POSTS AND TELECOMMUNICATIONS INSTITUTE OF TECHNOLOGY

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Chapter 4: DIFFERENTIAL EQUATIONS

CALCULUS 2

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Outline of differential equations

4.1.1. Definition

Definition 1

A differential equation is an equation involving an unknown function and its derivatives of the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \text{ or } F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) = 0,$$

where x is an independent variable, $y = y(x)$ is the function to find, $y', y'', ..., y^{(n)}$ are the derivatives of the function must find.
The order of a differential equation is the largest derivative present in the differential equation.

Example 1

$y' - x^4y^3 = 5, \ (x^3 + y^2)dx - (x^2 + y^2)dy = 0$ are first-order differential equations.
$y'' - 2x^3(y')^3 = 5$ is a second-order differential equation.

4.1.1. Definition

A differential equation is called an n th-order linear differential equation if the function F is first order with respect to $y, y', \ldots, y^{(n)}$, that is, the equation has the form:
$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x)$, where $a_1(x), \ldots, a_n(x),\ f(x)$ are continuous functions on (a, b).
If $f(x) \equiv 0, \forall x \in (a, b)$ then it is called an n-th order linear homogeneous differential equation.
If $f(x) \neq 0,\ x \in (a, b)$ then it is called a non-homogeneous.

Example 2

$y'' - x^2y = 5x^2 + 1$ is a second-order linear non-homogeneous differential equation and $y' - x^2y = 0$ is a first-order linear homogeneous differential equation.

Definition 2

A solution of a differential equation in the unknown function y and the independent variable x on the interval J is a function y(x) that satisfies the differential equation identically for all x in J.

+) The solution is an explicit function $y = y(x, C_1, C_2, ..., C_n)$ depends on the constants $C_1, C_2, ..., C_n$ is called the general solution.

Example 3

The function $y(x) = C_1 \sin 2x + C_2 \cos 2x$, where C_1 and C_2 are arbitrary constants, is a general solution of $y'' + 4y = 0$ in the interval $(-\infty, +\infty)$.

+) The solution is the implicit function $\Phi(x, C_1, C_2, ..., C_n) = 0$ depends on which constants $C_1, C_2, ..., C_n$ are called is the general integral.

Example 4

The function $2e^{y^3} + xy^2 = C$ with C is arbitrary constants, is a general integral of $6y'ye^{y^3} + y^2 + 2xy'y = 0$.

+) If we give the constants $C_1, C_2, ..., C_n$ determined values, then the general solution (general integral) is called a particular solution (particular integral).

4.2.1 Introduction to first-order differential equation

Standard form for a first-order differential equation in the unknown function $y(x)$ is

$$F(x, y, y') = 0 \text{ or } F(x, y, \frac{dy}{dx}) = 0 \text{ or } M(x, y)dx + N(x, y)dy = 0 \quad (1)$$

If from (1) we can solve for y', then the first-order differential equation has been solved for the derivative:

$$y' = f(x, y) \quad (2)$$

Example 1

$y' = 3x^5y^3 + 2, \quad (2x - 3y)dx - (2x^2 - y)dy =$ are the first-order differential equations.

Cauchy-Peano's theorem (Existence and uniqueness theorem)

Consider the differential equation (2): $y' = f(x, y)$ and $M_0(x_0, y_0) \in D \subset \mathbb{R}^2$.

Theorem 4.1 If $f(x, y)$ is continuous on the domain D in the plane of Oxy, then there exists a solution $y = y(x)$ in the neighborhood x_0 satisfy

$$y_0 = y(x_0) \quad (3)$$

In addition, if $\dfrac{\partial f}{\partial y}(x, y)$ is also continuous on domain D, then the found solution is unique.

The problem of finding a solution of a differential equation satisfying the condition (3) is called a Cauchy problem. Condition (3) is called the initial condition.

4.2.2 Separable equations

Definition 1

Consider a differential equation in differential form (1). If $M(x, y) = f_1(x)$ (a function only of x) and $N(x, y) = f_2(y)$ (a function only of y), differential equation is separable, or has its variables separated.

