arXiv:2206.02477v1 [econ.TH] 6 Jun 2022

Optimal Stopping Theory for a Distributionally Robust Seller

Pieter Kleer and Johan S.H. van Leeuwaarden
Tilburg University
Department of Econometrics and Operations Research
{p.s.kleer,j.s.h.vanleeuwaarden}@tilburguniversity.edu

June 7, 2022

Abstract

Sellers in online markets face the challenge of determining the right time to sell in view of uncertain future offers. Classical stopping theory assumes that sellers have full knowledge of the value distributions, and leverage this knowledge to determine stopping rules that maximize expected welfare. In practice, however, stopping rules must often be determined under partial information, based on scarce data or expert predictions. Consider a seller that has one item for sale and receives successive offers drawn from some value distributions. The decision on whether or not to accept an offer is irrevocable, and the value distributions are only partially known. We therefore let the seller adopt a robust maximin strategy, assuming that value distributions are chosen adversarially by nature to minimize the value of the accepted offer. We provide a general maximin solution to this stopping problem that identifies the optimal (threshold-based) stopping rule for the seller for all statistical information structures. We then perform a detailed analysis for when the seller knows the common mean, dispersion (variance or mean absolute deviation) and support of the distributions. We show for this information structure that the seller's stopping rule consists of decreasing thresholds converging to the common mean, and that nature's adversarial response, in the long run, is to always create an all-or-nothing scenario. The maximin solutions also reveal what happens as dispersion or the number of offers grows large.

1 Introduction

A major challenge faced by sellers in online markets is to determine the right time to sell in view of uncertain future offers. We consider the classical setting of a seller that has one item for sale and receives sequentially n offers. This setting is studied since the 1960s [19] and gained renewed relevance due to the rise of online algorithms. The seller decides to sell or not, by comparing the current offer to possibly higher future offers. Rejecting the current offer holds the promise of receiving a higher offer but comes with the risk of receiving lower future offers only. To provide the seller a guideline for when to accept an offer, some knowledge about future offers is required. Therefore, in the area of (Bayesian) optimal stopping theory, the distributions of future offers are assumed to be known, and leveraged to obtain stopping rules for the seller.

Settings with such full distributional knowledge, however, are often criticized for being unrealistic. What is nowadays known as "Wilson's doctrine" in economics and game theory [42] (see also, e.g., [30, Section 2]) calls for reducing the common knowledge that is taken as given, so that the resulting analysis becomes more robust and valuable in realistic problems where only limited information is available. Inspired by this, we study a robust version of the stopping problem that assumes the seller has very limited distributional knowledge, for example in the form of moment information or maximal values. Optimizing a seller's welfare (or revenue) under limited information

is a rich area of research in the fields of theoretical economics, computer science and operations research, dating back to the work of Scarf [37] and falling within the scope of distributionally robust optimization. We elaborate on this literature, and other robust approaches to stopping problems, in Section 1.2.

We continue with a formal description of the Bayesian stopping problem, and then explain in more detail our robust version with limited information. Consider the stopping problem in which a seller receives for an item a sequence of n offers. The seller gets to see the offers one by one, in the order $(1, \ldots, n)$, and upon seeing an offer, has to decide irrevocably whether or not to accept (a rejected offer cannot still be accepted at a later point in time). The seller can accept at most one offer, after which the selling process stops. The goal of the seller is to come up with a stopping rule that maximizes the expected payoff of the selected offer (i.e., we consider the problem of so-called online welfare maximization).

