CHAPTER 1:
BASIC CONCEPTS IN GRAPH THEORY

Discrete Mathematics 2

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Graphs

Definition

A *graph* $G = \langle V, E \rangle$ consists of V, a nonempty set of *vertices* (or nodes) and E, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

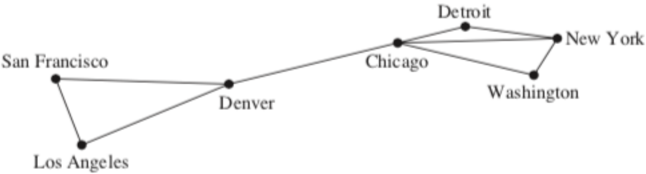

Figure: A Computer Network.

Contents

Some types of Graphs

- Undirected graph
  - Undirected Simple Graph
  - Undirected Multigraph
  - Undirected Pseudograph
- Directed graph
  - Directed Simple Graph
  - Directed Multigraph

Undirected Simple Graph

Undirected simple graph $G = \langle V, E \rangle$:

- $\star$ V is the set of vertices
- $\star$ E is the set of edges, consisting of unordered pairs of two distinct vertices in V
- $\star$ There is at most one edge connecting two vertices.

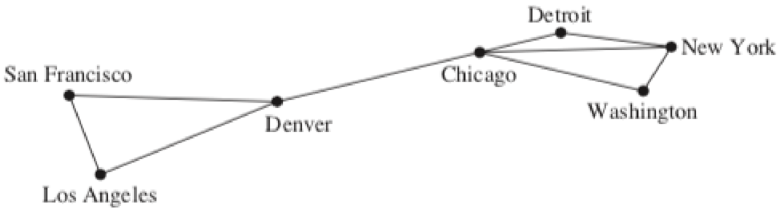

Figure: Undirected Simple Graph.

Undirected Multigraph

Undirected multigraph $G = \langle V, E \rangle$:

- V is the set of vertices
- E is the set of edges, consisting of unordered pairs of two distinct vertices in V
- $e_1, e_2 \in E$ are called *multiple edges* if they connect the same two vertices, $e_1 = (u, v), e_2 = (v, u)$.

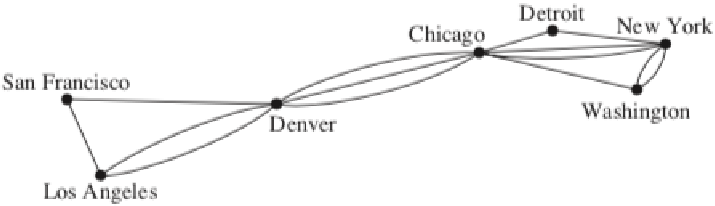

Figure: Undirected multigraph.

Undirected Pseudograph

Undirected Pseudograph $G = \langle V, E \rangle$:

- ⋆ V is the set of vertices
- ⋆ E is the set of edges, consisting of unordered pairs of two vertices (maybe the same) in V
- ⋆ The graph includes edges that connect a vertex to itself. Such edges are called *loops*, and sometimes we may even have more than one loop at a vertex.

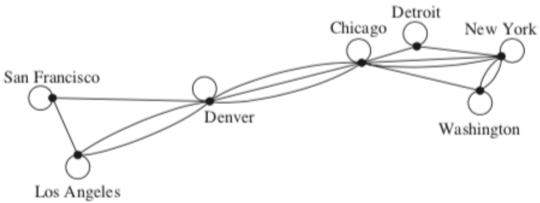

Figure: Undirected Pseudograph.

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Directed Simple Graph

Directed Simple Graph $G = \langle V, E \rangle$:

- ⋆ V is the set of vertices
- ⋆ E is the set of *directed edges (or arcs, arrows)*, consisting of ordered pairs of two distinct vertices in V
- ⋆ There is at most one directed edges (or arcs) from a vertex u to another one v. The directed edge associated with the ordered pair (u, v) is said to *start* at u and *end* at v.

Figure: Directed Simple Graph.

Directed Multigraph

Directed Multigraph $G = \langle V, E \rangle$:

- ⋆ V is the set of vertices
- ⋆ E is the set of *directed edges (or arcs)*, consisting of ordered pairs of two distinct vertices in V
- ⋆ $e_1, e_2 \in E$ are called *multiple directed edges* if they connect the same two vertices, $e_1 = (u, v), e_2 = (v, u)$.