Solution

The solution to the first-order separable differential equation $f_1(x)dx + f_2(y)dy = 0 \ \ (1.1)$ is $\int f_1(x)dx + \int f_2(y)dy = C \ \ (1.2)$ where C represents an arbitrary constant.

Example 2

Solve the equation: $\dfrac{dx}{dy} = \dfrac{3x^2 + 1}{2y}$. This equation may be rewritten in the differential form $(3x^2 + 1)dx = 2ydy$. Its solution is $\int(3x^2 + 1)dx - \int 2ydy = C$ or $x^3 + x - y^2 = C$.

4.2.3 Homogeneous equations

Definition 2

A differential equation in standard form $y' = f(x,y) \quad (1.3)$ is homogeneous if $f(tx,ty) = f(x,y)$ for every real number $t \neq 0$.
Consider $x \neq 0$. Then, we can write
$f(x,y) = f(x, x\frac{y}{x}) = f(1, \frac{y}{x}) := g(\frac{y}{x})$ for a function g depending only on the ratio $\frac{y}{x}$.

Solution

The homogeneous differential equation can be transformed into a separable equation by making the substitution $y = xu$ along with its corresponding derivative $y' = u + xu'$. This can be rewritten as
$\dfrac{du}{g(u) - u} = \dfrac{dx}{x}$ if $g(u) - u \neq 0$. The resulting equation in the variables u and x is solved as a separable differential equation.

4.2.3 Homogeneous equations

Example 3

Solve $y' = \dfrac{y+x}{x}$ for $x \neq 0$.
This differential equation is not separable. Instead it has the form $y' = f(x,y)$, with $f(x,y) = \dfrac{y+x}{x}$, where $f(tx,ty) = f(x,y)$, so it is homogeneous.

Example 4

Integral equation $(y-x+1)dx = (x+y+3)dy$.
Solution: $\dfrac{dy}{dx} = \dfrac{y-x-1}{x+y+3}$.

4.2.4 Linear equations

Definition 3

A first-order linear differential equation has the form

$$y' + p(x)y = q(x), \quad (1.4)$$

where $p(x),\ q(x)$ are continuous on (a, b). In other words, a linear differential equation of the first order is a differential equation in which the function to be found and its derivative are both in first order form.
- If $q(x) \neq 0,\ x \in (a, b)$ then (1.4) is called a non-homogeneous linear differential equation.
- If $q(x) \equiv 0,\ \forall x \in (a, b)$ then call it a homogeneous linear differential equation.

Method of solutions:

The general solution for equation (1.4) is

$$y = e^{-\int p(x)dx}\left(C + \int q(x)e^{\int p(x)dx}dx\right),$$

where C represents an arbitrary constant.

Example 5

Find the solution of the given initial value problem
$y' - \dfrac{3}{x}y = x^3, y(1) = 2.$

Example 6

Solve $e^y dx + (xe^y - 1)dy = 0.$
Solution: $x' - x = e^{-y}.$

4.2.5 Bernoulli equations

Definition 4

A Bernoulli differential equation has the form

$$y' + p(x)y = q(x)y^{\alpha}, \quad (1.5)$$

where α is a real number $\alpha \neq 0,\ \alpha \neq 1$.

Solution

If $\alpha > 0$, then $y = 0$ is a solution of (1.5). Otherwise, if $\alpha < 0$, then the condition is $y \neq 0$. In both cases, we now find the solutions $y \neq 0$. To do this we divide both sides by y^{α} to obtain $y^{-\alpha}y' + p(x)y^{1-\alpha} = q(x)$. The substitution $z = y^{1-\alpha}$ now transforms (1.5) into a linear differential equation in the unknown function $z = z(x)$.

Example 7

Solve $y' + xy = xy^2$.

