POSTS AND TELECOMMUNICATIONS INSTITUTE OF TECHNOLOGY

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Chapter 2: MULTIPLE INTEGRALS

CALCULUS 2

Faculty of Fundamental Science 1

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2.1.1 Definite integral depends on a parameter

Definition 2.1

Let $f : [a, b] \times [c, d] \to \mathbb{R}$, if for each fixed $y \in [c, d]$ the function $f(x, y)$ is integral over $[a, b]$ on the x variable, we define the following function $F : [a, b] \to \mathbb{R}$ as

$$F(y) = \int_a^b f(x, y)dx$$

is called an integral depending on a parameter. The function $F(y)$ has the following properties:

Theorem 2.1 (Continuity)

If the function $f(x, y)$ is continuous on $[a, b] \times [c, d]$ then $F(y)$ is continuous on $[c, d]$.

Note 2.1

If $f(x,y)$ is continuous on $[a,b]\times[c,d]$, and $\alpha(y),\beta(y)$ are continuous on $[c,d]$ with $a\le\alpha(y),\beta(y)\le b$, $\forall y\in[c,d]$ then $F(y)=\int\limits_{\alpha(y)}^{\beta(y)} f(x,y)dx$ is continuous on $[c,d]$.

Example 2.1

Let the function $f(x)$ be continuous on $[0,1]$. Prove that

$$F(y)=\int\limits_0^1 \frac{y^2 f(x)}{x^2+y^2}dx$$

is continuous on $(0,+\infty)$.

Theorem 2.2 (Differentability)

If $f(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ are continuous on $[a,b]\times[c,d]$, then $F(y)$ differentiable on $[c,d]$ and $F'(y)=\int\limits_a^b \dfrac{\partial f}{\partial y}(x,y)dx.$

Theorem 2.3 (Leibniz's Theorem)

Let $f(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ be continuous functions on $[a,b]\times[c,d]$, and $\alpha(y),\beta(y)$ are differentiable functions on $[c,d]$ with image on $[a,b]$, that is, $\alpha,\beta:[a,b]\to[c,d]$, $\forall x\in[\alpha(y),\beta(y)]\subset[c,d]$. We define $F(y)=\int\limits_{\alpha(y)}^{\beta(y)} f(x,y)dx$, then $F(y)$ is differentiable on $[c,d]$ and

$$F'(y)=f(\beta(y),y).\beta'(y)-f(\alpha(y),y).\alpha'(y)+\int\limits_a^b \frac{\partial f}{\partial y}(x,y)dx.$$

2.1.1 Definite integral depends on a parameter

Example 2.2

Calculate the derivative of the following function

$$F(y) = \int_0^1 \arctan \frac{x}{y} dx, \; y > 0.$$

Theorem 2.4 (Integral)

Let $f(x, y)$ be integrable over $[a, b] \times [c, d]$, then $F(y)$ is integrable on $[c, d]$ and

$$\int_c^d F(y) dy = \int_c^d \left(\int_a^b f(x, y) dx \right) dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx.$$

Example 2.3

Calculating integrals

$$I = \int\limits_0^1 \frac{x^b - x^a}{\ln x} dx, \ b > a > 0.$$

2.1.2 Improper integral depends on parameter

Definition 2.2

1. Let $f : D := [a, +\infty) \times [c, d] \to \mathbb{R}$, if for each fixed $y \in [c, d]$ the function $f(x, y)$ is integrable over $[a, +\infty)$ on the x variable, we define
$$F(y) = \int_a^{+\infty} f(x, y)dx$$
is called an improper integral of depending on a parameter of y.
2. The function $F(y)$ is called uniformly converge for each $y \in [c, d]$, if $\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon, y) > 0,\ \forall b \geq n_0 \Rightarrow \left| \int_b^{+\infty} f(x, y)dx \right| < \varepsilon.$
3. The function $F(y)$ is called uniformly converge on the interval $[c, d]$, if $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}^*,\ \forall b \geq n_0 \Rightarrow \left| \int_b^{+\infty} f(x, y)dx \right| < \varepsilon,\ \forall y \in [c, d].$

Theorem 2.5 (Weierstrass' theorem)

If the function $\int\limits_a^{+\infty} h(x)dx$ converges and $|f(x,y)| \le h(x),\ \forall (x,y) \in D$ then the function $F(y)$ is uniformly converge on $[c,d]$.

Example 2.4

Prove that

$$\int\limits_1^{+\infty} \frac{\cos(x+2y)}{x^2+y^2}dx$$

is continuous on $\mathbb{R}$.

2.1.2 Improper integral depends on parameter

Theorem 2.6

If the function $f(x, y)$ is continuous on $[a, +\infty) \times [c, d]$ and the function $F(y)$ is uniformly convergent on $[c, d]$ then then $F(y)$ is continuous on $[c, d]$.

Example 2.5

Prove that

$$\int_{1}^{+\infty} \frac{x}{2 + x^y} dx$$

is continuous on $(2, +\infty)$.