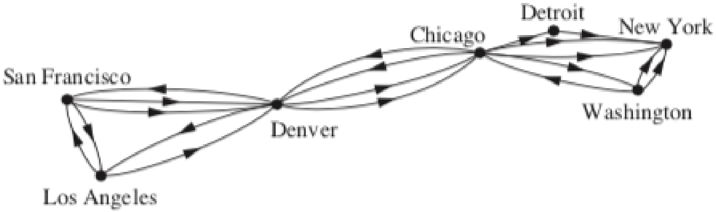


Figure: Directed Simple Graph.

Convention

- ⋆ We will focus on Undirected Simple Graph and Directed Simple Graph
- ⋆ "Undirected Graph" means "Undirected Simple Graph"
- ⋆ "Directed Graph" means "Directed Simple Graph"

Contents

Vertex Degree

Definition

Two vertices u and v in an undirected graph G are called *adjacent (or neighbors)* in G if u and v are endpoints of an edge e of G. Such an edge e is called *incident* with the vertices u and v and e is said to *connect* u and v.

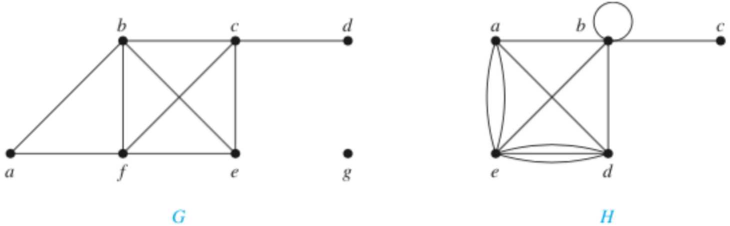
Definition

The *degree of a vertex* in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

Vertex Degree

Example

What are the degrees in the graphs G and H displayed in following figure?



- ⋆ A vertex of degree zero is called *isolated*.
- ⋆ A vertex is *pendant* if and only if it has degree one.

Vertex Degree

Theorem

- *Let* $G = \langle V, E\rangle$ *be an undirected graph with m edges. Then*

$$2m = \sum_{v \in V} deg(v)$$

- *An undirected graph has an even number of vertices of odd degree.*

Example

How many edges are there in a graph with 10 vertices each of degree six?

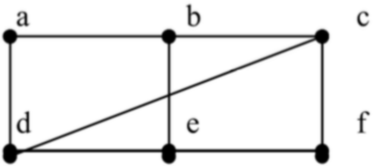
Path and Circuit

- A *path with length n* from vertex u to vertex v in undirected graph $G = \langle V, E >$ is sequence $x_0, x_1, \ldots, x_{n-1}, x_n$ in which n is a positive integer, $x_0 = u, x_n = v, (x_i, x_{i+1}) \in E$, $i = 0, 1, 2, \ldots, n-1$.
- The above path can be represented as a sequence of edges $(x_0, x_1)\,(x_1, x_2), \ldots, (x_{n-1}, x_n)$.
- u is the starting point and v is the ending point of the path
- A *circuit* is a path ending at the starting point $(u = v)$
- A path or a circuit is said to be *simple* if there is no repetition of edges
- A *cycle* is a *simple circuit* with no repeated vertices other than the first and last ones

Path and Circuit

Example

- a, d, c, f, e is a simple path with length 4
- d, e, c, b is not a path because (e, c) is not an edge
- b, c, f, e, b is a circuit with length 4
- Path with length 5 : a, b, e, d, a, b is not simple because (a, b) appears twice



Connected Graph

Definition

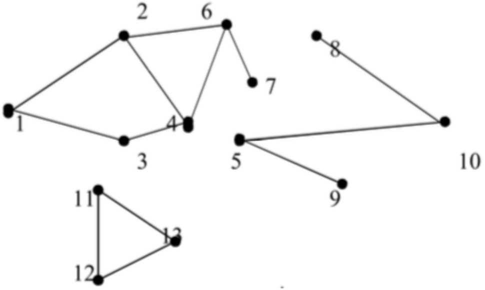
- ⋆ An undirected graph is said to be *connected* if there is a path between every pair of vertices
- ⋆ If is not connected, G consists of several *connected subgraphs* (two subgraphs do not share any vertex)
    - Each such subgraph is called a *connected component* of G.
    - An undirected graph is *connected* if and only if it has only one connected component

Theorem

In an undirected graph, if there exist a vertex $v \in V$ such that there is a path from v to all the other vertices of V, the graph is connected.