Solution: This equation is not linear. It is, however, a Bernoulli differential equation having the form of equation (1.5) with $p(x) = q(x) = x$, and $\alpha = 2$. First, we can see that $y = 0$ is a solution of the equation. We now find the solution $y \neq 0$. To do so, we make the substitution: $z = y^{1-2} = y^{-1}$, from which follow $y = 1/z$ and $y' = -\frac{z'}{z^2}$. Substituting these equations into the given differential equation, we obtain the equation $z' - xz = -x$ which is linear for the unknown function $z(x)$.

Example 8

Solve $y' + y = e^{\frac{x}{2}}\sqrt{y}$.

4.2.6 Exact equations

Definition 5

A differential equation in differential form

$$M(x,y)dx + N(x,y)dy = 0 \quad (1.6)$$

is exact if there exists a function $g(x,y)$ such that
$dg(x,y) = M(x,y)dx + N(x,y)dy$ or $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}, \forall (x,y) \in D \subset \mathbb{R}^2 \quad (1.7)$.

Solution

To solve equation (1.6), assuming that it is exact, first solve the equations $\dfrac{\partial g(x,y)}{\partial x} = M(x,y), \quad \dfrac{\partial g(x,y)}{\partial y} = N(x,y)$ for g(x, y). We have $g(x,y) = \int\limits_{x_0}^{x} M(x,y)dx + \int\limits_{y_0}^{y} N(x_0,y)dy$. The solution to (1.6) is then given implicitly by $g(x,y) = C$, where $(x_0,y_0) \in D, C$ represents an arbitrary constant.

Example 9

Solve $2xydx + (1 + x^2)dy = 0$.

Solution: This equation has the form of equation (1.6) with $M(x, y) = 2xy$ and $N(x, y) = 1 + x^2$ are determined on $D = \mathbb{R}^2$. Since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y} = 2x$, the differential equation is exact. The solution to the differential equation, which is given implicitly by (1.7) as $g(x, y) = C$, is $x^2y + y = C$.

Example 10

Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$.

Integrating Factors

In general, equation (1.6) is not exact. Occasionally, it is possible to transform (1.6) into an exact differential equation by a judicious multiplication. A function $I(x,y)$ is an integrating factor for (1.6) if the equation

$$I(x,y)\big(M(x,y)dx + N(x,y)dy\big) = 0 \quad (1.8)$$

is exact. Some of the following special form integral factors:

1. if $\dfrac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv g(x)$, a function of x alone, then $I(x,y) = e^{\int g(x)dx}$.
2. item if $\dfrac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv g(y)$, a function of y alone, then $I(x,y) = e^{-\int g(y)dy}$.
3. item if $M = yf(x,y)$ and $N = xg(x,y)$, then $I(x,y) = \frac{1}{xM - yN}$.

Remark

In general, integrating factors are difficult to uncover. If a differential equation does not have one of the forms given above, then a search for an integrating factor likely will not be successful, and other methods of solution are recommended.

Example 11

Solve $ydx - xdy = 0$.

Solution: This equation is not exact. It is easy to see that an integrating factor is $I(x,y) = \dfrac{1}{x^2}$. Therefore, we can rewrite the given differential equation as $\dfrac{ydx - xdy}{x^2} = 0$ which is exact.

4.3.1 Introduction to second-order differential equation

Definition 1. A second-order differential equation has the form

$$F(x, y, y', y^{''}) = 0 \text{ or } F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0, \qquad (4.3.1a)$$

If from (4.3.1) we can solve for $y^{''}$, then the second-order differential equation has been solved for the derivative:

$$y^{''} = f(x, y, y'), \qquad (4.3.1b)$$

where f is some given function. Usually, we will denote the independent variable by t since time is often the independent variable in physical problems, but sometimes we will use x instead.