2.1.2 Improper integral depends on parameter

Theorem 2.7

If the function $f(x,y)$ is continuous on $[a,+\infty)\times[c,d]$ and the function $F(y)$ is uniformly convergent on $[c,d]$ then $F(y)$ is differentiable on $[c,d]$ and

$$\int\limits_c^d F(y)dy = \int\limits_c^d \left(\int\limits_a^{+\infty} f(x,y)dx \right) dy = \int\limits_a^{+\infty} \left(\int\limits_c^d f(x,y)dy \right) dx.$$

Example 2.6

Given $b>a>0$, calculate the following integral:

$$I = \int\limits_0^{+\infty} \frac{e^{-ax}-e^{-bx}}{x} dx.$$

2.1.2 Improper integral depends on parameter

Theorem 2.8

Let $f(x,y)$ be define on the D satisfying the following assumptions:
1) the function $f(x,y)$ is continuous in the variable x on $[a,+\infty)$ for each $y \in [c,d]$,
2) a function $f'_y(x,y)$ is continuous in domain D,
3) the funtion $F(y)$ converges for each $y \in [c,d]$,
4) a integral $\int\limits_a^{+\infty} f'_x(x,y)dx$ converges uniformly on the $[c,d]$

Therefor $F'(y) = \int\limits_a^{+\infty} f'_x(x,y)dx.$

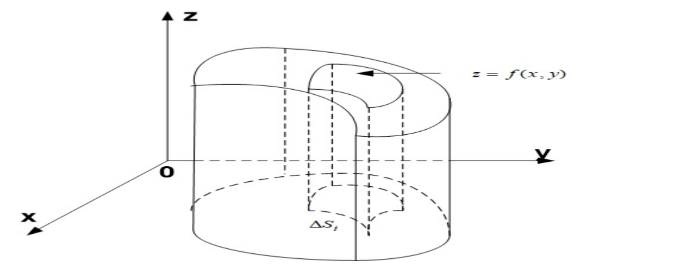
Example 2.6

Find the derivative of the function
$F(y) = \int\limits_0^{+\infty} \dfrac{1-\cos xy}{xe^{2x}}dx, y \in (0,+\infty)$, and find the function $F(y)$.

Problem

Calculate the volume of the bounded domain V is given by:

+ (Oxy) is a plane.
+ The axis Oz and the standard curve L is the boundary of the finite closed domain $D \subset (Oxy)$.
+ The curved surface is the graph of a function of two variables $z = f(x, y), (x, y) \in D$.



2.2.1 Definition of double integrals

Definition

Let the function $z = f(x, y)$ define on a closed domain $D \subset R^2$.

+ Divide D into n small regions by a grid of curves, name and area the domains as $\Delta S_i (i = 1, \ldots, n)$ and denoted d_i is the diameter of the second piece i.
+ Choose an arbitrary point $M_i\left(x_j, y_j\right) \in \Delta S_i$.
  Then $I_n = \sum_{i=1}^{n} f\left(x_i, y_i\right) \Delta S_{\mathrm{i}}$ is called the sum of the integrals of $f(x, y)$ on the domain D D corresponds to the partition and how to choose the points $M_1, M_2, \ldots M_n$ as above when $n \to \infty$ so that $\max d_i \to 0$ but I_n does not depend on the partition ΔS_i and how to choose $M_i\left(x_i, y_j\right) \in \Delta S_i$ then number I is called the double integrals of $f(x, y)$ on the domain D and the symbol is

$$\iint_D f(x, y) dS \quad \text{So} \quad \iint_D f(x, y) dS = I = \lim_{\max d_i \to 0} \sum_{i=1}^{n} f\left(x_i, y_i\right) \Delta S_i$$

Note

1) Since the double integral does not depend on the division of the domain D should be able to divide D by a grid of lines parallel to the coordinate axes Ox, Oy . Then $dS = dx \cdot dy$. Therefore, the double integral is denoted by

$$I = \iint_D f(x, y) dx dy$$

2) Like definite integrals, the symbol of a variable that is double integrated does not change the double integral, that is,

$$\iint_D f(x, y) dx dy = \iint_D f(u, v) du dv = I.$$

2.2.2 Integral conditions of double integrals

- If the function $f(x, y)$ is integrable over the domain D then $f(x, y)$ is bounded on the domain D (necessary condition of integrable function).
- If the function $f(x, y)$ is continuous on the D , more general: If the function $f(x, y)$ is only a discontinuity of type 1 on domain D, then it is integrable on the domain D.

2.2.3 Properties of double integrals

Let $f(x, y), g(x, y)$ be integrable on D. Then, we have

1) $\iint_D [f(x, y) \pm g(x, y)] dxdy = \iint_D f(x, y) dxdy \pm \iint_D g(x, y) dxdy.$
2) $\iint_D k \cdot f(x, y) dxdy = k \iint_D f(x, y) dxdy, \ \forall k.$
3) If $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$ then

$$\iint_D f(x, y) dxdy = \iint_{D_1} f(x, y) dxdy + \iint_{D_2} f(x, y) dxdy$$

2.2.3 Properties of double integrals

4) If $f(x,y) \le g(x,y), \forall (x,y) \in D$ then

$$\iint_D f(x,y)dxdy \le \iint_D g(x,y)dxdy$$

5) , If $f(x,y)$ is integral on D then $|f(x,y)|$ is also integrable on D and

$$\left|\iint_D f(x,y)dxdy\right| \le \iint_D |f(x,y)|dxdy$$

6) If $f(x,y)$ is integral on D and satisfies $m \le f(x,y) \le M, \forall (x,y) \in D$ then

$$mS \le \iint_D f(x,y)dxdy \le MS.$$

where S is the area of the domain D.

