POSTS AND TELECOMMUNICATIONS INSTITUTE OF TECHNOLOGY

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Chapter 3: LINE AND SURFACE INTEGRALS

CALCULUS 2

Faculty of Fundamental Science 1

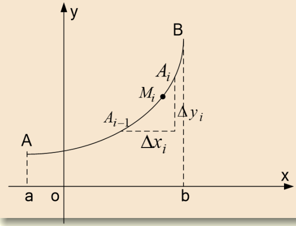
Hanoi - 2022

1. Definition

Let the function f(x,y) define on a plane arc $\overset{\frown}{AB}$.

Divide the arc $\overset{\frown}{AB}$ by n points

$$A_0 \equiv A, A_1, \ldots, A_{i-1}, A_i, \ldots, A_n \equiv B$$



Set

$\Delta s_i,\ i = \overline{1,n}$ is called the arc length of $\overset{\frown}{A_{i-1}A_i}$.

Choose the

arbitrary points $M_i\left(x_i, y_i\right) \in \overset{\frown}{A_{i-1}A_i},\ (i = \overline{1,n})$.

Then, $I_n = \sum_{i=1}^{n} f\left(x_i, y_i\right) \Delta S_i$

is called the sum of first-order

line integrals of the function $f(x,y)$ on arc $\overset{\frown}{AB}$.

I. Line integral of type 1

If $n \to \infty$ so that $\Delta s_i \to 0, I_n$ converges to I regardless of the division of arc $\overset{\frown}{AB}$ and the choice of $M_i\left(x_i, y_i\right) \in \overset{\frown}{A_{i-1}A_i}, (i = \overline{1, n})$, then the number I is called the first-order line integral of $f(x, y)$ along arc $\overset{\frown}{AB}$ and the symbol $\int\limits_{\overset{\frown}{AB}} f(x, y) ds$.

So $I = \lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} f\left(x_i, y_i\right) \Delta S_i = \int\limits_{\overset{\frown}{AB}} f(x, y) ds$, where dS denotes the length factor of the arc or the differential of arc.

If the function $f(x, y, z)$ is integrable on arc $\overset{\frown}{AB} \subset \mathrm{R}^3$ then the first-order line integral of f(x,y,z) on arc AB denoted is

$$I = \int\limits_{\overset{\frown}{AB}} f(x, y, z) ds.$$

I.Line integral of type 1

- The arc $\overset{\frown}{AB}$ is called smooth if its tangent is variable continuous.
- The arc $\overset{\frown}{AB}$ is called a segmented smooth arc if arc $\overset{\frown}{AB}$ can be divided into a finite number of smooth arcs.
- It can be proved: If arc $\overset{\frown}{AB}$ is smooth or smooth each segment and f(x,y) is continuous on arc $\overset{\frown}{AB}$, then $f(x,y)$ is integrable on arc $\overset{\frown}{AB}$.
- The first-order line product has the same properties as the product definite stool

$$\int\limits_{\overset{\frown}{AB}} (\alpha f + \beta g)(x,y)dS = \alpha \int\limits_{\overset{\frown}{AB}} f(x,y)ds + \beta \int\limits_{\overset{\frown}{AB}} g(x,y)ds.$$

$$\int\limits_{\overset{\frown}{AC}} f(x,y)ds = \int\limits_{\overset{\frown}{AB}} f(x,y)ds + \int\limits_{\overset{\frown}{BC}} f(x,y)ds.$$

Remark 1

a) From the above definition, we see the direction of arc $\overset{\frown}{AB}$ plays no role because I_n does not depend on the direction of arc AB. So

$$\int\limits_{\overset{\frown}{AB}} f(x,y)ds = \int\limits_{\overset{\frown}{BA}} f(x,y)ds$$

b) If l is the length of arc $\overset{\frown}{AB}$, then $l = \int\limits_{\overset{\frown}{AB}} ds$

Remark 1

c) If a material wire has arc $\overset{\frown}{AB}$ and mass density is $\rho(x,y)$, then the mass of the material wire is calculated according to the formula

$$m = \int\limits_{\overset{\frown}{AB}} \rho(x,y)ds.$$

The center of mass of the wire with density function $\rho(x,y)$ is located at the point $G(x_G, y_G)$, where

$$x_G = \frac{1}{m}\int\limits_{\overset{\frown}{AB}} x\rho(x,y)ds, \;\; y_G = \frac{1}{m}\int\limits_{\overset{\frown}{AB}} y\rho(x,y)ds.$$

2. The formula for first-order line integral

a. The arc $\overset{\frown}{AB}$ has the general form:

Case 1: Let $\overset{\frown}{AB}$ be smooth segmented arc of the form $y = y(x),\ x \in [a, b]$ and the function $f(x, y)$ is continuous on $\overset{\frown}{AB}$. Then

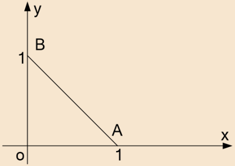
$$I = \int\limits_{\overset{\frown}{AB}} f(x,y)ds = \int\limits_a^b f(x, y(x))\sqrt{1 + y'^2(x)}\, dx \quad (3.1)$$

Case 2: Let $\overset{\frown}{AB}$ be smooth segmented arc of the form $x = x(y),\ y \in [c, d]$ and the function $f(x, y)$ is continuous on $\overset{\frown}{AB}$. Then

$$I = \int\limits_{\overset{\frown}{AB}} f(x,y)ds = \int\limits_c^d f(x(y), y)\sqrt{1 + x'^2(y)}\ dy \quad (3.2)$$

Example 1.

Calculate $\int\limits_C (x+y)ds$, where C is the boundary of the triangle with points $O(0,0), A(1,0), B(0,1)$.



Solution

$$\int\limits_C = \int\limits_{\overline{OA}} + \int\limits_{\overline{AB}} + \int\limits_{\overline{BO}} .$$

The segment $\overline{OA}$ has the equation $y = 0, 0 \le x \le 1$

$$\int\limits_{\overline{OA}} (x+y)ds = \int\limits_0^1 x\sqrt{1+0}dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}.$$

Continuity example 1.

