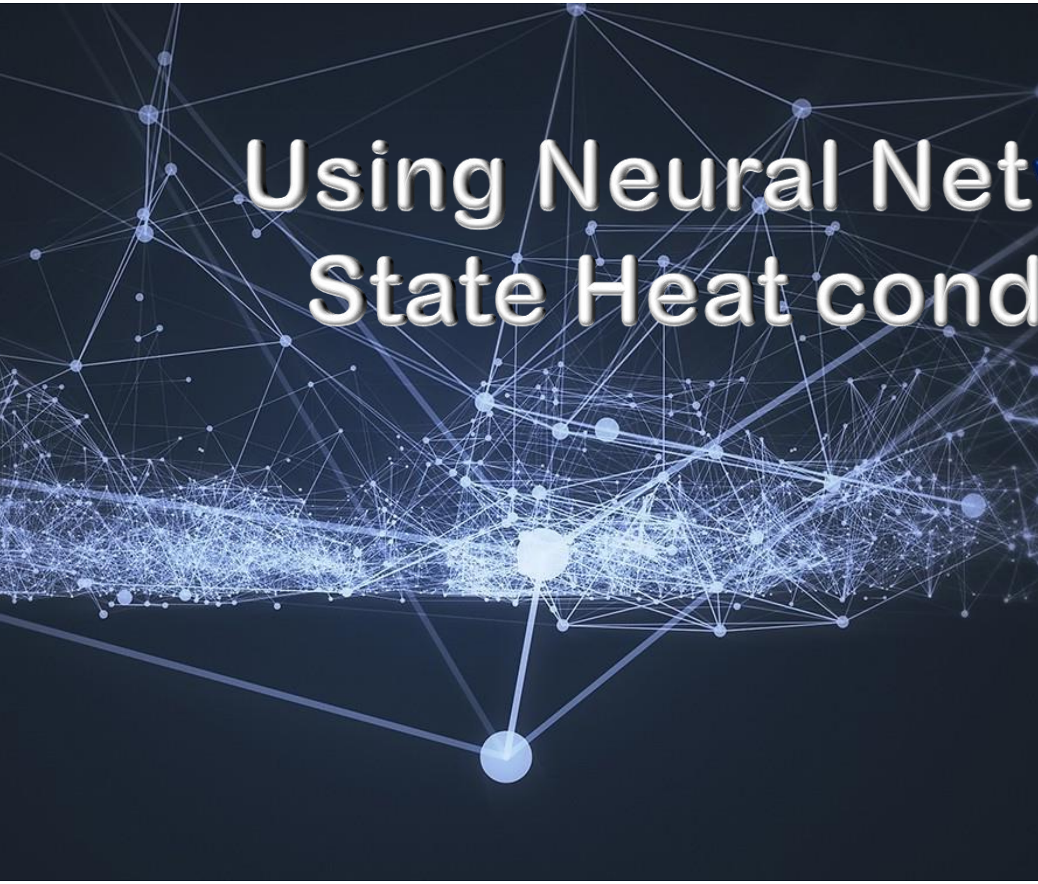
Using Neural Networks in Steady State Heat conduction in 2 - D



Under Guidance of
Professor P. Srinivasan

by
Somya Upadhyay
(2017B5A40962P)

Course Type :
Special Project (ME F 491)

Introduction and Theory

EQUATIONS :STEADY STATE EQUATION IN 2-D

Theory

Steady State Equation in 2-D

Heat Equation : $\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}+\dfrac{\partial^2 T}{\partial z^2}+\dfrac{\dot{e}_{gen}}{k}=\dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$

Assuming 2-D: $\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}+\cancel{\dfrac{\partial^2 T}{\partial z^2}}+\dfrac{\dot{e}_{gen}}{k}=\dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$

Steady State:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\cancel{\frac{\partial^2 T}{\partial z^2}}+\frac{\dot{e}_{gen}}{k}=0$$

Steady state implies with no heat generation :

Therefore, for no heat generation, in 2-D Steady State for which analytical solution is available in 2 dimensions is:

$$\cancel{\frac{\partial T}{\partial t}=}\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\cancel{\frac{\partial^2 T}{\partial z^2}}\right)=\frac{k}{\rho c_p}\nabla^2 T=c^2\nabla^2 T$$

;

Methods to solve complex partial Differential equations

Integration Techniques

- **Analytical methods** (not possible for functions except very simplified cases that rarely occur in nature)
- **Finite element method** (FEM)
- **Artificial Neural Network method** (ANN)

FEM AND ANN techniques were compared to weigh their advantages and disadvantages.

Comparison

Comparison of Neural Network and Finite Element method

Reference	FEM	ANN
Solving PDE problems with uncertainty using neural-networks Yuehaw Khoo , Jianfeng Lu, Lexing Ying May 24, 2018	Although being applicable in many situations, the **computed quantity is inherently noisy** due to approximation.	choosing a sufficiently large class of approximation functions without the issue of **over-fitting** remains a delicate business,
Hermann G Matthies and Andreas Keese. Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. Computer methods in applied mechanics and engineering	Monte-Carlo algorithm (based on FEM)' is on one hand the strength of this approach—its **general applicability**, but on the other hand it does not take into account any special properties of the problem. Galerkin method' **statistics like the mean, covariance, or probabilities are very cheap.**	

Reference	FEM	ANN
,A. G. Baydin, B. A. Pearlmutter, A. A. Radul, and J. M. Siskind, Automatic differentiation in machine learning: a survey, The Journal of Machine Learning	To mitigate approximation errors in numerical differentiation, use a center difference approximation to correct error. With increasing dimensionality, a **trade-off between accuracy and performance** is faced.	(Automatic differentiation) allows **accurate** evaluation of derivatives at machine precision with only a **small constant factor of overhead** and ideal asymptotic efficiency. It has a two-sided nature that is partly symbolic and partly numerical. (Griewank, 2003)
Deepxde: A Deep Learning Library For Solving Differential Equations Lu Lu, Xuhui Meng, Zhiping Mao, And George Em Karniadakis	Requires a mesh to compute with **dependence on the mesh** shape for error/nature of result.	'deep learning could be a **mesh-free approach** by taking advantage of the automatic differentiation and could break the curse of dimensionality.'

