POSTS AND TELECOMMUNICATIONS INSTITUTE OF TECHNOLOGY

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Chapter 1: MULTIVARIABLE CALCULUS

CALCULUS 2

Faculty of Fundamental Science 1

Hanoi - 2022

1.1.1. Set $\mathbb{R}^n$, distance, neighborhood, open set, closed set, bounded set

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2. Given $M(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,\ N(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. The distance between a point M and N is denoted by $d(M, N)$, calculating by the formula

$$d(M, N) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

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3. Given $M(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\varepsilon > 0$. The set $\{M \in \mathbb{R}^n : d(M, M_0) < \varepsilon\}$ is called the ε- neighborhood of the point M_0.

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Example 1

Given $M_0 \in \mathbb{R}^n$, $\varepsilon > 0$. Set $\{M \in \mathbb{R}^n : \ d(M, M_0) < \varepsilon\}$ is an open set (called an open sphere with center M_0, radius ε).
Set $\{M \in \mathbb{R}^n : \ d(M, M_0) \leq \varepsilon\}$ is a closed set (called a closed sphere M_0, radius ε.

Definition 1

Given $\emptyset \neq D \subset \mathbb{R}^n$, mapping $f : D \to \mathbb{R}$ such that $M(x_1, x_2, \cdots, x_n) \in D \mapsto z = f(M) = f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$ is called a function of n variables defined on D.
Where, the D is called the **domain** of the function f and $x_1, x_2, \cdots, x_n$ are called independent variables.

1.1.2. Functions of several variables

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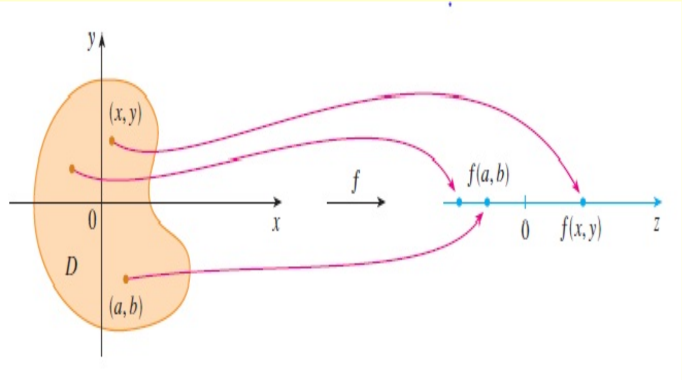
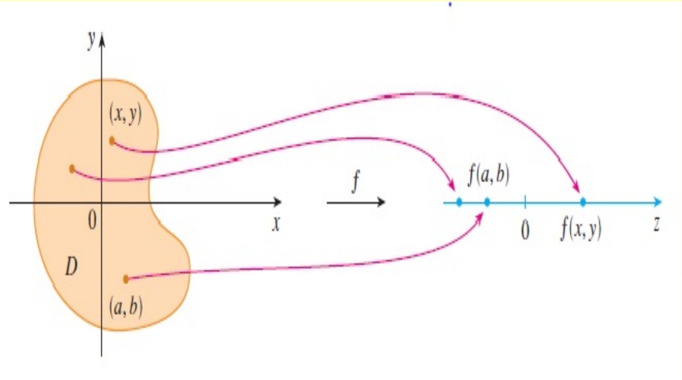
Given $\emptyset \neq D \subset \mathbb{R}^n$, mapping $f : D \to \mathbb{R}$ such that $M(x_1, x_2, \cdots, x_n) \in D \mapsto z = f(M) = f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$ is called a function of n variables defined on D.
Where, the D is called the **domain** of the function f and $x_1, x_2, \cdots, x_n$ are called independent variables.
Note: A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set $D \subset \mathbb{R}^2$ a unique real number denoted by $z = f(x, y)$. The set D is the **domain** of and its **range** is the set of values f that takes on, that is, $\{f(x, y) | (x, y) \in D\}$.



