{\nabla} \mathbf{N}_s)^T\ \mathbf{D}_s\ \widetilde{\nabla} \mathbf{N}_s\ dV \\ \mathbf{f}_f &= \int_{\Gamma_s} \mathbf{N}_s^T\ \mathbf{t}_s\ dA \\ \mathbf{F}_{bs} &= \int_{\Omega_s} \mathbf{N}_s^T\ \mathbf{f}_{bs}\ dV \end{aligned} \tag{33}$$

3. Interface kernel coupling fluid and structural domains (no fluid body forces and compressible fluid)

The boundary between the fluid and structure is denoted as Γ_{sf}. At this boundary, the displacements in the normal direction for the fluid and structural domains are the same:

$$\mathbf{u}_s \cdot \mathbf{n} = \mathbf{u}_f \cdot \mathbf{n} \tag{34}$$

where $\mathbf{n}$ is the normal vector given by $\mathbf{n} = \mathbf{n}_f = -\mathbf{n}_s$. In addition, there is continuity of pressure at the boundary as expressed by the structural Cauchy stress tensor:

$$[\mathbf{S}_s]_{\Gamma_{sf}} = -p\ \mathbf{I} \tag{35}$$

where $\mathbf{I}$ is an identity matrix. The structural forcing term coming from the fluid is now given by:

$$\begin{aligned}
\mathbf{f}_f &= \int_{\Gamma_{sf}} \mathbf{N}_s^T \ \mathbf{t}_s \ dA \\
&= \int_{\Gamma_{sf}} \mathbf{N}_s^T \ \mathbf{S}_s \ \mathbf{n}_s \ dA \\
&= \int_{\Gamma_{sf}} \mathbf{N}_s^T \ \ -p\mathbf{I} \ \mathbf{n}_s \ dA \\
&= \int_{\Gamma_{sf}} \mathbf{N}_s^T \ p \ \mathbf{n} \ dA \\
&= \int_{\Gamma_{sf}} \mathbf{N}_s^T \ \mathbf{n} \ p \ dA \\
&= \int_{\Gamma_{sf}} \mathbf{N}_s^T \ \mathbf{n} \ \mathbf{N}_f \ dA \ \mathbf{p}_f
\end{aligned} \tag{36}$$

where the relation $\mathbf{n} = -\mathbf{n}_s$ is used. Without body forces, the fluid momentum equation is given by:

$$\nabla p = -\rho_o \ \frac{\partial^2 \mathbf{u}_f}{\partial t^2} \tag{37}$$

At the fluid-structure boundary, the fluid displacements can be expressed in terms of structural accelerations as:

$$\begin{aligned}
\mathbf{n}^T [\nabla p]_{\Gamma_{sf}} &= -\rho_o \ \mathbf{n}^T \ \big[\frac{\partial^2 \mathbf{u}_f}{\partial t^2}\big]_{\Gamma_{sf}} \\
&= -\rho_o \ \mathbf{n}^T \ \big[\frac{\partial^2 \mathbf{u}_s}{\partial t^2}\big]_{\Gamma_{sf}} \\
&= -\rho_o \ \mathbf{n}^T \ \big[\mathbf{N}_s \ \ddot{\mathbf{d}}_s\big]_{\Gamma_{sf}}
\end{aligned} \tag{38}$$

The fluid forcing vector coming from the structure is now expressed as:

$$\begin{aligned}
\mathbf{f}_s &= c_o^2 \int_{\Gamma_{sf}} \mathbf{N}_f^T \nabla p \cdot \mathbf{n}_f \ dA \\
&= c_o^2 \int_{\Gamma_{sf}} \mathbf{N}_f^T \nabla p \cdot \mathbf{n} \ dA \\
&= c_o^2 \int_{\Gamma_{sf}} \mathbf{N}_f^T \ \mathbf{n}^T \nabla p \ dA \\
&= -\rho_o c_o^2 \int_{\Gamma_{sf}} \mathbf{N}_f^T \ \mathbf{n}^T \ \mathbf{N}_s \ dA \ \ddot{\mathbf{d}}_s
\end{aligned} \tag{39}$$

Introducing the interface matrix $\mathbf{H}$:

$$\mathbf{H} = \int_{\Gamma_{sf}} \mathbf{N}_s^T \ \mathbf{n} \ \mathbf{N}_f \ dA \tag{40}$$

The structural and fluid forcing functions are given by:

$$\begin{aligned}
\mathbf{f}_f &= \mathbf{H}\mathbf{p}_f \\
\mathbf{f}_s &= -\rho_o c_o^2 \mathbf{H}^T \ddot{\mathbf{d}}_s
\end{aligned} \tag{41}$$

The combined finite element equations for fluid and structural domains is:

$$\begin{bmatrix} \mathbf{M}_s & 0 \\ \rho_o c_o^2 \mathbf{H}^T & \mathbf{M}_f \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{d}}_s \\ \ddot{\mathbf{p}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{K}_s & -\mathbf{H} \\ 0 & \mathbf{K}_f \end{bmatrix} \begin{bmatrix} \mathbf{d}_s \\ \mathbf{p}_f \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{bs} \\ \mathbf{0}_f \end{bmatrix} \tag{42}$$

where $\mathbf{0}_f$ is a vector of zeros.