It is well-known that an optimal stopping rule for the seller is to use a threshold-based strategy, that can be computed using dynamic programming (or backwards induction), and which we explain next. Let $v_i \geq 0$ denote the (initially unknown) value of offer i. The seller only knows the distribution of an offer, so v_i should be considered a realization of a random variable $X_i \sim \mathbb{P}_i$, where $\mathbb{P}_i$ is a nonnegative probability distribution (or measure), for $i = 1, \ldots, n$. The seller knows $\mathbb{P}_1, \ldots, \mathbb{P}_n$, but not the actual values v_i. We further assume that the random variables $X_1, \ldots, X_n$ are pairwise independent. Based on the known distributions $\mathbb{P}_1, \ldots, \mathbb{P}_n$, the seller computes a threshold $T(i)$ for every offer i, and selects the first offer for which $v_i \geq T(i)$. This strategy yields the maximum expected payoff (i.e., the value of the selected offer), with the expectation taken with respect to the random variables $X_1, \ldots, X_n$. How does one compute the thresholds? First note that once the seller reaches the last offer n, it is always optimal to accept it, because all values are nonnegative. Knowing that $T(n) = 0$, what is the optimal stopping rule for selecting offer $n-1$? If the seller does not select offer $n-1$, the expected payoff from the last offer is $\mathbb{E}[X_n]$. Therefore, the optimal policy is to select offer $n-1$ if and only if $v_{n-1} \geq \mathbb{E}[X_n]$. Hence, we set the threshold $T(n-1) = \mathbb{E}[X_n]$. Generalizing this line of thought, the threshold $T(i)$ is set to be the expected payoff would the seller not select offers $k = 1, \ldots, i$, but instead continue. These thresholds can be computed recursively, in a backwards fashion, by

$$T(i) = \mathbb{E}[\max\{T(i+1), X_{i+1}\}] = \int_0^\infty \max\{T(i+1), x\} \, \mathrm{d}\mathbb{P}_{i+1}(x). \tag{1.1}$$

In our robust version of the problem, the seller does not know the full distributions $\mathbb{P}_1, \ldots, \mathbb{P}_n$, but only that $\mathbb{P}_i$ belongs to some ambiguity set $\mathcal{P}(I_i)$ containing all distributions that satisfy the limited information I_i. For instance, if $I_i = \{\mathbb{E}[X_i]\}$, the ambiguity set $\mathcal{P}(I_i)$ contains all nonnegative distributions with mean $\mathbb{E}[X_i]$. Distributional information, like moments or maximal values, can typically be learned or estimated based on historical data. The information structure $\{I_1, \ldots, I_n\}$, however, is usually not sufficient for computing the thresholds in (1.1), and instead the seller will have to play a maximin game against nature. In this game, the seller first decides on a stopping rule τ for which offer to select, after which nature gets to choose from the ambiguity sets the worst-case distributions $\{\mathbb{P}_1, \ldots, \mathbb{P}_n\}$ that under the stopping rule τ minimize the expected payoff of the seller. Maximizing the (robust) expected payoff of the seller then corresponds to solving the maximin problem

$$r^* = \max_{\tau} \min_{\forall i: \mathbb{P}_i \in \mathcal{P}(I_i)} \mathbb{E}_{\mathbb{P}_1, \ldots, \mathbb{P}_n}[X_\tau], \tag{1.2}$$

where X_τ is the random variable denoting the value of the offer that was selected under stopping rule τ. In this work, we first determine the optimal stopping rule τ^* solving (1.2), which can be seen

as a robust counterpart for the threshold-based strategy as described by the thresholds in (1.1). Secondly, we give explicit analytical expressions for the robust thresholds for various ambiguity sets in terms of the informational parameters describing them. We continue with a more detailed description of these results in Section 1.1.

1.1 Results

We always assume that the set $\mathcal{P}(I_i)$ is non-empty for every $i = 1, \ldots, n$, and that the random variables $X_1, \ldots, X_n$ are pairwise independent. In case correlations are allowed, the problem is not very interesting; see Appendix A for a discussion. Our first main result, Theorem 1.1, concerns the robust counterpart of the backwards recursion (1.1). We elaborate on related robust stopping and dynamic programming results in Section 1.2.