The arc $\overset{\frown}{AB}$ has the equation $y = 1 - x, 0 \leq x \leq 1$

$$\Rightarrow \int\limits_{\overset{\frown}{AB}} (x+y)ds = \int\limits_0^1 1\sqrt{1+1}dx = \sqrt{2}$$

The segment $\overline{BO}$ has the equation $x = 0, 0 \leq y \leq 1$

$$\int\limits_{\overline{BO}} (x+y)ds = \int\limits_0^1 y\sqrt{1+0}dy = \left.\frac{1}{2}y^2\right|_0^1 = \frac{1}{2}$$

$$\Rightarrow \int\limits_C (x+y)ds = 1 + \sqrt{2}.$$

2. The formula for first-order line integrals

b. The arc $\overset{\frown}{AB}$ has parametric form in the plane

Let $\overset{\frown}{AB}$ be smooth segmented arc of the form

$$\overset{\frown}{AB}:\begin{cases} x = x(t) \\ y = y(t) \end{cases}, t_1 \leq t \leq t_2$$

and the function $f(x, y)$ is continuous on $\overset{\frown}{AB}$.Then

$$I = \int\limits_{\overset{\frown}{AB}} f(x,y)ds = \int\limits_{t_1}^{t_2} f[x(t), y(t)]\sqrt{x'^2(t) + y'^2(t)}dt$$

Note: The curves in space

$$\overset{\frown}{AB} \subset \mathbb{R}^3 : \begin{cases} x = x(t) \\ y = y(t), t_1 \leq t \leq t_2 \\ z = z(t) \end{cases}$$

$$\int\limits_{\overset{\frown}{AB}} f(x,y,z)ds = \int\limits_{t_1}^{t_2} f(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}dt.$$

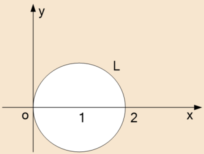
c. The curve in polar coordinates

$$\overset{\frown}{AB} : r = r(\varphi), \varphi_1 \leq \varphi \leq \varphi_2 \Rightarrow x'^2(\varphi) + y'^2(\varphi) = r^2(\varphi) + r'^2(\varphi)$$

$$I \int\limits_{\overset{\frown}{AB}} f(x,y)ds = \int\limits_{\varphi_1}^{\varphi_2} f[r(\varphi)\cos\varphi, r(\varphi)\sin\varphi]\sqrt{r^2(\varphi)+r'^2(\varphi)}d\varphi$$

Example 2.

Calculating
$I = \int\limits_L \sqrt{x^2+y^2}ds$, where L is the circle $x^2+y^2=2x$.



The equation
of the line L in polar coordinates has the form

$$r = 2\cos\varphi, -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$$

$$I = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\varphi\sqrt{4\cos^2\varphi + 4\sin^2\varphi}d\varphi = 8\int\limits_0^{\frac{\pi}{2}} \cos\varphi d\varphi = 8\sin\varphi|_0^{\frac{\pi}{2}} = 8$$

It is possible to integrate as a parameter $\begin{cases} x = 1+\cos t \\ y = \sin t \end{cases}, 0 \le t \le 2\pi$

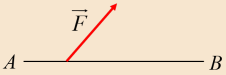
$$I = \int\limits_0^{2\pi} \sqrt{(1+\cos t)^2 + \sin^2 t}dt = \int\limits_0^{2\pi} \sqrt{2+2\cos t}dt = \int\limits_0^{2\pi} \sqrt{4\cos^2\frac{t}{2}}dt = 8.$$

II. Line integral of type 2

1. Problem: Calculate the power of the transformed force

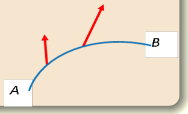
A power produced by force $\vec{F}$ move on the arc L from A to B is

$$W = \vec{F} \cdot \overrightarrow{AB}$$



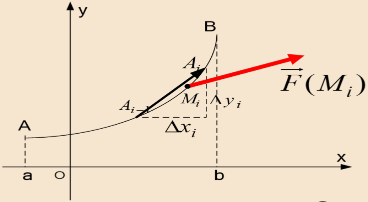
Calculate the power W of the force $\vec{F}(M)$ born while moving on the arc AB from point A to point B

$$\vec{F}(M) = P(M)\vec{i} + Q(M)\vec{j} = (P, Q); M \in AB$$



II. Line integral of type 2

- Divide arc $\overset{\frown}{AB}$ by n the points $A_0, A_1, \ldots, A_n$. Let the coordinates of the vector $\overset{\frown}{A_{i-1}A_i}$ be $\Delta x_i,\ \Delta y_i$ and the arc length $\overset{\frown}{A_{i-1}A_i}$ is $\Delta s_i,\ i = \overline{1,\ n}$.



- Choose the arbitrary points $M_i\,(x_i, y_i) \in \overset{\frown}{A_{i-1}A_i},\ i = \overline{1,\ n}$.
- So that the power W of the force produced from A to B on arc AB approximately $W \approx \sum_{i=1}^{n} P\,(x_i, y_i)\,\Delta x_i + Q\,(x_i, y_i)\,\Delta y_i$

$$\Rightarrow W = \lim_{\max \Delta S_i \to 0} \sum_{i=1}^{n} P\,(x_i, y_i)\,\Delta x_i + Q\,(x_i, y_i)\,\Delta y_i.$$

2. Definition of Line integral of type 2

Given two functions $P(x,y), Q(x,y)$ defined on arc L (or arc $\overset{\frown}{AB}$)

- Divide arc L by points

$$A \equiv \mathrm{A}_0, A_1, \ldots, A_{i-1}, A_i, \ldots, A_n \equiv B.$$

- Call the coordinates of the vector $\overrightarrow{A_{i-1}A_i}$ is $\Delta x_i, \Delta y_i$ and arc length $A_{i\overset{\frown}{-1}}A_i$ is $\Delta S_i, i = \overline{1,n}$
- Choose the arbitrary points $M_i\left(x_i, y_i\right) \in A_{i\overset{\frown}{-1}}A_i$.
- Set up totals $I_n = \sum_{i=1}^{n} P\left(M_i\right)\Delta x_i + Q\left(M_i\right)\Delta y_i$ it's called the sumthe second-order line segment of the function $P(x,y), Q(x,y)$ along L going from A to B corresponds to a partition of L and a choice $M_i \in A_{i\overset{\frown}{-1}}A_i$

