Multi-learner risk reduction under endogenous participation dynamics

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Abstract

Prediction systems face exogenous and endogenous distribution shift—the world constantly changes, and the predictions the system makes change the environment in which it operates. For example, a music recommender observes exogeneous changes in the user distribution as different communities have increased access to high speed internet. If users under the age of 18 enjoy their recommendations, the proportion of the user base comprised of those under 18 may endogeneously increase. Most of the study of endogenous shifts has focused on the single decision-maker setting, where there is one learner that users either choose to use or not.

This paper studies participation dynamics between sub-populations and possibly many learners. We study the behavior of systems with *risk-reducing* learners and sub-populations. A risk-reducing learner updates their decision upon observing a mixture distribution of the sub-populations $\mathcal{D}$ in such a way that it decreases the risk of the learner on that mixture. A risk reducing sub-population updates its apportionment amongst learners in a way which reduces its overall loss. Previous work on the single learner case shows that myopic risk minimization can result in high overall loss (Perdomo et al., 2020; Miller et al., 2021) and representation disparity (Hashimoto et al., 2018; Zhang et al., 2019). Our work analyzes the outcomes of multiple myopic learners and market forces, often leading to better global loss and less representation disparity.

1 Introduction

The behavior of machine learning systems in settings with shifting distributions is of considerable interest for both practical and theoretical work. Whether or not accounting for these distribution shifts result in a robustness vs. accuracy trade-off depends on several factors. These include whether the shifts are adversarial or have a particular form, whether the shifts are exogeneous (fixed but unknown) or endogeneous (the result of the system's behavior), and whether robustness must be

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guaranteed during a single round of training or through some sequence of interactions with an environment.

In this work we study a form of endogeneously shifting distributions over multiple rounds, motivated by the observation that individuals will choose to use a service only if its predictions are sufficiently accurate. Consider as an example a music recommendation platform. If the platform recommends playlists that do not appeal to the tastes of users under 18, this group of users is likely to leave the system. Such effects result in positive (i.e. self-reinforcing) feedback loops, where a system's poor performance on young customers dissuades young customers from using the system, leading to less data and smaller objective value placed on making better predictions for young customers in the future. Within a single recommendation system, these effects may lead to representation disparity (Hashimoto et al., 2018).

In a broader recommendation ecosystem, customers have yet more choices than whether or not to participate in one particular recommendation system X: they can choose *amongst* systems $X_1, \ldots, X_m$. Then a given service will observe user distribution shifts as a function of both of their own behavior as well as the choices of other recommendation systems. Each service's choices affect the others' user distributions, which in turn affect the services' choices in models. This problem formulation captures more than just interactions between recommendation systems and their users. In all but perfectly monopolistic markets, customers interested in a service can choose between firms who provide that service with different qualities: an employer can choose amongst temporary employment firms to fill open positions, a person interested in a new email service can choose amongst different spam filters, and a shopper can choose amongst stores to purchase their items.

In this paper, we study the dynamics of such an ecosystem of populations apportioning themselves amongst firms, and firms that choose predictors based on their observed user population. Section 2 formally presents this setting, in which both populations and firms choose their actions myopically, incrementally improving their utility based on current market conditions. For this general class of dynamics, which we term *risk reducing*, we present a complete characterization of stable fixed points in Section 3. By drawing a connection between the dynamics and the *total risk*, we further characterize the implications of this dynamic in terms of a utilitarian notion of *social welfare*, and argue that increasing the number of competing firms leads to strictly better outcomes in terms of accurate predictions and user experience. Finally, in Section 4 we illustrate our theory with numerical examples and in Section 5 we conclude with a discussion of implications and ideas for future work.

1.1 Related Work

It would be remiss to not point out that optimization over decision-dependent probabilities arising from either exogenous or endogenous uncertainties has classical roots in stochastic optimization and control, and in particular, the application to operations research—see, e.g., the review article by Hellemo et al. (2018) and references therein. However, we narrow our focus the most relevant literature on decision-dependent optimization as it arises in machine learning systems.

Endogenous Distribution Shifts. In the study of machine learning systems, a large body of literature studies exogenous distribution shifts such as covariate, label, or concept drift (Quiñonero-Candela et al., 2008). A more recent trend is to study shifts in the underlying data distribution due to explicit or implicit endogenous reactions. Such reactions may arise due to strategic behavior exhibited by a user population. For instance, the work of Perdomo et al. (2020) introduces *performative prediction*, an abstraction capturing user reaction via endogenous distribution shifts. In particular, the abstraction models a single decision-maker facing a risk minimization problem subject to an underlying decision-dependent data distribution. Following its introduction, several relevant solution concepts have been explored and algorithms for achieving them proposed (Izzo et al., 2021; Drusvyatskiy and Xiao, 2020; Mendler-Dünner et al., 2020; Miller et al., 2021). A variant of single decision-maker performative prediction problem studies time-dependent dynamics of the data distribution; both exogenous (Wood et al., 2021; Cutler et al., 2021) and endogenous (Ray et al., 2022; Brown et al., 2022; Ginart et al., 2021) sources of the dynamic behavior have been considered. The focus of this work, however, has been on a single distribution; in contrast, we consider a mixture of distributions capturing different populations of strategic users.

Retention. User retention in machine learning systems is closely related to the population distribution dynamics we consider in this paper (Hashimoto et al., 2018; Zhang et al., 2019). In settings with multiple populations of users of different types, the question of retention has been explored in parallel with the issue of fairness. Hashimoto et al. (2018) coins the term *representation disparity* for the phenomenon in which the traditional approach of minimizing average performance leads to high overall accuracy coupled with low accuracy on minority populations, causing an exodus of said populations. Consequentially, myopic risk minimizing systems can quickly reach undesirable states, and may never recover. For single learners, systems which instead perform robust risk minimization avoid such feedback loops.

Our work, in contrast, analyzes the fixed points of dynamics between multiple systems and populations. This interaction can lead to non-monotone population dynamics: if a population has many choices of systems, it might drift towards a particularly good one; if at some point that system's subpopulation performance suffers or another system improves, the subpopulation might shift again. Our focus is on *myopic risk reducers*, a generalization of risk minimizers. Risk reducing learning systems are common in practice, so understanding the behavior of these rich environments is of high importance.

Multiple Decision-Makers. Endogenous distribution shift has also been studied in settings with multiple decision-makers as a continuous game. For instance, the multi-player performative prediction problem extends the original problem by allowing for multiple competing decision-makers (Narang et al., 2022; Piliouras and Yu, 2022; Wood and Dall'Anese, 2022). This line of work is related, but differs in that the population is modelled as homogeneous and stateless. These works focus on characterizing the existence and uniqueness of different types of competitive equilibria for the continuous game, and analyze how different learning dynamics lead to different equilibrium concepts. In contrast, in this paper our focus is on asymptotically stable points (equilibrium) for the combined dynamical system resulting from user participation updates, and the decision-maker's iterative optimization process.

2 Problem Setting

We consider a setting in which the overall population of individuals is composed of n subpopulations choosing between m learners (corresponding to a firm, service provider, decision-maker, or the like). Each subpopulation $i \in \{1, \dots, n\} =: [n]$ is distributed according to a fixed distribution $\mathcal{D}_i$ and makes up β_i proportion of the total population, so that $\sum_{i=1}^{n} \beta_i = 1$. An α_{ij} proportion of subpopulation i is associated to each learner $j \in [m]$, normalized so that $\sum_{j=1}^{m} \alpha_{ij} = 1$. Thus subpopulations do not opt-out, only redistribute their participation among the various learners. We remark that opting-out could be modelled by a "null learner."

The learner j selects parameter or decision: $\theta_j \in \mathbb{R}^d$. The learners are not aware of which subpopulation individuals are from. They only observe a mixture distribution determined by the participation and the relative subpopulations sizes: $(\sum_{i=1}^n \alpha_{ij}\beta_i)^{-1} \sum_{i=1}^n \alpha_{ij}\beta_i \mathcal{D}_i$. Throughout, we take the number of learners to be smaller than the number of subpopulations, $m \leq n$.

The quality of a parameter $\theta \in \mathbb{R}^d$ for an individual z is quantified by the loss $\ell(\theta; z)$. The *risk* is the average subpopulation loss $\mathcal{R}_i(\theta) = \mathbb{E}_{z\sim\mathcal{D}_i}[\ell(\theta; z)]$. Throughout we will make the additional assumption that the risk function for each subpopulation $\mathcal{R}_i(\theta)$ is strongly convex, which implies that there is a unique optimal parameter for each subpopulation.

Given all the parameters $\Theta = \{\theta_j\}_{j=1}^m$ chosen by the learners and the participation α, the average risk experienced by subpopulations and learners are:

$$\bar{\mathcal{R}}_i^{\mathsf{subpop}}(\alpha_{i,:}, \Theta) := \sum_{j=1}^m \alpha_{ij}\mathcal{R}_i(\theta_j) \quad \text{and} \quad \bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha_{:,j}, \theta_j) := \frac{1}{\sum_{i=1}^n \alpha_{ij}\beta_i} \sum_{i=1}^n \alpha_{ij}\beta_i\mathcal{R}_i(\theta_j),$$

where by $\alpha_{i,:} \in \Delta_m$ we denote the vector of allocations from the subpopulation i to all learners. On the other hand, by $\alpha_{:,j} \in \mathbb{R}^n$ we denote the vector of allocations from all subpopulations to learner j.

