Mathematical thinking
Graded assignment
Week 4
Total marks: 20

1. Which of the following option(s) is (are) true?

   (a) If p is a prime and $p \mid a^7$ then $p \mid a$.

   (b) If p is a prime and $p \mid a$ and $p \mid (a^2 + b^2)$ then $p \mid b$.

   (c) If $a \equiv b \pmod{m}$, then $\gcd(a, m) = \gcd(b, m)$.

   (d) If $ax \equiv ay \pmod{m}$ and $\gcd(a, m) = 2$, then $x \equiv y \pmod{m}$.

2. Which of the following options is/are true?

   (a) $\displaystyle\sum_{n=1}^{18} n^{19} \equiv 0 \pmod{19}$.

   (b) $\displaystyle\sum_{n=1}^{18} n^{19} \equiv 1 \pmod{19}$.

   (c) $\displaystyle\sum_{n=1}^{18} n^{18} \equiv 18 \pmod{19}$.

   (d) $\displaystyle\sum_{n=1}^{19} n^{18} \equiv 18 \pmod{19}$.

3. If $2^{1000} \equiv x \pmod{13}$ where $0 \leq x \leq 12$, then find the value of x.

4. Consider a set $S = \{n \mid n \in \mathbb{N}, \sqrt{n} \notin \mathbb{N}, n \leq 200\}$. How many composite odd numbers are there in the set S?

5. Prove that any prime of the form $3k + 1$ is of the form $6k + 1$.
6. Let $a \equiv b \pmod{12}$ and $m \mid 12$, then prove that $a \equiv b \pmod{m}$.
7. Let a be a perfect square. Prove that all powers in prime factorization of a are even.
8. Suppose $a = a_1\ a_2\ a_3 \ldots a_n$ with all $a_i \geq 2$. If p is a prime number which divides a, prove that $p \mid a_i$ for some i.