II. Line integral of type 2

When $n \to \infty$ so that $\max \Delta s_i \to 0$ $(\max \Delta x_i \to 0, \max \Delta y_i \to 0)$ that I_n converge to a number I regardless of the division of the arc L and the arbitrary choice $M_i \in \overset{\frown}{A_{i-1}A_i}$ then the number I is called a line integral of the second type of functions $P(x, y), Q(x, y)$ along arc L go from A. Denote by

$$\int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy.$$

Thus

$$I = \int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_i \to 0}} \sum_{i=1}^{n} P(x_i, y_i) \Delta x_i + Q(x_i, y_i) \Delta y_i.$$

II. Line integral of type 2

Remark

- Unlike the first-order line integral, in the first-order line integral two, the direction of integration of L is important If integrating along arc $\overset{\frown}{AB}$ going from B to A, the vectors silk $\overrightarrow{A_{i-1}A_i}$ change direction. So the sum of the integrals will change sign, so

$$\int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy = -\int\limits_{\overset{\frown}{BA}} P(x,y)dx + Q(x,y)dy.$$

- The power produced by force $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ for the point to move from A to B along the arc $\overset{\frown}{AB}$ will be

$$W = \int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy.$$

Remark

- If the $\overset{\frown}{AB}$ is a curve in space and the functions $P(x,y,z), Q(x,y,z), R(x,y,z)$ define on the arc $\overset{\frown}{AB}$ then product the second-order segmentation of these three functions is also denoted by

$$\int\limits_{\overset{\frown}{AB}} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz.$$

- Let L be a plane curve and closed curve.we use the convention that the positive orientation of a simple closed curve L refers to a single counterclockwise traversal of L. That is a person walking along L in that direction will see the domain bounded by the L closest to me is on the left. The integral taken in the positive direction is denoted by $\oint\limits_{L} P(x,y)dx + Q(x,y)dy$.

II. Line integral of type 2

Remark

- If two functions $P(x,y), Q(x,y)$ are continuous on smooth arc $\overset{\frown}{AB}$ or segmented smooth, then there exists a line integral of the second-order

$$I = \int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy$$

Line integrals of the second-order have the same properties as definite integrals.

Note:

- Sum, difference, multiply a number of the second-order line integrals

$$\int\limits_{\overset{\frown}{AC}} Pdx + Qdy = \int\limits_{\overset{\frown}{AB}} Pdx + Qdy + \int\limits_{\overset{\frown}{BC}} Pdx + Qdy.$$

3.The formula for calculating line integrals of type 2

Let the two functions $P(x,y), Q(x,y)$ be continuous on smooth arc $\overset{\frown}{AB}$ is given by the parametric equation $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$; $A = (x(t_A), y(t_A)), B = (x(t_B), y(t_B))$. Then

$$I = \int_{t_A}^{t_B} \left[P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\right] dt,$$

where arc $\overset{\frown}{AB}$ is planar given by an equation of the form

$$y = y(x); A(x_A, y(x_A)), B = (x_B, y(x_B))$$

$$I = \int P(x,y)dx + Q(x,y)dy = \int^{b} \left[P(x, y(x)) + Q(x, y(x))y'(x)\right] dx.$$

Example 1.

Calculating work done by force $\vec{F} = -y\vec{i} + x\vec{j}$ born along the road ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in its positive orientation.

Solution

Parametric equation of the ellipse

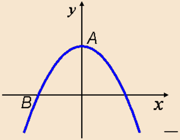
$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, 0 \le t \le 2\pi$$

$$A = \int\limits_L xdy - ydx = \int\limits_0^{2\pi} (a\cos t \cdot b\cos t + b\sin t \cdot a\sin t)dt = ab\int\limits_0^{2\pi} dt = 2\pi ab$$

Example 2.

Calculating $I = \int\limits_L \left(2xy - x^2\right)dx + \left(x + y^2\right)dy$

where L is the arc of the parabola
$y = 1 - x^2$ go from point A(0,1) to point B (-1,0)



$$y = 1 - x^2 \Rightarrow dy = -2xdx$$

$$I = \int\limits_0^{-1} \left[2x\left(1 - x^2\right) - x^2 + \left(x + 1 - 2x^2 + x^4\right)(-2x)\right]dx$$

$$= \int\limits_0^{-1} \left(-2x^5 + 2x^3 - 3x^2\right)dx$$

$$= \left(-\frac{1}{3}x^6 + \frac{1}{2}x^4 - x^3\right)\Big|_0^{-1} = -\frac{1}{3} + \frac{1}{2} + 1 = \frac{7}{6}$$

3. Green's formula

Theorem 1. (Green's formula)

Let L be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region that contains D, then

$$\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint\limits_L Pdx + Qdy$$

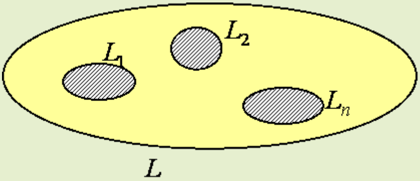
Corollary 1

If D is a simple domain with boundary L, then

$$S_D = \frac{1}{2} \oint\limits_L xdy - ydx = -\oint\limits_L ydx = \oint\limits_L xdy$$

Corollary 2

If D is multidomain, with outer boundary L and inner boundary $L_1, L_2, \ldots, L_n$, then



$$\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint\limits_L Pdx + Qdy - \sum_{k=1}^{n} \oint\limits_{L_k} Pdx + Qdy$$

Example 3.

Evaluate $I = \oint\limits_L (3y - e^{\sin x})dx + (7x + \sqrt{y^4+1})dy$, with L is circle $x^2 + y^2 = 4$.

Example 4.