4. Summary of the fluid and structural domains finite element equations

The structural domain finite element equation in matrix form is:

$$\mathbf{M}_s \ \ddot{\mathbf{d}}_s + \mathbf{K}_s \ \mathbf{d}_s - \mathbf{H} \ \mathbf{p}_f = \mathbf{F}_{bs} \tag{43}$$

The different terms in the above equation are expanded below:

- $\mathbf{M}_s = \int_{\Omega_s} \mathbf{N}_s^T \ \rho_s \ \mathbf{N}_s \ dV$ is the structural mass matrix which integrates over the density of the structural domain.
- $\mathbf{K}_s = \int_{\Omega_s} (\widetilde{\nabla}\mathbf{N}_s)^T \ \mathbf{D}_s \ \widetilde{\nabla}\mathbf{N}_s \ dV$ is the structural stiffness matrix which integrates over the elasticity tensor of the structural domain.
- $\mathbf{H} = \int_{\Gamma_{sf}} \mathbf{N}_s^T \ \mathbf{n} \ \mathbf{N}_f \ dA$ is the interface matrix which transfers the displacements from structure to fluid and pressures from fluid to structure.
- $\mathbf{F}_{bs} = \int_{\Omega_s} \mathbf{N}_s^T \ \mathbf{f}_{bs} \ dV$ is the structural body force vector.
- $\mathbf{d}_s$ is the structural displacement vector at the nodes in the finite-element mesh.
- $\mathbf{p}_f$ is the pressure vector at the nodes.

The fluid domain finite element equation in matrix form is:

$$\rho_o c_o^2 \mathbf{H}^T \ \ddot{\mathbf{d}}_s + \mathbf{M}_f \ \ddot{\mathbf{p}}_f + \mathbf{K}_f \ \mathbf{p}_f = \mathbf{0}_f \tag{44}$$

- ρ_o is the static density of the fluid.
- c_o is the speed of wave propagation through the fluid.
- $\mathbf{M}_f = \int_{\Omega_f} \mathbf{N}_f^T \ \mathbf{N}_f \ dV$ is similar to the mass matrix for the fluid domain.
- $\mathbf{K}_f = c_o^2 \int_{\Omega_f} (\nabla \mathbf{N}_f)^T \nabla \mathbf{N}_f \ dV$ is similar to the stiffness matrix for the fluid domain.
- $\mathbf{0}_f$ is a vector of zeros.

5. Equivalence between acoustic and structural mechanics governing equations

Consider a structural domain. If the displacements in two directions are zero everywhere in the domain, one governing equation is sufficient to describe the domain behavior:

$$\frac{\lambda^e + 2G^e}{G^e} \frac{\partial^2 u_{s,1}^e}{\partial x_1^2} + \frac{\partial^2 u_{s,1}^e}{\partial x_2^2} + \frac{\partial^2 u_{s,1}^e}{\partial x_3^2} = \frac{\rho^e}{G^e} \ \frac{\partial^2 u_{s,1}^e}{\partial t^2} \tag{45}$$

where, the superscript indicates a structural domain that is equivalent to the acoustic domain, and λ, G, and ρ are the Lamé's constant, shear modulus, and density, respectively. The above equation assumes that the equivalent structural domain has no body force. Further, if we set $\lambda^e = -G^e$ in the above equation:

$$\frac{\partial^2 u_{s,1}^e}{\partial x_1^2} + \frac{\partial^2 u_{s,1}^e}{\partial x_2^2} + \frac{\partial^2 u_{s,1}^e}{\partial x_3^2} = \frac{\rho^e}{G^e} \ \frac{\partial^2 u_{s,1}^e}{\partial t^2} \tag{46}$$

Now, expanding the acoustic domain governing equation [equation (13)]:

$$\frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial x_3^2} = \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} \tag{47}$$

We see that equations (46) and (47) are equivalent if $\rho^e = \frac{G^e}{c_o^2}$, $\lambda^e = -G^e$, and Young's modulus $E^e \to +\infty$.

5.1. Implementing acoustics and equivalent structural mechanics approaches in MASTODON

The equivalence between acoustics and structural mechanics governing equations is presented below:

$$\begin{aligned} &\nabla^2 p = \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} \iff \nabla^2 u_{s,1}^e = \frac{\rho^e}{G^e}\frac{\partial^2 u_{s,1}^e}{\partial t^2} \\ &\text{such that, } \rho^e = \frac{G^e}{c_o^2} \text{ and } \lambda^e = -G^e \end{aligned} \tag{48}$$

In MASTODON, the acoustics approach is implemented by using the `Diffusion` kernel for the $\nabla^2 p$ term and the `InertialForce` kernel with density $\frac{1}{c_o^2}$ for the $\frac{\partial^2 p}{\partial t^2}$ term in equation (48). The equivalent structural mechanics approach is implemented by using the `DynamicTensorMechanics` kernel for the $\nabla^2 u_{s,1}^e$ term and the `InertialForce` kernel with density $\frac{G^e}{c_o^2}$ for the $\frac{\partial^2 u_{s,1}^e}{\partial t^2}$ term in equation (48). Furthermore, the `ComputeIsotropicElasticityTensor` material is used for the structural mechanics approach with material parameters defined as per the second line of equation (48). It is noted that G^e can be set to any arbitrary value.

