Optimal Adjacency Labels for Subgraphs of Cartesian Products

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Abstract

For any hereditary graph class \mathcal{F} , we construct optimal adjacency labeling schemes for the classes of subgraphs and induced subgraphs of Cartesian products of graphs in \mathcal{F} . As a consequence, we show that, if \mathcal{F} admits efficient adjacency labels (or, equivalently, small induced-universal graphs) meeting the information-theoretic minimum, then the classes of subgraphs and induced subgraphs of Cartesian products of graphs in \mathcal{F} do too.

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1 Introduction

Adjacency labeling. A class of graphs is a set \mathcal{F} of graphs closed under isomorphism, where the set $\mathcal{F}_n \subseteq \mathcal{F}$ of graphs on n vertices has vertex set [n]. It is hereditary if it is also closed under taking induced subgraphs, and monotone if it is also closed under taking subgraphs. An adjacency labeling scheme for a class \mathcal{F} consists of a decoder $D: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ such that for every $G \in \mathcal{F}$ there exists a labeling $\ell: V(G) \to \{0,1\}^*$ satisfying

$$\forall x, y \in V(G) : D(\ell(x), \ell(y)) = 1 \iff xy \in E(G).$$

The size of the adjacency labeling scheme (or labeling scheme for short) is the function $n \mapsto \max_{G \in \mathcal{F}_n} \max_{x \in V(G)} |\ell(x)|$, where $|\ell(x)|$ is the number of bits of $\ell(x)$. Labeling schemes have been studied extensively since their introduction by Kannan, Naor, & Rudich [KNR92] and Muller [Mul89]. If \mathcal{F} admits a labeling scheme of size s(n), then a graph $G \in \mathcal{F}_n$ can be recovered from the $n \cdot s(n)$ total bits in the adjacency labels of its vertices, so a labeling scheme is an encoding of the graph, distributed among its vertices. The information-theoretic lower bound on any encoding is $\log |\mathcal{F}_n|$, so the question is, when can the distributed adjacency labeling scheme approach this bound? In other words, which classes of graphs admit labeling schemes of size $O(\frac{1}{n}\log |\mathcal{F}_n|)$?

Cartesian products. Write $G \square H$ for the Cartesian product of G and H, write G^d for the d-wise Cartesian product of G, and for any class \mathcal{F} write $\mathcal{F}^{\square} = \{G_1 \square G_2 \square \cdots \square G_d : d \in \mathbb{N}, G_i \in \mathcal{F}\}$ for the class of Cartesian products of graphs in \mathcal{F} . Write $\mathsf{mon}(\mathcal{F}^{\square})$ and $\mathsf{her}(\mathcal{F}^{\square})$, respectively, for the monotone and hereditary closures of this class, which are the sets of all graphs G that are a subgraph (respectively, induced subgraph) of some $H \in \mathcal{F}^{\square}$.

Cartesian products have appeared several times independently in the recent literature on labeling schemes [CLR20, Har20, AAL21] (and later in [HWZ22, AAA⁺22, EHK22]), and are extremely natural for the problem of adjacency labeling. For example, taking \mathcal{F} to be the class of complete graphs, a labeling scheme for $\ker(\mathcal{F}^{\square})$ is equivalent to an encoding $\ell: T \to \{0,1\}^*$ of a set of strings $T \subseteq \Sigma^*$, with Σ being an arbitrarily large finite alphabet, such that a decoder who doesn't know T can decide whether $x, y \in T$ have Hamming distance 1, using only the encodings $\ell(x)$ and $\ell(y)$. Replacing complete graphs with, say, paths, one obtains induced subgraphs of grids in arbitrary dimension. Switching to $\mathsf{mon}(\mathcal{F}^{\square})$ allows arbitrary edges of these products to be deleted.

Motivated by the problem of designing labeling schemes, Chepoi, Labourel, & Ratel [CLR20] studied the structure of general Cartesian products. For the subgraphs and induced subgraphs of hypercubes (i.e. $her(\{K_2\}^{\square})$ and $mon(\{K_2\}^{\square})$), there is a labeling scheme of size $O(\log^2 n)$ that follows from a folklore bound of $\log(n)$ on their degeneracy (see [Gra70]) and a general $O(k \log n)$ labeling scheme for classes of degeneracy k [KNR92]. [CLR20] give upper bounds on the label size for a number of special cases but do not improve in general upon the $O(\log^2 n)$ bound for subgraphs of hypercubes. Designing a labeling scheme for *induced* subgraphs of hypercubes (rather, the weaker question of proving bounds on $|\mathcal{F}_n|$) was an open problem of Alecu, Atminas, & Lozin [AAL21], answered concurrently in independent work [Har20] (see the clearer exposition [Har22]) using a probabilistic argument, which also answered a question of [AAL21] about *graph functionality*. Finding an efficient *deterministic* labeling scheme is an open problem of [AAA+22].

