Exercise set 7: Image formation

Bonus submission deadline: 5.05.2021 23:59

The exercises marked with **BONUS** should be returned in Moodle by the submission deadline. Successfully solving it will give you bonus points to boost your final grade. During the exercise session, the solution of the exercises will be discussed.

Exercise 1: Lines in the projective plane (BONUS)

1. The equation of a line in the Cartesian plane is

$$Ax + By + C = 0. \tag{1}$$

   - Prove that using homogeneous coordinates this equation can be written as
$$\mathbf{x} \cdot \mathbf{l} = 0, \tag{2}$$
where $\mathbf{l} = \begin{bmatrix} A & B & C \end{bmatrix}^\mathsf{T}$ and $\mathbf{x} \cdot \mathbf{l}$ is the dot product.
This implies that in the projective plane a line can be represented with homogeneous coordinates $\mathbf{l}$.
   - Show that $\mathbf{l}$ is unique up to scale, i.e. that $\mathbf{l}' = \lambda \mathbf{l}$ with $\lambda \neq 0$ defines the same line of $\mathbf{l}$.

2. Show that the intersection of two lines $\mathbf{l}_1$ and $\mathbf{l}_2$ is the point $\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$, where $\times$ denotes the cross product. What happens if the lines are parallel and distinct?
Hint: You may use that the triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is zero if any two of the three vectors are parallel. Note also that the dot product is commutative.

3. Show that the line through two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is the line $\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$.
Hint: same hint as before.

Exercise 2: Dissecting camera matrix

1. Draw the setup for the pinhole camera model and derive its equations. What are the assumption of the pinhole camera model?

2. The previous equations give the the image point coordinates in the same unit of f, generally millimeters. However we would like to express the coordinates in pixels. Suppose the image plane has m_x pixels per

unit length in the x-direction and m_y pixels per unit length in the y direction. Modify the previous model to return the image coordinates in pixels. What does it mean if we have $m_x = m_y$?

3. Suppose that axes x and y are not perpendicular, but have an angle θ between them. How does this modify the previously derived model? The phenomenon when $\theta \neq \frac{\pi}{2}$ is called *skew*. For most modern cameras, $\theta = \frac{\pi}{2}$ and hence we have no skew. Can you think of some situations where we have skew?

4. Write this generalized camera model in matrix form using homogeneous coordinates. How many degrees of freedom do you have so far?

5. Finally, modify the previous model to take into account the situation when the camera center $\mathbf{C}$ is not at the origin and the image plane axes are not aligned with the world axes. Write your final camera model in matrix form. How many degrees of freedom do you have in total?

Exercise 3: Least squares and overfitting

- **Linear LS**: Prove that the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ can be obtained as $\mathbf{x} = (A^\mathsf{T} A)^{-1} A^\mathsf{T}\mathbf{b}$.
  Hint: $\frac{\partial}{\partial \mathbf{x}}\mathbf{x}^\mathsf{T} A\mathbf{x} = 2A\mathbf{x}$ if A is symmetric and $\frac{\partial}{\partial \mathbf{x}}\mathbf{x}^\mathsf{T} B = B$

- **Ridge Regression**: Given the linear system $A\mathbf{x} = \mathbf{b}$, prove that the value of $\mathbf{x}$ minimizing the cost function $\|A\mathbf{x} - \mathbf{b}\|^2 + \lambda\|\mathbf{x}\|^2$ can be obtained as $\mathbf{x} = (A^\mathsf{T} A + \lambda I)^{-1} A^\mathsf{T}\mathbf{b}$

- **Homogeneous Least Squares** Given a $m \times n$ matrix with $m < n$, find a LS square solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$ subject to the constraint $\|\mathbf{x}\| = 1$.

- Implement a function `th = poly_ls(x, y, n)`. Which fits a polynomial of degree n to the data points in the arrays x and y. You can assume the arays contain at least $n+1$ points. The output array should contain the coefficients $th = [a_n \; a_{n-1} \; \ldots \; a_0]^\mathsf{T}$ so that $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$.
  Hint: Vandermonde matrix

- Implement a similar function `th = poly_ridge(x, y, n, lam)`, which takes an extra parameter λ and returns the polynomial coefficients using ridge regression.

- When you are done, run the script `task3.m`. In the script, we are using both linear LS and ridge regression to fit a polynomial equation to given data. Try using different values of λ. Explain what you see.