5.1.1. Finite element mesh

Figure 1 presents the mesh for performing the simulations. It is a 2D mesh with a time varying pressure boundary (Dirichlet) condition applied at the bottom edge. Pressure is recorded at the middle of the top edge. Also, for the equivalent structural mechanics approach, the displacements in the y-direction are constrained to zero along the four edges and the pressure boundary condition is equivalent to applying a displacement in the x-direction at the bottom edge.

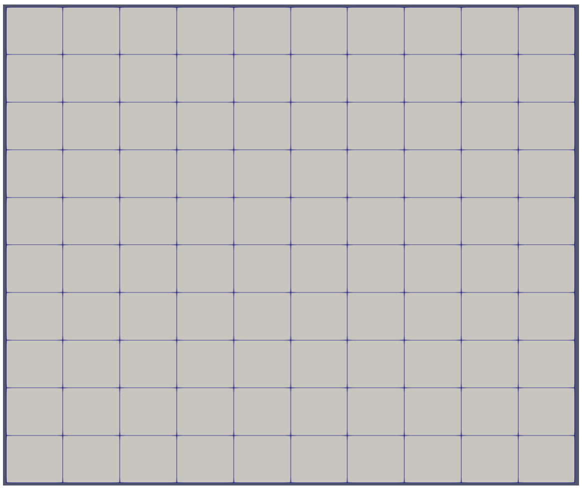


Figure 1: A 2D mesh for simulating acoustics.

5.1.2. *Results*

The input pressure at the bottom edge of the mesh is presented in Figure 2. Figure 3 presents the pressures at the center of the top edge for the acoustics and equivalent structural mechanics approaches considering three c_o^2 values: 1 m/s, 200 m/s, and 1450 m/s. It is seen that the results using the two approaches are identical.

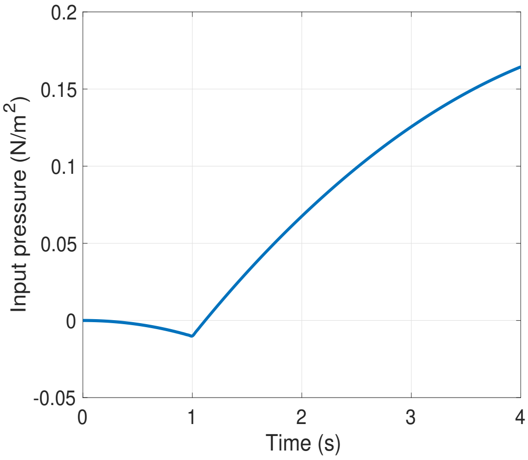
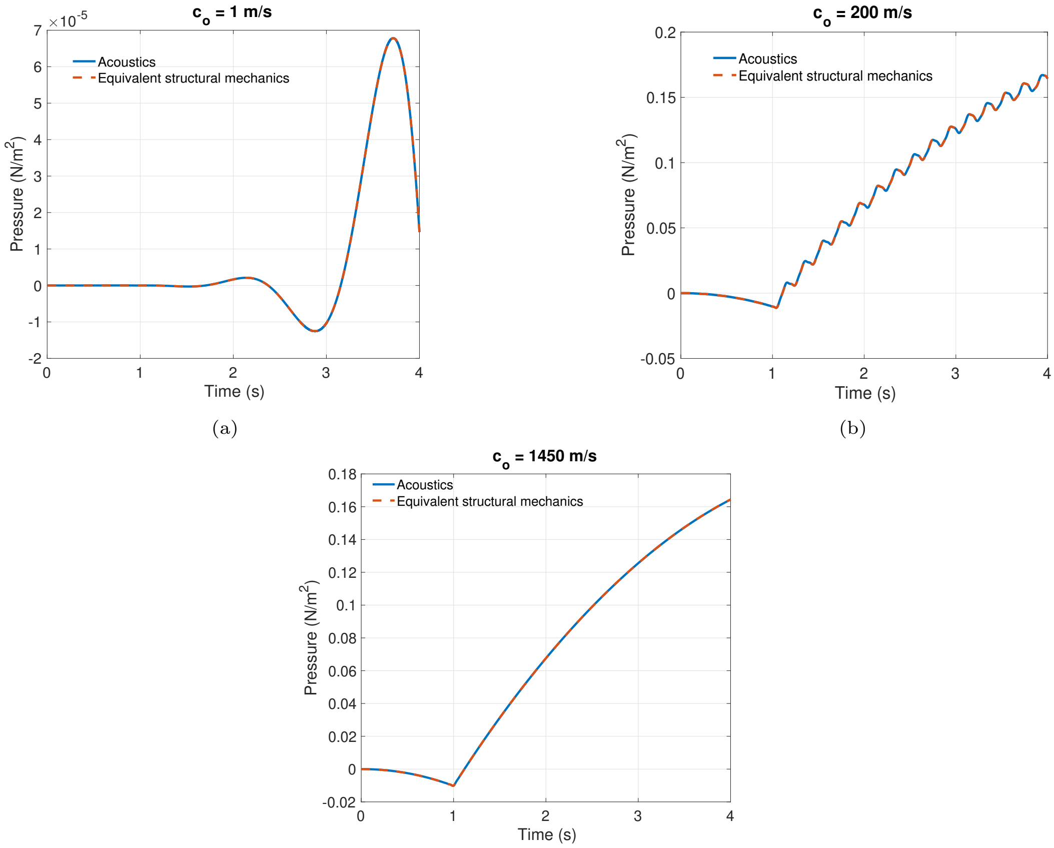