FEM	ANN
Numerical approximations of derivatives are inherently ill-conditioned and unstable, with the exception of complex variable methods that are applicable to a **limited set of holomorphic functions**. *(C. C. Margossian, A review of automatic differentiation and its efficient implementation, (2019),*	It has **accurate application in higher order** PDE. In certain cases, AD libraries are implemented as **black boxes** which support statistical and machine learning softwares, such as the python package PyTorch *(Paszke et al., 2017) or the probabilistic programing language Stan (Carpenter et al., 2017).*

Summary

FEM	ANN
• **General applicability**, • **Computed quantity is inherently noisy** • Galerkin method: **statistics like the mean, covariance, or probabilities are very cheap.** • **dependence on the mesh** shape • Applicable to a **limited set of holomorphic functions**. • During correction, **trade-off between accuracy and performance**.	• Initial **extensive computation** process. • issue of **over-fitting** • **Accurate** evaluation of derivatives ,only a **small constant factor of overhead** • **mesh-free approach** • **accurate application in higher order** • AD libraries are implemented as **black boxes** • No correction terms required(self **correcting algorithm)**

Problem Statement

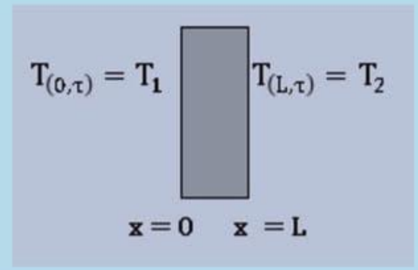
Types of Boundary Conditions

Boundary Condition of 1st kind – Dirichlet conditions

Constant surface temperature is specified at the boundary surface

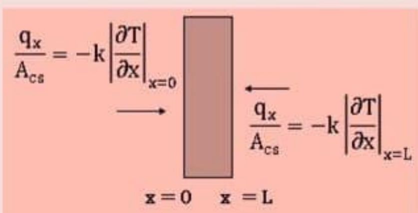
i.e T = T (x,τ)

$$T(x,\tau) = \text{Constant value}$$



Boundary Condition of 2nd kind – Neumann conditions

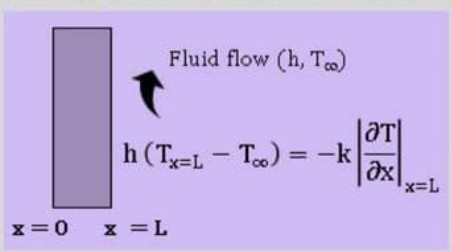
Constant surface heat flux is specified at the boundary surface



Finite heat flux $$\frac{q_x}{A_{cs}} = -k\left|\frac{\partial T}{\partial x}\right|_x$$

Adiabatic or Insulated surface $$\frac{q_x}{A_{cs}} = 0 = -k\left|\frac{\partial T}{\partial x}\right|_x$$

Boundary Condition of 3rd kind – Robin or Mixed conditions



Convective heat transfer condition is specified at the boundary surface

$$h\,(T_{x=L} - T_\infty) = -k\left|\frac{\partial T}{\partial x}\right|_{x=L}$$

Sachin Anant Telang

Problem statement

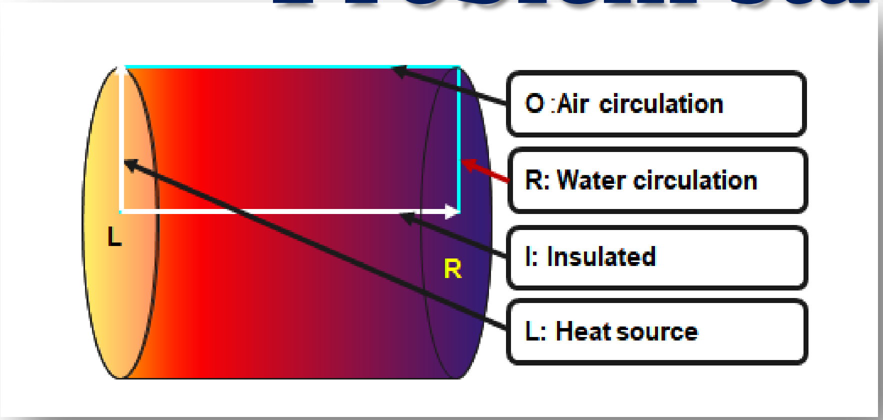


Figure 1:Fixed external boundaries temperature indicative heat color map.

The **Direchlet boundary condition** requirement on the temperature function ψ(x,y) is given for all four edges. The steady values are based off realistic imaginary setup considering convection calculations .

- **L:Left wall** is insulated with constant heat flux
- **R:Right wal**l has a constant coolant circulation
- **I:Inner wall** is insulated (because it simulated the inner radial space
- **O:Outer wal**l is exposed to the environment.