Hình 1.1: *Graph of the function* $z = f(x, y)$



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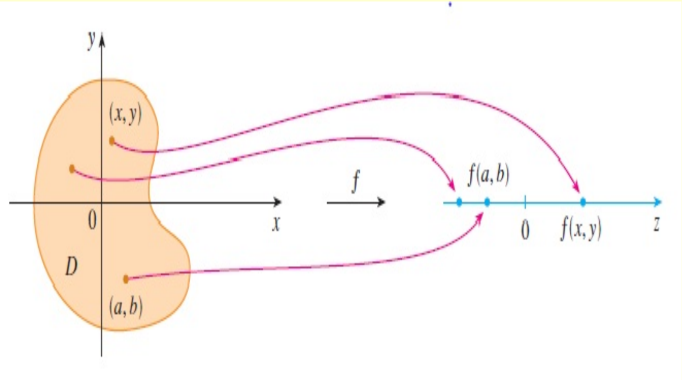
Example 2

Find the domains of the following functions and calculate $f(3, 2)$.

a) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$; b) $f(x, y) = x \ln(y^2 - x)$



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Behavior of a function of two variables

If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$.

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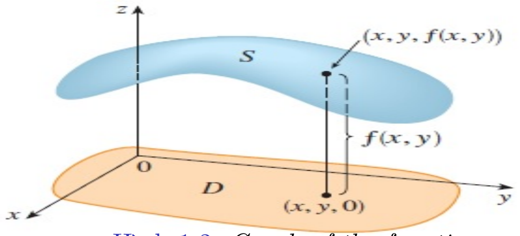

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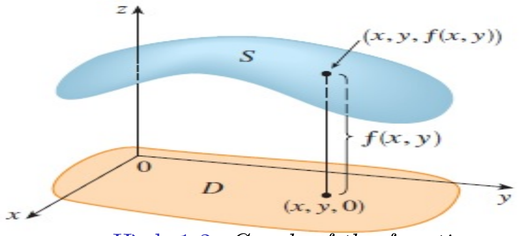

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Example 3

Find the domain and range and sketch the graph of the following functions.

a) $f(x, y) = \sqrt{1 - x^2 - y^2}$; b) $f(x, y) = x^2 + y^2$

1.2.1. Limits

Definition 1

The sequence of points $M_n(x_n, y_n)$ is said to approach the point $M_0(x_0, y_0)$, $n \to \infty$ if $\lim\limits_{n\to\infty} d(M_n, M_0) = 0$ or $\begin{cases} \lim\limits_{n\to\infty} x_n = x_0 \\ \lim\limits_{n\to\infty} y_n = y_0, \end{cases}$

denoted by $\lim\limits_{n\to\infty} M_n = M_0$ or $M_n \to M_0$ when $n \to \infty$.

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Definition 2

Given the function $f(M) = f(x, y)$ defined on the domain D. The point $M_0(x_0, y_0)$ may or may not be in D. The function $f(M)$ has a limit $L \in \mathbb{R},\ M \to M_0$ if

$$\forall \varepsilon > 0,\ \exists \delta > 0 : (\forall M \in D),\ 0 < d(M, M_0) < \delta \Rightarrow |f(M) - L| < \varepsilon,$$

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Remark

If any small interval $(L-\varepsilon, L+\varepsilon)$ is given around L, we can find a disk D_δ with center (a,b) and radius $\delta > 0$ such that f maps all the points in D_δ (except possibly (a,b)) into the interval $(L-\varepsilon, L+\varepsilon)$.

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Find the following limits

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2+y^2}$; b) $\lim\limits_{(x,y)\to(1,2)} (x^3 - 2xy)$

c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$; d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$.

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Example 4

Consider the continuity of the following function:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

1.2.2 Continuity

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Definition 1.3.1

Given the function $z = f(x, y)$ defined in D and $M_0(x_0, y_0) \in D$.