Example

How many connected components are there in graph G?



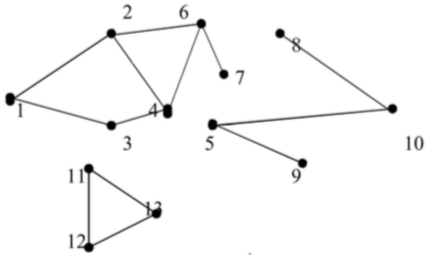
Bridge and Cut Vertex

Definition

In an undirected graph, a *bridge* is an edge of the graph whose deletion increases its number of connected components. A *cut vertex* is a vertex whose deletion (with its boundary edges) increases its number of connected components.

Example

Find the bridges and cut vertices in the graph below



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In-degree and Out-degree

Definition

When (u, v) is an edge of the graph G with directed edges, u is said to be *adjacent to v* and v is said to be *adjacent from u*. The vertex u is called the *initial vertex of (u, v)*, and v is called the *terminal or end vertex of (u, v)*.

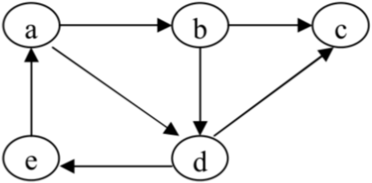
Definition

In a graph with directed edges the *in-degree* of a vertex v, denoted by $\deg^- v$, is the number of edges with v as their terminal vertex. The *out-degree* of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex.

In-degree and Out-degree

Example

Find the in-degree and out-degree of each vertex in the following graph with directed edges.



In-degree and Out-degree

Theorem

For any directed graph $G = \langle V, E \rangle$, we have

$$\sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v) = |E|.$$

Notation

- ⋆ *Many properties of directed graphs do not depend on directions. In some cases, we can ignore the directions on directed edges.*
- ⋆ *The undirected graph receiving by removing directions on directed edges is called the corresponding undirected graph.*

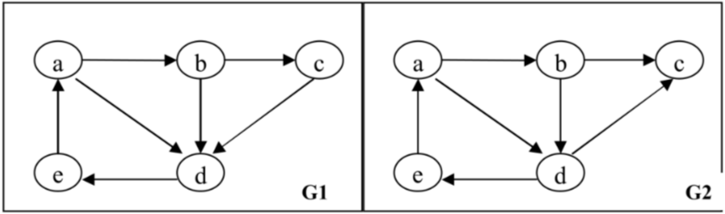
Path and Circuit

- ⋆ A *path with length n* from vertex u to vertex v in directed graph $G = \langle V, E \rangle$. is sequence $x_0, x_1, \ldots, x_{n-1}, x_n$ in which n is a positive integer, $x_0 = u, x_n = v, (x_i, x_{i+1}) \in E$, $i = 0, 1, 2, \ldots, n-1$.
- ⋆ The above path can be represented as a sequence of edges $(x_0, x_1)\,(x_1, x_2), \ldots, (x_{n-1}, x_n)$.
- ⋆ u is the starting point and v is the ending point of the path
- ⋆ A *circuit* is a path ending at the starting point $(u = v)$
- ⋆ A path or a circuit is said to be *simple* if there is no repetition of directed edges.

Strongly Connected Graph, Weakly Connected Graph

Definition

- ⋆ Directed graph $G = \langle V, E \rangle$ is said to be *strongly connected* if there is a path between every pair of vertices.
- ⋆ Directed graph $G = \langle V, E \rangle$ is said to be *weakly connected* if its corresponding undirected graph is connected.



Orientation

Definition

An *orientation* of an undirected graph is an assignment of a direction to each edge, turning the initial graph into a directed graph. A *strong orientation* is an orientation that results in a strongly connected graph.