(left)$\Psi(r)=100$ at $x=0$

(right)$\Psi(r)=20$ at $x=$ 2units(length)

(bottom)$\Psi(x)=100$ at $r=0$

(top)$\Psi(x)=40$ at $r=$ 1 unit(radius)

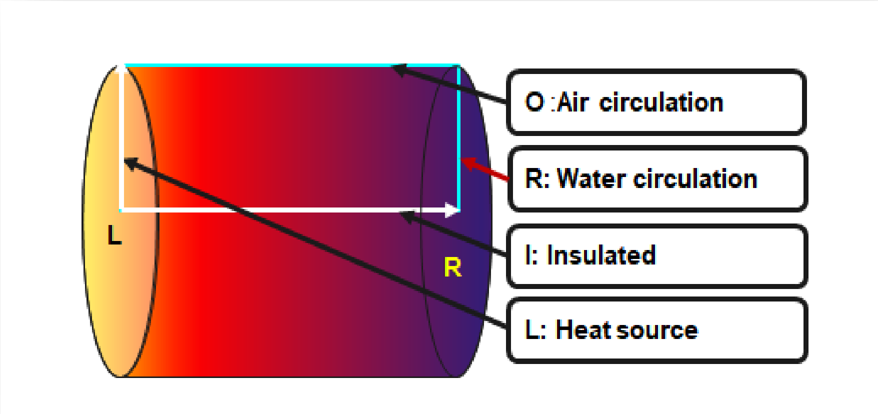
Heat Generation term

Constant heat generation at source up till 0.1 per cent of distance.

$$\frac{\dot{e}_{gen}}{k} = \begin{cases} 100 & x < 0.1 \text{ x length} \\ 0 & x \geq 0.1 \text{ x length} \end{cases}$$

Assuming cylindrical system that is symmetric about the tangential projection (azimuthal) at a fixed radial distance(r) and fixed length (x)from the constant heat source surface (L).

The heat equation solutions are closely holomorphic (infinitely differentiable everywhere in a given domain)*(Numerical study on uniformity of temperature difference field in a spiral tube heat exchanger 2021 ,Hechang Caia,b,1, Yuling Zhaib,c,1, Yao Chend,*, Fanhan Liue,*, Hua Wangb,c, Jianxin Xu)*



Machine Learning Model

Machine Learning Model

One model was made for FEM

Two training models were made for ANN

a) Model size 1[2:32:32:1]

b) Model size 2:[2:16:16:1]

- **Optimizer Function**

The optimizer used is Adaptive Moment Estimation (Adam) optimiser. Adam uses Momentum and Adaptive Learning Rates to converge faster. The intuition for adaptive learning rates, is that we start off with big steps and finish with small steps – like mini-golf. We are then allowed to move faster initially. As the learning rate decays, we take smaller and smaller steps, allowing us to converge faster, since we don't overstep the local minimum with equally large steps.

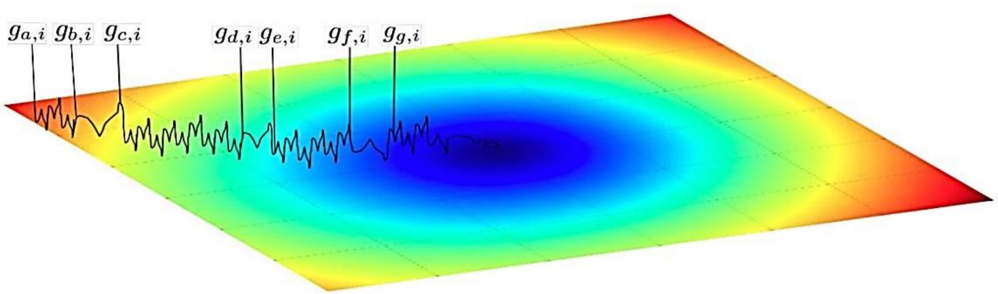


Figure 3:Intution for Adams optimizer (source:https://forums.fast.ai/)

Machine Learning Model

- **Differential equation (r is denoted as y in the model)**

Constant heat generation at source up till 0.1 per cent of distance x(energy array). Laplace is sum of arrays dTdy and dTdx (double derivatives of T with r and x) .

```
##differential equation diffeq###
diffeq = array(dTdy) + array(dTdx) + array(energy.reshape([divs*divs,1]))
```

- **Loss Function**

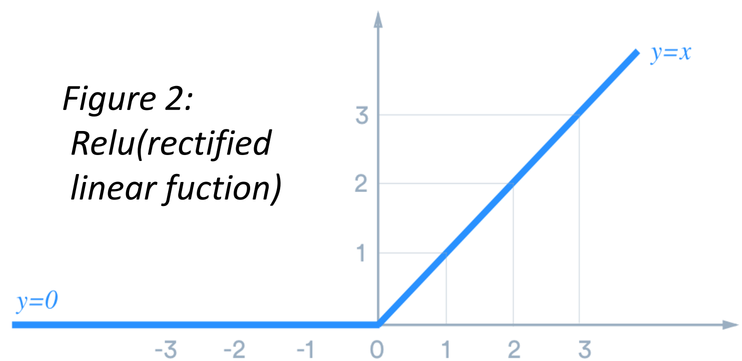
The loss is sum of mean(diffeq) and boundary conditions.

```
loss = mean  + boundry2(T.reshape([divs,divs]))
```

- **Activation Function**

It defines the output of each neuron.
The activation function used is a relu function (rectified linear unit) ,normalized to values between 0 and 1 for optimizer.

Figure 2: Relu(rectified linear fuction)