1) Suppose we let only x vary while keeping $y = y_0$, say $y = y_0$, where y_0 is a constant. Then we are really considering a function of a single variable x, namely, $f(x, y_0)$. If $f(x, y_0)$ has a derivative at x_0 then we call it the partial derivative of f with respect to variable x at M_0 and denote it by $z'_x(M_0) = f'_x(M_0)$ or $\dfrac{\partial z}{\partial x}(M_0) = \dfrac{\partial f}{\partial x}(M_0)$, is defined by

$$\frac{\partial f}{\partial x}(M_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

2) Similarly, the partial derivative of with respect to the variable y at M_0, denoted by $z'_y(M_0) = f'_y(M_0)$ or $\dfrac{\partial z}{\partial y}(M_0) = \dfrac{\partial f}{\partial y}(M_0)$, is defined by $\dfrac{\partial f}{\partial y}(M_0) = \lim\limits_{\Delta y \to 0} \dfrac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$.

Example 1

For $z = f(x, y) = 2x^3 - 5x^2y + 3y^2 + 1$, find

$$\frac{\partial z}{\partial x}(x, y);\ \frac{\partial z}{\partial y}(x, y);\ \frac{\partial z}{\partial x}(2, 1);\ \frac{\partial z}{\partial y}(-1, 3).$$

Example 2

The productivity of an airplane-manufacturing company is given approximately by the Cobb–Douglas production function $f(x, y) = 40x^{0,3}y^{0,7}$, with the utilization of x units of labor and y units of capital.

a) Find $f'_x(x, y)$ and $f'_y(x, y)$.

b) If the company is currently using 1,500 units of labor and 4,500 units of capital, find the marginal productivity of labor and the marginal productivity of capital.

1.3.2 The chain rule

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

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The chain rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

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Example 3

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\dfrac{dz}{dt}(0)$.

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Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

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$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Example 4

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\dfrac{\partial z}{\partial s}$, $\dfrac{\partial z}{\partial t}$.

The chain rule (General version)

Suppose that u is a differentiable function of the n variables $x_1, x_2, ..., x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, ..., t_m$. Then u is a function of $t_1, t_2, ..., t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i},$$

for each $i = 1, 2, ..., m$.

1.3.2 The chain rule

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Example 5

a) If $u = x^4y + y^2z^3$, where $x = rse^t$ and $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find $\dfrac{\partial u}{\partial s}$ when $r = 2, s = 1, t = 0$.

b) If $u = f(x^2 - y^2, y^2 - x^2)$ and f is differentiable, show that u satisfies the equation $y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0$.

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Theorem 1

We suppose that an equation of the form $F(x, y) = 0$ defines implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x))$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the chain rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x. Since both x and y are functions of x, we obtain $\dfrac{\partial F}{\partial x}\dfrac{dx}{dx} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dx} = 0$. But $\dfrac{dx}{dx} = 1$, so if $\dfrac{\partial F}{\partial y} \neq 0$ we solve for $\dfrac{dy}{dx}$ and obtain

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Find y' if $x^5-y^5=5xy$.

1.3.3 Implicit differentiation

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Find y' if $x^5 - y^5 = 5xy$.

Theorem 2

we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the chain rule to differentiate the equation as follows $\dfrac{\partial F}{\partial x}\dfrac{\partial x}{\partial x} + \dfrac{\partial F}{\partial y}\dfrac{\partial y}{\partial x} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial x} = 0$. If $\dfrac{\partial F}{\partial x} \neq 0$, we solve for $\dfrac{\partial z}{\partial x}$ and obtain $\dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\dfrac{F'_x}{F'_z}$. Similarly, we obtain $\dfrac{\partial z}{\partial y} = -\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\dfrac{F'_y}{F'_z}$.

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Example 7

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^4 + y^3 + z^5 - 6xyz = 2$.

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If $z = f(x, y)$, then f is differentiable at $M_0(x_0, y_0)$ if $\Delta f(M_0)$ can be expressed in the form

$$\Delta f(M_0) = f'_x(M_0)\Delta x + f'_y(M_0)\Delta y + \varepsilon_1 \Delta x + +\varepsilon_2 \Delta y,$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

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Definition 1.3.5

Assume the function f is differentiable at $M_0(x_0, y_0)$, the expression $f'_x(M_0)\Delta x + f'_y(M_0)\Delta y$ is called the total differential of f at $M_0(x_0, y_0)$, denoted by $df(M_0) = df(x_0, y_0)$. Thus,
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Note: If we consider the functions $f(x, y) = x$ and $g(x, y) = y$ in $\mathbb{R}^2$, then $df(x, y) = dx = \Delta x$, $dg(x, y) = dy = \Delta y$. So the total differential of the function $f(x, y)$ at M_0 is also written as:

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Given the function $f(x, y) = x^4 + 4xy^2 - 3x2y^3$. Calculate
a) $df(x, y)$; b) $df(1, 2)$ with $\Delta x = 0,1$; $\Delta x = 0,2$.