Example 1

$y^{''} = x^2y^5 + 2y^4, \quad (2x - 3y)y^{''} - (2x^2 - y)\sqrt[3]{y}^{''} = x^2y'$ are the

4.3.1 Introduction to second-order differential equation

Theorem 1 (Cauchy-Peano's theorem (Existence and uniqueness theorem))

Consider the differential equation (4.3.1b): $y^{''} = f(x, y, y')$ and $M_0(x_0, y_0, y_0') \in V \subset \mathbb{R}^3$.
If $f(x, y)$ is continuous on the domain V in the plane of $Oxyy'$, then there exists a solution $y = y(x)$ in the neighborhood x_0 satisfy

$$y_0 = y(x_0),\ y_0' = y'(x_0), \qquad (4.3.1c)$$

In addition, if $\dfrac{\partial f}{\partial y}(x, y, y'),\ \dfrac{\partial f}{\partial y'}(x, y, y')$ are also continuous on domain V, then the find solution is unique.

4.3.2 Second-order linear differential equations

Definition 2

A second-order differential equation is called linear if it can be written in the form

$$y^{''} + p(x)y' + q(x)y = f(x), \quad (4.3.2a)$$

where $p(x),\ q(x),\ f(x)$ are continuous on (a, b). In other words, a linear differential equation of the first order is a differential equation in which the function to be found and its derivative are both in first order form.
- If $f(x) \neq 0,\ x \in (a, b)$ then the second-order linear equation (4.3a) is said to be non-homogeneous.
- If $f(x) \equiv 0,\ \forall x \in (a, b)$ then the second-order linear equation (4.3a) is said to be homogeneous has form $y^{''} + p(x)y' + q(x)y = 0,\ (4.3.2b)$

Example 2

The following equations: $y^{''} + 2y = e^x \sin x$ and $y^{''} + 3xy' + 5y = 0$ are examples of nonhomogeneous and homogeneous second-order linear equations, respectively.

Theorem 3

If y_1 and y_2 are two solutions of the differential equation $(4.3.2b)$, then the linear combination $y = C_1y_1 + C_2y_2$ is also a solution for any values of the constants C_1 and C_2.

Proof

$y' = C_1y_1' + C_2y_2'$ and $y'' = C_1y_1'' + C_2y_2''$. We obtain

$$\begin{aligned}&(C_1y_1 + C_2y_2)'' + p(x)\left(C_1y_1 + C_2y_2\right)' + q(x)\left(C_1y_1 + C_2y_2\right)\\&+ C_1\left[y_1'' + p(x)y_1' + q(x)y_1\right] + C_2\left[y_2'' + p(x)y_2' + q(x)y_2\right] \equiv 0.\end{aligned}$$

The two functions $y_1(x)$ and $y_2(x)$ are called linearly independent on (a, b), if $\alpha_1 y_1(x) + \alpha_2 y_2(x) = 0$ for all $x \in (a, b)$ implies $\alpha_1 = \alpha_2 = 0$, and we call $y_1(x),\ y_2(x)$ linearly dependent on (a, b) if there exist $\alpha_1,\ \alpha_2$ not both zero such that $\alpha_1 y_1(x) + \alpha_2 y_2(x) = 0,\ t \in (a, b)$.

The $y_1(x) = e^x$ and $y_2(x) = e^{-2x}$ are linearly independent, because they are not proportional.

Wronski determinant (or Wronskian)

The Wronski determinant (or Wronskian) of the two solutions $y_1(x), y_2(x)$ of the equation $(4.3.2b)$ is defined by

$$W\left[y_1, y_2\right] = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Theorem 4

The two solutions $y_1(x)$ and $y_2(x)$ of Eq. (4.3.2b) are linearly dependent on (a, b) if and only if their Wronskian $W[y_1, y_2]$ is zero at for all points $x \in (a, b)$. That is,

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \equiv 0, \ \forall x \in (a, b).$$

If there exists $x_0 \in (a, b)$ such that $W[y_1(x_0), y_2(x_0)] \neq 0$, then the $y_1(x), y_2(x)$ are independent.
On the contrary, if the two solutions $y_1(x), y_2(x)$ of Eq. (4.3.2b) are linearly independent on (a, b) then

$$W(x) = W[y_1, y_2] \neq 0, \forall x \in (a, b).$$

Theorem 5 (Structure of solutions to homogeneous equations)

If there exists two linearly independent solutions y_1, y_2 on (a, b) of Eq.(4.3.2b), then the general solution (the set of all solutions) of Eq.(4.3.2b) is

$$y = C_1 y_1 + C_2 y_2,$$

where C_1, C_2 are arbitrary constants.

