Chapter 4: Euler and Hamilton Graphs

Discrete Mathematics 2

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Euler Graph

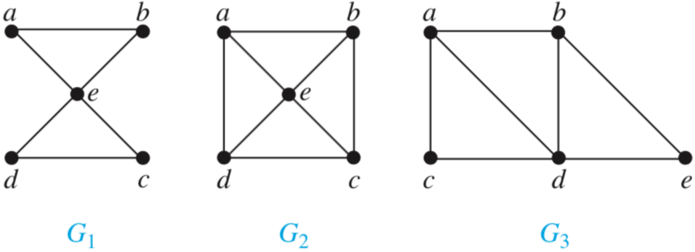
Definition

- A simple circuit in graph G is said to be Euler circuit if it passes through all the edges of the graph
- A simple path in graph G is said to be Euler path if it passes through all the edges of the graph
- A graph is said to be Euler graph if it contains an Euler circuit
- A graph is said to be Semi-Euler graph if it contains an Euler path

Adjacency Matrix of Undirected Graph

Example

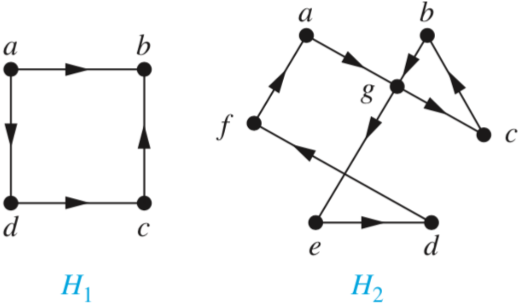
Which of the undirected graphs has an Euler circuit? Of those that do not, which have an Euler path?



Adjacency Matrix of Undirected Graph

Example

Which of the directed graphs has an Euler circuit? Of those that do not, which have an Euler path?



Necessary and Sufficient Conditions for Euler Graph

- Undirected graphs: Connected undirected graph $G = \langle V, E \rangle$ is Euler graph if and only if every vertex of G has even degree.
- Directed graphs: Weakly-connected directed graph $G = \langle V, E \rangle$ is Euler graph if and only if in-degree of each vertex equals to its out-degree (it makes the graph strongly-connected).

Euler Graph Proof

Undirected graphs

- Check whether the graph is connected?
  Check DFS(u) $= V$ or BFS(u) $= V$?
- Check whether the degree of each vertex is even?
  For adjacency matrix, sum of all elements in u^{th} row (u^{th} column) is the degree of u

Euler Graph Proof

Directed graphs

- Check whether the graph is weakly connected?
  - Check whether the corresponding undirected graph is connected, or
  - Check if exist vertex $u \in V$ such that DFS$(u) = V$ or $BFS(u) = V$?
- Check whether the out-degree and the in-degree of each vertex are equal?
  - For adjacency matrix, the out-degree of vertex $u, \deg^{+}(u)$ is the number of 1 in u^{th} row, the in-degree of vertex $u, \deg^{-}(u)$ is the number of 1 in u^{th} column.

Exercise 1.

Given undirected graph G represented as the adjacency matrix below. Prove that G is an Euler graph.

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

Exercise 2.

Given directed graph G represented as the adjacency matrix below. Prove that G is an Euler graph.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Algorithm for Finding an Euler Circuit

```
Euler-Cycle(u){
        Step 1: Initialize
        stack = ∅; //initialize stack ∅
        CE = ∅; //initialize CE ∅
        push(stack,u); //push u to the stack
        Step 2: Loop
        while(stack ≠ ∅){
                s = get(stack);
                if(Adj(s) ≠ ∅){
                        t=<the first vertex in Adj(s)>;
                        push(stack,t);//push t to stack
                        E = E\{(s,t)}; //remove edge (s,t)
                }
                else{
                        s = pop(stack); //remove s from stack
                        s ⇒ CE; //move s to CE
                }
        }
        Step 3: Result
        <Overturning vertices in CE we get an Eulerian circuit>;
}
```

Example

Find an Euler circuit starting from vertex 1 for the undirected graph represented as the adjacency matrix below.