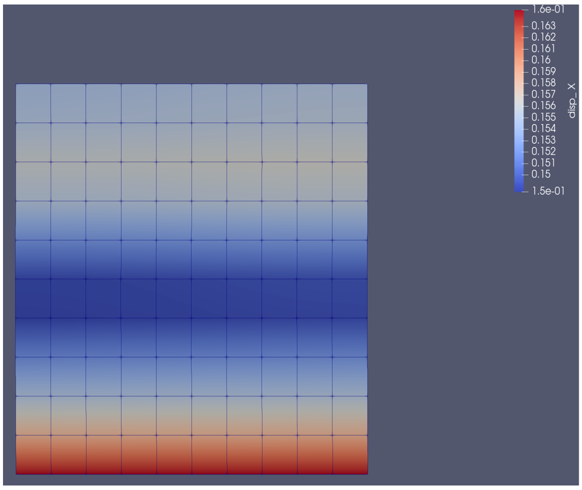
Figure 2: Input pressure at the bottom edge of the mesh.



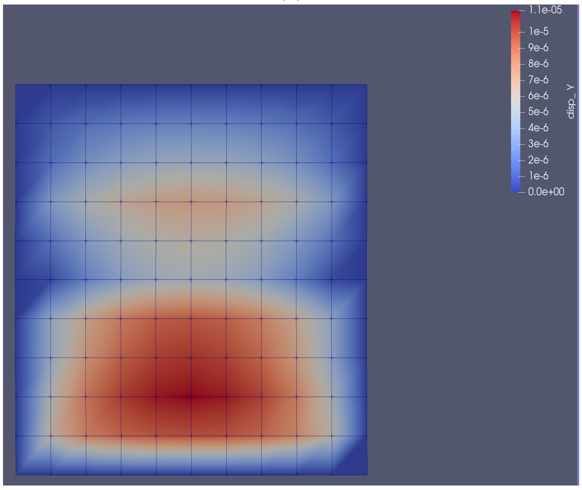
(a) (b) (c)

Figure 3: Comparison between acoustics and equivalent structural mechanics approaches for different c_o values.

Figure 4 presents the distribution of x-direction displacements (i.e., pressure) and y-direction dummy displacements in the equivalent structural mechanics mesh. It is seen that the y-direction dummy displacements are very close to zero.



(a)



(b)

Figure 4: Distribution of (a) x-direction displacements (i.e., pressure) and (b) y-direction dummy displacements in the equivalent structural mechanics mesh.)

6. Verification of the numerical diffusion approach with analytical solutions in 1D

The acoustic equation with both initial and boundary conditions assuming $c_o = 1$ can be written as:

$$\begin{aligned} &\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} \qquad 0 < x < 1 \\ &\text{BC}: \;\; p(0,\; t) = p(1,\; t) = 0 \qquad t > 0 \\ &\text{IC}: \;\; p(x,\; 0) = f(x) \qquad 0 < x < 1 \end{aligned} \tag{49}$$

where, it is assumed that the domain is between $[0,\; 1]$ and the pressures are always zero at the ends of the domain. The function $f(x)$ can be arbitrarily defined. A closed form solution to equation (49) using Fourier series expansion is given by:

$$
\begin{aligned}
p(x,\ t) &= \sum_{n=1}^{+\infty} \alpha_n\ cos(n\pi t)\ sin(n\pi x) \\
\alpha_n &= 2\int_0^1\ f(x)\ sin(n\pi x)\ dx
\end{aligned}
\tag{50}
$$

The initial condition we use is given by:

$$
f(x) = sin(\pi x) + sin(3\pi x) + sin(5\pi x) + sin(7\pi x) + sin(9\pi x) \tag{51}
$$

Figure 5 presents the initial condition.

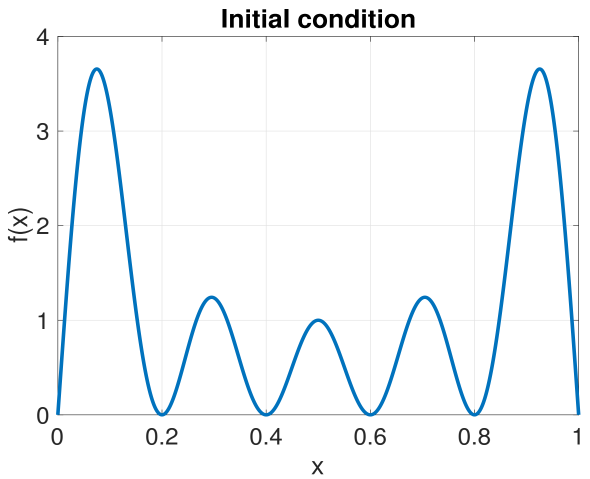


Figure 5: Initial condition for the 1D wave problem.