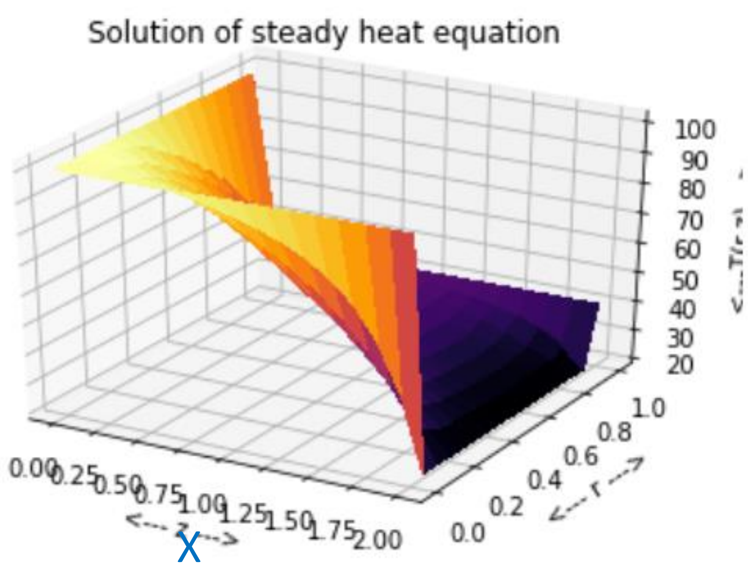


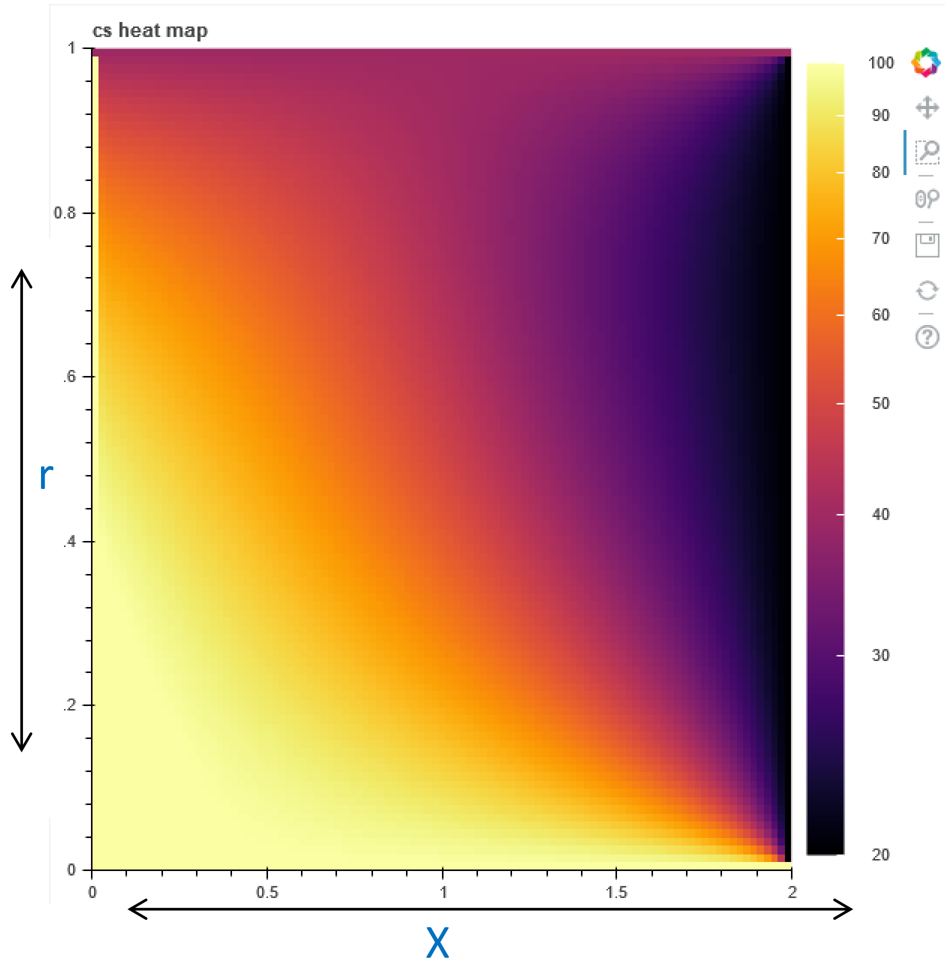
Results

FEM

Results

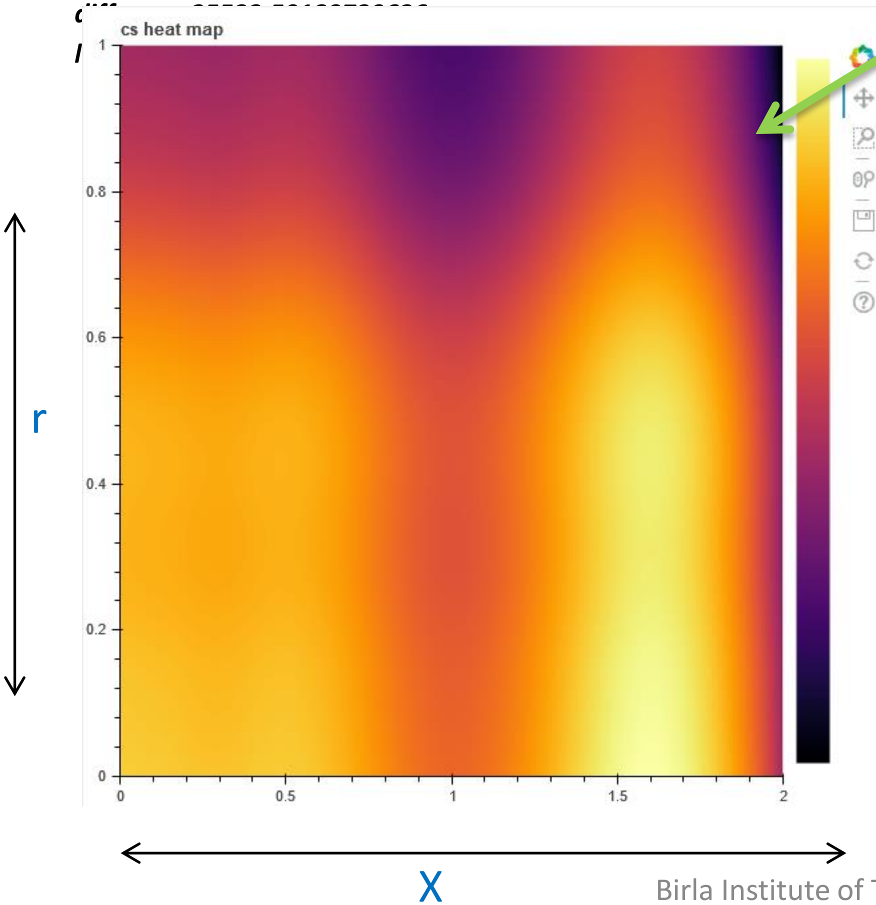
Figure 4:
FEM
Mesh size= 101x101

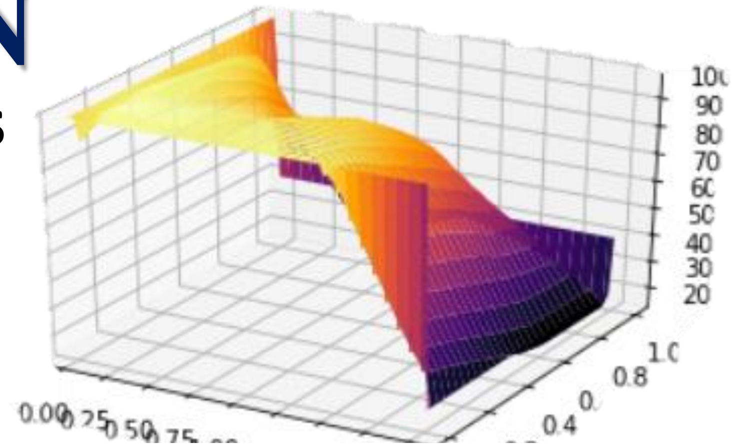