1.3.4 Total differentials

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Definition 1.4.1

If $f(x, y)$ is a function of two variables, then its partial derivatives f'_x, f'_y are also function of two variables, so we can consider their partial derivatives $(f'_x)'_x$, $(f'_x)'_y$, $(f'_y)'_x$, $(f'_y)'_y$, which are called second order partial derivatives of $f(x, y)$. If $z = f(x, y)$, we use the following notation:

$$f''_{xx} = f''_{x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad f''_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y},$$

$$f''_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x}, \quad f''_{yy} = f''_{y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}.$$

Theorem 1.4.1 (Schwarz's theorem)

Suppose $f(x, y)$ is defined on a disk D that contains the point $M_0(x_0, y_0)$. If the functions $f"_{xy}$ and $f"_{yx}$ are both continuous on D, then $f''_{xy}(M_0) = f''_{yx}(M_0)$. Partial derivatives of order 3 or higher can also be defined. For instance $f'''_{xyy} = (f''_{xy})'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$.

Example 2

Given the function $f(x, y) = x^3 y - 3x^2 y^2 + 5y^4$.

a) Calculate first and second partial derivatives of $f(x, y)$.

b) Calculate $f^{(3)}_{x^2 y}$, $f^{(3)}_{xyx}$, $f^{(4)}_{x^2 y^2}$.

1.4.2 Higher differentials

Definition 1.4.2

If $f(x,y)$ is a differentiable at (x,y) then $df = f'_x dx + f'_y dy$ is called a first-order differential of $f(x,y)$.
If $df(x,y)$ is differentiable at (x,y) then the total differential of $df(x,y)$ is called the second-order differential of $f(x,y)$, denoted by $d^2f(x,y)$ defined by

$$d^2f(x,y) = d(df(x,y)) = d(f'_x dx + f'_y dy).$$

Similarly, we have $d^n f(x,y) = d(d^{n-1}f(x,y))$, $n \in \mathbb{N}^*$. The second order differential formula is as follows:

$$d^2 f = d(df) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)dy$$

$$= \frac{\partial^2 f}{\partial x^2}dx^2 + \left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial x}\right)dxdy + \frac{\partial^2 f}{\partial y^2}dy^2$$

.

Definition 1.4.2

According to Schwarz theorem we have:

$$d^2f(x,y) = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2.$$

Example 3

Given the function $f(x,y) = x^2y^3 + 2xy^2 - xsy^4$. Calculate
a) $d^2f(x,y)$.
b)$d^2f^(2,1)$.

Definition 1.5.1

The directional derivative of f at $M_0(x_0, y_0, z_0)$ in the direction of a unit vector $\vec{l} = (a, b, c)$ is

$$\frac{\partial f}{\partial \vec{l}}(M_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + ah, z_0 + ah) - f((x_0, y_0, z_0))}{h},$$

if this limit exists.

Theorem 1.5.1

If f is a differentiable function of x, y and z, then f has a directional derivative in the direction of any unit vector $\vec{l} = (a, b, c)$ and

$$\frac{\partial f}{\partial \vec{l}}(x, y, z) = f'_x(x, y, z)a + f'_y(x, y, z)b + f'_z(x, y, z)c.$$

Example 1

Find the directional derivative $f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\vec{l}$ is the unit vector given by the angle $\frac{\pi}{6}$. What is $\dfrac{\partial f}{\partial \vec{l}}(1,2)$?