Example

Solve the following differential equation: $y'' - 3y' + 2y = 0$, knowing that $y_1 = e^x$ and $y_2 = e^{2x}$ are solutions.

4.3.2 Second-order linear differential equations

Theorem 6 (Liouville's theorem)

If a nontrivial solution y_1 is known, then we can find the solution y_2 linearly independent with y_1 of Eq. (4.3.2b) by formula

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} e^{-\int p(x)dx} dx.$$

Example 8

Solve the differential equation: $y'' + \frac{2}{x}y' + y = 0$ know a solution $y_1 = \dfrac{\sin x}{x}$.

Solution: $y_2 = \dfrac{\sin x}{x} \displaystyle\int \frac{x^2 \cdot e^{-\int \frac{2}{x}dx}}{\sin^2 x} dx = \dfrac{\sin x}{x} \displaystyle\int \frac{x^2 \cdot e^{-2\ln x}}{\sin^2 x} dx = \dfrac{\sin x}{x} \displaystyle\int \frac{dx}{\sin^2 x} = \dfrac{\sin x}{x}(-\cot x) = -\dfrac{\cos x}{x}$.

Therefore, the general solution equqtion is $y = \frac{1}{x}\left(C_1 \sin x + C_2 \cos x\right)$.

Example 9

Solve the equation $x^2(\ln x - 1)y'' - xy' + y = 0$ know that it has a solution that is a power function.

Solution: Put $y_1 = x^a$ in the equation, we have

$$x^2(\ln x - 1)\alpha(\alpha-1)x^{\alpha-2} - \alpha x^\alpha + x^\alpha = 0, \forall x \in (a,b)$$

$$\Rightarrow \alpha(\ln x - 1)(\alpha - 1) - \alpha + 1 = 0, \forall x \in (a,b)$$

$$\Rightarrow \begin{cases} \alpha(\alpha-1) = 0 \\ -\alpha + 1 = 0 \end{cases} \Rightarrow \alpha = 1 \Rightarrow y_1 = x$$

$$y_2 = x\int \frac{e^{\int \frac{x\,dx}{x^2(\ln x - 1)}dx}}{x^2} = x\int \frac{e^{\int \frac{d\ln x}{\ln x - 1}}}{x^2}dx = x\int \frac{e^{\ln(\ln x - 1)}}{x^2}dx$$

$$= x\int \frac{\ln x - 1}{x^2}dx = x\left[-\frac{1}{x}(\ln x - 1) + \int \frac{dx}{x^2}\right] = -\ln x$$

Therefore, so the general solution is $y = C_1 x + C_2 \ln x$.

4.3.2 Second-order linear differential equations

Theorem 7 (Structure of solutions to non-homogeneous equations)

The general solution of the non-homogeneous equation (4.3.2a) is equal to the general solution $\bar{y}$ of the homogeneous equation (4.3.2b) plus some particular solution y^* of the equation (4.3.2a). That is $y = \bar{y} + y^*$.

Theorem 8 (Principle of superposition of solutions)

Now suppose that $f(x)$ is the sum of two terms, $f(x) = f_1(x) + f_2(x)$, and suppose that y_1^* and y_2^* are solutions of the equations

$$y^{''} + py^{'} + qy = f_1(x), \quad (4.3.2a1)$$

and

$$y^{''} + py^{'} + qy = f_1(x), \quad (4.3.2a2)$$

respectively. Then $y^* = y_1^* + y_2^*$ is a particular solution of the equation (4.3.2a)(i.e. $y^{''} + py^{'} + qy = f_1(x) + f_2(x)$).