In MOOSE, a 1D mesh of length 1 meter with 500 elements was created. Pressures at the two ends of the mesh were fixed to zeros. An initial condition representing equation (51) was applied. The pressures as a function of time were recorded at 0.5 meters in the mesh. Figure 6 presents the pressures simulated using MOOSE, and they compare well with the analytical solution.

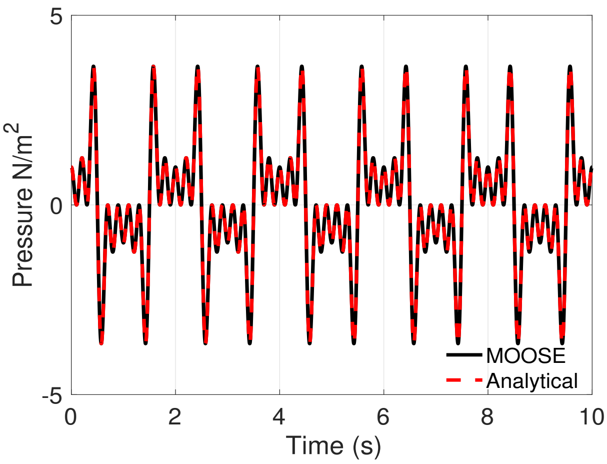


Figure 6: Solution comparison between MOOSE and analytical of the 1D wave problem with an $f(x)$ initial condition.

7. Fluid free surface boundary condition

Fluid, when subjected to shaking, experiences waves on the surface due to changes in pressures. These waves are called as gravity waves. Acoustics formulation alone cannot capture these gravity waves. An additional boundary condition is needed to simulate gravity waves called the free surface boundary condition. The pressure at the free surface of a fluid because of waves generated due to dynamic action is given by:

$$p = \rho\ g\ d_w \tag{52}$$

where, d_w is the height of the wave with reference to the initial free surface before applying the dynamic action. The height can be further expressed as:

$$\begin{aligned} d_w &= \mathbf{n} \cdot \mathbf{u} = u_z \\ \nabla d_w &= \nabla\big(\mathbf{n} \cdot \mathbf{u}\big) = \frac{\partial u_z}{\partial z} \end{aligned} \tag{53}$$

where, u_z represents the normal component of fluid displacement above/below the free surface. The pressure in equation (52) can be further expressed using equation (1) as (no body force vector):

$$\begin{aligned} &\nabla p = \rho\ g\ \frac{\partial u_z}{\partial z} = -\rho\ \frac{\partial^2 u_z}{\partial t^2} \\ &\frac{\partial^2 u_z}{\partial t^2} + g\ \frac{\partial u_z}{\partial z} = 0 \end{aligned} \tag{54}$$

The above equation is the free surface gravity condition in terms of vertical displacements. Because $u_z = \frac{p}{\rho\ g}$, it can be expressed in terms of pressures as well:

$$\frac{\partial^2 p}{\partial t^2} + g\ \frac{\partial p}{\partial z} = 0 \tag{55}$$

The above boundary condition is like a coupled Dirchlet and Neumann conditions. Term $\frac{\partial^2 p}{\partial t^2}$ represents a Dirichlet condition and term $\frac{\partial p}{\partial z}$ represents a Neumann condition.

7.1. Implementation in MOOSE

A new class called `FluidFreeSurfaceBC` has been created to impose equation (55) at the free surface of the fluid. This class has `FluidFreeSurfaceBC::computeQpResidual()` and `FluidFreeSurfaceBC::computeQpJacobian()` parts. The `.C` file of the `FluidFreeSurfaceBC` class is presented below:

Listing 1: FluidFreeSurfaceBC.C

```
#include "FluidFreeSurfaceBC.h"

registerMooseObject("MooseApp", FluidFreeSurfaceBC);

defineLegacyParams(FluidFreeSurfaceBC);

InputParameters
FluidFreeSurfaceBC::validParams()
{
  InputParameters params = IntegratedBC::validParams();
  params.addParam<Real>("alpha", 1, "No idea.");
  return params;
}
```

```
FluidFreeSurfaceBC::FluidFreeSurfaceBC(const InputParameters & parameters)
  : IntegratedBC(parameters), _alpha(getParam<Real>("alpha")),
  _u_dotdot(dotDot()),
  _du_dotdot_du(dotDotDu())
{
}

Real
FluidFreeSurfaceBC::computeQpResidual()
{
  return _test[_i][_qp] * _alpha * _u_dotdot[_qp] / 1.;
}

Real
FluidFreeSurfaceBC::computeQpJacobian()
{
  return _test[_i][_qp] * _alpha * _du_dotdot_du[_qp] * _phi[_j][_qp] / 1.;
}
```