Calculate $J = \int\limits_C \left(x \arctan x + y^2\right) dx + \left(x + 2yx + y^2 e^{-y^3}\right) dy$, with C is given by the equation $OA : x^2 + y^2 = 2x, y \geq 0$ going from origin to $A(0, 2)$.

4. Equivalence propositions for line integrals of type 2

Theorem 2.

Assume that the functions $P(x,y), Q(x,y)$ are continuous with the derivatives their first-order exclusivity in the simple domain D, then the following propositions are equivalent

- $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \forall (x,y) \in D$
- $\oint\limits_L Pdx + Qdy = 0$, where L is any closed curve in the domain D.
- The $\int\limits_{\overset{\frown}{AB}} Pdx + Qdy$, depends only on 2 points A and B but it does't depend on arc type $\overset{\frown}{AB}$ on the domain D.
- the $Pdx + Qdy$ is the total differential of the function some $u(x,y)$ on the domain D.

II. Line integral of type 2

Corollary 1

If $du(x,y) = Pdx + Qdy$ in the D domain, then

$$\int\limits_{\overset{\frown}{AB}} Pdx + Qdy = u(B) - u(A)$$

Corollary 2

If $Pdx + Qdy$ is the total differential of the function $u(x,y)$ on the domain $D \in\subset \mathbb{R}^2$ then the function $u(x,y)$ is given by the formula:

$u(x,y) = \int\limits_{x_0}^{x} P(x,y)dx + \int\limits_{y_0}^{y} Q(x_0,y)dy + C$ or

$u(x,y) = \int\limits_{x_0}^{x} P(x,y_0)dx + \int\limits_{y_0}^{y} Q(x,y)dy + C$, where

$M_0(x_0,y_0),\ M(x,y) \in D.$

Example 5.

Prove that the expression

$$\left(x^2 - 2xy^2 + 3\right) dx + \left(y^2 - 2x^2y + 4y - 5\right) dy$$

is the total differential of the function $u(x, y)$ on the R^2 and find the function $u(x, y)$.

$$\frac{\partial Q}{\partial x} = -4xy = \frac{\partial P}{\partial y}, \forall (x, y) \in \mathrm{R}^2$$

$$\Rightarrow \exists u(x,y) : \begin{cases} \frac{\partial u}{\partial x} = P(x,y) = x^2 - 2xy^2 + 3 \\ \frac{\partial u}{\partial y} = Q(x,y) = y^2 - 2x^2y + 4y - 5 \end{cases} \Rightarrow \begin{cases} u = \frac{x^3}{3} - x^2y^2 + \\ \Rightarrow \frac{\partial u}{\partial y} = -2x^2y \end{cases}$$

$$\Rightarrow f'(y) = y^2 + 4y - 5 \Rightarrow f(y) = \frac{y^3}{3} + 2y^2 - 5y + C$$

$$\Rightarrow u = \frac{x^3}{3} - x^2y^2 + 3x + \frac{y^3}{3} + 2y^2 - 5y + C.$$

Example 6.

Evaluate $I = \int\limits_{\overset{\frown}{AB}} \frac{xdy - ydx}{x^2+y^2}$, where $A(1,1), B(\sqrt{3}, 3)$.

a) The arc $\overset{\frown}{AB}$ is given by the equation: $y = x^2, 1 \le x \le \sqrt{3}$.

b) Let the arc $\overset{\frown}{AB}$ make the segment AB a closed curve that does not cover the origin.

1. Definition of surface integral of the of type 1

Let the function $f(M) = f(x,y,z)$ be define on the curved surface S.
- Divide the surface S into n pieces that do not step on each other, name and the symbol for the area, the diameter of the i-th piece is $\Delta S_i, d_i; i = \overline{1,n}$.
- Choose the arbitrary points $M_i\left(x_i, y_i, z_i\right) \in \Delta S_i, i = \overline{1,n}$.
- The totals $I_n = \sum_{i=1}^{n} f\left(M_i\right)\Delta S_i$ is called the total surface integral of type one for a division of the surface S and choice of the points $M_i\left(x_i, y_i, z_i\right) \in \Delta S_i, i = \overline{1,n}$.

If when $n \to \infty$ such that $\max d_i \to 0$ that I_n converges to the number I depends on the division of the surface S and selects the points $M_i \in \Delta S_i$, then the number I is called the first-order surface integral of $f(M)$ on the surface S, denoted by $\iint\limits_S f(x,y,z)dS$.

So $I = \iint\limits_S f(x,y,z)dS = \lim\limits_{\max d_i \to 0} \sum\limits_{j=1}^{n} f\left(x_i, y_i, z_i\right)\Delta S_i$.

2. Integral conditions and properties of surface integral of type 1.

- If the surface S is smooth (the surface S has a normal variation continuous) or piecewise smooth (dividing S into a finite number of smooth surfaces and the function $f(x,y,z)$ is continuous or piecewise on the surface S, then there exists a first-order surface integral of that function on S.
- A face integral of the first kind has the same properties as a double integral

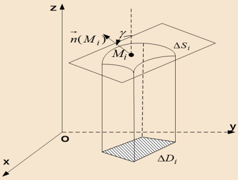
$$\iint\limits_S (\alpha f + \beta g) dS = \alpha \iint\limits_S f dS + \beta \iint\limits_S g dS.$$

$$\iint\limits_S f dS = \iint\limits_{S_1} f dS + \iint\limits_{S_2} f dS; \; S = S_1 \cup S_2, S_1 \cap S_2 = \varnothing.$$

3. Therem (How to calculate surface integrals of type 1)

Let the function $f(x,y,z)$ be continuity on a smooth surface S given by equation $z = z(x,y), (x,y) \in D$. Then

$$\iint\limits_S f(x,y,z)dS = \iint\limits_D f(x,y,z(x,y))\sqrt{1 + z'_{x^2}(x,y) + z'_{y^2}(x,y)}dxdy$$



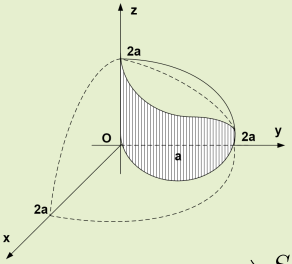
$$0 = dF(x,y,z) = F'_x dx + F'_y dy + F'zdz$$

$$\Rightarrow \cos\gamma = \pm\frac{F_z}{\sqrt{F_x'^2 + F_y'^2 + F_z'^2}} = \pm\frac{1}{\sqrt{1 + z_x'^2 + z_y'^2}}$$

$$I_n = \sum_{i=1}^{n} f(M_i)\,\Delta S_i \approx \sum_{i=1}^{n} f(x_i, y_i, z_i)\sqrt{1 + z_x'^2 + z_y'^2}.\Delta D_i; \Delta S_i \approx \frac{\Delta D_i}{|\cos\gamma_i|}.$$

Example 1.