Meanwhile, the subgraphs of hypercubes are a counterexample [EHK22] to a conjecture in [HWZ22] about adjacency *sketches* (a randomized variant of labeling schemes). Cartesian products are important for understanding the relationship between sketches and labels: a constant-size sketch and an $O(\log n)$ labeling scheme exist for *induced* subgraphs of the hypercube, but constant-size sketches for *subgraphs* do not exist [EHK22]. This prompted [EHK22] to ask if an $O(\log n)$

labeling scheme for subgraphs also does not exist, since the earlier work considering hypercubes [CLR20, Har20, HWZ22, AAL21, AAA $^+$ 22] had not improved on the trivial $O(\log^2 n)$ bound. This would be the first natural and explicit counterexample to the Implicit Graph Conjecture of [KNR92, Spi03], which was recently refuted by a non-constructive proof [HH21]. That possibility was one motivation for this work; alas, it is not so.

An equivalent problem to designing labeling schemes is constructing induced-universal graphs (or simply universal graphs) for a hereditary class \mathcal{F} . A sequence of graphs $(U_n)_{n\in\mathbb{N}}$ are universal graphs of size $n\mapsto |V(U_n)|$ if every $G\in\mathcal{F}_n$ is an induced subgraph of U_n . [KNR92] observed that a labeling scheme of size s(n) is equivalent to a universal graph of size $2^{s(n)}$. In this terminology, the goal is to find universal graphs of size $poly(|\mathcal{F}_n|^{1/n})$; the case where this is poly(n) is of special interest, which corresponds to labels of size $O(\log n)$ [KNR92, Spi03, HH21]. If U_n is a universal graph for \mathcal{F} then for large enough d, U_n^d is a universal graph for $her(\mathcal{F}^\square)$, but in general it has at least exponential size: a star with n-1 leaves is a member of $her(\{K_2\}^\square)$ but the smallest product it can be embedded into is K_2^{n-1} . It is not clear a priori how to get a universal graph for $mon(\mathcal{F}^\square)$ in a similar way from U_n even when \mathcal{F} is monotone, since U_n^d is not universal for $mon(\mathcal{F}^\square)$ (we emphasize that $mon(\mathcal{F}^\square)$ is not the same as $her(mon(\mathcal{F}^\square))$). Our proof shows that one can derive small universal graphs for $her(\mathcal{F}^\square)$ and $mon(\mathcal{F}^\square)$ from U_n , although the transformation is not clearly interpretable from a graph theory perspective.

Results. We improve the best-known $O(\log^2 n)$ bound for subgraphs of hypercubes to the optimal $O(\log n)$, and in general show how to construct optimal labels for all subgraphs and induced subgraphs of Cartesian products.

Theorem 1.1. Let \mathcal{F} be a hereditary class with an adjacency labeling scheme of size s(n). Then:

- 1. $her(\mathcal{F}^{\square})$ has a labeling scheme of size at most $4s(n) + O(\log n)$.
- 2. $\operatorname{mon}(\mathcal{F}^{\square})$ has a labeling scheme where each $G \in \operatorname{mon}(\mathcal{F}^{\square})$ on n vertices is given labels of size at most $4s(n) + O(k(G) + \log n)$, where k(G) is the degeneracy of G.

All of the labeling schemes of Chepoi, Labourel, & Ratel [CLR20] are obtained by bounding k(G) and applying the black-box $O(k(G) \cdot \log n)$ bound of [KNR92]. For example, they get labels of size $O(d \log^2 n)$ when the base class \mathcal{F} has degeneracy d, by showing that $\mathsf{mon}(\mathcal{F}^{\square})$ has degeneracy $O(d \log n)$. Our result can substituted for that black-box, replacing the multiplicative $O(\log n)$ with an $additive\ O(\log n)$, thereby improving all of the results of [CLR20]; for example, achieving $O(d \log n)$ when \mathcal{F} has degeneracy d.