1.5.2 Gradient vector

Definition 1.5.2 If f is a function of three variables x, y and z, then the gradient of f is the vector function $\overrightarrow{grad} f$ defined by

$$\overrightarrow{grad} f(x,y,z) = (f'_x, f'_y, f'_z) = f'_x\vec{i} + f'_y\vec{j} + f'_z\vec{k}.$$

Remark

With this notation for the gradient vector, we have

$$\frac{\partial f}{\partial \vec{l}}(x,y,z) = \overrightarrow{grad} f(x,y,z).\vec{l}$$

Definition 1.6.1

A function $f(M) = f(x, y)$ has a local maximum (local manimum) at $M_0(x_0, y_0) \in D$ if $f(M) \leq f(M_0)$ $(f(M) \geq f(M_0))$ when M is near M_0 (i.e $f(M) \leq f(M_0)(f(M) \geq f(M_0))$ for all points M in some disk with center M_0). The number $f(M_0)$ is called a **local maximum value (local manimum value)**. The maximum or minimum point are called the **critical point (or stationary point)**.

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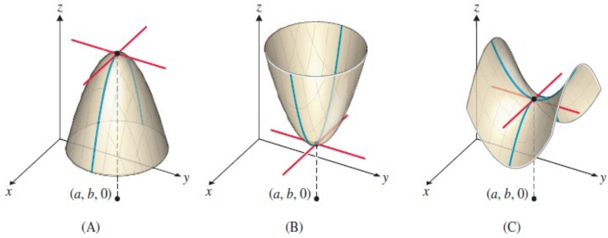
Note: If the inequalities in Definition 1.6.1 hold for all points $M(x, y)$ in the domain of $f(M)$, then $f(M)$ has an **absolute maximum (or absolute minimum)** at M_0.

Example 1.6.1



Hình 6.1: *Graph of the function $z = f(x, y)$ has extreme point*

Theorem 1.6.1 (Necessary condition)

If $f(x, y)$ has a local maximum or minimum at M_0 and the first order partial derivatives of $f(x, y)$ exist there, then

$$f'_x(M_0) = 0 \text{ and } f'_y(M_0) = 0.$$

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Note

Theorem 1.6.1 gives us necessary (but not sufficient) conditions for $f(M_0)$ to be a local extremum. We find all points M_0 such that $f'_x(M_0) = 0$ and $f'_y(M_0) = 0$ and test these further to determine whether $f(M_0)$ is a local extremum or a saddle point. Points M_0 such that conditions $f'_x(M_0) = 0$ and $f'_y(M_0) = 0$, or if one of these partial derivatives does not exist are called **critical points.**

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Theorem 1.6.2 (Second-Derivative Test for Local Extrema (Sufficient condition)

Suppose the second partial derivatives of $f(M) = f(x, y)$ are con tinuous on a disk with center M_0, and suppose that the point M_0 is a critical point of $f(M)$. Put
$A = f"_{xx}(M_0),\ B = f"_{xy}(M_0),\ C = f"_{yy}(M_0)$, and $\Delta = B^2 - AC$.

1. if If $\Delta < 0$ and $A < 0$, then $f(M_0)$ is a local maximum.
2. if If $\Delta < 0$ and $A > 0$, then $f(M_0)$ is a local minimum.
3. if If $\Delta > 0$ then $f(x, y)$ has a saddle point at M_0 (i.e then $f(M_0)$ is not local maximum or minimum).

Remark

1. In case (3) the point $M_0(x_0, y_0)$ is called a **saddle point** of f.
2. If $\Delta = 0$, the test gives no information: f could have a local maximum or local minimum at $M_0(x_0, y_0)$, or M_0 could be a saddle point of f.
3. To remember the formula for Δ, it's helpful to write it as a determinant $\Delta = f"_{xx} f"_{yy} - (f"_{xy})^2$.

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Example 1

Find the local maximum and minimum values of the following functions:
a) $f(x, y) = -x^2 - y^2 + 6x + 8y - 21$.
b) $f(x, y) = x^3 + y^3 - 3xy + 2$.
c) $f(x, y) = x^4 + y^4 - 4xy + 1$; d) $f(x, y) = x^3 + y^2$.