2.1 Decision dynamics of learners and subpopulations

We consider a dynamic feedback setting where learner decisions lead to changes in participation for the subpopulations. Changes in the participation then lead to updates in the learners' decisions. Throughout we will use the superscript t to denote the decisions at time t. We consider *myopic* learners which aim simply to reduce the risk on the mixture distribution that they observe. Such incremental updates are reflective of real decision dynamics, as firms often focus on continually improving their services and are not aware of the subpopulation structure of their user base. As individuals in a subpopulation compare the quality of their services with those of their peers, their participation naturally shifts to better performing services. Then the firms react to these changes by updating their decisions incrementally to better serve their current customers or users.

2.1.1 Dynamics of subpopulations

Given current allocations $\alpha_{i,:}^t$ and risk values $r_{i,:}(\Theta^t) := \{\mathcal{R}_i(\theta_j^t)\}_{j=1}^m$, we define an allocation function $\nu_i : \Delta_m \times \mathbb{R}^m \to \Delta_m$ which describes the retention update

$$\alpha_{i,:}^{t+1} = \nu_i\left(\alpha_{i,:}^t, r_{i,:}(\Theta^t)\right) \ .$$

The general form of this update allows the allocation function to depend on the current allocations as well as current risks. In this case, the subpopulation update is *stateful*. In the special case that the allocations at the next time step depend only on the risk, we call the subpopulation update *stateless*. A stateful update is particularly relevant in the scenario that learners retrain models relatively frequently compared to the behavior of subpopulations. On the other hand, a stateless update corresponds to subpopulations reacting to changing models quickly relative to the rate at which they are retrained.

Definition 2.1 (Risk-Reducing Allocations)**.** *A subpopulation is* risk reducing *with respect to learner's decisions* Θ *when the allocation function is such that:*

$$\bar{\mathcal{R}}_i^{\mathsf{subpop}}(\alpha_{i,:}^{t+1}, \Theta) \leq \bar{\mathcal{R}}_i^{\mathsf{subpop}}(\alpha_{i,:}^{t}, \Theta).$$

It is further risk minimizing in the limit *if the inequality is strict and* $\alpha_{i,:}^{t+1} = \alpha_{i,:}^{t}$ *if and only if* $\alpha_{i,:}^{t} \in \arg\min_{\alpha_{i,:} \in \Delta^m} \bar{\mathcal{R}}_i^{\mathsf{subpop}}(\alpha_{i,;}, \theta)$.

Risk reducing can be viewed as better response, whereas risk minimizing in the limit can be viewed as a eventually approaching the best response.

Example 2.2 (Multiplicative Weights Update Dynamics (MWUD))**.** *MWUD is a stateful risk-reducing allocation function such that:* $\alpha_{ij}^{t+1} \propto \alpha_{ij}^{t} \exp\left(-\gamma c_j \left(r_{i,:}(\Theta^t)\right)\right)$ *where* γ *is an update parameter and* $c_j(\cdot)$ *is a comparison function. For instance, we can use the absolute risk or the relative risk:* $c_j\left(r_{i,:}(\theta^t)\right) = \mathcal{R}_i(\theta_j^t)$ *or* $c_j\left(r_{i,:}(\theta^t)\right) = \frac{\mathcal{R}_i(\theta_j^t)}{\bar{\mathcal{R}}_i^{\mathsf{subpop}}(\alpha^t, \theta^t)}$. *As long as* $\gamma > 0$ *and* c_j *is strictly monotonically increasing with respect to* $\mathcal{R}_i(\theta_j^t)$*, MWUD are risk minimizing in the limit.*

This multiplicative weight update is similar to the retention function studied by Hashimoto et al. (2018) in expectation. There is an equivalence between MWUD and *replicator dynamics*, which is a fundamental evolutionary dynamic rate of growth in which the use of a certain strategy (here learner) is proportional to the relative performance of the strategy (learner). Economically, this can be interpreted as a process of information diffusion and imitation (Sandholm, 2020).

2.1.2 Dynamics of learners

At each time step the mixture distribution of the subpopulations changes a bit, and subsequently each learner updates their decision $\theta_j^{t+1} = \mu_j(\theta_j^t, \alpha_{:,j})$. The learners are *myopic* rather than explicitly competitive, and we consider *risk reducing* learners, defined as follows.

Definition 2.3 (Risk-Reducing learner)**.** *A learner is* risk reducing *with respect to the mixture distribution induced by the subpopulations' allocations* $\{\alpha_{ij}\}_{i=1}^n$ *when it updates the decision to reduce the risk on the current observed distribution—i.e.,*

$$\bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha_{:,j}, \theta_j^{t+1}) \leq \bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha_{:,j}, \theta_j^{t}).$$

It is further risk minimizing in the limit *if the inequality is strict and* $\theta_j^{t+1} = \theta_j^t$ *if and only if* $\theta_j^{t+1} \in \arg\min_{\theta_j} \bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha_{:,j}, \theta_j)$.

The notion of risk minimizing in the limit is well motivated due to the assumption that risks are strongly convex (and hence the learner average risk is also strongly convex).

Example 2.4 (Repeated gradient descent)**.** *In repeated gradient descent dynamics, learners incrementally update their parameter in the negative direction of the gradient of the current risk function:* $\theta_j^{t+1} := \theta_j^t - \gamma^t \nabla_\theta \bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha_{:,j}, \theta_j^t)$*, where step size* γ^t *satisfies the standard assumptions of decreasing step size and infinite travel:* $\lim_{t\to\infty} \gamma^t = 0$ *and* $\lim_{t\to\infty} \sum_{\tau=1}^t \gamma^\tau = \infty$*. For strongly convex risk functions, these learner dynamics are risk minimizing in the limit.*

We remark that our definition of risk reduction does not translate well to the stateless update setting, since there is not a clear connection between time steps. However, there is a straightforward notion of stateless risk minimization which is akin to repeated retraining dynamics which have been studied in the single learner case (Hashimoto et al., 2018; Perdomo et al., 2020).

Example 2.5 (Full risk minimization)**.** *Under these dynamics, learners fully update their parameter to minimize the current risk function:* $\theta_j^{t+1} := \arg\min \bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha_{:,j}, \theta_j^{t+1})$.

Remark. *For both learners and subpopulations, risk-reducing is a property defined with respect to the distribution observed at a previous time step. Thus it does not necessarily hold that* $\bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha_{:,j}^{t+1}, \theta_j^{t+1}) \leq \bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha_{:,j}^{t}, \theta_j^{t})$ *nor that* $\bar{\mathcal{R}}_i^{\mathsf{subpop}}(\alpha_{i,:}^{t+1}, \Theta^{t+1}) \leq \bar{\mathcal{R}}_i^{\mathsf{subpop}}(\alpha_{i,:}^{t}, \Theta^{t})$.

2.2 Equilibria and stability

We are concerned with the study of the equilibrium states of the dynamics defined in Section 2.1. The state consists of the parameters Θ of learners and allocations α of the subpopulations.

Definition 2.6 (Equilibrium)**.** *The state* $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ *is an equilibrium state if it is stationary under the dynamics update; i.e. the following conditions hold:*

$$\alpha_{i,:}^{\mathsf{eq}} = \nu_i(\alpha_{i,:}^{\mathsf{eq}}, \Theta^{\mathsf{eq}}) \quad \forall \quad i \in [n] \quad \textit{and} \quad \theta_j^{\mathsf{eq}} = \mu_j(\alpha_{:,j}^{\mathsf{eq}}, \theta_j^{\mathsf{eq}}) \quad \forall \quad j \in [m] \,.$$

Thus, if learners and subpopulations are in an equilibrium state, they will remain that way indefinitely. However, one can imagine that the stationarity of certain configurations is fragile. The following definition formalizes equilibria which are persistent under perturbations.

Definition 2.7 (Stable Equilibrium)**.** *The state* $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ *is a* stable *equilibrium state if it is an equilibrium and for each* $\epsilon_\alpha, \epsilon_\theta > 0$*, there exist* $\delta_\alpha, \delta_\theta > 0$ *such that if* $\|\alpha^0 - \alpha^{\mathsf{eq}}\| < \delta_\alpha$ *and* $\|\Theta^0 - \Theta^{\mathsf{eq}}\| < \delta_\theta$*, then* $\|\alpha^t - \alpha^{\mathsf{eq}}\| \leq \epsilon_\alpha$ *and* $\|\Theta^t - \Theta^{\mathsf{eq}}\| \leq \epsilon_\theta$ *for all* $t \geq 0$*. It is an* asymptotically stable *equilibrium state if furthermore* $\lim_{t\to\infty} \|\alpha^t - \alpha^{\mathsf{eq}}\| = 0$ *and* $\lim_{t\to\infty} \|\Theta^t - \Theta^{\mathsf{eq}}\| = 0$.

The next section presents our main results, which characterize the stable equilibria of the learner and subpopulation dynamics. The study of stable equilibria is relevant for qualitatively understanding system behavior. For most initializations (a set with probability 1 under uniform measure), the allocations and parameters will eventually converge to a stable equilibrium. Furthermore, the results we present can be used to understand high probability behavior of systems under noisy updates which are risk reducing only in expectation. This particularly relevant for extensions which model realistic phenomena—for example, learners observing only a finite sample of the population, or subpopulation shifts being driven by discrete individual decisions.