Calculate the area of the upper part of the sphere $x^2 + y^2 + z^2 = 4a^2$ inside the cylinder $x^2 + y^2 \leq 2ay, a > 0$ The upper sphere has the equation $z = \sqrt{4a^2 - x^2 - y^2}$.



$$D : x^2 + (y-a)^2 \leq a^2, x \geq 0$$

$$S = \iint\limits_S dS = \iint\limits_D \sqrt{1 + z_x'^2 + z'^2} dxdy$$

$$z'x = -\frac{x}{z}, z'y = -\frac{y}{z} \Rightarrow \sqrt{1 + z_x'^2 + z'^2} = \frac{2a}{|z|}$$

$$\Rightarrow S = \iint\limits_D \frac{-2a}{\sqrt{4a^2 - x^2 - y^2}} dxdy.$$

Converting to polar coordinates, we get

$$S = 2a \int_0^{\pi} d\varphi \int_0^{2a\sin\varphi} \frac{r\,dr}{\sqrt{4a^2 - r^2}} = 8a^2 \left(\frac{\pi}{2} - 1\right).$$

Remark

- The case of surface S is given by the equation $y = y(z, x)$ or $x = x(y, z)$ then we have to project S onto the Ozx or Oyz to find the corresponding double integral.
- In the case of a curved surface of any shape, we must divide it into a number finite parts that satisfy the above theorem, then apply the formula.

4. Applications

- From the definition, we get the formula for surface area curvature S thanks to surface integrals of the first-order surface is $S = \iint\limits_S dS$.
- If S is a material surface, the mass density function is $\rho(x, y, z)$ then the mass of that material surface will be

$$m = \iint\limits_S \rho(x, y, z) dS.$$

- Formula for determining the center of gravity of a curved surface

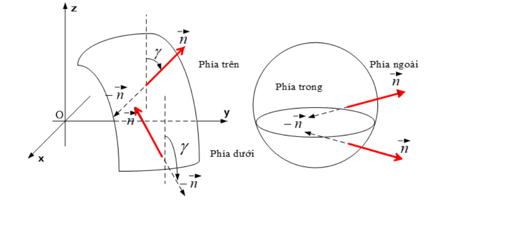
$$x_G = \frac{1}{m} \iint\limits_S x\rho(M) dS, y_G = \frac{1}{m} \iint\limits_S y\rho(M) dS, z_G = \frac{1}{m} \iint\limits_S z\rho(M) dS.$$

1. Oriented surfaces

- A smooth S-curve is called an oriented if the normal vector unit line $\vec{n}(M)$ completely determined at every $M \in S$ (can subtract the boundary of S) and transform continuously as M runs over S.
- The set $\vec{n}(M), \forall M \in S$ of a oriented curved surface define one side of the surface. Because $-\vec{n}(M)$ is also a normal vector, so the oriented surface always has two sides.

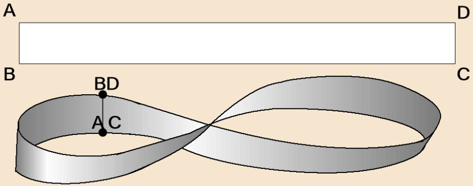
1. Oriented surfaces

- When the S-curve is not closed and oriented, one usually used from above and below to indicate a specified direction determined by $\vec{n}(M)$. The top of the S-face is the side that $\vec{n}(M)$ with angle Oz axis pointed, and the bottom is the side $\vec{n}(M)$ with Oz axis obtuse angle.
- When the closed S-curve is oriented, one uses the side in and out to describe the specified direction.



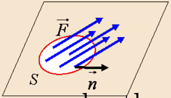
1. Oriented surfaces

- Outside is the side $\vec{n}(M)$ outward of the object V surrounded by the S-curve, inside is the opposite side.
- There is a curved surface that cannot be oriented, for example, the following surface is called Möbius strip.



2. Calculate the flux of the vector field through a surface

The Flux of the vector field silk is constant across the plane



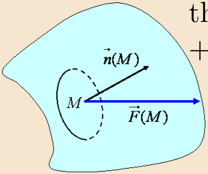
$$\Phi = S \cdot \vec{F} \cdot \vec{n}$$

Divide the surface S into n pieces that do not step on each other, name and the symbol for the area, the diameter of the i-th piece is $\Delta S_i, d_i; i = \overline{1, n}$

+ Choose

the arbitrary points $M_i\left(x_i, y_i, z_i\right) \in \Delta S_i, i = \overline{1, n}$.

+ Throughput approx



$$\Phi \approx \Phi_n = \sum_{i=1}^{n} \Delta S_i \cdot \vec{F}\left(M_i\right) \cdot \vec{n}\left(M_i\right)$$

Suppose

$$\vec{F}\left(M_i\right)=\left(P\left(M_i\right);Q\left(M_i\right);R\left(M_i\right)\right),\vec{n}\left(M_i\right)=\left(\cos\alpha_i;\cos\beta_i;\cos\gamma_i\right)$$

$$\begin{aligned}\Phi_n&=\sum_{i=1}^{n}\Delta S_i\vec{F}\left(M_i\right)\cdots\vec{n}\left(M_i\right)\\&=\sum_{i=1}^{n}\left(P\left(M_i\right)\cos\alpha_i+Q\left(M_i\right)\cos\beta_i+R\left(M_i\right)\cos\gamma_i\right)\Delta S_i\end{aligned}$$

The flux of the vector field $\vec{F}$ through the S -curve in the direction $\vec{n}$

$$\Phi=\lim_{\max d_i\to0}\sum_{i=1}^{n}\left(P\left(M_i\right)\cos\alpha_i+Q\left(M_i\right)\cos\beta_i+R\left(M_i\right)\cos\gamma_i\right)\Delta S_i$$

3. Definition of the surface integral of type 2

Let the surface S oriented along the normal vector $\vec{n}(M)$ and three functions $P(x,y,z), Q(x,y,z), R(x,y,z)$ determined on S.