For subgraphs of hypercubes, [CLR20] observed that a bound of $O(\mathsf{vc}(G)\log n)$ follows from the inequality $k(G) \le \mathsf{vc}(G)$ due to Haussler [Hau95], where $\mathsf{vc}(G)$ is the VC dimension¹, which can be as large as $\log n$ but is often much smaller; they generalize this inequality in various ways to other Cartesian products. Our result supercedes the VC dimension result for hypercubes.

Theorem 1.1 is optimal up to constant factors (which we have not tried to optimize), and we get the following corollary (see Section 3 for proofs). Say that a hereditary class \mathcal{F} admits an *efficient* labeling scheme if it either admits a *constant-size* labeling scheme (i.e. it satisfies $\log |\mathcal{F}_n| = o(n \log n)$ [Sch99], including the case where \mathcal{F} is finite), or it admits a labeling scheme of size $O(\frac{1}{n} \log |\mathcal{F}_n|)$. Equivalently, if it admits a universal graph of size $poly(|\mathcal{F}_n|^{1/n})$. Then:

Corollary 1.2. If a hereditary class \mathcal{F} has an efficient labeling scheme, then so do $her(\mathcal{F}^{\square})$ and $mon(\mathcal{F}^{\square})$.

¹See [CLR20] for the definition of VC dimension

2 Adjacency Labeling Scheme

Notation. For two binary strings x, y, we write $x \oplus y$ for the bitwise XOR. For two graphs G and H, we will write $G \subset H$ if G is a subgraph of H, and $G \subset_I H$ if G is an *induced* subgraph of H. We will write V(G) and E(G) as the vertex and edge set of a graph G, respectively. All graphs in this paper are simple and undirected. A graph G has degeneracy K if all subgraphs of G have a vertex of degree at most K.

For graphs G_1, \ldots, G_d , the Cartesian product graph $G = G_1 \square \cdots \square G_d$ is the graph with vertices $V(G) = V(G_1) \times \cdots \times V(G_d)$, where each $x \in V(G)$ can be written as $x = (x_1, \ldots, x_d)$ for $x_i \in V(G_i)$; there is an edge between x and y if and only if there is a unique $i \in [d]$ where $x_i \neq y_i$, and, for this coordinate, $x_i y_i \in E(G_i)$.

Strategy. Suppose $G \subset G_1 \square \cdots \square G_d$ is a subgraph of a Cartesian product. Then $V(G) \subseteq V(G_1) \times \cdots \times V(G_d)$. Let $H \subset_I G_1 \square \cdots \square G_d$ be the subgraph induced by V(G), so that $E(G) \subseteq E(H)$. One may think of G as being obtained from the induced subgraph H by deleting some edges. Then two vertices $x, y \in V(G)$ are adjacent if and only if:

- 1. There exists exactly one coordinate $i \in [d]$ where $x_i \neq y_i$;
- 2. On this coordinate, $x_i y_i \in E(G_i)$; and,
- 3. The edge $xy \in E(H)$ has not been deleted in E(G).

We construct the labels for vertices in G in three phases, which check these conditions in sequence.

2.1 Phase 1: Exactly One Difference

This phase is an easy extension of the simple proof in [Har22] of the labeling scheme for induced subgraphs of hypercubes, derived from [Har20, HWZ22]. The labels are obtained by a simple probabilistic method (and are efficiently computable by a randomized algorithm).

For any alphabet Σ and any two strings $x, y \in \Sigma^d$ where $d \in \mathbb{N}$, write $\mathsf{dist}(x, y)$ for the Hamming distance between x and y, i.e. $\mathsf{dist}(x, y) = |\{i \in [d] : x_i \neq y_i\}|$.

Proposition 2.1. For any set $S \subseteq \{0,1\}^d$, there exists a random function $\ell: S \to \{0,1\}^4$ such that, for all $x, y \in S$,

- 1. If $\operatorname{dist}(x,y) \leq 1$ then $\underset{\ell}{\mathbb{P}}\left[\operatorname{dist}(\ell(x),\ell(y)) \leq 1\right] = 1$, and
- 2. If $\operatorname{dist}(x,y) > 1$ then $\mathbb{P}_{\ell}[\operatorname{dist}(\ell(x),\ell(y)) \leq 1] \leq 3/4$.

Proof. Choose a uniformly random map $p:[d] \to [4]$ and partition [d] into four sets $P_j = p^{-1}(j)$. For each $i \in [4]$, define $\ell(x)_i := \bigoplus_{j \in P_i} x_j$.