Theorem

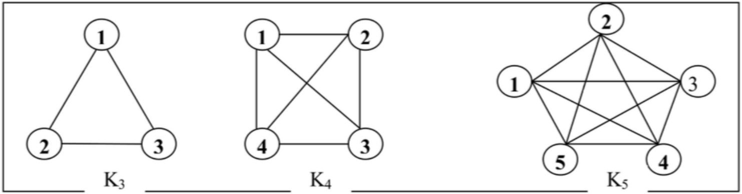
For any undirected graph $G = \langle V, E \rangle$, there exists a strong orientation on G if and only if all its edges are not bridge.

Contents

Complete Graph

Definition

- *Complete* graph n vertices, denoted by K_n, is a simple undirected graph that exists an edge connecting between two every vertices
- Number of edges: $\frac{n(n-1)}{2}$

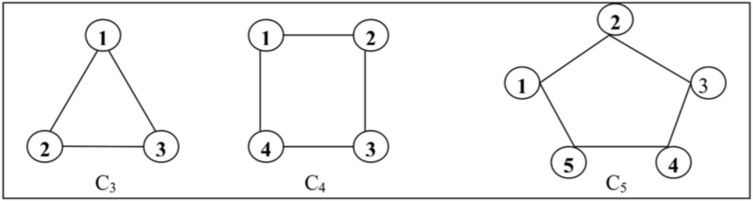


Cycle Graph

Definition

Cycle Graph C_n, $n \geq 3$, consists of n vertices

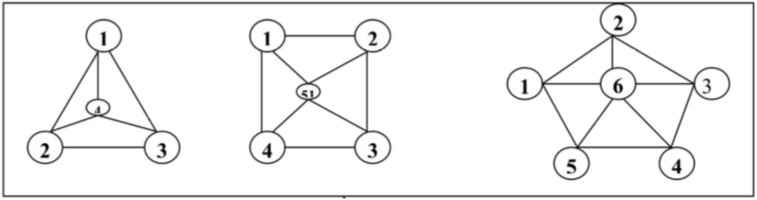
$$\{v_2, v_3\}, ..., \{v_{n-1}, v_n\}, \text{ and } \{v_n, v_1\}.$$



Wheel Graph

Definition

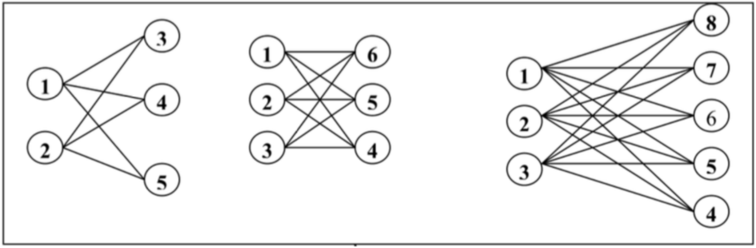
Wheel graph n vertices, denoted by W_n is a graph formed by connecting a single vertex to all vertices of a cycle graph C_{n-1}.



Bipartite Graph (Bigraph)

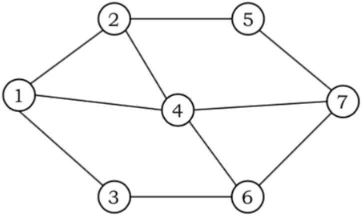
Definition

Bigraph $G = \langle V, E \rangle$ is a graph whose vertices can be divided into two disjoint sets X and Y and such that every edge connects a vertex in to one in, i.e. (x, y), in which $x \in X, y \in Y$



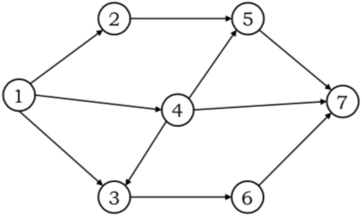
Exercises

Exercise 1. Determine the degree of each vertex in the below undirected graph



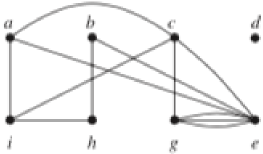
Exercises

Exercise 2. Determine the in-degree and out-degree of each vertex in the below directed graph



Exercises

Exercise 3. Determine the degree of each vertex in the below undirected graph



Exercises

Exercise 4. Determine the in-degree and out-degree of each vertex in the below directed graph