$$\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}$$

Verification

#	Stack	CE	#	Stack	CE
1	1	∅	14	1,2,3,4,7,5,2,6,5,3,11,4	1
2	1,2	∅	15	1,2,3,4,7,5,2,6,5,3,11,4,8	1
3	1,2,3	∅	16	1,2,3,4,7,5,2,6,5,3,11,4,8,7	1
4	1,2,3,4	∅	17	1,2,3,4,7,5,2,6,5,3,11,4,8,7,6	1
5	1,2,3,4,7	∅	18	1,2,3,4,7,5,2,6,5,3,11,4,8,7	1,6
6	1,2,3,4,7,5	∅	19	1,2,3,4,7,5,2,6,5,3,11,4,8	1,6,7
7	1,2,3,4,7,5,2	∅	20	1,2,3,4,7,5,2,6,5,3,11,4,8,9	1,6,7
8	1,2,3,4,7,5,2,6	∅	21	1,2,3,4,7,5,2,6,5,3,11,4,8,9,10	1,6,7
9	1,2,3,4,7,5,2,6,1	∅	22	1,2,3,4,7,5,2,6,5,3,11,4,8,9,10,8	1,6,7
10	1,2,3,4,7,5,2,6	1	23	1,2,3,4,7,5,2,6,5,3,11,4,8,9,10	1,6,7,8
11	1,2,3,4,7,5,2,6,5	1	24	1,2,3,4,7,5,2,6,5,3,11,4,8,9,10,11	1,6,7,8
12	1,2,3,4,7,5,2,6,5,3	1	25	1,2,3,4,7,5,2,6,5,3,11,4,8,9,10,11,12	1,6,7,8
13	1,2,3,4,7,5,2,6,5,3,11	1	26	1,2,3,4,7,5,2,6,5,3,11,4,8,9,10,11,12,9	1,6,7,8

Verification

#	Stack	CE
27	1,2,3,4,7,5,2,6,5,3,11,4,8,9,10,11,12,9,13	1,6,7,8
28	1,2,3,4,7,5,2,6,5,3,11,4,8,9,10,11,12,9,13,12	1,6,7,8
29	1,2,3,4,7,5,2,6,5,3,11,4,8,9,10,11,12,9,13,12,10	1,6,7,8
Move vertices in *Stack* to *CE* one by one until *Stack* = ∅		
30-...	*CE* = 1,6,7,8,10,12,13,9,12,11,10,9,8,4,11,3,5,6,2,5,7,4,3,2,1	
Overturning vertices in *CE* we get an Eulerian circuit		
1-2-3-4-7-5-2-6-5-3-11-4-8-9-10-11-12-9-13-12-10-8-7-6-1		

Necessary and Sufficient Conditions for Semi-Euler Graph

Undirected graph: Connected undirected graph $G = \langle V, E \rangle$ is semi-Euler if and only if G has 0 or 2 vertices with odd degree

- G has 2 vertices with odd degree: Euler path starts at an odd degree vertex and ends at the other odd-degree vertex
- G does not have any odd-degree vertex: G is Euler graph

Directed graph: Weakly-connected directed graph $G = \langle V, E > $ is semi-Euler if and only if:

- There exits exactly two vertices $u, v \in V$ such that $\deg^+(u) - \deg^-(u) = \deg^-(v) - \deg^+(v) = 1$
- For every vertices $s \neq u, s \neq v$, we have $\deg^+(s) = \deg^-(s)$
- Euler path starts at u and ends at v

Semi-Euler Graph Proof

Undirected graph:

- Prove that the graph is connected: Using DFS(u) or $BFS(u)$
- Has 0 or 2 vertices with odd degree: G Using properties of graph representation methods to determine the degree of each vertex

Directed graph:

- Prove that the graph is weakly-connected: Using DFS (u) or $BFS(u)$
- Has $u, v \in V$ such that

$$\deg^+(u) - \deg^-(u) = \deg^-(v) - \deg^+(v) = 1$$

- For other vertices $s \neq u, s \neq v$, we have $\deg^+(s) = \deg^-(s)$

Exercise 3

Given undirected graph G represented as the adjacency matrix below. Prove that G is semi- Euler graph.