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Definition

The point $M_0(x_0, y_0) \in \mathbb{R}^2$ is called the maximum (minmum) point of the function $f(x, y)$ subject to a constraint (or or side condition) $\varphi(x, y) = 0$ if satisfying $\varphi(M_0) = 0$ also exists a small enough neighborhood of M_0 on the constraint curve $\varphi(x, y) = 0$, there is inequality $f(M) < f(M_0)(f(M) > f(M_0)$.

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Maxima using Lagrange multipliers

This problem is one of a general class of problems of the form:
Maximize or minimize $f(x, y) \quad (a)$
subject to $\varphi(x, y) = 0 \quad (b)$
Now to the method: We form a new function F, using functions f and φ in equations (a) and (b), as follows:
$F(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y), \quad (c)$
Here, λ (the Greek lowercase letter lambda) is called a **Lagrange multiplier**.

1.7 Conditional extremes

Definition

The point $M_0(x_0, y_0) \in \mathbb{R}^2$ is called the maximum (minmum) point of the function $f(x, y)$ subject to a constraint (or or side condition) $\varphi(x, y) = 0$ if satisfying $\varphi(M_0) = 0$ also exists a small enough neighborhood of M_0 on the constraint curve $\varphi(x, y) = 0$, there is inequality $f(M) < f(M_0)(f(M) > f(M_0)$.

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$F(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y), \quad (c)$
Here, λ (the Greek lowercase letter lambda) is called a **Lagrange multiplier**.

Theorem (Method of Lagrange multipliers for functions of two variables)

Any local maxima or minima of the function $z = f(x, y)$ subject to the constraint $\varphi(x, y) = 0$ will be among those points (x_0, y_0) for which (x_0, y_0, λ_0) is a solution of the system $F'_x(x, y, \lambda) = 0$, $F'_y(x, y, \lambda) = 0$, and $F'_\lambda(x, y, \lambda) = 0 \quad (d)$, where $F(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)$ provided that all the partial derivatives exist.

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Example 1

Find maximize

a) $f(x, y) = xy$ subject to $3x + y - 720 = 0$.

b) $f(x, y) = 6 - 4x - 3y$ subject to $x^2 + y^2 - 1 = 0$.

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Example 2

Find maximize $f(x, y, z) = x + y + z^2$ subject to $z - x = 1$ and $y - xz = 1$.

1.7 Conditional extremes

Theorem (Method of Lagrange multipliers for functions of two variables)

Any local maxima or minima of the function $z = f(x, y)$ subject to the constraint $\varphi(x, y) = 0$ will be among those points (x_0, y_0) for which (x_0, y_0, λ_0) is a solution of the system $F'_x(x, y, \lambda) = 0$, $F'_y(x, y, \lambda) = 0$, and $F'_\lambda(x, y, \lambda) = 0 \quad (d)$, where $F(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)$ provided that all the partial derivatives exist.

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Find maximize

a) $f(x, y) = xy$ subject to $3x + y - 720 = 0$.

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Example 2

Find maximize $f(x, y, z) = x + y + z^2$ subject to $z - x = 1$ and $y - xz = 1$.

1.8 Maximum and minimum values

For a function f of one variable the extreme value theorem says that if f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value. There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, a closed set in $\mathbb{R}^2$ is one that contains all its boundary points (A boundary point of D is a point such that every disk with center contains points in D and also points not in D).

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Definition 1.8.1

If $f(M)$ is continuous on a closed, bounded set D in$\mathbb{R}^n (n \in N^*)$, then $f(M)$ attains an absolute maximum value $f(M_1)$ and an absolute minimum value $f(M_2)$ at some points M_1 and M_2 in$D$.

1.8 Absolute maximum and minimum values

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Step 1. Find the values of f at the critical points of f in D.
2. Step 2. Find the extreme values of f on the boundary of D.
3. Step 3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example

Find the absolute maximum and minimum values of the following functions:

a) $f(x,y) = x^2 - 2xy + 2y$, where $D = \{(x,y)|0 \le x \le 3,\ 0 \le y \le 2\}$.

b) $f(x,y) = x^2 + y^2 - 2x^2y$, where $D = \{(x,y)|x^2 + y^2 \le 1\}$.