Theorem 1.1 (Optimal robust thresholds)**.** *The following threshold-based stopping rule τ^* solves the minimax problem (1.2) of the seller within the class of randomized stopping rules (formal definition is given in Section 2.1): Select v_i if and only if $v_i \geq T(i)$ where $T(i)$ is recursively defined by*

$$\begin{aligned} T(i) &= \min_{\mathbb{P}_{i+1} \in \mathcal{P}(I_{i+1})} \mathbb{E}_{\mathbb{P}_{i+1}}[\max\{T(i+1), X_{i+1}\}] \\ &= \min_{\mathbb{P}_{i+1} \in \mathcal{P}(I_{i+1})} \int_0^\infty \max\{T(i+1), x\} \, \mathrm{d}\mathbb{P}_{i+1}(x) \end{aligned} \tag{1.3}$$

for $i = 0, \ldots, n-1$ with $T(n) = 0$. The robust expected payoff r^* of the seller is $T(0)$. In case the minimum in the definition of $T(i)$ does not exist, one should replace it by the infimum (which we will assume to exist).

An equivalent formulation of (1.3), that we will often use, is given by

$$T(i) = T(i+1) + \mathbb{E}_{\mathbb{P}_{i+1}}[X_i] - \max_{\mathbb{P}_{i+1} \in \mathcal{P}(I_{i+1})} \int_0^\infty \min\{T(i+1), x\} \, \mathrm{d}\mathbb{P}_{i+1}(x). \tag{1.4}$$

Although Theorem 1.1 might seem intuitive, its proof requires careful reasoning. We present the formal proof, including a formal definition of stopping rules, in Section 2, and just sketch the main ideas here. The proof uses induction and consists of two main steps: First we argue that, based on the induction hypothesis, it is always optimal for the seller to use the thresholds in (1.3) for the offers $i = 2, \ldots, n$. After that, we argue that it is then also optimal to use (1.3) for the first offer. All this requires a lot of care because of the minimum involved in the definition of $T(i)$.

Theorem 1.1 is rather general, only assuming independence of the offer values. The optimal robust strategy defined by the recursive relation (1.3) applies to all forms of ambiguity of the offer value distributions. However, when we consider specific ambiguity sets, we might be able to solve the recursive relation explicitly. Using Theorem 1.1 to derive closed-form stopping rules is the main theme of this paper. Notice that this can only work if the maximin problem in Theorem 1.1 can be solved. To do so, one needs to find the worst-case distribution in the ambiguity set that gives a tight bound on $\mathbb{E}[\max\{T(i+1), X\}]$. We shall focus on ambiguity sets that fix the mean and dispersion of the offer value distribution, where dispersion is measured in terms of variance or mean absolute devation (MAD). These intuitive summary statistics are easy to estimate when data is available, but without data can also be tuned by experts to assess risk-return tradeoffs. Despite the generality of Theorem 1.1, we restrict to ambiguity sets in which each offer has the same distributional information, as if the offers were generated by a sample from a large population of

statistically indistinguishable buyers. For the canonical choice $\mathcal{P}(\mu, \sigma^2)$, the set of all distributions with mean μ and variance σ^2, we will show that nature chooses as worst-case distribution a two-point distribution with one value just below μ and one extremely large value. This worst-case scenario forces the seller to set all thresholds equal to μ, which generates an expected payoff μ. To avoid such degenerate results, we take as primary example the ambiguity set $\mathcal{P}_2(\mu, \sigma^2, L)$, which contains all two-point distributions with mean μ, variance σ^2 and support upper bound L. In this way nature still creates a good and bad scenario, just as for $\mathcal{P}(\mu, \sigma^2)$, but the scenarios will be less extreme due to the support upper bound L. We solve the maximin problem for $\mathcal{P}_2(\mu, \sigma^2, L)$ explicitly. We show that the robust max-min analysis does not become overly conservative, and instead leads to insightful optimal stopping strategies with non-trivial thresholds and expected pay-off larger than μ.

It turns out that there are two cases to consider. The first case, in which $L \leq 2\mu$, actually yields in every step of the backwards recursion the same worst-case two-point distribution f^*, with a good value $\mu + \sigma^2/\mu$ and a bad value 0. This good-bad scenario results in the threshold-based strategy as summarized in Theorem 1.2.