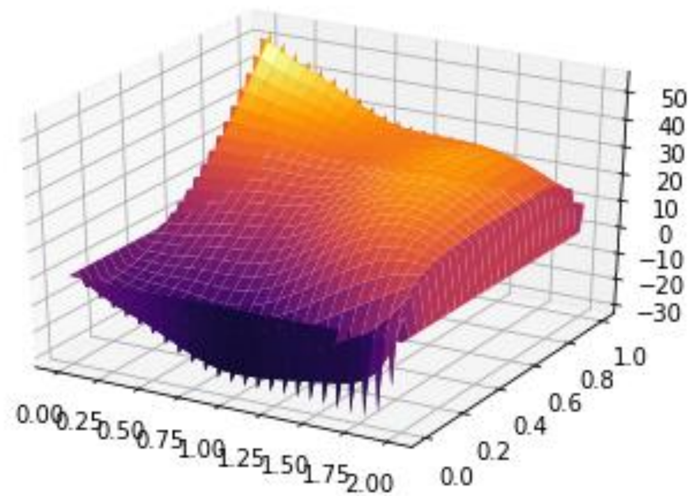


ANN Results

Figure 5:
Grid plot using ANN model:
Network size =[2,16,16,1]
Mesh size= 600x600





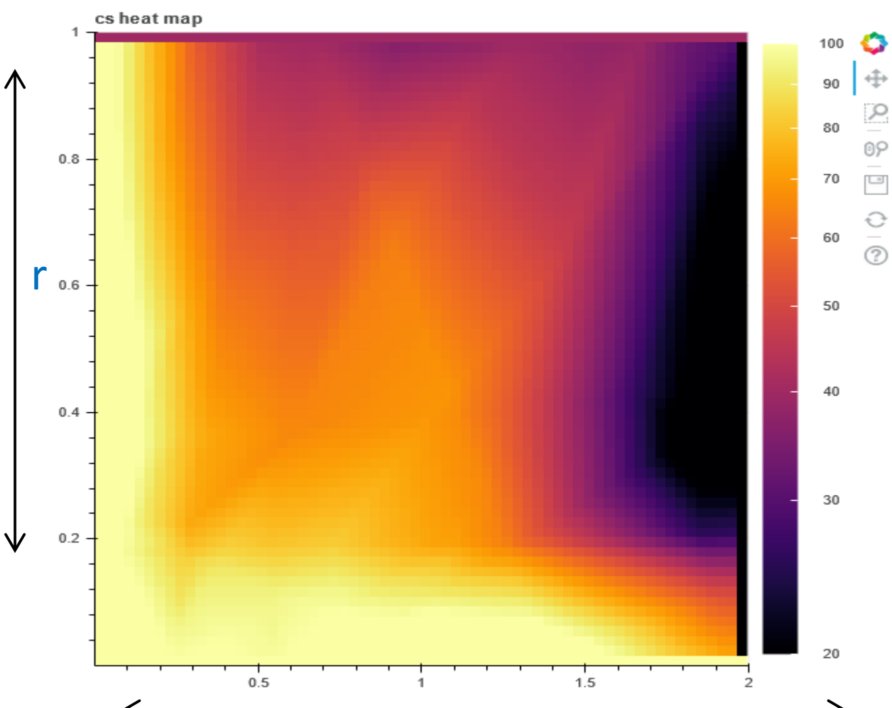


Difference matrix between FEM and ANN[2:16:16:1] :