3 Main Results

We study the feedback dynamics between learners and subpopulations under the assumption that they update their decisions in a sequential fashion:

$$\alpha^{t+1} = \nu(\alpha^t, \Theta^t), \quad \Theta^{t+1} = \mu(\alpha^{t+1}, \Theta^t) \,.$$

The class of sequential updates satisfying the risk reducing and sequential assumptions is quite large. For example, our analysis does not require that every learner or every subpopulation updates their parameter or allocation at every step. This allows, for example, any number of round-robin style updates. The order or number of updates need not be consistent, so long as the risk reducing or minimizing properties are satisfied. The only situation that our analysis does not trivially extend to is the one where updates are precisely synchronous.

3.1 Total Risk Reduction

The total risk of all subpopulations over all learners $\mathcal{R}^{\mathsf{total}} : \Delta_m^n \times \mathbb{R}^{m\times d} \to \mathbb{R}$ is defined as

$$\mathcal{R}^{\mathsf{total}}(\alpha, \Theta) := \sum_{i=1}^{n}\sum_{j=1}^{m} \beta_i \alpha_{ij} \mathcal{R}_i(\theta_j).$$

Under the general assumption that the subpopulation participation and learner update dynamics are risk reducing, our first result shows that the total risk $\mathcal{R}^{\mathsf{total}}(\alpha^t, \Theta^t)$ is non-increasing over time. This means that the total risk acts as a potential function for the feedback dynamics between learners and subpopulations.

Proposition 3.1. *Under any risk-reducing subpopulation dynamics and any risk-reducing learner dynamics, the total risk is non-increasing in every timestep.*

$$\mathcal{R}^{\mathsf{total}}(\alpha^{t+1}, \Theta^{t+1}) \le \mathcal{R}^{\mathsf{total}}(\alpha^t, \Theta^t).$$

Further, if subpopulations and learners are risk minimizing in the limit, then the total risk is strictly decreasing unless (α^t, Θ^t) *is an equilibrium.*

Proof Sketch. First note that the total risk can be decomposed into either a weighted sum of average subpopulation risk or average learner risk. Thus the fact that learner and subpopulation dynamics are risk reducing ensures that the total risk is decreasing after the sequential updates. □

When the subpopulation and learner dynamics are risk minimizing in the limit, there is a strong connection between properties of the total risk function and equilibria of the dynamics.

Theorem 3.2. *When learners and subpopulations are risk minimizing in the limit, an equilibrium* $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ *is asymptotically stable if and only if it is an isolated local minimizer of the total risk* $\mathcal{R}^{\mathsf{total}}$. *If it is not a local minimizer, then it is not stable.*

Proof Sketch. Supposing that the equilibrium is not stable, and using that the total risk decreases along trajectories (Proposition 3.1), it must not be a local minimum. To argue the implications in the other direction, we show that we can use the total risk to construct a Lyapunov function. □

The connection described in Theorem 3.2 is meaningful in at least two ways: first, it means that under general classes of myopic and self-interested behaviors on the part of subpopulations and learners, the total risk is driven to at least a local minimum. Second, it is a technically useful connection that will enable us to characterize and classify the stable equilibria. We briefly comment on the requirement that the dynamics are risk minimizing. The necessity of this condition can be understood by analogy to optimization of a smooth function via gradient descent (GD). If the step size decays too fast, the GD dynamics converge to an equilibria without necessarily being at a local minimum of the function. Such equilibria will not correspond to the underlying function in a meaningful way; rather, they are sensitive to the initialization and decay rate.

3.2 Split-Market and Balanced Equilibria

We consider "split market" allocations α where the learners segment the subpopulations. Formally, a split market occurs when α_{ij} takes values only in $\{0, 1\}$ for all i, j. For allocation dynamics like MWUD, such configurations will clearly be equilibria. We thus consider the set of possible split-market equilibria and characterize which are asymptotically stable.

Theorem 3.3. *Suppose learners and subpopulations are risk minimizing in the limit, α^{eq} is split-market and $\Theta^{\mathsf{eq}} = \arg\min_\Theta \mathcal{R}^{\mathsf{total}}(\alpha^{\mathsf{eq}}, \Theta)$. Define a mapping $\gamma : [n] \to [m]$ such that $\gamma(i) = j$ is the learner with nonzero mass in $\alpha^{\mathsf{eq}}_{i,:}$. Then $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ is an asymptotically stable equilibrium if and only if each learner is associated to at least one subpopulation, and*

$$\mathcal{R}_i(\theta^{\mathsf{eq}}_{\gamma(i)}) < \mathcal{R}_i(\theta^{\mathsf{eq}}_j) ,$$

for all subpopulations i and learners $j \neq \gamma(i)$, i.e. no subpopulation would prefer to switch learners. Furthermore, such split market equilibria are the only asymptotically stable equilibria.

Remark that Θ^{eq} is well defined because we consider strongly convex risks. In particular, in a split market, each θ^{eq}_j will minimize the average loss over the group of subpopulations assigned to them, i.e. the jth partition of the total population. Denote by ϕ_i the parameter which minimizes the subpopulation risk $\phi_i := \arg\min_{\theta\in\mathbb{R}^d} \mathcal{R}_i(\theta)$. Then each θ^{eq}_j is a convex combination of ϕ_i for i in jth partition. Using this perspective, we provide an intuitive necessary (but not sufficient) condition for a class of symmetric risk functions.

Corollary 3.4. *Suppose that risk functions satisfy $\mathcal{R}_i(\theta) < \mathcal{R}_i(\theta') \iff \|\theta - \phi_i\| < \|\theta' - \phi_i\|$ for ϕ_i the subpopulation optimal parameter. Then in an asymptotically stable split market equilibrium, the convex hulls of the grouped subpopulations optimal parameters $\{\phi_i\}$ are non-intersecting.*

Proof Sketch. Consider a partition where the convex hulls intersect for some pair of learners. Then there exists at least one subpopulation who would be better off switching to the other learner, and thus the risk condition in Theorem 3.3 cannot hold. □

One can also imagine an equilibrium where a subpopulation has support over multiple learners, so long as they are "balanced" in the sense that they provide the same risk to the subpopulation. Though the previous result rules out asymptotic stability for such points, the next result provides a condition under which they may be stable.

Theorem 3.5. *Consider dynamics which are risk minimizing in the limit and an α^{eq} with any subpopulation i having nonzero support on set of two or more learners $j \in \mathcal{J}$. Define $\Theta^{\mathsf{eq}} = \arg\min \mathcal{R}^{\mathsf{total}}(\alpha^{\mathsf{eq}}, \Theta)$. Then $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ cannot be stable unless it is "balanced" in the sense that learners in $\mathcal{J}$ are* risk equivalent *and* optimal *for i, i.e. for all $j, j' \in \mathcal{J}$,*

$$\mathcal{R}_i(\theta^{\mathsf{eq}}_j) = \mathcal{R}_i(\theta^{\mathsf{eq}}_{j'}) \quad \text{and} \quad \nabla\mathcal{R}_i(\theta^{\mathsf{eq}}_j) = 0 \,.$$

If it is balanced, so are all allocations for subpopulation i with support over $\mathcal{J}$. Finally, such balanced equilibria are the only ones besides split-market equilbria that can be stable.

This result characterizes a set of possibly stable (but not asymptotically stable) equilibria which are not split market. Guaranteeing the stability requires further information about the subpopulation dynamics. For example, specific instantiations of subpopulation dynamics that encode a preference for homogeneity, e.g. those that incorporate tie-breaking, would preclude the stability of these the balanced equilibria. Theorem 3.5 also sheds light on a class of split market equilibria that may be stable but are not asymptotically stable. It implies that for any balanced equilibrium with support over some set of learners, the split-market equilibria putting mass on the singletons are also balanced. Then the risks must be equal, so Theorem 3.3 rules out asymptotic stability due to the existence of risk-equivalent options for some subpopulation.

We further remark that the balance condition is fragile in the sense that it would not hold under small perturbations to the underlying risk functions. While the number of possible balanced equilibria is combinatorial in the number of learners and subpopulations, risk functions are continuous, so it is possible to imagine arbitrarily small perturbations to any the risk functions that would destabilize all balanced equilibria.

The proofs of Theorems 3.3 and 3.5 make use of the connection between minima of the total risk and stable equilibria presented in Theorem 3.2. The following lemma plays a crucial role by facilitating the analysis of local minimizers of the total risk. It simplifies the total risk by "minimizing out" the dependence on parameters Θ.

Lemma 3.6. *Define the function* $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ *as* $F(\alpha) = \min_\Theta \mathcal{R}^{\mathsf{total}}(\alpha, \Theta)$. *This function is concave and a point* (α^0, Θ^0) *is a local minimum (resp. isolated local minimum) of* $\mathcal{R}^{\mathsf{total}}$ *over the domain* $\mathcal{X} = \mathcal{X}_\alpha \times \mathbb{R}^{m\times d}$ *if and only if* α^0 *is a local minimum (resp. isolated local minimum) of* F *over the domain* $\mathcal{X}_\alpha$ *and* $\Theta^0 = \arg\min_\Theta \mathcal{R}^{\mathsf{total}}(\alpha^0, \Theta)$.

We present full proofs in the appendix, and sketch the arguments here.