8. Rigid tank with fluid subjected to dynamic pressures on side wall

8.1. Problem description and kernels

A rigid tank with fluid height 1.2m and radius 0.79m is subjected to dynamic pressures on its side wall. As the tank is rigid, modeling only fluid domain is sufficient. The fluid domain is modeled using `Diffusion` and `InertialForce` kernels which represent $\nabla^2 p$ and $\frac{\partial^2 p}{\partial t^2}$ terms in the governing equation [equation (13)], respectively. The velocity of sound c_o is represented as the density term for the `InertialForce` kernel by taking $\rho_o = 1/c_o^2$. Figure 7 presents the fluid domain and meshing. The points in this domain represent locations at which the pressures are inferred. At the fluid surface, pressures are inferred at 21 locations at 0.07m spacing along the diameter. Along the height of the domain, pressures are inferred at seven locations with 0.2m spacing.

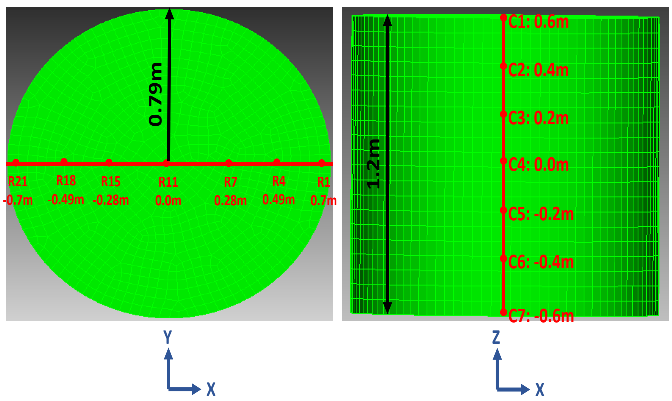


Figure 7: Plan and elevation views of the fluid domain. The points represent the locations at which the pressures are inferred.

8.2. Boundary conditions

The outer surface of the fluid domain along its height is treated as a Neumann boundary with the gradients of the pressures being zero. At the top surface, the fluid free surface condition discussed in the previous section is applied. Fluid outer surface along the height is subjected to a dynamic pressure input for about 0.4s. Figure 8 presents this pressure input.

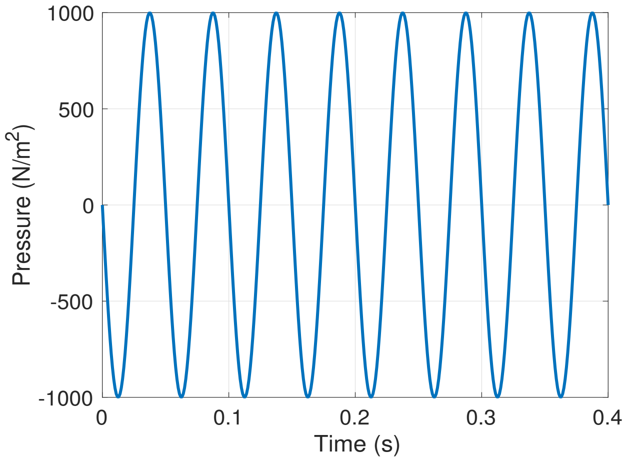
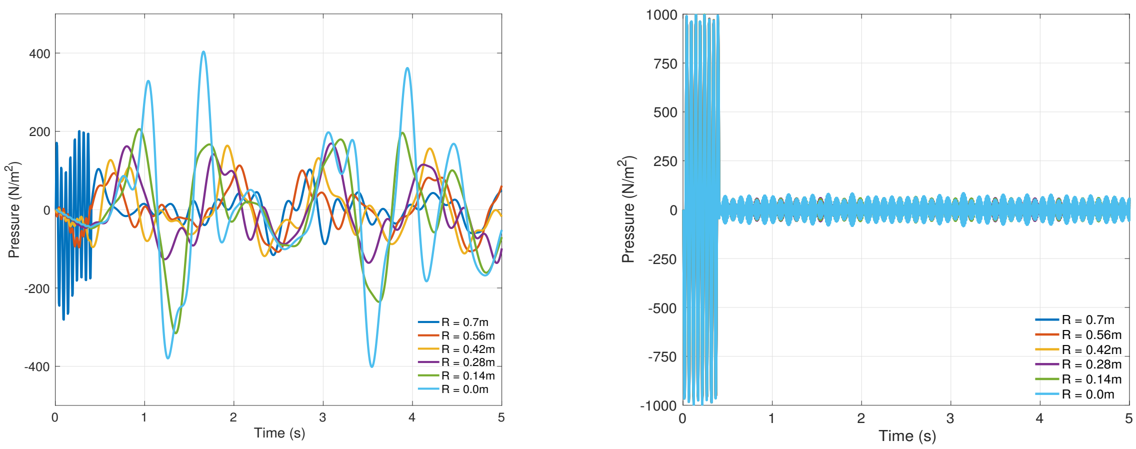


Figure 8: Input pressure history on the sides of the tank.