Let $x, y \in S$ and write $w = \ell(x) \oplus \ell(y)$. Note that $\operatorname{dist}(\ell(x), \ell(y)) = |w|$, which is the number of 1s in w. If $\operatorname{dist}(x, y) = 0$ then $\operatorname{dist}(\ell(x), \ell(y)) = 0 \le 1$. Now suppose $\operatorname{dist}(x, y) = 1$. For any choice of $p : [d] \to [4]$, one of the sets P_i contains the differing coordinate and will have $w_i = 1$, while the other three sets P_j will have $w_j = 0$, so $\mathbb{P}_{\ell}[\operatorname{dist}(\ell(x), \ell(y)) \le 1] = 1$.

Now suppose $\operatorname{dist}(x,y)=t\geq 2$. We will show that $|w|\leq 1$ with probability at most 3/4. Note that w is obtained by the random process where $\vec{0}=w^{(0)}, w=w^{(t)},$ and $w^{(i)}$ is obtained from $w^{(i-1)}$ by flipping a uniformly random coordinate.

Observe that, for $i \geq 1$, $\mathbb{P}\left[w^{(i)} = \vec{0}\right] \leq 1/4$. This is because $w^{(i)} = \vec{0}$ can occur only if $|w^{(i-1)}| = 1$, so the probability of flipping the 1-valued coordinate is 1/4. If $|w^{(i-1)}| \geq 1$ then

 $\mathbb{P}\left[|w^{(i)}| \leq 1 \quad | \quad |w^{(i-1)}| \geq 1\right] \leq 1/2 \text{ since either } |w^{(i-1)}| = 1 \text{ and then } |w^{(i)}| = 0 \leq 1 \text{ with probability } 1/4, \text{ or } |w^{(i-1)}| \geq 2 \text{ and } |w^{(i)}| = 1 \text{ with probability at most } 1/2. \text{ Then, for } t \geq 2,$

$$\mathbb{P}\left[|w^{(t)}| \leq 1\right] = \mathbb{P}\left[w^{(t-1)} = \vec{0}\right] + \mathbb{P}\left[|w^{(t-1)}| \geq 1\right] \cdot \mathbb{P}\left[|w^{(t)}| = 1 \mid |w^{(t-1)}| \geq 1\right] \leq \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \cdot \square$$

Proposition 2.2. There exists a function $D: \{0,1\}^4 \times \{0,1\}^4 \to \{0,1\}$ such that, for any countable alphabet Σ , any $d \in \mathbb{N}$, and any $S \subseteq \Sigma^d$ of size n = |S|, there exists a random function $\ell: S \to \{0,1\}^4$ such that, for all $x, y \in S$,

- 1. If $\operatorname{dist}(x,y) \leq 1$, then $\mathbb{P}\left[D(\ell(x),\ell(y)) = 1\right] = 1$, and
- $\text{2. If } {\rm dist}(x,y) > 1, \ then \ \mathop{\mathbb{P}}_{\ell} \left[D(\ell(x),\ell(y)) = 1 \right] \leq 15/16.$

Proof. For each $\sigma \in \Sigma$ and $i \in [d]$, generate an independently and uniformly random bit $q_i(\sigma) \sim \{0,1\}$. Then for each $x \in S$ define $p(x) = (q_1(x_1), \ldots, q_d(x_d)) \in \{0,1\}^d$ and $S' = \{p(x) : x \in S\}$, and let ℓ' be the random function $S' \to \{0,1\}^4$ guaranteed to exist by Proposition 2.1. We define the random function $\ell : S \to \{0,1\}^4$ as $\ell(x) = \ell'(p(x))$. We define $D(\ell(x), \ell(y)) = 1$ if and only if $\operatorname{dist}(\ell'(p(x)), \ell'(p(y))) \leq 1$.