Theorem 1.2 (Ambiguity set $\mathcal{P}_2(\mu, \sigma^2, L)$ with $L \leq 2\mu$)**.** *Let $n \in \mathbb{N}$, and let $\mu, L > 0$ and $\sigma^2 \geq 0$. Assume that $L \leq 2\mu$. Let $\mathcal{P}_2(\mu, \sigma^2, L)$ be the common ambiguity set consisting of all two-point distributions with mean μ, variance σ^2 and support contained in $[0, L]$. For $i = 1, \ldots, n-1$, the optimal robust threshold in (1.3) equals*

$$T(i) = \mu + \frac{\sigma^2}{\mu}\left[1 - \left(\frac{\sigma^2}{\mu^2 + \sigma^2}\right)^{n-1-i}\right] \tag{1.5}$$

and $T(n) = 0$. As $n \to \infty$, the seller's expected robust payoff $r^* = T(0)$ approaches $\mu + \sigma^2/\mu$.

When we consider the case $L > 2\mu$, things become more involved. It is no longer the case that every step of the backwards recursion yields the same worst-case distribution for (1.3). There turn out to be two worst-case distributions, f^* (same as before) and g^* and a "turning point" n_0, such that g^* is the worst case for $i = n - n_0, \ldots, n$, and f^* the worst case for $i = 1, \ldots, n - n_0 - 1$. In another words, if n is large enough we essentially end up in the same scenario as for the case $L \leq 2\mu$. Our analysis leads to the threshold-based strategy as summarized in Theorem 1.3 for two-point distributions in combination with the assumption that $L \geq 2\mu$.

Theorem 1.3 (Ambiguity set $\mathcal{P}_2(\mu, \sigma^2, L)$ with $L \geq 2\mu$)**.** *Let $n \in \mathbb{N}$, and let $\mu, L > 0$ and $\sigma^2 \geq 0$. Assume that $L \geq 2\mu$. Let $\mathcal{P}_2(\mu, \sigma^2, L)$ be the common ambiguity set consisting of all two-point distributions with mean μ, variance σ^2 and support contained in $[0, L]$. There exists an $n_0 = n_0(\mu, \sigma^2, L) \in \{3, \ldots, n\}$ (independent of n) such that the optimal robust threshold in (1.3) is given by $T(n) = 0$,*

$$T(i) = L\left[1 - \left(1 - \frac{\mu}{L}\right)\left[\frac{(L-\mu)^2}{(L-\mu)^2 + \sigma^2}\right]^{n-1-i}\right] \tag{1.6}$$

for $i = n - n_0 + 1, \ldots, n-1$, and by

$$T(i) = \mu + \frac{\sigma^2}{\mu}\left[1 - \left(\frac{\sigma^2}{\mu^2 + \sigma^2}\right)^{n-n_0-i}\right] + T(n - n_0 + 1)\left(\frac{\sigma^2}{\sigma^2 + \mu^2}\right)^{n-n_0+1-i} \tag{1.7}$$

for $i = 1, \ldots, n - n_0$. The optimal robust payoff of the seller approaches $\mu + \sigma^2/\mu$ as $n \to \infty$.

In Section 3.3 we explain how to compute n_0 in terms of μ, σ^2 and L. We have provided an illustration of the behavior of the thresholds in Figure 1. Roughly speaking, our analysis in Section 3.3 shows that in every step the distributions f^* and g^* are the only candidates for yielding the worst-case threshold in (1.3) (the red squares and blue circles, respectively). Nature then chooses in every step the distribution yielding the minimum of the two (indicated by the green line). This results in the policy as described in Theorem 1.3. We have also indicated the turning point n_0 in Figure 1, as well as the asymptotic payoff $\mu + \sigma^2/\mu$.