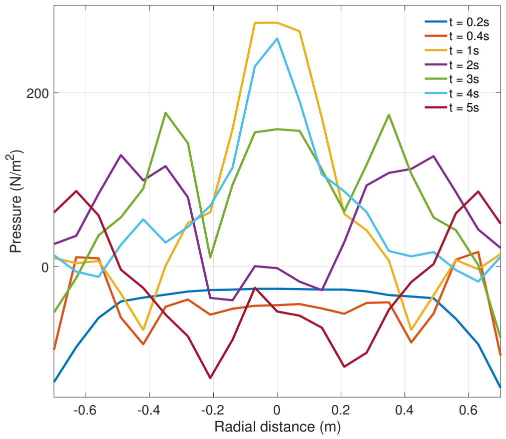
8.3. Results: Speed of sound = 1500 m/s



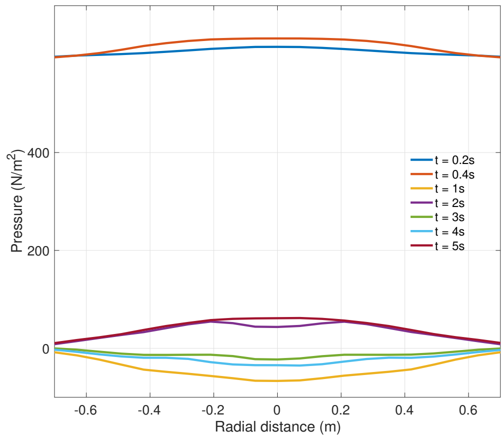
(a)

(b)

Figure 9: Time histories of surface pressures at different radial distances for $c_o = 1500m/s$: (a) with the free surface BC; (b) without the free surface BC



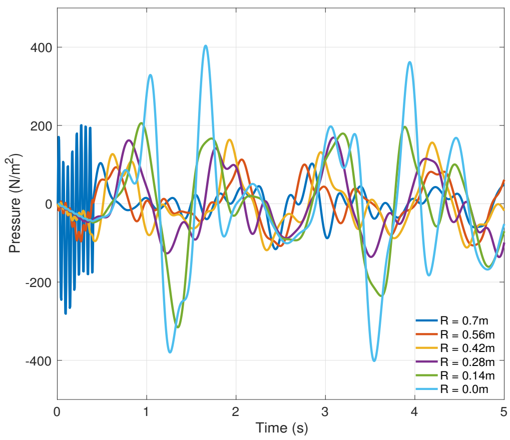
(a)



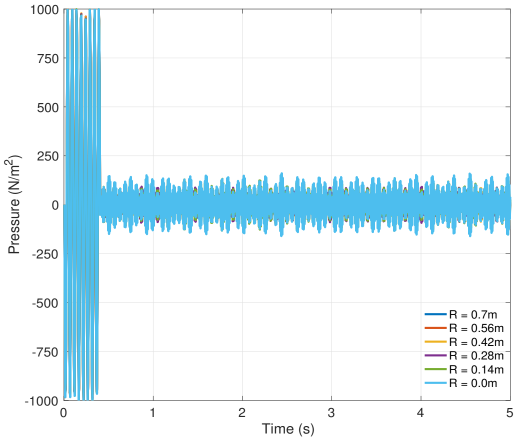
(b)

Figure 10: Wave profiles of surface pressures at different times for $c_o = 1500m/s$: (a) with the free surface BC; (b) without the free surface BC

8.4. Results: Speed of sound = 1000 m/s

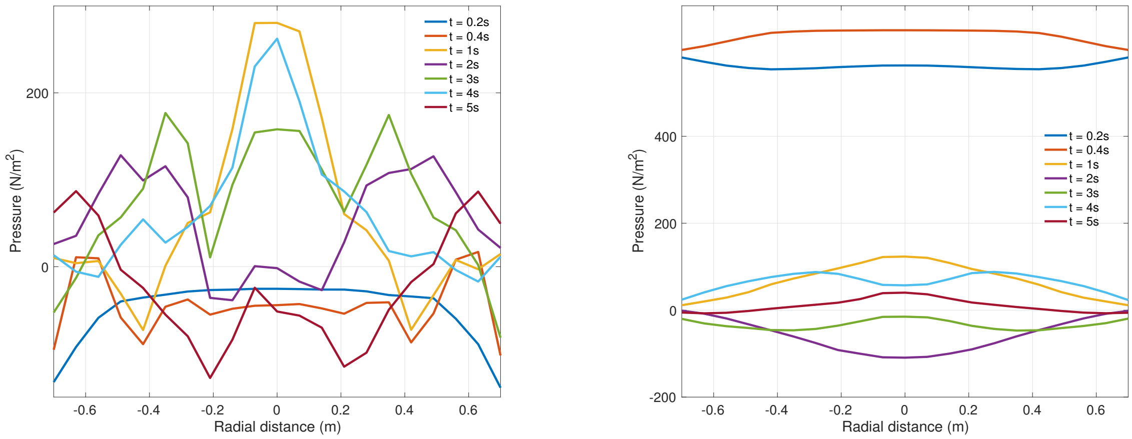


(a)



(b)

Figure 11: Time histories of surface pressures at different radial distances for $c_o = 1000m/s$: (a) with the free surface BC; (b) without the free surface BC