2.2.4 Double integrals over rectangles

Theorem 2.2.1 (Fubini's Theorem)

Let $f(x, y)$ be continuous on $D = [a, b] \times [c, d]$ (the domain D is a rectangular domain). We have

$$\iint\limits_D f(x,y)dxdy = \int\limits_a^b dx\left(\int\limits_c^d f(x,y)dy\right) = \int\limits_c^d dy\left(\int\limits_a^b f(x,y)dx\right).$$

Example 2.2.1

Calculating integrals $I = \iint\limits_D (2x + y)dxdy$, where $D = [1, 2] \times [0, 2]$.

Example 2.2.2

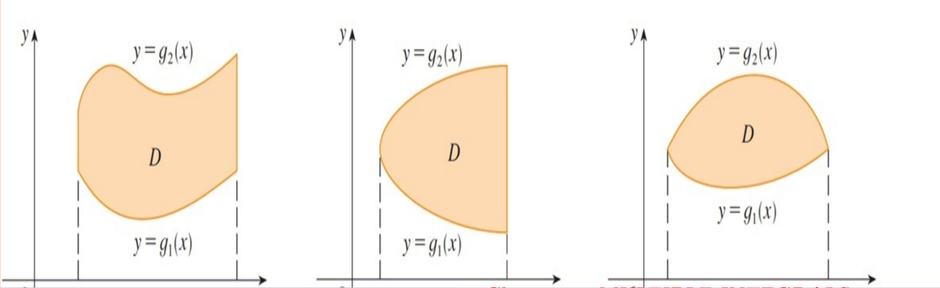
Calculating integrals $I = \iint\limits_D xy^2 dxdy$, where $D = [0, 2] \times [0, 3]$.

2.2.4 Double integrals over general regions

Type I regions

A plane region D is said to be of type I if it lies between the graphs of two continuous functions of x, that is
$D = \left\{(x,y) \in \mathbb{R}^2 \middle| a \le x \le b,\ g_1(x) \le y \le g_2(x)\right\}$. If $f(x,y)$ is continuous on a type I region D then

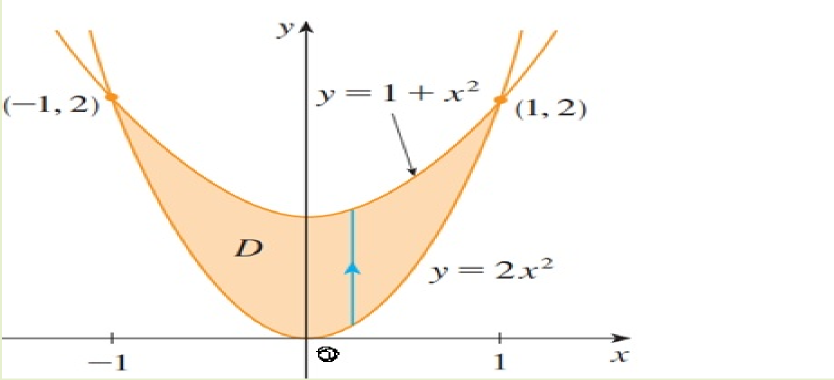
$$I = \iint\limits_D f(x,y)dxdy = \int\limits_a^b dx\left(\int\limits_{g_1(x)}^{g_2(x)} f(x,y)dy\right).$$



Example 2.2.3

Evaluate $I = \iint\limits_{D} (x^2 + 2y)dxdy$, where D is a region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

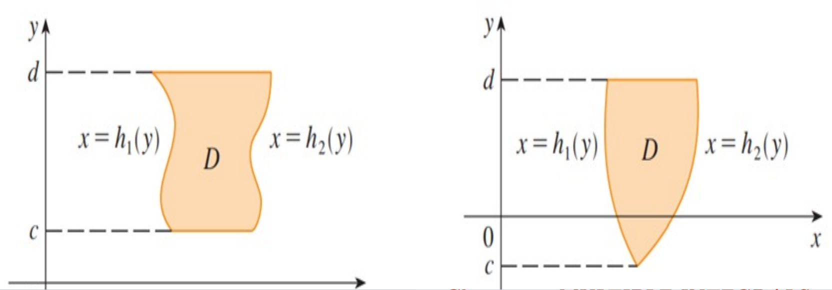
Solution



Type II regions

A plane region D is said to be of type II if it lies between the graphs of two continuous functions of y, that is
$D = \{(x, y) \in \mathbb{R}^2 \big| c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$. If $f(x, y)$ is continuous on a type II region D then