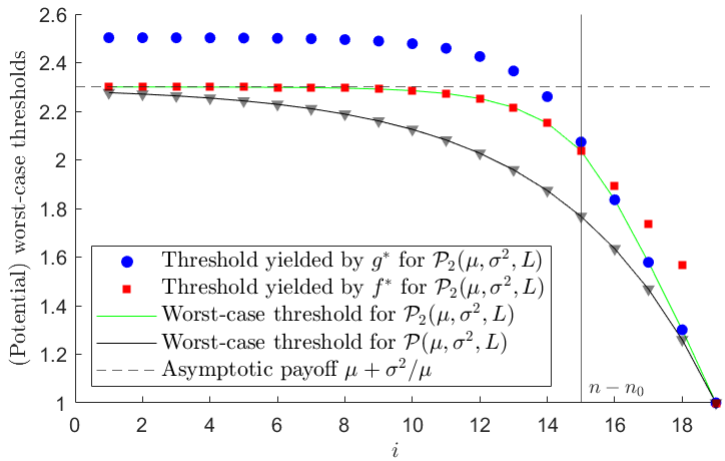

Figure 1: Sketch of potential worst-case thresholds yielded by f^* and g^* for $\mu = 1$, $\sigma^2 = 1.3\mu$ and $L = 5\mu$. The turning point $n - n_0 = 15$ is indicated with a vertical line, and the asymptotic payoff $\mu + \sigma^2/\mu$ with a horizontal dashed line. Note that both f^* and g^* yield the same threshold for $i = n - 1$ (namely $\mu = 1$). The robust threshold $T(i)$ as described in Theorem 1.3 is given by the minimum in every step i (indicated by the green line). That is, in steps $i > 15$ the minimum is attained by the function g^*, but for $i \leq 15$, it is attained by f^*.

Equipped with a full understanding of the ambiguity set $\mathcal{P}_2(\mu, \sigma^2, L)$, we further develop the connection between ambiguity and optimal stopping, by considering two more related ambiguity sets.

The first is $\mathcal{P}(\mu, \sigma^2, L)$, the set of *all* distributions with mean μ, variance σ^2 and whose support is a subset of the interval $[0, L]$. For this ambiguity set, like for $\mathcal{P}(\mu, \sigma^2)$, the maximin problem gives rise to a so-called moment problem. There is a rich literature on solving such moment problems in terms of semi-infinite linear programs (LPs) and using duality theory, see, e.g., [31] and references therein. The novelty of our work here lies in using mathematically tractable moment problems as input for the threshold recursion in (1.3) to obtain explicit stopping rules. We will explain how to use moment bound techniques, and show, based on a result of Cox [12], that for $\mathcal{P}(\mu, \sigma^2, L)$ the worst-case distribution is in fact a three-point distribution on $\{0, T(i+1), L\}$, which in turn gives an explicit non-trivial stopping strategy. The thresholds in this case are given by (see Theorem 4.1)

$$T(i) = \mu + \frac{\sigma^2}{\mu}\left[1 - \left(1 - \frac{\mu}{L}\right)^{n-1-i}\right] \tag{1.8}$$

for $i = 1, \ldots, n-1$, and gradually decrease from (asymptotically) $\mu + \frac{\sigma^2}{\mu}$ to μ, similar as in the analysis for $\mathcal{P}_2(\mu, \sigma^2, L)$. We have added these thresholds to Figure 1 for comparison with the

thresholds of the ambiguity set $\mathcal{P}_2(\mu, \sigma^2, L)$. We emphasize that the probability mass assigned to the points $\{0, T(i+1), L\}$ is different for every i. In fact, it turns out that when we move back in time, the three-point distribution gradually starts to mimic the two-point distribution f^* on $\{0, \mu + \sigma^2/\mu\}$. To be more precise, the point $T(i)$ gets closer and closer to $\mu + \sigma^2/\mu$ and the probability mass put on L approaches 0. This is illustrated in Figure 6 in Section 4.2. This means that for large n, nature will, informally speaking, eventually resort to a good/bad scenario in order to establish the worst possible situation for the seller. For both ambiguity sets $\mathcal{P}_2(\mu, \sigma^2, L)$ and $\mathcal{P}(\mu, \sigma^2, L)$, we also show (Theorems 3.4 and 4.4) that the seller can (almost) obtain the asymptotic payoff of $\mu + \sigma^2/\mu$ by using the simple static threshold $T \approx \mu + \sigma^2/\mu$ in every step, if n is large. In other words, in case there are many offers and the seller is not interested in choosing the optimal threshold in every individual step of the game, then this static threshold policy is a good alternative to obtain approximately the same asymptotic payoff as for the optimal policy.