4.3.3. Second-order linear differential equations with constant coefficients

Definition 3

The second-order differential equation is called the linear second-order differential equations with constant coefficients of the form

$$y'' + py' + qy = f(x), \quad (4.3.3a)$$

where p, q are real constants, $f(x)$ is continuous on (a, b).
- If $f(x) \equiv 0, \forall x \in (a, b)$ then $(4.3.3a)$ is called the linear second-order differential honogeneous equations with constant coefficients of the form

$$y'' + py' + qy = 0, \quad (4.3.3b).$$

If $f(x) \neq 0, x \in (a, b)$ then $(4.3.3a)$ is called the linear second-order differential non-honogeneous equations with constant coefficients.

4.3.3. Second-order linear differential equations with constant coefficients

a. General solutions of homogeneous equations with constant coefficients

To seek exponential solutions, we suppose that $y = y^{kx}$, where k is a constant.Then it follows that $y' = ke^{kx}$ and $y'' = k^2e^{kx}$. By substituting these expression for y, y', and y'' in Eq. $(4.3.3b)$, we obtain $(k^2 + pk + q)e^{kx} = 0$ or, since e^{kx} is never zero,

$$k^2 + pk + q = 0, \quad (4.3.3c)$$

is called the characteristic equation for the Eq.(4.3.3b).

a.Homogeneous equations with constant coefficients

We now consider solution of the characteristic equation

$$k^2 + pk + q = 0(4.3.3c)$$

1^{st} **Case: Distinct real roots.**

Assuming that the roots of the characteristic equation (4.3.3c) are real and different, let them be denoted by k_1 and k_2, where $k_1 \neq k_2$. Then $y_1 = e^{k_1 x}$ and $y_2 = e^{k_2 x}$ are two linearly dependent solutions of Eq.(4.3.3b). Therefore, we obtain the general solution of Eq.((4.3.3b)):$\bar{y} = C_1 e^{k_1 x} + C_2 e^{k_2 x}$.

Example 12

Find the general solution of $y'' + 5y' + 6y = 0$. The characteristic equation is $k^2 + 5k + 6 = 0$. It has two distinct real roots: $k_1 = -2$ and $k_2 = -3$, then the general solution is $\bar{y} = C_1 e^{-2x} + C_2 e^{-3x}$

a.Homogeneous equations with constant coefficients

2^{nd} **Case: Double real root.**

We consider the second possibility, namely, that the two real roots k_1 and k_2 are equal. This case occurs when the discriminant $\Delta = p^2 - 4q$ is zero, and it follows from the quadratic formula that $k_1 = k_2 = -\frac{p}{2}$.The difficulty is immediately apparent; both roots yield the same solution $y_1 = e^{k_1 x} = e^{-\frac{p}{2}x}$ of the differential equation (4.3.3c). We now find a second solution y_1 which is linearly independent to y_2, we have $y_2 = e^{-\frac{px}{2}} \int \frac{e^{-\int p dx}}{e^{-px}} dx = x e^{-\frac{px}{2}} = x.y_1$. Therefore, the general solution of Eq. (4.3.3b) in this case is $\bar{y} = e^{kx}\left(C_1 + C_2 x\right)$.

Example 13

Solve the differential equation $y'' + 4y' + 4y = 0$, The characteristic equation is $k^2 + 4k + 4 = 0$, which has a double real root $k_1 = k_2 = -2$. Therefore, the general solution of given differential equation is

$$y = \left(C_1 + C_2 x\right) e^{-2x}.$$

a.Homogeneous equations with constant coefficients

3^{rd} **Case: Complex conjugate roots.**

If the roots of characteristic $(4.3.3c)$ are conjugate complex numbers $k = \alpha \pm i\beta$, then the general solution of Eq. $(4.3.3b)$ is

$$\bar{y} = e^{\alpha x} \left(C_1 \cos \beta x + C_2 \sin \beta x \right),$$

where $y_1 = \frac{e^{\alpha x + i\beta x} + e^{\alpha x - i\beta x}}{2}$, $y_2 = \frac{e^{\alpha x - i\beta x} + e^{\alpha x - i\beta x}}{2i}$.