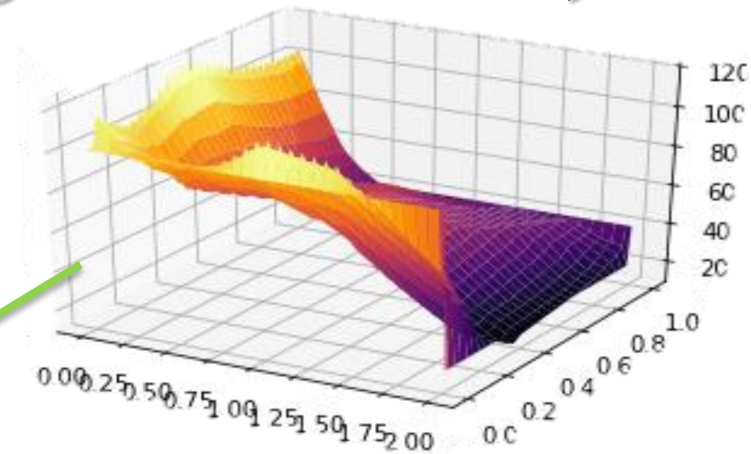
mean difference=19.578

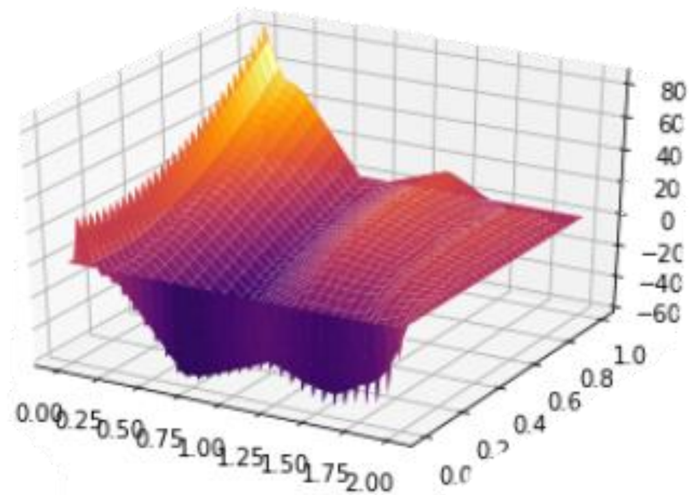
ANN (bigger network)

Figure 6:
Grid plot using ANN model:
Network size = [2,32,32,1]
Mesh size= 64x64
diff. eq.= 3533.50189739626
loss = 310028.65328



X





Difference matrix between FEM and ANN [2:32:32:1]:

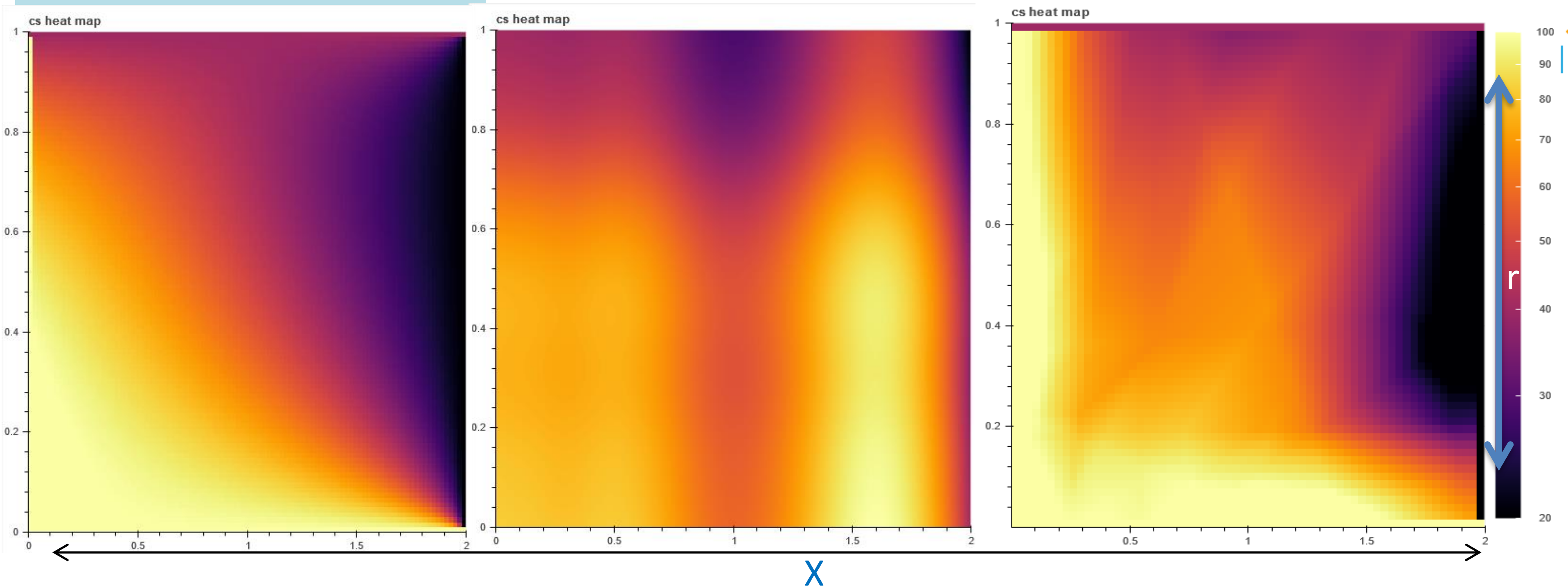
mean difference=12.566

Heat Map

FEM

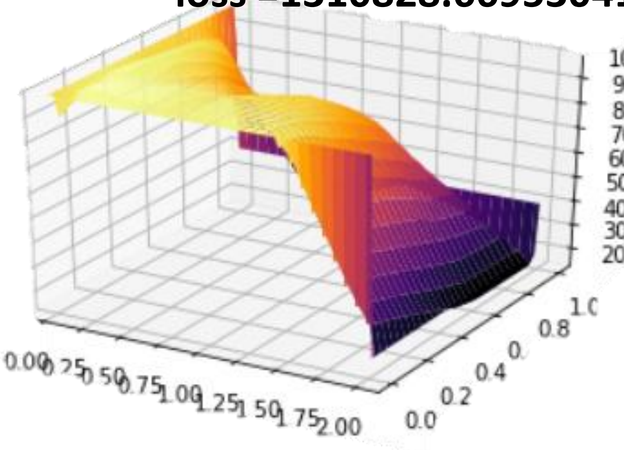
linear model [2:16:16:1]
loss =1310828.6695304194