Let $x, y \in S$. If $dist(x, y) \leq 1$, so there is a unique $i \in [d]$ with $x_i \neq y_i$, then

$$\mathbb{P}\left[\mathsf{dist}(p(x),p(y))=1\right]=\mathbb{P}\left[q_i(x_i)\neq q_i(y_i)\right]=\mathbb{P}\left[\mathsf{dist}(p(x),p(y))=0\right]=1/2\,,$$

so $\mathbb{P}\left[\mathsf{dist}(p(x), p(y)) \leq 1\right] = 1$. Then by Proposition 2.1,

$$\mathbb{P}\left[D(\ell(x), \ell(y)) = 1\right] = \mathbb{P}\left[\mathsf{dist}(\ell'(p(x)), \ell'(p(y))) \le 1\right] = 1.$$

If dist(x,y) > 1 so that there are distinct $i, i' \in [d]$ such that $x_i \neq y_i$ and $x_{i'} \neq y_{i'}$, then

$$\mathbb{P}\left[\mathsf{dist}(p(x), p(y)) \ge 2\right] \ge \mathbb{P}\left[q_i(x_i) \ne q_i(y_i) \land q_{i'}(x_{i'}) \ne q_{i'}(y_{i'})\right] = 1/4 \,.$$

Then by Proposition 2.1,

$$\begin{split} \mathbb{P}\left[D(\ell(x),\ell(y)) = 1\right] &= \mathbb{P}\left[\mathsf{dist}(\ell'(p(x)),\ell'(p(y))) \leq 1\right] \\ &= \mathbb{P}\left[\mathsf{dist}(p(x),p(y)) \leq 1 \lor \mathsf{dist}(\ell'(p(x)),\ell'(p(y))) \leq 1\right] \\ &\leq 3/4 + (1-3/4)(3/4) = 15/16 \,. \end{split}$$

Lemma 2.3. There exists a function $D: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ and a constant c > 0 such that, for any countable alphabet Σ any $d \in \mathbb{N}$, and any set $S \subseteq \Sigma^d$ of size |S| = n, there exists a function $\ell: S \to \{0,1\}^k$ for $k \le c \log n$, where $D(\ell(x), \ell(y)) = 1$ if and only if $\operatorname{dist}(x,y) = 1$.

Proof. Since $\lceil \log n \rceil$ bits can be added to any $\ell(x)$ to ensure that each $\ell(x)$ is unique, it suffices to construct functions D, ℓ where $D(\ell(x), \ell(y)) = 1$ if and only if $\operatorname{dist}(x, y) \leq 1$, instead of $\operatorname{dist}(x, y) = 1$.

Let q be a parameter to be fixed later. Let $D': \{0,1\}^4 \times \{0,1\}^4 \to \{0,1\}$ be the function from Proposition 2.2. Choose a random function $\ell: S \to \{0,1\}^k$ with k=4q as follows. For each $x \in S$, let $\ell(x) = (\ell_1(x), \ldots, \ell_q(x))$ where each $\ell_i: S \to \{0,1\}^4$ is an independent copy of the random function from Proposition 2.2. Define

$$D(\ell(x), \ell(y)) = \bigwedge_{i=1}^{q} D'(\ell_i(x), \ell_i(y)).$$

Fix any pair $x, y \in S$. If $\operatorname{dist}(x, y) \leq 1$ then $\mathbb{P}[D(\ell(x), \ell(y)) = 1] = 1$ since $D'(\ell_j(x), \ell_j(y)) = 1$ with probability 1. If $\operatorname{dist}(x, y) > 1$ then $\mathbb{P}[D'(\ell_j(x), \ell_j(y)) = 1] \leq 15/16$. Then

$$\mathbb{P}\left[D(\ell(x), \ell(y)) = 1\right] \le (15/16)^q \le 1/n^2$$
,

when we set $q = \frac{2}{\log(16/15)}\log(n)$. Therefore, by the union bound, the probability that there exist two $x, y \in S$ such that $D(\ell(x), \ell(y))$ is incorrect is less than 1. So there exists a fixed function ℓ with the desired properties.

2.2 Phase 2: Induced Subgraphs

After the first phase, we are guaranteed that there is a unique coordinate $i \in [d]$ where $x_i \neq y_i$. In the second phase we wish to determine whether $x_i y_i \in E(G_i)$. It is convenient to have labeling schemes for the factors G_1, \ldots, G_d where we can XOR the labels together while retaining the ability to compute adjacency. Define an XOR-labeling scheme the same as an adjacency labeling scheme, with the restriction that for each $s \in \mathbb{N}$ there is some function $g_s : \{0,1\}^s \to \{0,1\}$ such that on any two labels $\ell(x), \ell(y)$ of size s, the decoder outputs $D(\ell(x), \ell(y)) = g_s(\ell(x) \oplus \ell(y))$. Any labeling scheme can be transformed into an XOR-labeling scheme with at most a constant-factor loss:

Lemma 2.4. Let \mathcal{F} be any hereditary class of graphs with an adjacency labeling scheme of size s(n). Then \mathcal{F} admits an XOR-labeling scheme of size at most 4s(n).