Proof Sketch of Theorems 3.3 and 3.5. By Theorem 3.2, characterizing the stable equilibria is equivalent to characterizing isolated and non-isolated local minima of the total risk. By Lemma 3.6, it suffices to characterize local minima of F over the simplex product $\Delta_n \times \cdots \times \Delta_n$. Since F is concave, all minima will occur on the boundary, i.e. a face or a vertex. Since F is still concave when restricted to a face of the simplex, the same argument shows the minima are on the boundary, hence vertices, except for the degenerate case where F takes a constant value over the face.

Thus, the only isolated local minima will occur at vertices of the simplex product, which correspond to split markets. Further analysis of F yields the conditions presented in Theorem 3.3. The local minima in the degenerate case are characterized by the balanced equilibria conditions presented in Theorem 3.5. □

3.3 Social Welfare in Split Markets

We now turn to the broader implications of the risk reducing dynamics with a notion of *social welfare*.

Definition 3.7 (Social Welfare)**.** *The* social welfare *of a state* (α, Θ) *is a strictly decreasing function of the total risk* $\mathcal{R}^{\mathsf{total}}(\alpha, \Theta)$.

Thus social welfare is utilitarian in the sense that it depends on the cumulative quality of individuals' experiences. Maximizing the social welfare corresponds to minimizing the total risk, which can be posed as the following optimization problem

$$(\alpha^*, \Theta^\star) = \min_{\alpha,\Theta} \quad \mathcal{R}^{\mathsf{total}}(\alpha, \Theta) \quad \text{s.t.} \quad \alpha_{i,:} \in \Delta_n \;\; \forall \;\; i = \{1, \dots n\} \,. \tag{1}$$

Our discussion of stable equilibria has so far focused on only local minimizers of the total risk. In fact, global minimization of this objective (and therefore maximization of social welfare) is a hard problem. The total risk objective can be viewed as an instance of the k-means clustering problem with $k = m$. In the language of this literature (e.g., Selim and Ismail (1984)), each subpopulation is a data point or "pattern" and the parameter selected by each learner is a cluster center. The allocations described by α correspond to (fuzzy) cluster assignment and each risk function $\mathcal{R}_i(\theta_j)$ corresponds to a measure of "dissimilarity" between patterns (subpopulations) and cluster centers (learners).

The connection to k-means clustering elucidates the difficulty of minimizing the total risk. The "minimum sum-of-squares clustering" problem (i.e., squared Euclidean norm dissimilarity) is NP hard with general dimension even when $k = 2$ (Aloise et al., 2009). When the number of clusters and dimension are fixed, Inaba et al. (1994) present an algorithm for solving the minimum sum-of-squares clustering problem which is polynomial in the number of datapoints. Translated to our setting, its complexity is $O(n^{md})$. It is therefore overly optimistic to hope for straightforward social welfare maximization. However, due to the connections with total risk, risk reducing dynamics are at least well-behaved with regards to social welfare.

Proposition 3.8. *For risk reducing subpopulations and learners, social welfare is non-decreasing over time. If the dynamics are furthermore risk minimizing in the limit, social welfare is strictly increasing and asymptotically stable equilibria correspond to local social welfare maxima. As a result, the social welfare maximizing allocation $\alpha^\star$ and parameter $\Theta^\star$ is a stable equilibrium.*

Proof. Social welfare is non-decreasing (or increasing) if and only if total risk is non-increasing (or decreasing), as guaranteed by Proposition 3.1. Maxima of the social welfare are equivalent to minima of the total risk and therefore the connections to stable equilibria follow by Theorem 3.2. □

The following example illustrates that a local social welfare maximum can be much worse than the optimal value.

Example 3.9. *Suppose there are two learners and three subpopulations with sizes $\beta_1 = \beta_2 = \beta$ and $\beta_3 = 1 - 2\beta$ for some $0 < \beta < 1/2$. Consider the following:*

$$\mathcal{R}_1(\theta) = \theta^2, \quad \mathcal{R}_2(\theta) = (\theta - 1)^2, \quad \textit{and} \;\; \mathcal{R}_2(\theta) = \left(\theta - \frac{1-\beta}{1-2\beta} + \epsilon\right)^2 .$$

The social welfare optimizing decision $\Theta^\star = (1/2, \frac{1-\beta}{1-2\beta} - \epsilon)$ corresponds to total risk $\beta/2$. However, there is a stable equilibrium at $\Theta^{\mathsf{eq}} = (0, 1 + \epsilon)$ with total risk $\beta + \frac{(\beta-\epsilon)^2}{1-2\beta}$. For $\beta \to 1/2$, the gap becomes arbitrarily large.

In this example, the large gap between a stable local optimum and the global optimum arises in part due to subpopulations with very different sizes. We further remark that minority groups can be under-served particularly when considering not just total, but worst-case risk over subpopulations (Hashimoto et al., 2018). Even at the social welfare maximizer $(\alpha^\star, \Theta^\star)$, the worst-case

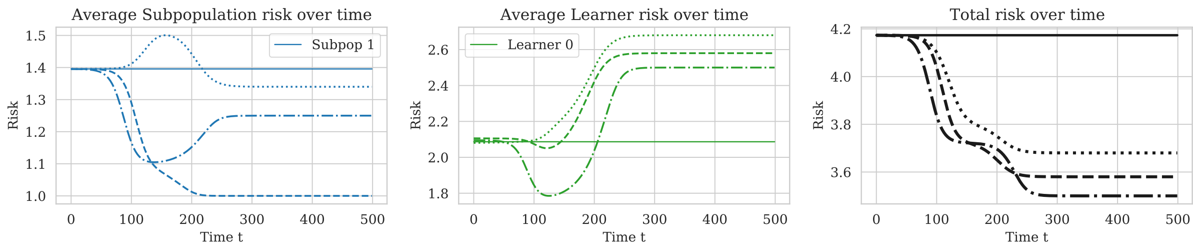

Figure 1: **Risk dynamics:** Setting with 3 subpopulation and 2 learners. The solid lines correspond to the unstable balanced equilibrium at initialization. Dotted lines illustrate system trajectories under three different slight perturbations from the initialization.

subpopulation risk can be arbitrarily bad. It is straightforward to construct such examples even in the single learner case: consider a minority group with vanishingly small population proportion and arbitrarily high risk at the optimal parameter for the majority group.

Despite these inherent difficulties, we now show that the situation improves as the number of learners increases. It is straightforward to see that the maximal social welfare will increase: any point which is optimal for m learners can be trivially transformed into a feasible point for $m+1$ learners which achieves the same social welfare, by allocating no subpopulations to the new learner. There is more nuance involved when considering any possible stable equilibria. Instead, we make a statement about a particular learner growth process which corresponds to existing learner m splitting in half.

Proposition 3.10. *Suppose that there are m learners, $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ is an equilibrium, and at least one subpopulation i allocated to learner m does not have optimal subpopulation risk, so $\nabla \mathcal{R}_i(\theta_m^{\mathsf{eq}}) \neq 0$. The state is amended to add an additional learner: $\tilde{\Theta}^{\mathsf{eq}} = [\Theta^{\mathsf{eq}}, \theta_m^{\mathsf{eq}}]$ and*

$$\tilde{\alpha}^{\mathsf{eq}}_{:,j} = \begin{cases} \alpha^{\mathsf{eq}}_{:,j} & j \leq m \\ \frac{1}{2}\alpha^{\mathsf{eq}}_{:,m} & j \in \{m, m+1\} \end{cases}$$

Under dynamics which are risk minimizing in the limit, the equilibrium $(\tilde{\alpha}^{\mathsf{eq}}, \tilde{\Theta}^{\mathsf{eq}})$ is not stable, so a small perturbation will send the system to a state with strictly lower total risk (higher social welfare).

Proof. By construction $(\tilde{\alpha}^{\mathsf{eq}}, \tilde{\Theta}^{\mathsf{eq}})$ is not a split market, and neither is it a stable balanced equilibrium (by the non-optimality assumption). Therefore, it is not stable (Theorem 3.5), and thus not a local minimum of the total risk (Theorem 3.2), so there are directions which reduce the total risk. □

4 Numerical Experiments

We illustrate the salient properties of the decision dynamics in simulation[1]. In Figure 1 we consider a simple scenario with 3 subpopulations of equal sizes $\beta_i = 1/3$, identical quadratic risk functions $\mathcal{R}_i = \|\phi_i - \theta\|^2 + 1$ with distinct risk minimizing decisions ϕ_j and 2 learners. The learners minimize their risk according to *full risk minimization* (Example 2.5) and the subpopulations update their participation via MWUD with identity comparison function (Example 2.2). When $\alpha^0_{i,j} = 1/2$ for all

[1]Code at `https://anonymous.4open.science/r/MultiPlayerRiskReduction-2C12/README.md`

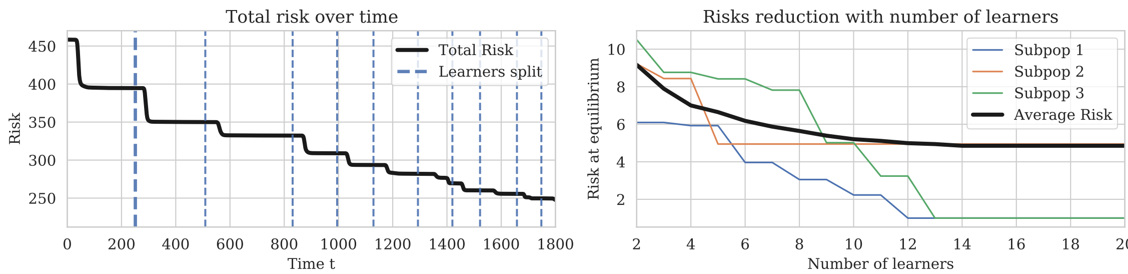

Figure 2: **Impact of competition on social welfare:** The left plot illustrates the reduction in total risk over time. The dashed blue lines indicate the times when a new learner joins. The right plot shows the equilibrium-risk reduction for a subset of the subpopulations as the number of learners increases.

i, j the risk equality condition from Theorem 3.5 is satisfied with $\theta_j^{\mathsf{eq}} = (\phi_1 + \phi_2 + \phi_3)/3$, however the optimality condition is not. We therefore observe that this equilibrium is not stable, and slightly perturbing the initial conditions leads to split-market equilibria. Figure 1 illustrates trajectories from three different perturbations. It demonstrates that the total risk is non-increasing whereas the average risks for both learners and subpopulations are not monotonic over time.