- Divide the curved surface S into n pieces that do not step on each other ΔS_i. The symbol for the diameter of the i-th piece is $d_i, i = \overline{1,n}$
- Choose the arbitrary points $M_i\left(x_i, y_i, z_i\right) \in \Delta S_i$. The normal vector offace S at point M_i is $\vec{n}\left(M_i\right) = \left(\cos\alpha_i; \cos\beta_i; \cos\gamma_i\right)$
- Set up totals

$$
\begin{aligned}
I_n &= \sum_{i=1}^{n} \Delta S_i \vec{F}\left(M_i\right) \cdot \vec{n}\left(M_i\right) \\
&= \sum_{i=1}^{n} \left[P\left(M_i\right)\cos\alpha_i + Q\left(M_i\right)\cos\beta_i + R\left(M_i\right)\cos\gamma_i\right]\Delta S_i
\end{aligned}
$$

3. Definition of the surface integral of type 2

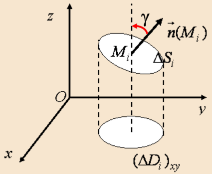
$$I_n = \sum_{i=1}^{n} \left(P\left(M_i\right)\cos\alpha_i + Q\left(M_i\right)\cos\beta_i + R\left(M_i\right)\cos\gamma_i\right)\Delta S_i$$

is called the sum of surface integrals of the second type of the three functions P, Q, R taken on the surface S oriented in $\vec{n}(M)$ with one way to divide and one way to choose $M_i \in \Delta S_i, i = 1, ..., n$.

- If when $n \to \infty$ so that $\max d_i \to 0$ but I_n converges to the number I regardless of the division of S and the choice of $M_i \in \Delta S_i$ then the number I is called the face integral of the second kind of the three functions P, Q, R, taken on the surface $I = \iint\limits_S [P(x, y, z)\cos\alpha + Q(x, y, z)\cos\beta + R(x, y, z)\cos\gamma]dS.$
- The integral surfaces of the second type of the vector field $\vec{F}(P, Q, R)$ pass curvature S in the direction $\vec{n}$ is the first-order surface integral of $\vec{F}$.

3. Definition of the surface integral of type 2

$$I = \iint_S [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma]dS$$



Calling $(\Delta D_i)_{xy}, (\Delta D_i)_{yz}, (\Delta D_i)_{zx}$
is in turn the projection
of ΔS_i onto the coordinate plane Oxy, Oyz, Ozx
$(\Delta D_i)_{xy} = \Delta S_i \cos\gamma_i \Rightarrow \cos\gamma dS = dxdy$
$(\Delta D_i)_{yz} = \Delta S_i \cos\alpha_i \Rightarrow \cos\alpha dS = dydz$
$(\Delta D_i)_{zx} = \Delta S_i \cos\beta_i \Rightarrow \cos\beta dS = dxdzdz$
Therefore, the face integral of the second kind of the functions P, Q, R on the surface S can sign
$I = \iint_S P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy$

Remark 1.

- It has been shown that, if the face S is oriented, smooth or fragmented smooth and the functions P, Q, R, are continuous on S then the second-order surface integral of P, Q, R exists.
- If the direction of the surface integral is changed, then the surface integral of second-order changes sign.
- The surface integrals of the second-order have the same properties as integrals dual.
- The flux of the vector field $\vec{F}(P, Q, R)$ past the curved surface S orientation is calculated by the formula
  $\Phi = \iint\limits_S Pdydz + Qdzdx + Rdxdy.$
- Assume that the liquid flows through the surface S with velocity $\vec{v}(M)$. Then the flux of the vector field $\vec{v}(M)$ overtaking S is amount of liquid flowing through S in a unit of time.

a. How to calculate the surface integrals of type 2

Theorem 1.

Suppose $R(x, y, z)$ is continuity on a smooth S-oriented surface for by equation $z = z(x, y), (x, y) \in D \subset (Oxy)$. Then

$$\iint\limits_S R(x, y, z)dxdy = \iint\limits_D R(x, y, z(x, y))dxdy$$

if the surface integral of the second-order is taken over the surface S

$$\iint\limits_S R(x, y, z)dxdy = -\iint\limits_D R(x, y, z(x, y))dxdy$$

if the surface integral of the second-order is taken over the surface S.

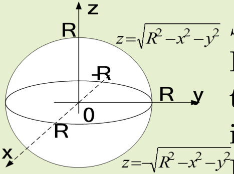
IV. Surface integral of type 2

Similarly, we also have

$$\iint\limits_S P(x,y,z)dydz = \begin{cases} \iint\limits_{D_{yz}} P(x(y,z),y,z)dydz & \text{khi } \cos\alpha \geq 0 \\ -\iint\limits_{D_{yz}} P(x(y,z),y,z)dydz & \text{khi } \cos\alpha \leq 0 \end{cases}$$

$$\iint\limits_S Q(x,y,z)dzdx = \begin{cases} \iint\limits_{D_{=x}} Q(x,y(z,x),z)dzdx & \text{khi } \cos\beta \geq 0 \\ -\iint\limits_{D_x} Q(x,y(z,x),z)dzdx & \text{khi } \cos\beta \leq 0 \end{cases}$$

Example 1.