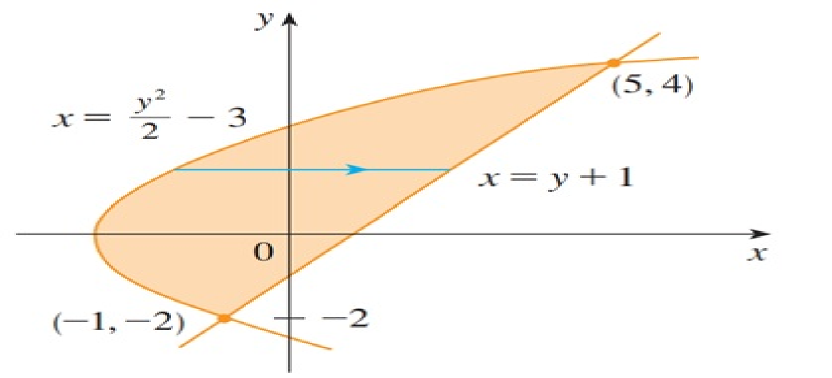
$$I = \iint\limits_D f(x, y)dxdy = \int\limits_c^d dx \left(\int\limits_{h_1(y)}^{h_2(y)} f(x, y)dx \right).$$



Example 2.2.4

Evaluate $I = \iint\limits_D xydxdy$, where D is a region bounded by the parabolas $y = x - 1$ and $y^2 = 2x + 6$.

Solution



Example 2.2.5

Change the order of integration in double integrals

$$a) I = \int_0^2 dx \int_x^{2x} f(x,y)dy.$$

.

$$b) J = \int_{-2}^{6} dy \int_{-\frac{y^2}{2}-1}^{2-y} f(x,y)dx.$$

$$c) K = \int_0^1 dx \int_x^{\sqrt{2-x^2}} f(x,y)dy.$$

2.2.5 Change of variables in double integrals

Let the function $f(x, y)$ be continuous on the domain $D \subset (Oxy)$ and assume the transformation $(x, y) \to (u, v) : \begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$ satisfying the condition

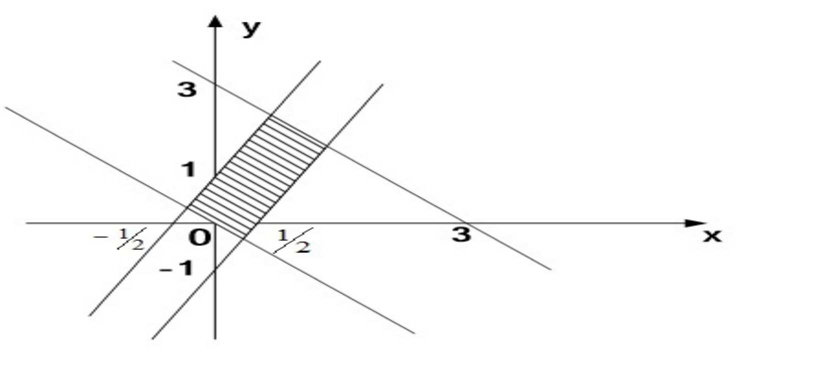
- The above transformation is a bijective from Δ to the domain D or $(x, y) \in D \Leftrightarrow (u, v) \in \Delta$.
- The $x(u, v), y(u, v)$ are the continuous partial derivatives on the domain $\Delta \subset (O'uv)$.
- The Jacobi determinant is $\frac{D(x,y)}{D(u,v)} \neq 0$ on the domain Δ (or just zero at some isolated point) then

$$I = \iint_D f(x, y)dxdy = \iint_\Delta f[x(u, v), y(u, v)] \cdot \left|\frac{D(x, y)}{D(u, v)}\right| dudv.$$

Example 2.2.6

Calculating integrals $I = \iint_D (x+y)dxdy$, where D is $y=-x, y=-x+3, y=2x-1, y=2x+1$.

Solution

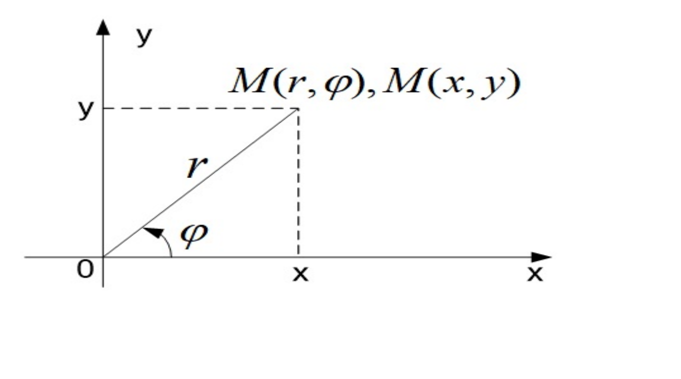


Example 2.2.7

Calculating integrals $I = \iint_D x^3 dxdy$, where D is $y = \frac{1}{x}, y = \frac{2}{x}, y = x^2, y = \frac{x^2}{2}$.

a. polar coordinate system

A polar coordinates are set of real numbers (r, φ) so that $r = |\overrightarrow{OM}|, \varphi = (Ox, \overrightarrow{OM})$



b. calculate the double integrals

Set

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\varphi = \frac{y}{x} \end{cases} \Rightarrow D \to \Delta = \left\{(r,\varphi) \mid \begin{cases} 0 \le \varphi < 2\pi \\ 0 \le r < +\infty \end{cases}\right.$$

$$J = \frac{D(x,y)}{D(r,\varphi)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