Another set of experiments in Figure 2 illustrates how more competitive settings, i.e. larger number of learners leads to better outcomes for the subpopulations. We consider a set of $m = 2$ learners and $n = 50$ subpopulations. We simulate the dynamics until the market has reached equilibrium, at which point one of the learners breaks up into two identical learners with half the user base. From this unstable equilibrium (Proposition 3.10) we slightly perturb the parameters of the two learners and allow the system to reach a new equilibrium state. The procedure repeats until the number of learners reaches number of subpopulations. These simulations illustrate how market diversity alleviates worst case outcomes: as more learners are added, not only does the total risk improve, but so does the risk of the worst-off subpopulation.

5 Discussion

In this paper, we study the feedback dynamics of user retention for loss minimizing learners, where subpopulations choose between providers. We introduce a formal notion of *risk reducing* and *minimzing* to capture this feedback, and show that there is a close connection between such dynamics and the *total risk* summed over subpopulations and learners. We provide a comprehensive characterization of stable equilibria and investigate the implications in terms of a utilitarian social welfare. Though not a panacea against all concerns of fairness and minority representation, we show that the multi-learner setting has many desirable characteristics compared with single-learner settings, as studied by Hashimoto et al. (2018); Zhang et al. (2019). Our results show that increasing the number of learners ameliorates such pitfalls, which resonates with recent work showing that monopolies have higher *performative power* and lead to lower individual utility (Hardt et al., 2022).

We highlight several directions for future work. Our results lay the groundwork for an investigation of the stochastic dynamics that occur for finite sample approximations to the risk or participation driven by decisions of individuals. So long as such behaviors are risk reducing in *expectation*, the noisy trajectories will converge with high probability to sets around the asymptotically stable equilbria we characterize. There are many interesting and relevant questions in the finite sample setting: What is the effect of sample size on the ability of new learners to enter a market and minority subpopulations to be adequately represented? Can we model heterogeneous learners who differ in which features they measure and with how much noise? Are there trade-offs between the expressivity of models and the practical difficulty of minimizing risk from finite samples in high dimensions?

It would also be interesting to consider extensions or alternative dynamics models for the learner and subpopulation decisions. One could investigate competitive learners who explicitly strategize to capture subpopulations; this setting is related to facility location and Hotelling games (Owen and Daskin, 1998; Hotelling, 1929). One might imagine that subpopulations do not act uniformly and may not even be entirely independent of each other. Instead, the participation update could be modelled as depending on some underlying social network. Finally, an investigation into further restricted classes of dynamics could yield interesting insights. The connections between total risk reduction and k-means clustering algorithms suggest variations (like k-means++) which have known convergence or stability properties. Results on "ground truth recovery" under particular generative models may yield insight into particular population structures that lead to simpler dynamics or restricted sets of equilibria.

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A Preliminaries

A.1 Notation

We introduce a compact notation. The simplex product is defined as

$$\Delta_m^n = \left\{ A \in \mathbb{R}^{n\times m} \mid \sum_{j=1}^{m} A_{ij} = 1 \right\}$$

so that the *rows* sum to 1. Then the state space of subpopulation allocations and learner parameters is $\mathcal{X} = \Delta_m^n \times \mathbb{R}^{m\times d}$. For a square matrix A, we use the notation $\mathrm{diag}(A)$ to represent the vector containing the diagonal entries of A. For a vector a, $\mathrm{Diag}(a)$ is a diagonal matrix with a along the diagonal. Furthermore we will say $a \le b$ for vectors a, b if the inequality holds elementwise.

Define a matrix valued risk function $R : \mathbb{R}^{m\times d} \to \mathbb{R}^{n\times m}$ so that $R_{ij}(\Theta) = \mathcal{R}_i(\theta_j)$. Recall that in Section 2.1, the subpopulation and learner risks played a key role. We therefore define vector valued functions $\bar{\mathcal{R}}^{\mathsf{subpop}} : \mathcal{X} \to \mathbb{R}^n$ and $\bar{\mathcal{R}}^{\mathsf{learner}} : \mathcal{X} \to \mathbb{R}^m$ as follows:

$$\bar{\mathcal{R}}^{\mathsf{subpop}}(\alpha, \Theta) = \mathrm{diag}(\alpha R(\Theta)^\top), \quad \bar{\mathcal{R}}^{\mathsf{learner}}(\alpha, \Theta) = \mathrm{diag}\left(\mathrm{Diag}(\alpha^\top \beta)^{-1} \alpha^\top \mathrm{Diag}(\beta) R(\Theta)\right) .$$

Then the definition of risk reducing dynamics for subpopulations and learners can be written as

$$\bar{\mathcal{R}}^{\mathsf{subpop}}(\alpha^{t+1}, \Theta) \le \bar{\mathcal{R}}^{\mathsf{subpop}}(\alpha^{t}, \Theta) \quad \text{and} \quad \bar{\mathcal{R}}^{\mathsf{learner}}(\alpha, \Theta^{t+1}) \le \bar{\mathcal{R}}^{\mathsf{learner}}(\alpha, \Theta^{t}) .$$

Risk minimizing in the limit is defined similarly, where the inequality is strict for at least one entry of the vectors unless the state is at an equilibrium.

The total risk can be written as

$$\mathcal{R}^{\mathsf{total}}(\alpha, \Theta) := \mathrm{tr}(\mathrm{diag}(\beta) \alpha R(\Theta)^\top) .$$

Lemma A.1. *Under the assumption that all loss functions are continuous, the risk function R is continuous w.r.t. to Θ, and thus $\mathcal{R}^{\mathsf{total}}$ is continuous w.r.t. α and Θ.*

The sequential dynamics updates described in Section 2.1 can be written as

$$\begin{bmatrix} \alpha^{t+1} \\ \Theta^{t+1} \end{bmatrix} = \begin{bmatrix} \nu(\alpha^t, \Theta^t) \\ \mu(\alpha^{t+1}, \Theta^t) \end{bmatrix} = \begin{bmatrix} \nu(\alpha^t, \Theta^t) \\ \mu(\nu(\alpha^t, \Theta^t), \Theta^t) \end{bmatrix} =: f(\alpha^t,\ \Theta^t) . \tag{2}$$

Lemma A.2. *As long as the subpopulation and learner updates described in Section 2.1 are locally Lipschitz, so is the dynamics function f defined in (2).*

A.2 Background

For completeness, we include important results and definitions that our proofs will make use of. First, we state two theorems about Lyapunov theory for stability.

Theorem A.3 (Theorem 1.2 in (Bof et al., 2018)). *Let $x_{\mathsf{eq}} \in \mathcal{D}$ be an equilibrium point for the autonomous systems $x_{t+1} = f(x_t)$ where $f : \mathcal{D} \to \mathcal{X}$ is locally Lipschitz in $\mathcal{D} \subseteq \mathcal{X}$. Suppose there exists a function $V : \mathcal{D} \to \mathbb{R}$ which is continuous and such that*

$$\begin{aligned} &V(x_{\mathsf{eq}}) = x_{\mathsf{eq}} \quad \textit{and} \quad V(x) > 0 \;\; \forall \;\; x \in \mathcal{D} - \{x_{\mathsf{eq}}\} \\ &V(f(x)) - V(x) \leq 0 \quad (\textit{resp.} \; < 0) \;\; \forall \;\; x \in \mathcal{D} \end{aligned}$$

Then x_{eq} is stable (resp. asymptotically stable).

Theorem A.4 (Theorem 1.5 in (Bof et al., 2018)). *Let $x_{\mathsf{eq}} \in \mathcal{D}$ be an equilibrium point for the autonomous systems $x_{t+1} = f(x_t)$ where $f : \mathcal{D} \to \mathcal{X}$ is locally Lipschitz in $\mathcal{D} \subseteq \mathcal{X}$. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuous function with $V(x_{\mathsf{eq}}) = 0$ and $V(x_0) > 0$ for some x_0 arbitrarily close to x_{eq}. Let $r > 0$ be such that $B_r(x_{\mathsf{eq}}) \subseteq \mathcal{D}$ and $\mathcal{U} = \{x \in B_r(x_{\mathsf{eq}}) \mid V(x) > 0\}$, and suppose that $V(f(x)) - V(x) > 0$ for all $x \in \mathcal{U}$. Then x_{eq} is not stable.*

Next, we state the definition of a (isolated) local minimum.