Calculate $I = \iint\limits_S zdxdy$, where

S is the outside of the sphere $x^2+y^2+z^2=R^2$
Divide the sphere into the upper half S_+ and the bottom half S_- there is a way program in turn is $z=\sqrt{R^2-x^2-y^2}$ and $z=-\sqrt{R^2-x^2-y^2}$
Projecting the halves of the sphere on Oxy, then we get $D: x^2+y^2 \le R^2$

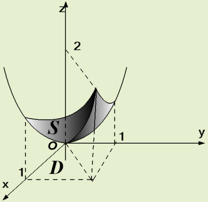
$$I = \iint\limits_{S_+} zdxdy + \iint\limits_{S_-} zdxdy = 2\iint\limits_D \sqrt{R^2-x^2-y^2}dxdy$$

$$I = 2\int\limits_0^{2\pi} d\varphi \int\limits_0^R \sqrt{R^2-r^2}\, rdr = \frac{4}{3}\pi R^3$$

Example 2.

Find the flux of the vector field over the top of the curved surface $z = x^2 + y^2, -1 \le x \le 1, -1 \le y \le 1$

$$\Phi = \iint_S z\,dydz + x^2 dxdy$$



Due to the S-curve against commensurate with the coordinate planes Oyz degrees so

$$\iint_S z\,dydz = 0$$

So that

$$\Phi = \iint_S x^2 dxdy = \iint_D x^2 dxdy; \quad D \begin{cases} -1 \le x \le 1 \\ -1 \le y \le 1 \end{cases}$$

$$\Phi = \iint_D x^2 dxdy = \int_{-1}^{1} x^2 dx \int_{-1}^{1} dy = \frac{4}{3}.$$

b. Convert to surface integrals of type 1

Suppose that the P(x,y,z) , Q(x,y,z) R(x,y,z) are integrable on the surface S has a normal vector $\vec{n} = (\cos\alpha; \cos\beta; \cos\gamma)$. Then

$$I = \iint\limits_S [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma]dS$$

$$= \iint\limits_S \vec{F}(P,Q,R).\vec{n}.dS$$

c. Ostrogradsky- Gauss' formula (O-G)

Let V be a simple solid region and let S be the boundary surface of V, given with positive (outward) orientation. Let P, Q, R be to have continuous partial derivatives on an open region that contains S. Then

$$\iint\limits_S Pdydz + Qdzdx + Rdxdy = \iiint\limits_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz.$$

Example 4.

Calculate $I = \iint\limits_S x^3dydz + y^3dzdx + z^3dxdy$, where S is the outer surface of the sphere $x^2 + y^2 + z^2 = R^2$.

IV. Surface integral of type 2

Remark 3.

- Consider $P = x, Q = y, R = z$, we get the formula calculate the volume of the body V is $V = \frac{1}{3}\iint_S xdydz + ydzdx + zdxdy$, where S is oriented outside the domain V.
- It can be considered that the Ostrogradsky-Gauss'formula is extended Green's formula from two-dimensional space to three-dimensional. Sometimes integrating on a non-closed surface S, we can add a curved surface somewhere to apply the Ostrogradsky-Gauss' formula.
- If $\vec{F} = (P, Q, R)$ is a vector field whose component functions have continuous partial derivatives on an open region that contains S, then the Ostrogradsky-Gauss' formula can be written in the form

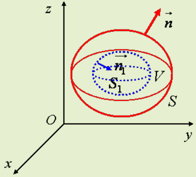
$$\iint\limits_S \vec{F} \cdot \vec{n} \cdot dS = \iiint_V \operatorname{div} \vec{F}.\, dxdydz$$

Corollary.

Assume the functions P, Q, R are continuous partial derivatives in the $V \subset \mathrm{R}^3$ whose outer boundary is a closed surface S ,the inner boundary is the closed surface S_1 which is smooth each piece. Then

$$\iint\limits_{S} Pdydz + Qdzdx + Rdxdy - \iint\limits_{S_1} Pdydz + Qdzdx + Rdxdy$$

$$= \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz.$$



Especially, if $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$, then

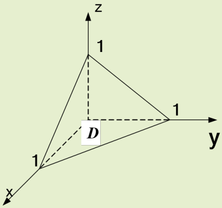
$$\iint\limits_{S} Pdydz+Qdzdx+Rdxdy = \iint\limits_{S_1} Pdydz+Qdzdx+R$$

Example 5.

Calculate $I = \iint_S xzdydz + yxdzdx + zydxdy$

take the outside

of the surface S as the boundary of the pyramid



$$x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$$

Applying the Odstrogradsky-Gauss' formula

$$I = \iiint\limits_V (z + x + y)dxdydz$$

$$I = \iint\limits_D dxdy \int\limits_0^{1-x-y} (x + y + z)dz$$

$$I = \iint\limits_D dxdy \left((x + y)z + \frac{1}{2}z^2 \Big|_{z=0}^{1-x-y} \right) = \int\limits_0^1 dx \int\limits_0^{1-x} \frac{1}{2}\left[1 - (x + y)^2\right] dy$$

$$= \frac{1}{2}\int\limits_0^1 dx \left(y - \frac{1}{3}(x + y)^3 \Big|_{y=0}^{y=1-x} \right) = \frac{1}{2}\int\limits_0^1 \left(1 - x - \frac{1}{3} + \frac{1}{3}x^3\right) dx = \frac{1}{8}$$

Example 6.

Calculating the flux of the vector field $\vec{F} = \frac{q\vec{r}}{r^3}$ across the surface $x^2 + y^2 + z^2 = R^2$ in which q is the charge at the root of the coordinates, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$P = q\frac{x}{r^3}, Q = q\frac{y}{r^3}, R = q\frac{z}{r^3}, \forall (x, y, z) \neq (0, 0, 0)$$

Note: We can not apply the Odstrogradsky-Gauss formula in the sphere

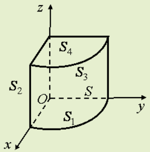
$$\Phi = q \iint\limits_S \frac{1}{r^3}(xdydz + ydzdx + zdxdy)$$

However, the Odstrogradsky-Gauss' formula can be applied to the integral

$$\Phi = \frac{q}{R^3} \iint\limits_S xdydz + ydzdx + zdxdy$$

$$= \frac{q}{R^3} \iiint_V 3dxdydz = \frac{q}{R^3} \cdot 3 \cdot \frac{4}{3}\pi R^3 = 4\pi q.$$

Example 7.