Example 14

Find the general solution of $y'' - 4y' + 13y = 0$, The characteristic equation is $k^2 - 4k + 13 = 0$, and its roots are $k = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$
Thus, the general solution is $\bar{y} = e^{2x} \left(C_1 \cos 3x + C_2 \sin 3x \right)$

$$y(0) = 1 = C_1; y'(0) = 1 = 2C_1 + 3C_2$$

$$\Rightarrow C_2 = -\frac{1}{3} \Rightarrow y = e^{2x} \left(\cos 3x - \frac{1}{3} \sin 3x \right)$$

Remark

- We can find the general solution of the homogeneous coefficient second order linear differential equation by solving the corresponding characteristic equation
- To find particular solution of non-homogeneous second order linear differential equations, we can use the method of variation of the Lagrange constant and the principle of superposition of solutions.

Example 15

Find the general solution of $y'' - y = \frac{e^x}{e^x+1}$.
The characteristic equation is
$k^2 - 1 = 0 \Rightarrow k = \pm 1 \Rightarrow y = C_1 e^{-x} + C_2 e^x$

$\begin{cases} C_1' e^{-x} + C_2' e^x = 0 \\ -C_1' e^{-x} + C_2' e^x = \frac{e^x}{1+e^x} \end{cases} \Rightarrow \begin{cases} C_1' = -\frac{1}{2}\frac{e^{2x}}{1+e^x} \\ C_2' = \frac{1}{2}\frac{1}{e^x+1} \end{cases}$ Thus, we obtain

$$y = \frac{e^{-x}}{2}\left[\ln\left(e^x+1\right) - e^x + C_1\right] + \frac{e^x}{2}\left[x - \ln\left(e^x+1\right) + C_2\right].$$

b. General solutions of non-homogeneous equations

The linear second-order differential non-homogeneous equations with constant coefficient of the special function $f(x)$ has the corresponding particular solutions formula.

FORM 1: $f(x) = e^{\alpha x} P_n(x)$

+ If the constant α is not a root of the characteristic equation, then $(4.3.3a)$ has a particular solution of the form $y^* = e^{\alpha x} Q_n(x)$.

+ If the constant α is a single root of the characteristic equation,then $(4.3.3a)$ has a particular solution of the form $y^* = x e^{\alpha x} Q_n(x)$.

+ If the constant α is the double root of the characteristic equation,then $(4.3.3a)$ has a particular solution of the form

$$y^* = x^2 e^{\alpha x} Q_n(x)$$

b. General solutions of non-homogeneous equations

FORM 2: $f(x) = e^{\alpha x}\left[P_n(x)\cos\beta x + P_m(x)\sin\beta x\right]$

If $\alpha \pm i\beta$ is not a root of the characteristic equation, then $(4.3.3a)$ has a particular solution of the form

$$y^* = e^{\alpha x}\left[Q_k(x)\cos\beta x + R_k(x)\sin\beta x\right]$$

where $Q_k(x), R_k(x)$ are polynomials of degree $k = max(m, n)$.
If $\alpha \pm i\beta$ is a root of the characteristic equation, then $(4.3.3a)$ has a particular solution of the form

$$y^* = xe^{\alpha x}\left[Q_k(x)\cos\beta x + R_k(x)\sin\beta x\right]$$

Example 16

Find the general solution of

$$y'' + 2y' - 3y = e^x x + x^2$$

The characteristic equation is $k^2 + 2k - 3 = 0 \Leftrightarrow \left[\begin{array}{l} k_1 = 1 \\ k_2 = -3 \end{array} \right.$

The general solution of the homogeneous equation is $\bar{y} = C_1 e^{-3x} + C_2 e^x$

+ particular solution $y_1^* = xe^x(ax+b) {y^*}'_1 = e^x \left(ax^2 + bx + 2ax + b\right)$

$${y^*}''_1 = e^x \left(ax^2 + bx + 4ax + 2b + 2a\right) \Rightarrow {y^*}_1 = \frac{x}{8} e^x (x-1).$$