Definition A.5. *The point $u_\star$ is a local minimum (resp. isolated local minimum) of a function h over a domain $\mathcal{U}$ if there is a $\delta > 0$ such that for any $u \in \mathcal{U}$ with $\|u - u_\star\| \leq \delta$, $h(u_\star) \leq h(u)$ (resp. $h(u_\star) < h(u)$).*

Next, we state the implicit function theorem.

Theorem A.6 (Implicit Function Theorem). *Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets and $f : U \times V \to \mathbb{R}$ is C^r for some $r \geq 1$. For some $x_0 \in U$, $y_0 \in V$ assume the partial derivative in the second argument $D_2 f(x_0, y_0) : \mathbb{R}^m \to \mathbb{R}$ is an isomorphism. Then there are neighborhoods U_0 of x_0 and W_0 of $f(x_0, y_0)$ and a unique C^r map $g : U_0 \times W_0 \to V$ such that for all $(x, w) \in U_0 \times W_0$, $f(x, g(x, w)) = w$.*

Finally we prove a property of intersecting convex hulls.

Lemma A.7. *Let $x_1, x_2, \cdots x_n$ and $y_1, y_2, \ldots, y_m$ be some points in $\mathbb{R}^d$. Define by $\mathcal{C}_x$ and $\mathcal{C}_y$ the convex hulls of $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^m$ respectively. Then there do not exist points $\bar{x} \in \mathbb{R}^d$ and $\bar{y} \in R^d$ such that the following inequalities are satisfied:*

$$\begin{aligned} &\|x_i - \bar{x}\| < \|x_i - \bar{y}\| \; \forall i = 1, 2, \ldots, n \\ &\|y_j - \bar{y}\| < \|y_j - \bar{x}\| \; \forall j = 1, 2, \ldots, m \end{aligned}$$

Proof. Assume by contradiction that the inequalities above hold. Define $\mathcal{H}_x := \{z \in R^d \mid \|z - \bar{x}\| < \|z - \bar{y}\|\}$ and $\mathcal{H}_y := \{z \in R^d \mid \|z - \bar{y}\| < \|z - \bar{x}\|\}$. The sets $\mathcal{H}_x$ and $\mathcal{H}_y$ are disjoint half-spaces (without boundary) then defined by the hyperplane bisecting the segment connecting $\bar{x}$ and $\bar{y}$. By assumption then we have that $x_i \in \mathcal{H}_x$ for all i and $y_j \in \mathcal{H}_y$ for all j; since $\mathcal{H}_x$ and $\mathcal{H}_y$ are convex, it follows that $\mathcal{C}_x \subset \mathcal{H}_x$ and $\mathcal{C}_y \subset \mathcal{H}_y$. Therefore $\mathcal{C}_x \cap \mathcal{C}_y = \emptyset$, which leads to a contradiction. □

A.3 Properties of partial minimization

In this section, we state a handful of important results about the partial minimization of the total risk. This is somewhat similar to the analysis presented by Selim and Ismail (1984) in the context of clustering algorithms.

Lemma 3.6. *Define the function $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ as $F(\alpha) = \min_\Theta \mathcal{R}^{\mathsf{total}}(\alpha, \Theta)$. This function is concave and a point (α^0, Θ^0) is a local minimum (resp. isolated local minimum) of $\mathcal{R}^{\mathsf{total}}$ over the domain $\mathcal{X} = \mathcal{X}_\alpha \times \mathbb{R}^{m\times d}$ if and only if α^0 is a local minimum (resp. isolated local minimum) of F over the domain $\mathcal{X}_\alpha$ and $\Theta^0 = \arg\min_\Theta \mathcal{R}^{\mathsf{total}}(\alpha^0, \Theta)$.*

Proof of Lemma 3.6. $F(\alpha)$ is well defined due to the strong convexity of the risk functions. Concavity follows from the observation that F is the point-wise minimum of a family of functions which are linear in α (since for every fixed Θ, the total risk is linear in α).

We break the proof of equivalence into two implications. The isolated local minimum case follows by the same arguments with all inequalities strict.

1. F minimized $\implies$ $\mathcal{R}^{\mathsf{total}}$ minimized
There is a $\delta > 0$ such that for any $\alpha \in \mathcal{X}_\alpha$ with $\|\alpha^0 - \alpha\| \le \delta$, $F(\alpha^0) \le F(\alpha)$, i.e.

$$\mathcal{R}^{\mathsf{total}}(\alpha^0, \Theta^0) \le \mathcal{R}^{\mathsf{total}}(\alpha, \Theta^*(\alpha))$$

For fixed allocation α define $\mathcal{R}^{\mathsf{total}}_\alpha(\Theta) = \mathcal{R}^{\mathsf{total}}(\alpha, \Theta)$. Since total risk function is a affine combination of strongly convex functions wrt Θ we have that $\mathcal{R}^{\mathsf{total}}_\alpha(\Theta)$ is strongly convex. The value of $\mathcal{R}^{\mathsf{total}}_\alpha(\Theta)$ is minimized at $\Theta^*(\alpha)$ and hence:

$$\mathcal{R}^{\mathsf{total}}(\alpha, \Theta^*(\alpha)) < \mathcal{R}^{\mathsf{total}}(\alpha, \Theta), \quad \forall \quad \Theta \ne \Theta^*(\alpha)$$

Combining the inequalities yields: $\mathcal{R}^{\mathsf{total}}(\alpha^0, \Theta^0) \le \mathcal{R}^{\mathsf{total}}(\alpha, \Theta)$, and thus (α^0, Θ^0) is a local minimum of $\mathcal{R}^{\mathsf{total}}$.

2. $\mathcal{R}^{\mathsf{total}}$ minimized $\implies$ F minimized
By strong convexity of the risks we can conclude that $\Theta^0 = \arg\min_\Theta \mathcal{R}^{\mathsf{total}}(\alpha^0, \Theta)$. We have that for any $\alpha \in \mathcal{X}_\alpha$ with $\|\alpha^0 - \alpha\| \le \delta_\alpha$ and any $\|\Theta^0 - \Theta\| \le \delta_\theta$, $\mathcal{R}^{\mathsf{total}}(\alpha^0, \Theta^0) \le \mathcal{R}^{\mathsf{total}}(\alpha, \Theta)$.

Additionally, due to the strong convexity and continuity of the total risk the function $\Theta(\alpha) = \arg\min_\Theta \mathcal{R}^{\mathsf{total}}(\alpha, \Theta)$ is Lipschitz, so there exists a $\delta_2 > 0$ such that $\|\Theta^0 - \Theta^\star(\alpha)\| \le \delta_\theta$ as long as $\|\alpha^0 - \alpha\| \le \delta_2$.

Combining these two observations, we can conclude that for any $\alpha \in \mathcal{X}_\alpha$ with $\|\alpha^0 - \alpha\| \le \min\{\delta_2, \delta_\alpha\}$, $F(\alpha^0) = \mathcal{R}^{\mathsf{total}}(\alpha^0, \Theta^0) \le \mathcal{R}^{\mathsf{total}}(\alpha, \Theta^\star(\alpha)) = F(\alpha)$. Thus α^0 is a local minimum of F. □

Lemma A.8. *For $F : \mathbb{R}^{n\times m} \to \mathbb{R}$ defined as in Lemma 3.6, define $\Theta^*(\alpha) = \arg\min_\Theta \mathcal{R}^{\mathsf{total}}(\alpha, \Theta)$. The gradient is*

$$\nabla_\alpha F(\alpha) = \mathrm{diag}(\beta) R(\Theta), \ \textit{i.e.} \quad \frac{\partial F(\alpha)}{\partial \alpha_{ij}} = \beta_i \mathcal{R}_i(\theta^*_j(\alpha))\,.$$

Second partial derivatives are given by

$$\frac{\partial^2 F(\alpha)}{\partial \alpha_{k\ell} \partial \alpha_{ij}} = \begin{cases} 0 & k \ne j \\ -\beta_i \nabla \mathcal{R}_i(\theta^\star_j)^\top \left(\sum_{\ell'} \beta_{\ell'} \alpha_{\ell' j} \nabla^2 \mathcal{R}_{\ell'}(\theta^\star_j)\right) \nabla \mathcal{R}_\ell(\theta^\star_j) & k = j \end{cases}.$$

Proof. Computing the gradient:

$$\nabla_\alpha F(\alpha) = \nabla_\alpha \mathcal{R}^{\mathsf{total}}(\alpha, \Theta^\star(\alpha)) + \nabla \Theta^*(\alpha) \nabla_\theta \mathcal{R}^{\mathsf{total}}(\alpha, \Theta^\star(\alpha)) = \mathrm{diag}(\beta) R(\Theta)\,.$$

The first equality follows by product rule. The second equality follows because 1) the total risk is linear in α and 2) the second term is zero due to the optimality of $\Theta^*(\alpha)$.