2D array model[2:32:32:1]
Loss= 47397.776049021086

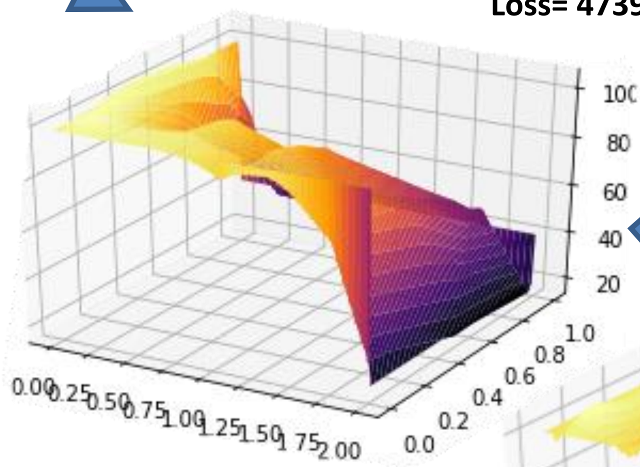


Chronology of model

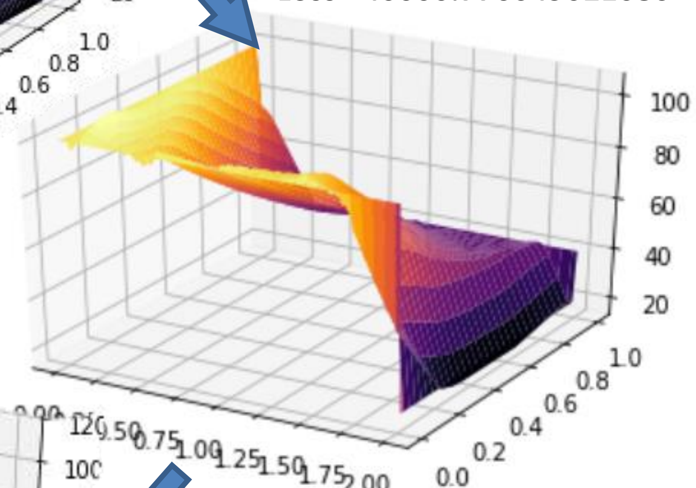
linear model
loss =1310828.6695304194



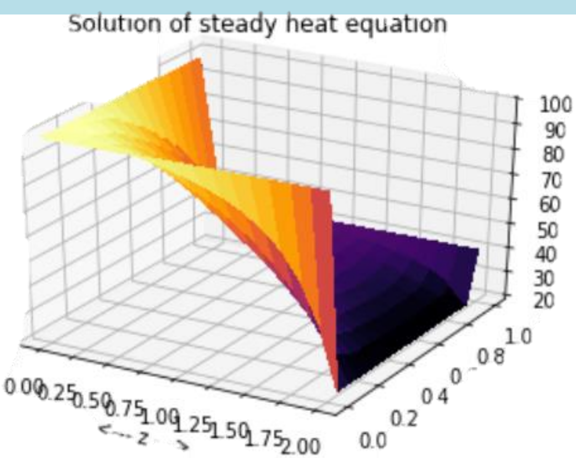
2D array model
Loss= 47397.776049021086

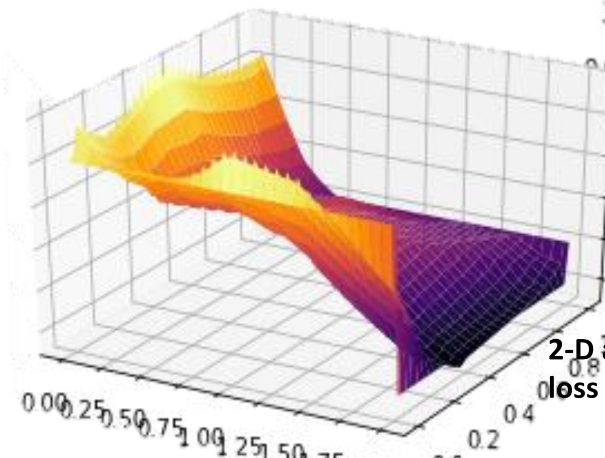


2D array model
Loss= 40000.776049021086



FEM





2-D array after more training
loss = 310028.65328

RESULT ANALYSIS

MATHEMATICAL VALIDITY

A. Based on the Partial Differential equation, at the top left corner the FEM model misses the continuity and appears to be quickly reducing at the source. Figure 7(a) :

B. The ANN overestimates the results at side centres on the insulated and source side Figure 7(b) It also causes an irregular noise at the (2,0)point.

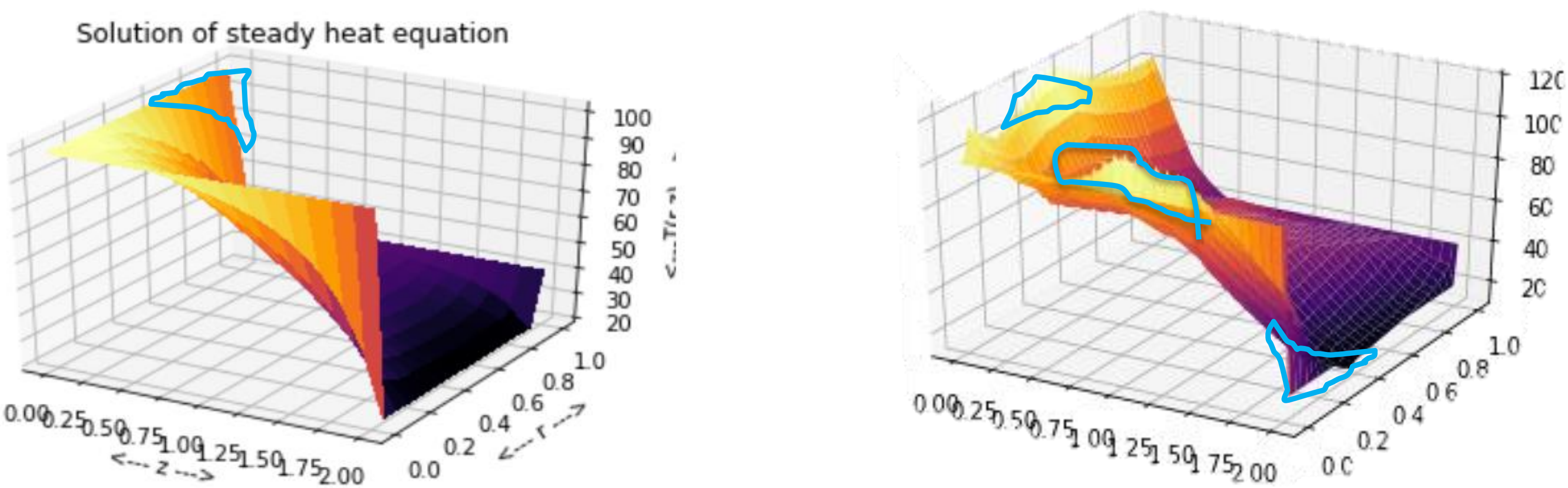


Figure 7:a)FEM inaccuracy b)ANN inaccuracy marked by blue marks.