+ particular solution ${y^*}_2 = ax^2 + bx + c \quad {y^*}'_2 = 2ax + b \quad {y^*}''_2 = 2a$

$$\Rightarrow {y^*}_2 = x^2 - 4x + 14.$$

The general solution of the nonhomogeneous equation is

$$y = \bar{y} + {y^*}_1 + {y^*}_2 = C_1 e^{-3x} + C_2 e^x + \frac{x}{8} e^x (x-1) + x^2 - 4x + 14$$

Example 17

Finding solutions to Cauchy's problem

$$y'' - 4y' + 4y = e^{2x}(x+1), y(0) = y'(0) = 1$$

The characteristic equation is $k^2 - 4k + 4 = 0 \Leftrightarrow k_1 = k_2 = 2$ The general solution of the homogeneous equation is $\bar{y} = C_1 + C_2 e^{2x}$
Particular solution:
$y^* = x^2 e^{2x}(ax+b) y^{*\prime} = e^{2x}\left(2ax^3 + 2bx^2 + 3ax^2 + 2bx\right)$

$y^{*\prime\prime} = e^{2x}\left(4ax^3 + 4bx^2 + 12ax^2 + 6ax + 8bx + 2b\right) \Rightarrow y^* = \dfrac{1}{6}x^2 e^{2x}(x+3)$

$y = \bar{y} + y^* = e^{2x}\left(C_1 + C_2 x\right) + \dfrac{1}{6}x^2 e^{2x}(x+3)$

$y' = e^{2x}\left(2C_1 + C_2 + 2C_2 x\right) + \dfrac{1}{6}e^{2x}\left(2x^3 + 9x^2 + 6x\right)$

$y(0) = C_1 = 1 \Rightarrow y'(0) = 2C_1 + C_2 = 1$

$C_1 = 1, C_2 = -1 \Rightarrow y = e^{2x}(1-x) + \dfrac{1}{6}x^2 e^{2x}(x+3)$

Example 18

Find the general solution of $y'' + y' = xcosx$.

The characteristic equation is $k^2 + k = 0 \Leftrightarrow \left[\begin{array}{l} k_1 = 0 \\ k_2 = -1 \end{array}\right.$

The general solution of the homogeneous equation is $\bar{y} = C_1 + C_2 e^{-x}$.

Particular solution: $y^* = (ax + b)\cos x + (cx + d)\sin x$

Substituting into the differential equation we have

$((c-a)x + a + 2c + d - b)\cos x + (-(a+c)x + c - 2a - b - d)\sin x = x\cos x.$

$$\begin{cases} c - a = 1 \\ a + 2c + d - b = 0 \\ a + c = 0 \\ c - 2a - b - d = 0 \end{cases} \Rightarrow a = -\frac{1}{2}, b = 1, c = \frac{1}{2}, d = \frac{1}{2}$$

Thus, the general solution is

$$y = C_1 + C_2 e^{-x} - \frac{1}{2}(x - 2)\cos x + \frac{1}{2}(x + 1)\sin x.$$

Example 19

Find the general solution of $y'' + 2y' + 2y = e^{-x}(1 + \sin x)$.
The characteristic equation is $k^2 + 2k + 2 = 0 \Leftrightarrow k = -1 \pm i$.
The general solution of the homogeneous equation is

$$\bar{y} = e^{-x}\left(C_1 \cos x + C_2 \sin x\right).$$

The particular solution of $y'' + 2y' + 2y = e^{-x} \sin x$ has the form $y_1^* = xe^{-x}(a \cos x + b \sin x)$. We have

$$2b \cos x - 2a \sin x = \sin x \Rightarrow b = 0, a = -\frac{1}{2} \Rightarrow y_1^* = -\frac{xe^{-x}}{2} \cos x.$$

The particular solution of $y'' + 2y' + 2y = e^{-x}$ has form $y_2^* = ce^{-x}$. We get $y_2^* = e^{-x}$.
Thus, the general solution is

$$y = \bar{y} + y_1^* + {y^*}_2 = e^{-x}\left(C_1 \cos x + C_2 \sin x\right) + e^{-x}\left(1 - \frac{x}{2} \cos x\right).$$