Calculating the flux of the vector field $\vec{F}\left(x^3, y^3, z^3\right)$ through the outside of the cylindrical part $x^2 + y^2 = R^2, x \geq 0, y \geq 0, 0 \leq z \leq h$.

$$\Phi = \iint\limits_S x^3 dydz + y^3 dxdz + z^3 dxdy$$

$$\Phi_k = \iint\limits_{S_k} x^3 dydz + y^3 dxdz + z^3 dxdy, \quad \Phi_1 = \Phi_2 = \Phi_3 = 0$$

$$\Phi_4 = \iint\limits_D h^3 dxdy = h^3 \frac{\pi R^2}{4}; D = \left\{(x, y) : x^2 + y^2 \leq R^2, x \geq 0, y \geq 0\right\}.$$

Example 7.

Applying the Odstrogradsky - Gauss' formula, we have

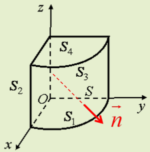
$$\Phi + \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 = 3 \iiint \left(x^2 + y^2 + z^2\right) dxdyd,$$

$$\iiint_V \left(x^2 + y^2 + z^2\right) dxdydz = \int_0^{\frac{\pi}{2}} d\varphi \int_0^R rdr \int_0^h \left(r^2 + z^2\right) dz$$

$$= \frac{\pi h R^4}{8} + \frac{\pi h^3 R^2}{12}$$

$$\Phi = 3\left(\frac{\pi h R^4}{8} + \frac{\pi h^3 R^2}{12}\right) - h^3 \frac{\pi R^2}{4} = \frac{3\pi h R^4}{8}.$$

Example 8.



Calculating the flux of the vector field $\vec{F}\left(x^3, y^3, z^3\right)$ through the outside of the cylindrical part $x^2+y^2=R^2, x \geq 0, y \geq 0, 0 \leq z \leq h$

We can calculate directly

$$\Phi = \iint\limits_S x^3dydz + y^3dxdz + z^3dxdy$$

$$\vec{n} = \left(\frac{x}{R}, \frac{y}{R}, 0\right) \Rightarrow \iint\limits_S z^3dxdy = 0$$

$$\iint\limits_S y^3dxdz = \iint\limits_D \left(\sqrt{R^2-x^2}\right)^3 dxdz$$

$$\iint\limits_D \left(\sqrt{R^2-x^2}\right)^3 dxdz = \int\limits_0^R \left(\sqrt{R^2-x^2}\right)^3 dx \int\limits_0^h dz = h\int\limits_0^R \left(\sqrt{R^2-x^2}\right)^3 dx$$

Example 8.

Set $x = R\sin t$, we obtain

$$h\int_0^R \left(\sqrt{R^2 - x^2}\right)^3 dx = hR^4 \int_0^{\pi/2} \cos^4 t dt = \frac{3\pi h R^4}{16}.$$

Thus

$$\iint_S x^3 dydz = \frac{3\pi h R^4}{16} \Rightarrow \Phi = \frac{3\pi h R^4}{8}$$

d. Stokes' formula

The Stokes' formula extends Green's formula, which is the relationship between the second-order line integral in space and the second-order surface integral.

Theorem 3 (Stokes' theorem)

Assuming that the segmented smooth, oriented S-curve has the boundary of the segmented smooth L and
Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let the functions P, Q, and R be continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\oint_C Pdx + Qdy + Rdz =$$

$$\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Remark 2.

- When substituting $z = 0, R(x, y, z) = 0$ into the Stokes formula, we get the Green's formula.
-Let the vector field $\vec{F} = (P, Q, R)$ and

$$\operatorname{rot} \vec{F} = [\vec{\nabla}; \vec{F}] = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right),$$

The Stokes' formula can be written in the form

$$\oint_L Pdx + Qdy + Rdz = \iint_S \operatorname{rot} \vec{F} \cdot n \cdot dS = \iint_S \begin{vmatrix} dydz & dxdz & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Theorem 4 (Theorem of equivalence statements)

Assume the functions P, Q, R are continuous partial derivatives in the simple domain V. Then the following propositions are similar.

(1) $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \forall (x, y, z) \in V.$

(2) $\iint\limits_{L} Pdx + Qdy + Rdz = 0, L$ is any closed curve in the domain V.

(3) $\int\limits_{\overset{\frown}{AB}} Pdx + Qdy + Rdz$, where $\overset{\frown}{AB} \subset V$ does't depend on the arc form $\overset{\frown}{AB}$.

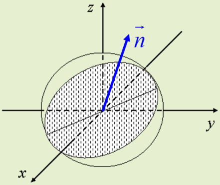
(4) Expression $Pdx + Qdy + Rdz$ is the total differential of some function $u(x, y, z)$ on the domain V and

$$\int\limits_{\overset{\frown}{AB}} Pdx + Qdy + Rdz = u(B) - u(A)$$

Example 9.

Calculate $I = \oint\limits_C ydx + zdy + xdz$, where C is the circle, intersection of the sphere $x^2 + y^2 + z^2 = R^2$ and plane $x + y + z = 0$ and direction of C is counterclockwise if looking towards $z > 0$.

Solution



The plane $x + y + z = 0$ passes through the center of the sphere. So the intersection is the great circle. Take the circle as a curved surface S with boundary C. The direction cosines of $\overrightarrow{n}$ oriented in the direction of C is $\vec{n} = (1, 1, 1)$. Apply Stokes' formula with $\overrightarrow{n_0} = \frac{1}{\sqrt{3}}(1, 1, 1)$

$$\operatorname{rot} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = (-1, -1, -1)$$

$$I = \iint\limits_S ydx + zdy + xdz = \iint\limits_S \operatorname{rot} \vec{F} \cdot \overrightarrow{n_0} \cdot dS = -\sqrt{3} \iint\limits_S dS = -\sqrt{3}\pi R^2.$$