$$\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}$$

Exercise 4

Given directed graph G represented as the adjacency matrix below. Prove that G is semi- Euler graph.

$$\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Algorithm for Finding an Euler Path

- Algorithm for finding an Euler path is similar to the one for finding Euler circuit
- Finding Euler circuit: Starting from any vertex $v \in V$
- Finding Euler path
  - Undirected graph: Starting from an odd-degree vertex $v \in V$(if there is no odd-degree vertex starting from any vertex)
  - Directed graph: Starting from vertex $v \in V$ such that $deg^{+}(u) - deg^{-}(u) = 1$.

Exercise 5

Find an Euler path of the undirected semi-Euler graph represented as the adjacency matrix below.

$$\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}$$

Contents

1 Euler Graph

2 Hamilton Graph

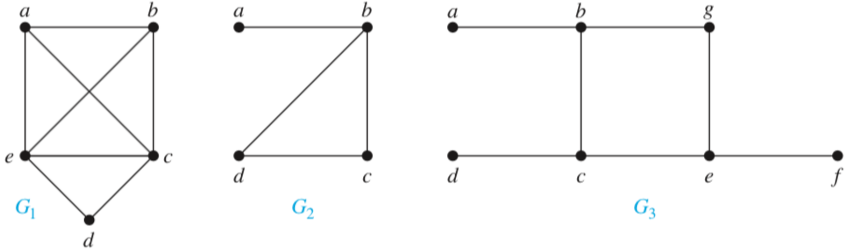
Hamilton Graph

Definition

- A path is said to be a Hamilton path if it passes through each vertex exactly once.
- A circuit is said to be a Hamilton circuit if it passes through each vertex exactly once
- A graph is said to be a Hamilton graph if it contains a Hamilton circuit
- A graph is said to be a semi-Hamilton graph if it contains a Hamilton path

Example

Which of the following simple graphs have a Hamilton circuit or, if not, a Hamilton path?



Sufficient Condition for Hamilton Graph

- Until now, there is no sufficient condition for Hamilton graph
- Until now, there is no efficient algorithm to check whether a graph is Hamilton or not

Algorithm for Finding Hamilton Circuits

Algorithm for listing all Hamilton circuits starting from vertex k.

Hamilton(int k){
    for($y \in Adj(X[k-1])$){
        if(($k == n+1$) && ($y == v_0$))
            Store_Hal_Cir($X[1], X[2], \ldots, X[n]$,
        else if($unCheck[y] == \boldsymbol{true}$){
            $X[k] = y$;
            $unCheck[y] = \boldsymbol{false}$;
            Hamilton($k+1$);
            $unCheck[y] = \boldsymbol{true}$;
        }
    }
}

Back Tracking !!!

Algorithm for Finding Hamilton Circuits

Listing all Hamilton circuits:

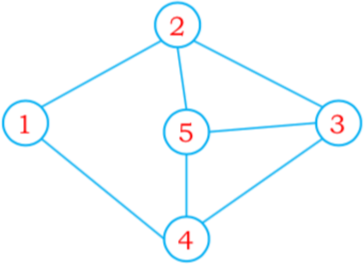
Hamilton-Cycle(v_0){
  //all vertices are unchecked
  for($v \in V$)
    $unCheck[v] = \boldsymbol{true}$;

  $X[1] = v_0$; //v_0 is a vertex of the graph
  $unCheck[v_0] = \boldsymbol{false}$;

  Hamilton(2); //call Hamilton algorithm
}

Example

Finding all Hamilton circuits of the undirected graph below:



Summary

- Basic concepts of Euler path, Euler circuit, semi- Euler graph, Euler graph
- Necessary and sufficient conditions for Euler graph, semi-Euler graph
- Algorithms for finding Euler circuits, Euler paths
- Basic concepts of Hamilton path, Hamilton circuit, semi-Hamilton graph, Hamilton graph
- Algorithm for finding Hamilton circuits