Now notice that

$$\frac{\partial}{\partial \alpha_{k\ell}} \mathcal{R}_i(\theta_j^\star(\alpha)) = \left\langle \frac{\partial \theta_j^\star(\alpha)}{\partial \alpha_{k\ell}}, \nabla_\theta \mathcal{R}_i(\theta_j^\star(\alpha)) \right\rangle$$

To compute the derivatives of $\theta_j^\star(\alpha)$ we use the implicit function theorem. We apply the implicit function theorem to the first order optimality condition

$$\theta_j^\star(\alpha) \in \arg\min_{\theta_j} \bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha_{:,j}, \theta_j)$$

The Hessian $\nabla_\theta^2 \bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha, \Theta))$ is non-degenerate due to strong convexity of the subpopulation risks. There exists a neighborhood U_0 of α and a unique (sufficiently smooth) map $\theta_j^*(\cdot)$ such that for all $\alpha \in U_0$, we have that $\nabla_\theta \bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha, \theta^*(\alpha)) = 0$. Then by implicit function theorem we obtain

$$\nabla \theta_j^\star(\alpha) = -\nabla_\theta^2 \bar{\mathcal{R}}_j^{\mathsf{learner}} \circ \nabla_{\alpha\theta} \bar{\mathcal{R}}_j^{\mathsf{learner}}(\alpha_{:,j}, \theta_j^\star(\alpha))$$

by taking the derivative of the first order condition differentiating through $\theta_j^\star(\cdot)$ and setting it to zero. We have that

$$\nabla_\theta^2 \bar{\mathcal{R}}_j^{\mathsf{learner}} = \sum_{\ell'} \beta_{\ell'} \alpha_{\ell' j} \nabla^2 \mathcal{R}_{\ell'}(\theta_j^\star), \quad \frac{\partial}{\partial \alpha_{k\ell}} \nabla_\theta \bar{\mathcal{R}}_j^{\mathsf{learner}} = \begin{cases} 0 & k \neq j \\ \nabla \mathcal{R}_\ell(\theta_j^{\mathsf{eq}}) & k = j \end{cases}.$$

The result follows by combining the expressions. $\square$

B Full Proofs of Main Results

B.1 Connections between dynamics and total risk

Proposition 3.1. *Under any risk-reducing subpopulation dynamics and any risk-reducing learner dynamics, the total risk is non-increasing in every timestep.*

$$\mathcal{R}^{\mathsf{total}}(\alpha^{t+1}, \Theta^{t+1}) \leq \mathcal{R}^{\mathsf{total}}(\alpha^t, \Theta^t).$$

Further, if subpopulations and learners are risk minimizing in the limit, then the total risk is strictly decreasing unless (α^t, Θ^t) is an equilibrium.

Proof of Proposition 3.1. The key to seeing that the total risk acts like a potential for the market dynamics is to note two equivalent decompositions of the total risk:

$$\mathcal{R}^{\mathsf{total}}(\alpha, \Theta) = \beta^\top \bar{\mathcal{R}}^{\mathsf{subpop}}(\alpha, \Theta) = \beta^\top \alpha \bar{\mathcal{R}}^{\mathsf{learner}}(\alpha, \Theta) .$$

Being risk-reducing learners' updates satisfy:

$$\bar{\mathcal{R}}^{\mathsf{learner}}(\alpha^t, \Theta^{t+1}) \leq \bar{\mathcal{R}}^{\mathsf{learner}}(\alpha^t, \Theta^t) \implies \mathcal{R}^{\mathsf{total}}(\alpha^t, \Theta^{t+1}) \leq \mathcal{R}^{\mathsf{total}}(\alpha^t, \Theta^t) .$$

Similarly risk reducing subpopulations satisfy:

$$\bar{\mathcal{R}}^{\mathsf{subpop}}(\alpha^{t+1}, \Theta^{t+1}) \leq \bar{\mathcal{R}}^{\mathsf{subpop}}(\alpha^t, \Theta^{t+1}) \implies \mathcal{R}^{\mathsf{total}}(\alpha^{t+1}, \Theta^{t+1}) \leq \mathcal{R}^{\mathsf{total}}(\alpha^t, \Theta^{t+1}) .$$

Finally, combining the two updates yields the desired inequality.

In the case that learners and subpopulations are risk minimizing in the limit, the same argument holds with strict inequality, unless $(\alpha^{t+1}, \Theta^{t+1}) = (\alpha^t, \Theta^t)$, i.e. we are at equilibrium. $\square$

Theorem 3.2. *When learners and subpopulations are risk minimizing in the limit, an equilibrium* $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ *is asymptotically stable if and only if it is an isolated local minimizer of the total risk* $\mathcal{R}^{\mathsf{total}}$. *If it is not a local minimizer, then it is not stable.*

Proof of Theorem 3.2. We break this proof into three implications.

1. Asymptotic stability $\implies$ Isolated local min
Assume $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ is an asymptotically stable equilibrium in the ball defined by δ_α and δ_θ. Let α^0 and Θ^0 be any nontrivial perturbation such that $\|\alpha^0 - \alpha^{\mathsf{eq}}\| \leq \delta_\alpha$ and $\|\Theta^0 - \Theta^{\mathsf{eq}}\| \leq \delta_\theta$. Asymptotic stability implies that $\alpha^t \to \alpha^{\mathsf{eq}}$ and $\Theta^t \to \Theta^{\mathsf{eq}}$. Since the dynamics are risk minimizing in the limit, Proposition 3.2 implies that the total risk is strictly decreasing along this trajectory. Thus $\mathcal{R}^{\mathsf{total}}(\alpha^{\mathsf{eq}}, \theta^{\mathsf{eq}}) < \mathcal{R}^{\mathsf{total}}(\alpha^0, \theta^0)$ and since this argument applies to arbitrary perturbations, $(\alpha^{\mathsf{eq}}, \theta^{\mathsf{eq}})$ is an isolated local minimizer for the total risk.

2. Isolated local min $\implies$ Asymptotic stability
Define $V(\alpha, \Theta) = \mathcal{R}^{\mathsf{total}}(\alpha, \Theta) - \mathcal{R}^{\mathsf{total}}(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$. The dynamics f are Lipschitz by Lemma A.2 and this V satisfies the conditions of Theorem A.3 with strict inequality, thus we conclude that $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ is an asymptotically stable equilibrium.

3. Not local min $\implies$ Not stable
Define $V(\alpha, \Theta) = \mathcal{R}^{\mathsf{total}}(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}}) - \mathcal{R}^{\mathsf{total}}(\alpha, \Theta)$ which will increase along trajectories. Since we are not at a local min, there must be some arbitrarily close α^0, Θ^0 such that $V(\alpha, \Theta) > 0$. Then we apply Theorem A.4 which guarantees that the equilibrium is not stable. $\square$

B.2 Stable equilibria

Theorem 3.3. *Suppose learners and subpopulations are risk minimizing in the limit,* α^{eq} *is split-market and* $\Theta^{\mathsf{eq}} = \arg\min_\Theta \mathcal{R}^{\mathsf{total}}(\alpha^{\mathsf{eq}}, \Theta)$. *Define a mapping* $\gamma : [n] \to [m]$ *such that* $\gamma(i) = j$ *is the learner with nonzero mass in* $\alpha^{\mathsf{eq}}_{i,:}$. *Then* $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ *is an asymptotically stable equilibrium if and only if each learner is associated to at least one subpopulation, and*

$$\mathcal{R}_i(\theta^{\mathsf{eq}}_{\gamma(i)}) < \mathcal{R}_i(\theta^{\mathsf{eq}}_j) ,$$

for all subpopulations i *and learners* $j \neq \gamma(i)$, *i.e. no subpopulation would prefer to switch learners. Furthermore, such split market equilibria are the only asymptotically stable equilibria.*

Proof of Theorem 3.3. We show an equivalence between the stated conditions and $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ being an isolated local minimum of the total risk. By Theorem 3.2, this is equivalent to asymptotic stability.

We first argue that it is necessary for every learner to have at least one subpopulation. If not, then the unassigned learner can be assigned the risk minimizing parameter of some subpopulation. Any small perturbation of that subpopulation's allocation towards the learner will not increase the total risk, and thus the point is not an isolated local minimum.

We now argue that the risk comparison condition is necessary. Suppose that for some subpopulation, it does not hold for some learner. Then any small perturbation of that subpopulations's allocation towards that learner will not increase the total risk, and thus the point is not an isolated local minimum.

We now argue that these conditions are sufficient for guaranteeing a local minimum with respect to $F(\alpha)$, appealing to Lemma 3.6. First notice that we have $\Theta^{\mathsf{eq}} = \arg\min_{\Theta} \mathcal{R}^{\mathsf{total}}(\alpha^{\mathsf{eq}}, \Theta)$ as required. Suppose by contradiction that there is some perturbation to α that causes $F(\alpha)$ to decrease. Equivalently, the projection of the negative gradient onto the simplex points towards some other vertex, i.e. the component of the gradient in the direction of learner j is smaller than in the direction of $\gamma(i)$ for some $j \neq \gamma(i)$. We can write this condition as

$$\frac{\partial F(\alpha)}{\partial \alpha_{i\gamma(i)}} \geq \frac{\partial F(\alpha)}{\partial \alpha_{ij}} \iff \mathcal{R}_i(\theta^{\mathsf{eq}}_{\gamma(i)}) \geq \mathcal{R}_i(\theta^{\mathsf{eq}}_j)$$

where we use Lemma A.8. This violates the risk comparison condition, and therefore there must be no such perturbation, and thus α^{eq} is a local minimum.