The next ambiguity set we consider in Section 4 trades variance for mean absolute deviation, and also gives a semi-infinite LP for the maximin problem, solvable with duality theory. In this case, as opposed to the ambiguity set $\mathcal{P}(\mu, \sigma^2)$, we already obtain a non-trivial solution for the set $\mathcal{P}(\mu, d)$ consisting of all distributions with mean μ and mean absolute deviation d, but *without* the assumption of a support upper bound L. This adds significantly to the relevance of studying the MAD instead of the variance, as in this case one only has to assume knowledge of two distributional properties, rather than three. The robust thresholds (see Theorem 4.2) in this case are given by

$$T(i) = \frac{2\mu^2}{2\mu - d} - \left[\frac{2\mu^2}{2\mu - d} - \mu\right]\left(\frac{d}{2\mu}\right)^{n-1-i} \tag{1.9}$$

for $i = 1, \ldots, n-1$. Perhaps surprisingly, these thresholds are also optimal for the ambiguity set $\mathcal{P}(\mu, d, L)$. That is, if one introduces an additional support constraint, then the same analysis remains valid. Furthermore, it turns out that in this case (as in Theorem 1.2) it is optimal for nature in every step to choose a fixed two-point distribution supported on $\{0, \mu + d\mu/(2\mu - d)\}$, which again gives rise to a good/bad scenario as the optimal strategy for nature. It is interesting to note though, that this worst-case scenario is no longer unique, leaving nature more choices. Our proof approach for Theorem 4.2 consists of using a mean-MAD-support analogue of the result of Cox [12], after which we apply our recursive reasoning. More details are given in Section 4.1.

A recurrent theme in our results is the prominent role that two-point distributions play for nature. In the mean-variance-support settings, the good/bad scenario supported on $\{0, \mu + \sigma^2/\mu\}$ becomes, sooner or later, the (approximately) optimal adversarial choice for nature. The most interesting observation here seems to be the abrupt phase transition towards this distribution for the ambiguity set $\mathcal{P}_2(\mu, \sigma^2, L)$ with $L \geq 2\mu$. In the mean-MAD setting, the two-point distribution supported on $\{0, \mu + d\mu/(2\mu - d)\}$ is always an optimal adversarial choice. In conclusion, the essence of nature's adversarial response to the seller's stopping rule, under mean-dispersion information, seems to lie in creating an all-or-nothing scenario for the seller.

The framework for robust stopping introduced in this paper can be widely applied. If we stay with the moment problems, then higher moments can be included in the ambiguity sets, or additional information such as skewness, unimodality and tail probabilities. But more generally, the robust framework works for all forms of ambiguity, and allows to condition on proximity to some reference distribution via statistical distance measures such as ϕ-divergence and Wasserstein distance. This more broader application of Theorem 1.1 we leave for future research.

1.2 Related work and further discussion

The areas of stopping theory and dynamic programming are vast. For an overview of classical stopping theory, we refer the reader to Ferguson [19]. Various robust approaches towards stopping problems have been introduced as well. Closest to our work is the robust stopping framework of Riedel [32], that also generalizes the classical Bayesian setting. He provides a similar result as Theorem 1.1, but much more general, and is based on martingale theory. However, this framework relies on some assumptions that seem not to be satisfied by our setting. We can circumvent those as we consider a simple, concrete stopping problem with independent distributions. This allows us to give a more elementary proof of Theorem 1.1, avoiding advanced martingale theory, which might be of independent interest. In the context of Markov decision processes, Iyengar [25], and independently Ghaoui and Nilim [17], provide a robust dynamic programming approach, leading to a robust Bellman equation, which is of a similar spirit as Theorem 1.1. In the Markov decision process literature, the robust perspective is introduced with respect to uncertainty arising in the transition matrix that determines with which probabilities the process moves to a new state.