Proof. Let $D: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ be the decoder of the adjacency labeling scheme for \mathcal{F} , fix any $n \in \mathbb{N}$, and write s = s(n). Note that D must be symmetric, so D(a,b) = D(b,a) for any $a,b \in \{0,1\}^s$. Let $\phi: \{0,1\}^s \to \{0,1\}^{4s}$ be uniformly randomly chosen, so that for every $z \in \{0,1\}^s$, $\phi(z) \sim \{0,1\}^{4s}$ is a uniform and independently random variable. For any two distinct pairs $\{z_1,z_2\},\{z_1',z_2'\} \in {\{0,1\}^s \choose 2}$ where $z_1 \neq z_2, z_1' \neq z_2'$, and $\{z_1,z_2\} \neq \{z_1',z_2'\}$, the probability that $\phi(z_1) \oplus \phi(z_2) = \phi(z_1') \oplus \phi(z_2')$ is at most 2^{-4s} , since at least one of the variables $\phi(z_1),\phi(z_2),\phi(z_1'),\phi(z_2')$ is independent of the other ones. Therefore, by the union bound,

$$\mathbb{P}\left[\exists \{z_1, z_2\}, \{z_1', z_2'\} : \phi(z_1) \oplus \phi(z_2) = \phi(z_1') \oplus \phi(z_2')\right] \le \binom{2^s}{2}^2 2^{-4s} \le \frac{1}{4}.$$

Then there is $\phi: \{0,1\}^s \to \{0,1\}^{4s}$ such that each distinct pair $\{z_1,z_2\} \in {\{0,1\}^s\} \choose 2}$ is assigned has a distinct unique value $\phi(z_1) \oplus \phi(z_2)$. So the function $\Phi(\{z_1,z_2\}) = \phi(z_1) \oplus \phi(z_2)$ is a one-to-one map ${\{0,1\}^s\} \choose 2} \to \{0,1\}^{4s}$. Then for any graph $G \in \mathcal{F}$ on n vertices, with labeling $\ell: V(G) \to \{0,1\}^s$, we may assign the new label $\ell'(x) = \phi(\ell(x))$. On labels $\phi(\ell(x)), \phi(\ell(y)) \in \{0,1\}^s$, the decoder for the XOR-labeling scheme simply computes $D(\Phi^{-1}(\phi(\ell(x)) \oplus \phi(\ell(y)))) = D(\ell(x), \ell(y))$.

We can now prove the first part of Theorem 1.1.

Lemma 2.5. Let \mathcal{F} be a hereditary class of graphs that admits an adjacency labeling scheme of size s(n). Then $her(\mathcal{F}^{\square})$ admits an adjacency labeling scheme of size $4s(n) + O(\log n)$.

Proof. By Lemma 2.4, there is an XOR-labeling scheme for \mathcal{F} with labels of size 4s(n). Let $D: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ be the decoder for this scheme, with $D(a,b) = g(a \oplus b)$ for some function g. Design the labels for $\ker(\mathcal{F}^{\square})$ as follows. Consider a graph $G \in \ker(\mathcal{F}^{\square})$, so that $G \subset_I G_1 \square G_2 \square \cdots \square G_d$ for some $d \in \mathbb{N}$ and $G_i \in \mathcal{F}$ for each $i \in [d]$. Since \mathcal{F} is hereditary, we may assume that each G_i has at most n vertices; otherwise we could simply replace it with the subgraph of G_i induced by the vertices $\{x_i : x \in V(G)\}$. For each $x = (x_1, \dots, x_d) \in V(G)$, construct the label as follows:

- 1. Treating the vertices in each G_i as characters of the alphabet [n], use $O(\log n)$ bits to assign the label given to $x = (x_1, \dots, x_d) \in [n]^d$ by Lemma 2.3.
- 2. Using 4s(n) bits, append the vector $\bigoplus_{i \in [d]} \ell_i(x_i)$, where $\ell_i(x_i)$ is the label of $x_i \in V(G_i)$ in graph G_i , according to the XOR-labeling scheme for \mathcal{F} .