Then the double integrals in polar coordinates has the form

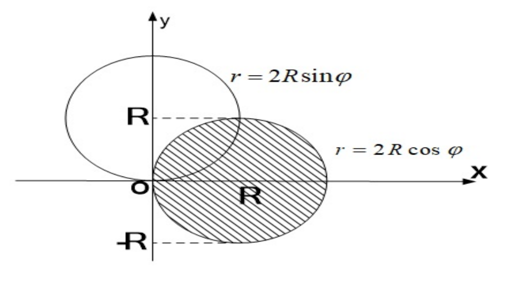
$$I = \iint_D f(x,y)dxdy = \iint_\Delta f(r\cos\varphi, r\sin\varphi)rdrd\varphi.$$

Example 2.2.8

Calculate $I = \iint_D \sqrt{x^2 + y^2}dxdy$, where, the domain D is defined by

$$D = \left\{(x, y) : x^2 + y^2 \leq 2Ry, x^2 + y^2 \geq 2Rx\right\}.$$

Solution



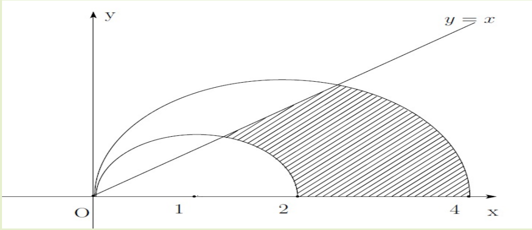
1. Finding area of a plane regions

If $f(x,y) = 1,\ \forall (x,y) \in D$ then the measure of the area of the domain D is calculated according to the formula $S_D = \iint_D dxdy$.

Example 2.2.9

Calculate the area of the domain D given by

$$D = \left\{(x,y) : (x-1)^2 + y^2 = 1,\ (x-2)^2 + y^2 = 4,\ y = x,\ y = 0\right\}.$$



2. Computing volumes

If $f(x, y) \geq 0, \ \forall (x, y) \in D$ then the volume of the curved cylinder bounded by the function graph is calculated by the formula

$$V = \iint_D f(x, y) dx dy.$$

Example 2.2.10

Calculate the volume of the figure V given by the following faces

$$z = x^2 + y^2, \ y = x^2, \ y = 1, \ z = 0.$$

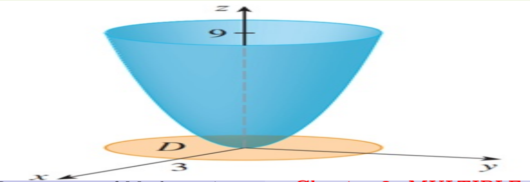
3. Surface area

For surface $(S) : z = f(x, y),\ (x, y) \in D$ has partial derivatives f'_x, f'_y exist and are continuous on domain D. Then the surface area of S is defined as

$$A(S) = \iint\limits_D \sqrt{1 + f'^2_x + f'^2_y}\,dxdy.$$

Example 2.2.11

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



4. Density mass

Suppose the lamina occupies a region D of the xy-plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. The density mass of the lamina is

$$m = \iint\limits_D \rho(x, y) dxdy.$$

If the plate is homogenous, that is $\rho(x, y) = \text{const}, \forall(x, y) \in D$, choose $\rho(x, y) = 1, \forall(x, y) \in D$ then the mass of the plate D is calculated by the formula $m = \iint_D dxdy = S_D$

5. Center of mass

$$x_G = \frac{1}{m} \iint_D x\rho(x, y)dxdy,\ y_G = \frac{1}{m} \iint_D y\rho(x, y)dxdy,$$

where $m = \iint_D \rho(x, y)dxdy$.

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where $m = \iint_D \rho(x, y)dxdy$.

6. Moment of inertia

According to the definition of the moment of inertia of the particle about the Ox, Oy -axis and the origin O, we have

$$I_{Ox} = my^2;\ I_{Oy} = mx^2; \quad I_O = m\left(x^2 + y^2\right)$$

Moment of inertia of the plate about the axes Ox, Oy and the origin O are

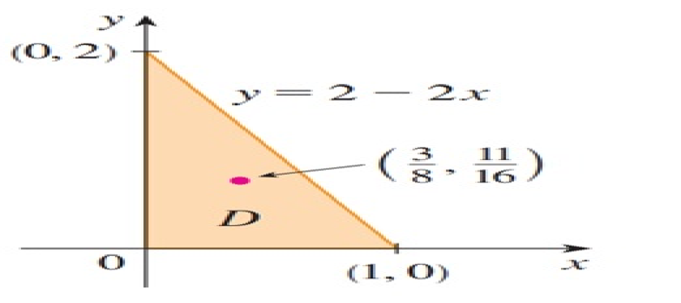
$$I_{Ox} = \iint_D y^2 \rho(x,y)dxdy; I_{Oy} = \iint_D x^2 \rho(x,y)dxdy;$$

$$I_O = \iint_D \left(x^2 + y^2\right)\rho(x,y)dxdy.$$

Example 2.2.12

Find the mass and center of mass of a triangular lamina with vertices $(0,0),(1,0), and (0,2)$ if the density function is $\rho(x,y) = 1 + 3x + y$.