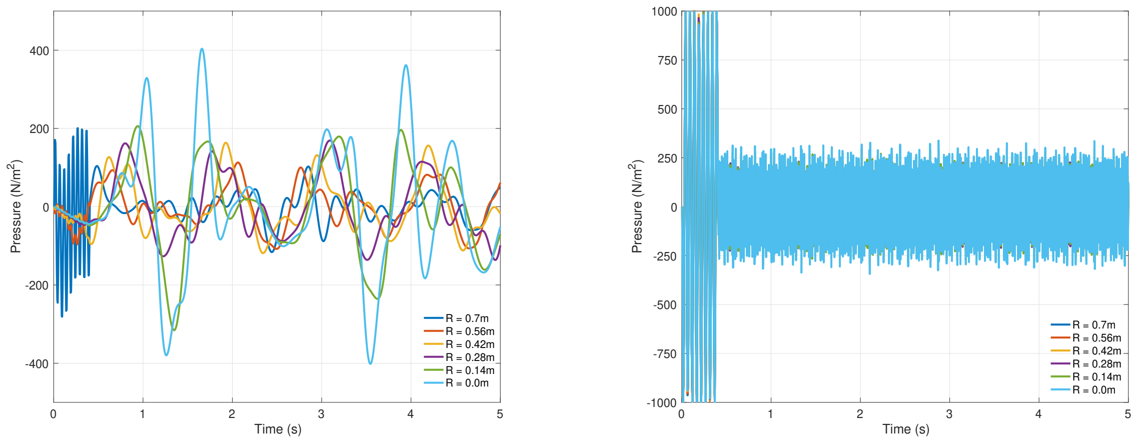


(a)

(b)

Figure 12: Wave profiles of surface pressures at different times for $c_o = 1000m/s$: (a) with the free surface BC; (b) without the free surface BC

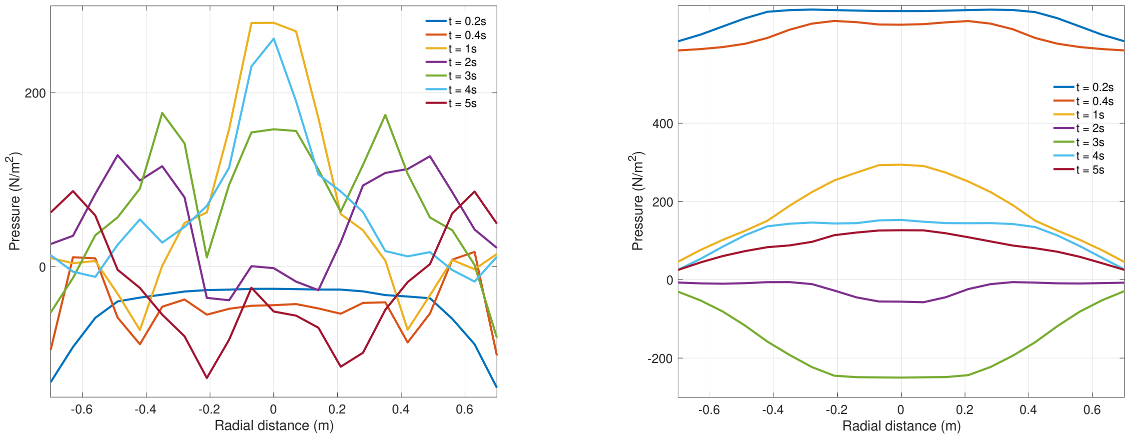
8.5. Results: Speed of sound = 500 m/s



(a)

(b)

Figure 13: Time histories of surface pressures at different radial distances for $c_o = 500m/s$: (a) with the free surface BC; (b) without the free surface BC



(a) (b)

Figure 14: Wave profiles of surface pressures at different times for $c_o = 500m/s$: (a) with the free surface BC; (b) without the free surface BC

8.6. *Natural frequencies of the surface waves* ($c_o = 1500m/s$)

Figure 15 presents the Fourier spectrum of the surface pressures with the free surface BC. The c_o of the fluid is $1500m/s$. First three frequencies are: $0.8Hz$, $1.4Hz$, and $1.8Hz$. Yu and Whittaker (2020) propose an analytical equation for the first three natural frequencies. These frequencies are: 0.768Hz, 1.296Hz, and 1.65Hz.

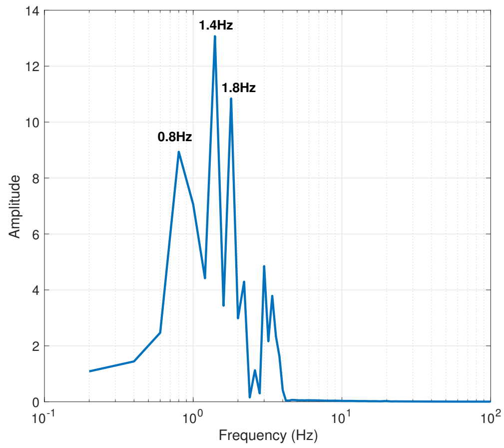


Figure 15: Natural frequencies of surface waves with $c_o = 1500m/s$

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