The decoder operates as follows. Given the labels for $x, y \in V(G)$:

- 1. If x and y differ on exactly one coordinate, as determined by the first part of the label, continue to the next step. Otherwise output "not adjacent".
- 2. Now guaranteed that there is a unique $i \in [d]$ such that $x_i \neq y_i$, output "adjacent" if and only if the following is 1:

$$D\left(\bigoplus_{j\in[d]} \ell_j(x_j), \bigoplus_{j\in[d]} \ell_j(y_j)\right) = g\left(\bigoplus_{j\in[d]} \ell_j(x_j) \oplus \bigoplus_{j\neq i} \ell_j(y_j)\right)$$
$$= g\left(\ell_i(x_i) \oplus \ell_i(y_i) \oplus \bigoplus_{j\neq i} \ell_j(x_j) \oplus \ell_j(y_j)\right) = g(\ell_i(x_i) \oplus \ell_i(y_i)),$$

where the final equality holds because $x_j = y_j$ for all $j \neq i$, so $\ell_j(x_j) = \ell_j(y_j)$. Then the output value is 1 if and only $x_i y_i$ is an edge of G_i ; equivalently, xy is an edge of G.

This concludes the proof.

The XOR-labeling trick can also be used to simplify the proof of [HWZ22] for adjacency sketches of Cartesian products. That proof uses a two-level hashing scheme to avoid destroying the labels of x_i and y_i with the XOR, with a high constant probability of success. That strategy does not work for us here, because we need to succeed with certainty.

2.3 Phase 3: Subgraphs

Finally, we must check whether the edge $xy \in E(H)$ in the *induced* subgraph $H \subset_I G_1 \square \cdots \square G_d$ has been deleted in E(G). There is a minimal and perfect tool for this task:

Theorem 2.6 (Minimal Perfect Hashing). For every $m, k \in \mathbb{N}$, there is a family $\mathcal{P}_{m,k}$ of hash functions $[m] \to [k]$ such that, for any $S \subseteq [m]$ of size k, there exists $h \in \mathcal{P}_{m,k}$ where the image of S under h is [k] and for every distinct $i, j \in S$ we have $h(i) \neq h(j)$. The function h can be stored in $k \ln e + \log \log m + o(k + \log \log m)$ bits of space and it can be computed by a randomized algorithm in expected time $O(k + \log \log m)$.

Minimal perfect hashing has been well-studied. A proof of the space bound appears in [Meh84] and significant effort has been applied to improving the construction and evaluation time. We take the above statement from [HT01]. We now conclude the proof of Theorem 1.1 by applying the next lemma to the class $\mathcal{G} = \text{her}(\mathcal{F}^{\square})$, using the labeling scheme for $\text{her}(\mathcal{F}^{\square})$ obtained in Lemma 2.5 (note that $\text{mon}(\text{her}(\mathcal{F}^{\square})) = \text{mon}(\mathcal{F}^{\square})$).

Lemma 2.7. Let \mathcal{G} be any graph class which admits an adjacency labeling scheme of size s(n). Then $mon(\mathcal{G})$ admits an adjacency labeling scheme where each $G \in mon(\mathcal{G})$ on n vertices has labels of size $s(n) + O(k(G) + \log n)$, where k(G) is the degeneracy of G.

Proof. Let $G \in \text{mon}(\mathcal{G})$ have n vertices, so that it is a subgraph of $H \in \mathcal{G}$ on n vertices. The labeling scheme is as follows.

- 1. Fix a total order \prec on V(H) such that each vertex x has at most k = k(G) neighbors y in H such that $x \prec y$; this exists by definition. We will identify each vertex x with its position in the order.
- 2. For each vertex x, assign the label as follows:
 - (a) Use s(n) bits for the adjacency label of x in H.
 - (b) Use $\log n$ bits to indicate x (the position in the order).
 - (c) Let $N^+(x)$ be the set of neighbors $x \prec y$. Construct a perfect hash function $h_x : N^+(x) \to [k]$ and store it, using $O(k + \log \log n)$ bits.
 - (d) Use k bits to write the function $\mathsf{edge}_x : [k] \to \{0,1\}$ which takes value 1 on $i \in [k]$ if and only if xy is an edge of G, where y is the unique vertex in $N^+(x)$ satisfying $h_x(y) = i$.

Given the labels for x and y, the decoder performs the following:

- 1. If xy are not adjacent in H, output "not adjacent".
- 2. Otherwise xy are adjacent. If $x \prec y$, we are guaranteed that y is in the domain of h_x , so output "adjacent" if and only if $\mathsf{edge}_x(h_x(y)) = 1$. If $y \prec x$, output "adjacent" if and only if $\mathsf{edge}_y(h_y(x)) = 1$.