Solution



Problem

Calculate the mass of the non-homogeneous body V, given that the density is $\rho = \rho(x, y, z), (x, y, z) \in V$. Similar to the double integral, we divide V arbitrarily into n parts that do not step on each other. Name and volume of the parts $\Delta V_i (i = \overline{1, n})$. Choose an arbitrary point $P_i\left(x_i, y_i, z_i\right) \in \Delta V_i$ and the $d_i, (i = \overline{1, n})$ are diameters of $\Delta V_i (i = \overline{1, n})$. We have

$$m \approx \sum_{i=1}^{n} \rho\left(P_i\right) \Delta V_i = \sum_{i=1}^{n} \rho\left(x_i, y_i, z_i\right) \Delta V_i$$

The mass of the object is

$$m = \lim_{\max d_i \to 0} \sum_{i=1}^{n} \rho\left(x_i, y_i, z_i\right) \Delta V_i$$

2.3.1 Definition of triple integrals

Definition

Let the function $f(x, y, z)$ define on the domain $V \subset R^3$.

- Divide V into n pieces, name and volume of the piece are $\Delta V_i(i = \overline{1, n})$, the piece diameter symbol ΔV_i is d_i, $i = \overline{1, n}$.
- Choose an arbitrary point $P_i\left(x_i, y_i, z_i\right) \in \Delta V_{i,}(i = \overline{1, n})$.
- The totals $I_n = \sum_{i=1}^{n} f\left(x_i, y_i, z_i\right) \Delta V_i$ is called the sum of integrals the triple of the function $f(x, y, z)$ taken over the domain V corresponds to a fraction plan and points $P_i \in \Delta V_i, (i = \overline{1, n})$.. When $n \to \infty$ such that $\max d_i \to 0$, we get I_n converges to $I \in \mathbb{R}$ regardless of the partition ΔV_i and how point $P_i\left(x_i, y_i, z_i\right) \in \Delta V_i$ is chosen, the number I is called a triple integral of $f(x, y, z)$ over the region V and denoted by

$$\iiint_V f(x, y, z) dV = I = \lim_{\max_i d_i \to 0} \sum_{i=0}^{n} f\left(x_i, y_i, z_i\right) \Delta V_i$$

2.3.1 Definition of triple integrals

Note

- Like the double integrals, the volume factor dV is replaced by $dxdvdz$ and then the triple integral is usually denoted by

$$I = \iiint_V f(x,y,z)dxdydz$$

- Similar to the double integrals, triple integrals do not depend on the notation of the variable being integrated

$$\iiint_V f(x,y,z)dxdydz = \iiint_V f(u,v,w)dudvdw$$

- If the function f(x, y, z) is continuous on the closed, bounded domain $V \in \mathbb{R}^3$, then it's integrable on V.
- The integral conditions and properties of triple integrals are similar to the double integrals.

Theorem 2.3.1 (Fubini's Theorem)

If $f(x, y, z)$ is continuous on the rectangular box $V = [a, b] \times [c, d] \times [r, s]$, then

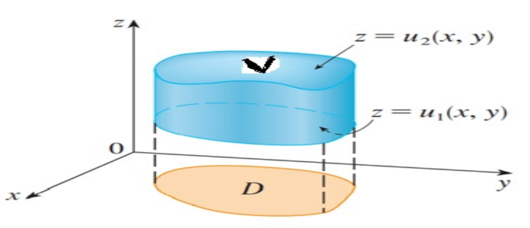
$$\iiint\limits_V f(x, y, z)dxdydz = \int\limits_a^b \int\limits_c^d \int\limits_r^s f(x, y, z)dxdydz.$$

Example 2.3.1

Calculating integrals $I = \iiint\limits_V xyz^2 dxdydz$, where

$$V = [0, 1] \times [-1, 2] \times [0, 3].$$

The region of type I



A solid region V is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,
$V = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$, then

$$I = \iiint\limits_V f(x, y, z)dxdydz = \iint\limits_D \left[\int\limits_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz \right] dxdy.$$

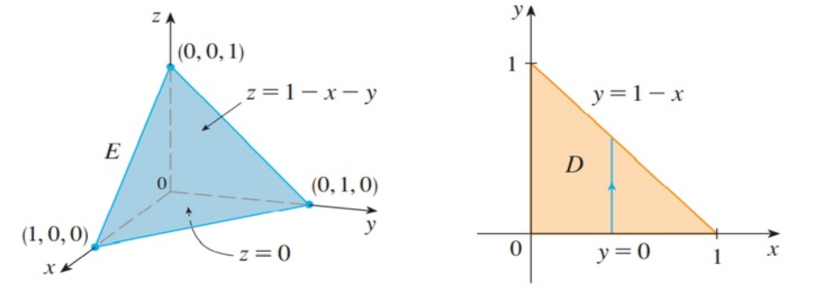
2.3.2 The triple integral over a general bounded region

where D is the projection of V onto xy-plane, $z = u_1(x, y)$ is the lower surface and $z = u_2(x, y)$ is the upper surface.

Example 2.3.2

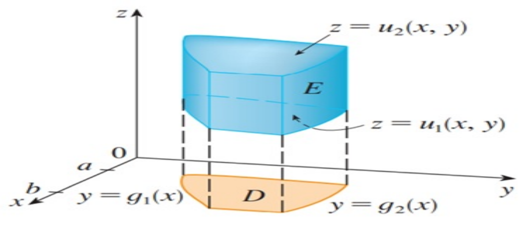
Evaluate $I = \iiint\limits_V z dx dy dz$, where V is the solid tetrahedron bounded by the four planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.