Finally, we argue that split market points are the only isolated local minima. Recall that $F(\alpha)$ is concave (Lemma 3.6), so all minima occur on the boundary of Δ^n_m. Thus the only *isolated* minima are vertices; in the degenerate case where $F(\alpha)$ takes a constant value over a face, the minima are not isolated. □

Corollary 3.4. *Suppose that risk functions satisfy $\mathcal{R}_i(\theta) < \mathcal{R}_i(\theta') \iff \|\theta - \phi_i\| < \|\theta' - \phi_i\|$ for ϕ_i the subpopulation optimal parameter. Then in an asymptotically stable split market equilibrium, the convex hulls of the grouped subpopulations optimal parameters $\{\phi_i\}$ are non-intersecting.*

Proof of Corollary 3.4. Let $\phi_1, \phi_2, \cdots, \phi_k \in \mathbb{R}^d$ be the optimal decisions for the subpopulations allocated to the first learner and $\psi_1, \psi_2, \cdots, \psi_l \in \mathbb{R}^d$ be the optimal decisions for the subpopulations allocated to the second learner. Let θ_1 and θ_2 be the decisions of each learner. Assume that the convex hulls of $\{\phi_i\}_{i=1}^k$ and $\{\psi\}_{i=1}^l$ intersect. By Lemma A.7, there exists i such that $\|\phi_i - \theta_2\| \leq \|\phi_i - \theta_1\|$. By the assumption about the risk runctions, this implies $\mathcal{R}_i(\theta_2) < \mathcal{R}_i(\theta_1)$. In other words, there exist a subpopulation that would prefer to switch learners. Thus by Theorem 3.3 these allocation of subpopulations to learner is not stable and so the convex hulls must not intersect. □

Theorem 3.5. *Consider dynamics which are risk minimizing in the limit and an α^{eq} with any subpopulation i having nonzero support on set of two or more learners $j \in \mathcal{J}$. Define $\Theta^{\mathsf{eq}} = \arg\min \mathcal{R}^{\mathsf{total}}(\alpha^{\mathsf{eq}}, \Theta)$. Then $(\alpha^{\mathsf{eq}}, \Theta^{\mathsf{eq}})$ cannot be stable unless it is "balanced" in the sense that learners in $\mathcal{J}$ are* risk equivalent *and* optimal *for i, i.e. for all $j, j' \in \mathcal{J}$,*

$$\mathcal{R}_i(\theta^{\mathsf{eq}}_j) = \mathcal{R}_i(\theta^{\mathsf{eq}}_{j'}) \quad \textit{and} \quad \nabla \mathcal{R}_i(\theta^{\mathsf{eq}}_j) = 0\,.$$

If it is balanced, so are all allocations for subpopulation i with support over $\mathcal{J}$. Finally, such balanced equilibria are the only ones besides split-market equilbria that can be stable.

Proof of Theorem 3.5. Theorem 3.2 shows that an equilibrium cannot be stable if it is not a local minimum of the total risk. We therefore develop conditions under which an equilibrium point will be a local minimum. By Lemma 3.6, it is equivalent to argue about the local minima of the concave function $F(\alpha)$ over the simplex product Δ_m^n. All minima of the total risk will occur on the boundary of the simplex product, i.e. a face or a vertex. Since F is still concave when restricted to a face of the simplex, the same argument shows the minima are on the boundary, hence vertices, except for the degenerate case where F takes a constant value over the face.

We now characterize this degenerate case. F takes a constant value over the face if and only if 1) the gradient of F is perpendicular to the face at α and 2) remains perpendicular along the face. The face is described by a set of indices $\mathcal{J} \subseteq [m]$. Mathematically, we write the two conditions as: for all pairs $j, j' \in \mathcal{J}$, $\ell \in [n]$, and $k \in [m]$,

$$\frac{\partial F(\alpha)}{\partial \alpha_{ij}} = \frac{\partial F(\alpha)}{\partial \alpha_{ij'}} \quad \text{and} \quad \frac{\partial}{\partial \alpha_{\ell k}} \left(\frac{\partial F(\alpha)}{\partial \alpha_{ij}} - \frac{\partial F(\alpha)}{\partial \alpha_{ij'}} \right) = 0 \tag{3}$$

Using Lemma 3.6, the first expression simplifies to the *risk equivalent* condition that $\mathcal{R}_i(\theta_j^{\mathsf{eq}}) = \mathcal{R}_i(\theta_{j'}^{\mathsf{eq}})$. Turning to the second expression in (3), we first use Lemma A.8 to compute

$$\frac{\partial}{\partial \alpha_{\ell k}} \frac{\partial F(\alpha)}{\partial \alpha_{ij}} = \begin{cases} 0 & k \neq j \\ -\beta_i \nabla \mathcal{R}_i(\theta_j^{\mathsf{eq}})^\top \left(\sum_{\ell'} \beta_{\ell'} \alpha_{\ell' j} \nabla^2 \mathcal{R}_{\ell'}(\theta_j^{\mathsf{eq}}) \right) \nabla \mathcal{R}_\ell(\theta_j^{\mathsf{eq}}) & k = j \end{cases}$$

Thus, the condition trivially holds for $k \notin \{j, j'\}$. Otherwise, when $\ell = i$, the condition in (3) requires that

$$\nabla \mathcal{R}_i(\theta_k^{\mathsf{eq}})^\top \left(\sum_{\ell'} \beta_{\ell'} \alpha_{\ell' k} \nabla^2 \mathcal{R}_{\ell'}(\theta_k^{\mathsf{eq}}) \right) \nabla \mathcal{R}_i(\theta_k^{\mathsf{eq}}) = 0, \quad k \in \{j, j'\}$$

Due to the strong convexity of the risks, the Hessian matrix is positive definite. Thus it must be that $\nabla \mathcal{R}_i(\theta_j^{\mathsf{eq}}) = 0$ for all $j \in \mathcal{J}$, i.e. the *risk optimal* condition. Risk optimality implies that the condition holds also when $\ell \neq i$ and thus the characterization is complete. □

C Detailed Examples

We begin with a somewhat generic example with $m = 2$ and $n = 3$ that illustrates the difference between stable equilibria and social welfare optima.

Example C.1 (Stability vs. optimality)**.** *Consider three subpopulations $i \in \{1, 2, 3\}$ with risks $\|\theta - \phi_i\|_2^2$, sizes β_i, and two learners $j \in \{1, 2\}$. Suppose that the α^{eq} is such that the subpopulations are partitioned into $\{1\}$ and $\{2, 3\}$. Then we have that*

$$\theta_1^{\mathsf{eq}} = \phi_1, \quad \theta_2^{\mathsf{eq}} = \frac{\beta_2}{\beta_2 + \beta_3} \phi_2 + \frac{\beta_3}{\beta_2 + \beta_3} \phi_3$$

By Theorem 3.3, this is stable if and only if

$$\|\phi_2 - \phi_3\|_2 \leq (\beta_2 + \beta_3) \min \left\{ \frac{\|\phi_2 - \phi_1\|_2}{\beta_3}, \frac{\|\phi_3 - \phi_1\|_2}{\beta_2} \right\} .$$

However, it is only social optimal if and only if ϕ_2 and ϕ_3 are relatively close to each other than to ϕ_1, i.e.

$$\|\phi_2 - \phi_3\|_2 \leq \min \{ \|\phi_2 - \phi_1\|_2, \|\phi_3 - \phi_1\|_2 \} .$$

The set of subpopulation optima $\{\phi_1, \phi_2, \phi_3\}$ satisfying the optimality condition are a subset of those satisfying the stability condition. As the difference between β_2 and β_3 becomes more extreme, the number of settings satisfying the stability but not optimality condition increases.

We use this generic example to illustrate a scenario in which the total risk can be arbitrarily high at a stable equilibria.

Example C.2 (Arbitrarily high total risk at local optimum)**.** *Consider three subpopulations with*

$$\mathcal{R}_1(\theta) = \theta^2, \quad \mathcal{R}_2(\theta) = (\theta - 1)^2, \quad \mathcal{R}_2(\theta) = (\theta - \phi)^2$$

for some $\phi > 2$. Suppose that subpopulation sizes are $\beta_1 = \beta_2 = \beta$ and $\beta_3 = 1 - 2\beta$ for some $0 < \beta < 1/2$. Further suppose that there are two learners. Up to permutation, the social welfare optimum is $\theta_1 = 1/2$ and $\theta_2 = \phi$, with total risk $\beta/2$. However, as long as $\phi < \frac{1-\beta}{1-2\beta}$, there is another stable equilibrium. Let $\phi = \frac{1-\beta}{1-2\beta} - \epsilon$. Then the following is a stable equilibrium: $\theta_1 = 0$ and $\theta_2 = 1 - \epsilon$. The total risk is $\beta + \frac{(\beta-\epsilon)^2}{1-2\beta}$. For β close to $1/2$, this risk can be arbitrarily larger than the social optimum.

Finally, we present an example which illustrates that even in the single learner setting, the risk of a subpopulation can be arbitrarily worse than the total risk.

Example C.3 (Arbitrarily high risk for minority subpopulation)**.** *Consider two subpopulations with $\mathcal{R}_1(\theta) = \theta^2$ and $\mathcal{R}_2(\theta) = (\theta - \phi)^2$ with $\beta_1 = \beta$ and $\beta_2 = 1 - \beta$ and a single learner. The single equilibrium and total risk minimizer is $\theta_1 = (1 - \beta)\phi$ with total risk $\beta(1 - \beta)\phi^2$ and $\mathcal{R}_2(\theta^\star) = \beta^2\phi^2$. The difference between the two quantities can be arbitrarily high as β gets close to 1.*

D Experimental Details

Full experimental details along with instructions for reproducing them can be found at `https://anonymous.4open.science/r/MultiPlayerRiskReduction-2C12`. The experiments used Python 3.9 on a MacBook Pro 2019.