This concludes the proof.

3 Optimality

We now prove the optimality of our labeling schemes, and Corollary 1.2. We require:

Proposition 3.1. For any hereditary class \mathcal{F} , let k(n) be the maximum degeneracy of an n-vertex graph $G \in \mathsf{her}(\mathcal{F}^{\square})$. Then $\mathsf{her}(\mathcal{F}^{\square})$ contains a graph H on n vertices with at least $n \cdot k(n)/4$ edges, so $\mathsf{mon}(\mathcal{F}^{\square})$ contains all $2^{n \cdot k(n)/4}$ spanning subgraphs of H.

Proof. Since G has degeneracy k = k(n), it contains an induced subgraph $G' \subset_I G$ with minimum degree k and $n_1 \leq n$ vertices. If $n_1 \geq n/2$ then G itself has at least $kn_1/2 \geq kn/4$ edges, and we are done. Now assume $n_1 < n/2$. Since $G \in \text{her}(\mathcal{F}^{\square})$, $G \subset_I H_1 \square \cdots \square H_t$ for some $t \in \mathbb{N}$ and $H_i \in \mathcal{F}$. So for any $d \in \mathbb{N}$, the graph $(G')^d \subset_I (H_1 \square \cdots \square H_t)^d$ belongs to $\text{her}(\mathcal{F}^{\square})$. Consider the graph $H \subset_I (G')^d$ defined as follows. Choose any $w \in V(G')$, and for each $i \in [d]$ let

$$V_i = \{(v_1, v_2, \dots, v_d) : v_i \in V(G') \text{ and } \forall j \neq i, v_j = w\},\$$

and let H be the graph induced by vertices $V_1 \cup \cdots \cup V_d$. Then H has dn_1 vertices, each of degree at least k, since each $v \in V_i$ is adjacent to k other vertices in V_i . Set $d = \lceil n/n_1 \rceil$, so that H has at least n vertices, and let $m = dn_1 - n$, which satisfies $m < n_1$. Remove any m vertices of V_1 . The remaining graph H' has n vertices, and at least $(d-1)n_1 \ge n - n_1 > n/2$ vertices of degree k. Then H' has at least kn/4 edges.

The next proposition shows that Theorem 1.1 is optimal up to constant factors. It is straightforward to check that this proposition implies Corollary 1.2.

Proposition 3.2. Let \mathcal{F} be a hereditary class whose optimal adjacency labeling scheme has size s(n) and which contains a graph with at least one edge. Then any adjacency labeling scheme for $\ker(\mathcal{F}^{\square})$ has size at least $\Omega(s(n) + \log n)$, and any adjacency labeling scheme for $\operatorname{mon}(\mathcal{F}^{\square})$ has size at least $\Omega(s(n) + k(n) + \log n)$, where k(n) is the maximum degeneracy of any n-vertex graph in $\operatorname{mon}(\mathcal{F}^{\square})$.

Proof. Since $\mathcal{F} \subseteq \operatorname{her}(\mathcal{F}^{\square})$ and $\mathcal{F} \subseteq \operatorname{mon}(\mathcal{F}^{\square})$, we have a lower bound of s(n) for the labeling schemes for both of these classes. Since \mathcal{F} contains a graph G with at least one edge, the Cartesian products contain the class of hypercubes: $\operatorname{her}(\{K_2\}^{\square}) \subseteq \operatorname{her}(\mathcal{F}^{\square}) \subseteq \operatorname{mon}(\mathcal{F}^{\square})$. A labeling scheme for $\operatorname{her}(\{K_2\}^{\square})$ must have size $\Omega(\log n)$ (which can be seen since each vertex of K_2^d has a unique neighborhood and thus requires a unique label). This establishes the lower bound for $\operatorname{her}(\mathcal{F}^{\square})$, since the labels must have size $\max\{s(n),\Omega(\log n)\}=\Omega(s(n)+\log n)$. Finally, by Proposition 3.1, the number of n-vertex graphs in $\operatorname{mon}(\mathcal{F}^{\square})$ is at least $2^{\Omega(nk(n))}$, so there is a lower bound on the label size of $\Omega(k(n))$, which implies a lower bound of $\max\{s(n),\Omega(\log n),\Omega(k(n))\}=\Omega(s(n)+k(n)+\log n)$ for $\operatorname{mon}(\mathcal{F}^{\square})$.

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