Solution



The region of type II

If the projection D of V onto the xy-plane is of type II plane region, then
$V = \{(x, y, z) | a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x),\ u_1(x, y) \leq z \leq u_2(x, y)\}$,
and we have



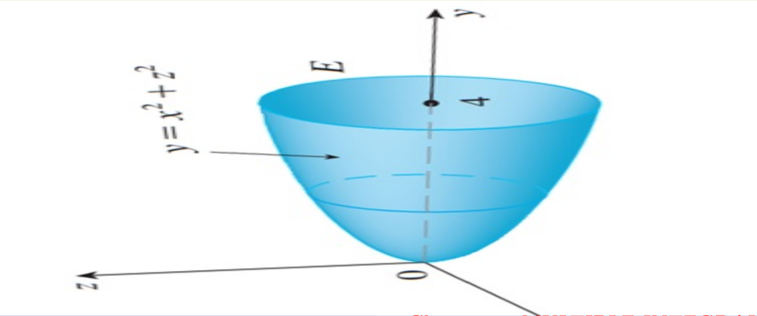
$$I = \iiint\limits_V f(x, y, z)dxdydz = \int_a^b dx \int_{g_1(x)}^{g_2(x)} dy \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz.$$

where D is the projection of V onto xy-plane, $z = u_1(x, y)$ is the lower surface and $z = u_2(x, y)$ is the upper surface.

Example 2.3.3

Evaluate $\iiint\limits_V \sqrt{x^2 + z^2}\,dxdydz$, where V is the solid tetrahedron bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Solution



2.3.3 Change of variables in triple integrals

For the function $f(x, y, z)$ to be continuous on the domain $V \subset Oxyz$ and and assume the transformation

$$(x, y, z) \to (u, v.w) : \begin{cases} x = x(u, v), w) \\ y = y(u, v, w), \quad (u, v, w) \in \Omega \\ z = z(u, v, w) \end{cases}$$ satisfy the

conditions

- The above transformation is a bijective from Ω to the domain V or $(x, y, z) \in V \Leftrightarrow (u, v, w) \in \Omega$.
- The $x(u, v, w), y(u, v, w), z(u, v, w)$ are the continuous partial derivatives on the domain $\Omega \subset (O'uvw)$.
- The Jacobi determinant is $J = \frac{D(x,y,z)}{D(u,v,w)} \neq 0$ on the domain Ω, then

$$\iiint_V f(x, y, z)dxdydz = \iiint\limits_{\Omega} f[x(u, v, w), y(u, v, w), z(u, v, w)]|J|du$$

Example 2.3.4

Evaluate $\iint\limits_V (x+y)(x-z)dxdydz$, where V is the bounded domain by the planes
$x+y=0,\ x+y=1;\ y+z=1,\ y+z=2;\ x+y-z=2,\ x+y-z=3.$.

Solution

Set

$$u = x+y, v = y+z, w = x+y-z$$

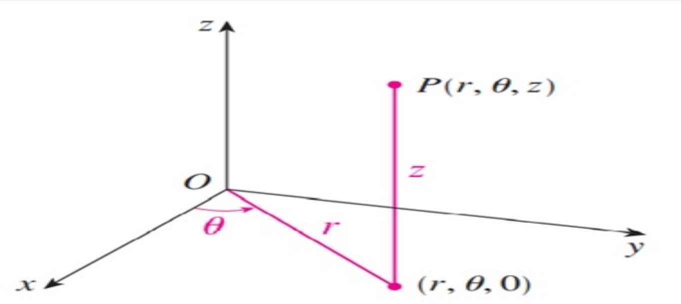
$$0 \le u \le 1, 1 \le v \le 2, 2 \le w \le 3$$

$$\frac{D(u,v,\mathrm{w})}{D(x,y,z)} = -1 \Rightarrow \frac{D(x,y,z)}{D(u,v,\mathrm{w})} = -1, (x+y)(x-z) = u(u-v)$$

$$I = \iiint_\Omega u(u-v)|-1|dudvdw = \int_0^1 udu \int_1^2 (u-v)dv \int_2^3 dw = -\frac{5}{12}.$$

Cylindrical coordinates

In the cylindrical coordinate system, a point P in three-dimensional space is represented by the ordered triple (r, θ, z) , where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane. The connections between cylindrical coordinates and rectangular coordinates are



Evaluating triple integrals with cylindrical coordinates

Set $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$ then $V \to \Omega : \begin{cases} r \geq 0 \\ 0 \leq \varphi < 2\pi \\ -\infty < z < +\infty \end{cases}$.