RESULT ANALYSIS

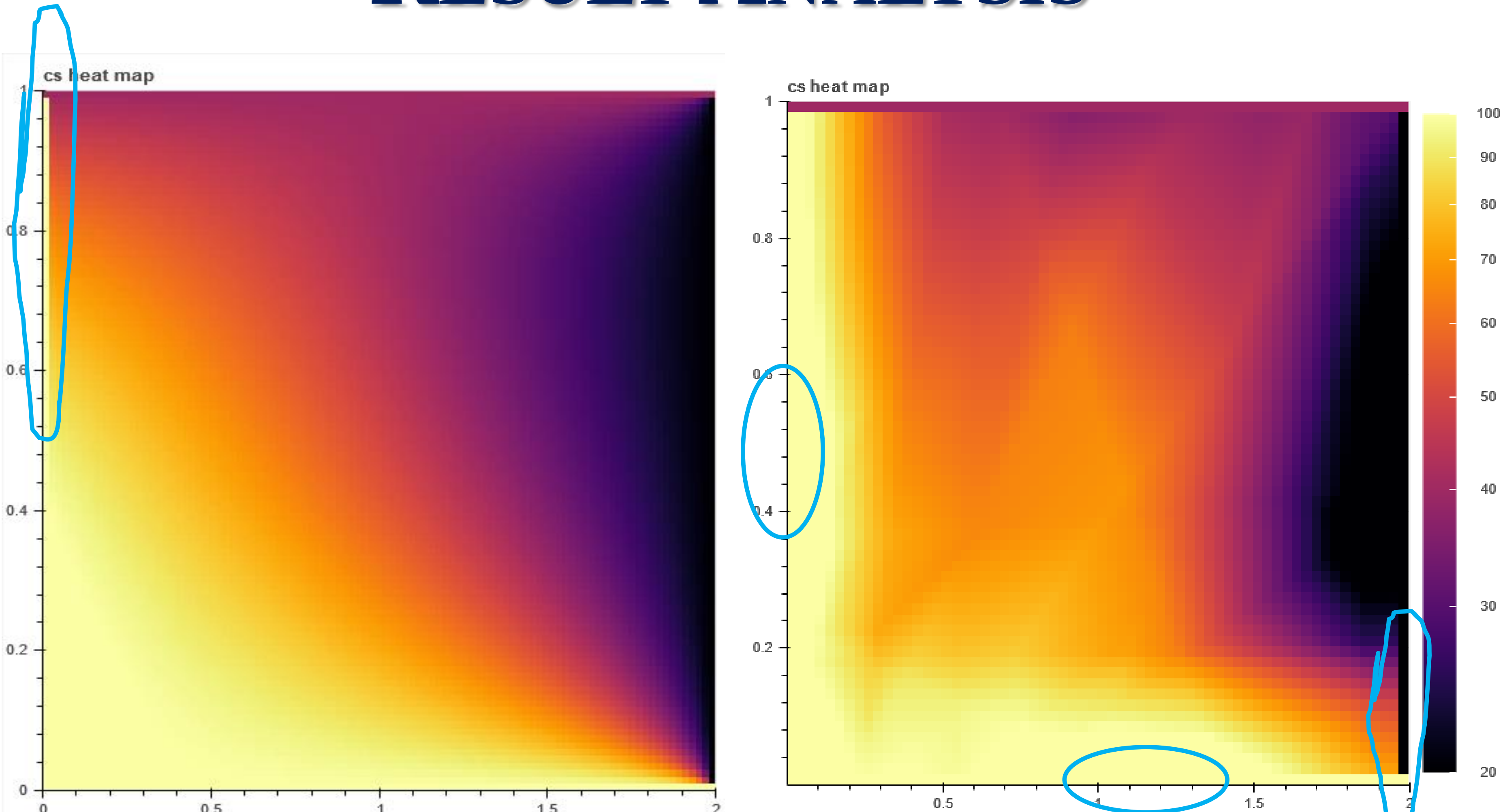


Figure 8:a)FEM inaccuracy b)ANN inaccuracy marked by blue marks.

*Note the grainy nature in ANN occurs because the mesh size is 64*64 compared to 601*601 for FEM.

CONCLUSION

FEM and ANN, both numerical techniques are potential candidates for likely temperature distribution. Analysing error, one can easily visualise the areas of problem.

The application for both methods is suitable in this heat conduction equation solving with different advantages. The mess-free ness of the trained weights allow s several grid dimensions to be able to adapt to the equation and generate point solutions with minimal error. However the initial processing time for the FEM and flexibility in choosing dimensions (symmetrical along each dimension) make it suitable for lower complexity heat generation function. For more complex heat generation function the ANN model becomes cheaper and give less approximate error according to above study. These conclusions validate and support claims that ANN is an accurate and precise alternative to FEM.

The error for moderate training indicate for ANN model strictly follows boundary condition with less approximation error than visible in FEM model. However, local regions of abnormal gradients and non-linearly that are impractical also occur in the central region of the plot that cannot be justified theoretically.

While the errors in FEM model are inherent and can be reduced at the cost of memory and time taken t process in each run. The errors have a scope for improvement in ANN with parameter optimisation experimenting with different model sizes and activation functions. For more complex shapes used in solvers like Ansys the intricate shapes can be application for ANN with lower long run cost.

Appendix

Link to final code :

https://github.com/somyaup/2-D-steady-state

References

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Thank You