The Jacobi determinant of the functions x, y, z in terms of r, φ, z are

$$J = \frac{D(x,y,z)}{D(r,\varphi,z)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

The formula for triple integration in cylindrical coordinates is

$$I = \iiint_V f(x,y,z)dxdydz = \iiint\limits_{\Omega} f(r\cos\varphi, r\sin\varphi, z)rdrd\varphi dz.$$

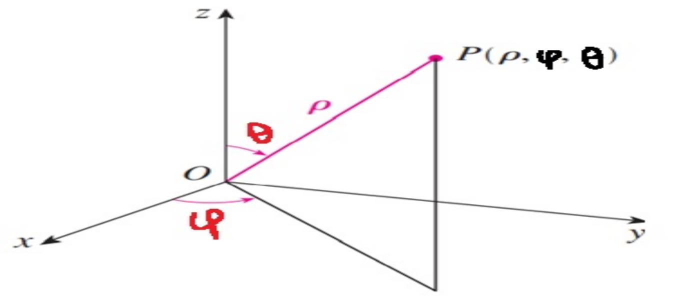
Example 2.3.5

Evaluate $I = \iiint\limits_V (x^2 + y^2 + 3z^2)dxdydz$, where

$$V = \{(x, y, z) | \sqrt{x^2 + y^2} \leq z \leq 2\}.$$

Spherical coordinates

The spherical coordinates of a point P in space are (ρ, φ, θ), where ρ is the distance from P to the origin, φ is the same angle as in cylindrical coordinates, and θ is the angle between the positive z-axis and the line segment OP. Note that $\rho \geq 0,\ 0 \leq \theta \leq \pi$.



Evaluating triple integrals with spherical coordinates

Set $\begin{cases} x = r\cos\varphi\cos\theta \\ y = r\cos\varphi\sin\theta \\ z = r\sin\theta \end{cases}$ then $V \to \Omega : \begin{cases} r \geq 0 \\ 0 \leq \varphi < 2\pi \\ 0 \leq \theta \leq \pi \end{cases}$.

The Jacobi determinant of the functions x, y, z in terms of r, φ, z are

$$J = \frac{D(x,y,z)}{D(r,\varphi,\theta)} = \begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi & r\cos\theta\sin\varphi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} = -r^2\sin\theta.$$

The formula for triple integration in spherical coordinates is

$$I = \iiint\limits_{\Omega} f(r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, r\sin\theta) r^2 \sin\theta dr d\varphi d\theta.$$

2.3.5 Triple integrals in spherical coordinates

Evaluating triple integrals with spherical coordinates

Set $\begin{cases} x = r\cos\varphi\cos\theta \\ y = r\cos\varphi\sin\theta \\ z = r\sin\theta \end{cases}$ then $V \to \Omega : \begin{cases} r \geq 0 \\ 0 \leq \varphi < 2\pi \\ 0 \leq \theta \leq \pi \end{cases}$.

The Jacobi determinant of the functions x, y, z in terms of r, φ, z are

$$J = \frac{D(x,y,z)}{D(r,\varphi,\theta)} = \begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi & r\cos\theta\sin\varphi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} = -r^2\sin\theta.$$

The formula for triple integration in spherical coordinates is

$$I = \iiint\limits_{\Omega} f(r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, r\sin\theta)r^2\sin\theta drd\varphi d\theta.$$

2.3.5 Triple integrals in spherical coordinates

Example 2.3.6

Evaluate $I = \iiint\limits_V \sqrt{x^2+y^2+z^2}\,dxdydz$, where

a) V is the unit ball.

b) $V = \{(x,y,z) | x^2+y^2+z^2 \leq 4\}$.

c) $V = \{(x,y,z) | 0 \leq z \leq \sqrt{4-x^2-y^2}\}$.

2.3.6 Applications of triple integrals

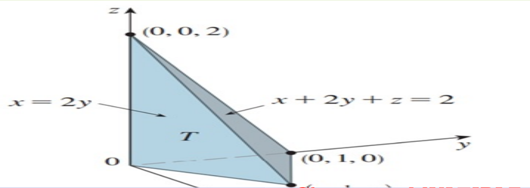
1. Volume

If $f(x, y, z) = 1$ for all point in V. Then the triple integral does represent the volume of V

$$V = \iiint\limits_V dV = \iiint\limits_V dxdydz.$$

Example 2.3.7

Use triple integral to find the volume of the tetrahedron V bounded by the planes $x + 2y + z = 2, x = 2y, x = 0$, and $z = 0$.



2. Mass of a solid object

If the density function of a solid object that occupies the region V is $\rho(x, y, z)$ (in units of mass per unit volume) at any given point (x, y, z), then its mass is

$$m = \iiint\limits_{V} \rho(x, y, z) dV = \iiint\limits_{V} \rho(x, y, z) dx dy dz.$$

3. Moments

Its moments about the three coordinate planes are

$$I_{Ox} = \iiint\limits_V (y^2 + z^2)\rho(x,y,z)dV, \quad I_{Oy} = \iiint\limits_V (x^2 + z^2)\rho(x,y,z)dV$$

$$I_{Oz} = \iiint\limits_V (x^2 + y^2)\rho(x,y,z)dV, \quad I_{O} = \iiint\limits_V (x^2 + y^2 + z^2)\rho(x,y,z)dV.$$

4. Center of mass

The center of mass is located at the point G, where

$$x_G = \frac{1}{m}\iiint_V x\rho(x,y,z)dV, \quad y_G = \frac{1}{m}\iiint_V y\rho(x,y,z)dV$$

$$z_G = \frac{1}{m}\iiint_V z\rho(x,y,z)dV, \qquad m = \iiint_V \rho(x,y,z)dV.$$

Example 2.3.8

Find the center of mass (if the density is constant, the center of mass is called the centroid) of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0$, and $x = 1$.