A closely related problem to maximizing the seller's expected payoff is the so-called prophet inequality problem. Here the payoff of the seller is compared to that of a prophet, who gets to see all the offers up front and simply can select the one with the highest value. Garling, Krengel and Sucheston [28] showed that the backwards induction procedure from the introduction yields an expected payoff for the seller that is at least half that of the prophet (such a result is called a prophet inequality). An elegant result by Samuel-Cahn [36] shows that the same guarantee can be achieved with a simple static threshold (see also the work of Kleinberg and Weinberg for an alternative static threshold approach [26]). Our asymptotic results in Theorems 3.4 and 4.4 illustrate (albeit only in the asymptotic case) a similar contrast between the optimal strategy for the seller and simple static threshold-based alternatives.

The prophet inequality literature has seen an immense surge in the last fifteen years; we refer to the nice survey of Lucier [29] for an overview. In particular, various robust/limited information alternatives for the classical setting have been proposed. One prominent direction here is that initiated by Azar, Kleinberg and Weinberg [5] where the seller only has access to samples from the underlying offer distributions, instead of the distribution itself. In this context, Rubinstein, Wang and Weinberg [35] showed that the seller can still guarantee in expectation half of the prophet's payoff in case the seller has access to one sample of every distribution. In the so-called i.i.d. setting, in which the offer distributions are distributed identically and independently, it is even possible to obtain non-trivial prophet inequalities when nothing is known about the underlying (common) distribution of the offers, see, e.g., the work of Correa et al. [11] and references therein.

In the sample-access setting, the performance of the seller is measured, in addition, with respect to the expected performance under the random samples. This yields an average-case analysis with respect to the limited information. Our framework, however, uses ambiguity sets and does not require additional randomness over the given limited information. Our framework thus yields a robust worst-case analysis with respect to the limited information. Moreover, to the best of our understanding, there is no way to model sample knowledge as an ambiguity set in our framework without having to introduce additional randomness in the performance guarantee of interest (in order to make the problem well-defined and nontrivial). So whereas sample access naturally ties in with average-case analysis, our robust approach presents a worst-case analysis.

A robust maximin approach, closer to our work, is that of Dütting and Kesselheim [16]. They study how robust various techniques for proving prophet inequalities are, with respect to inaccurate (prior) offer distributions. In particular, they also study how errors propagate in a prophet inequality approximation guarantee when using the backwards induction procedure for computing

the optimal strategy of the seller. Further robust prophet inequality-like approaches are, e.g., considered by Schlag and Zapoechelnyuk [39, 38]. An interesting direction for future work could be to investigate to what extent a distributionally robust prophet inequality is possible under mean-dispersion-support information. For example, as the seller can always achieve an (asymptotic) payoff of $\mu + \sigma^2/\mu$, and the prophet can clearly always get a payoff of at most L, the seller can obviously approximate the payoff of the prophet to within a factor $L/(\mu + \sigma^2/\mu)$. How much can this be improved?

Apart from the welfare maximization problem we study in this work, an interesting direction for future work we also plan to investigate is the revenue maximization version of the problem, in which the seller posts for every potential buyer a (randomized) price, and selects the first offer exceeding the posted price. In fact, in recent years there has been a large growth on literature concerning distributionally robust auction theory and mechanism design. This dates back to, e.g., the works of Bergemann and Schlag [7, 8] who study a robust monopoly pricing problem. Another prominent line of work is that initiated by Azar and Micali [6] who study parametric digital auctions under mean-variance ambiguity. It should be noted that they study the ratio between the revenue of a seller who only has access to limited distributional information, and a seller who has access to the full distributions. See also the works of Daskalakis and Pierrakos [13] and Azar et al. [4], and the more recent work of Giannakopoulos et al. [20]. The absolute revenue maximization problem is studied by Carrasco et al [9]. For further directions in distributionally robust mechanism design (some of which considering mean-dispersion information), see, e.g., [8, 2, 27, 10, 22, 41, 43, 34, 1, 3] and references therein.

Our work also opens up a plethora of opportunities to study other (absolute) online selection problems with mean-dispersion information where multiple offers may be accepted by the seller subject to a combinatorial feasibility constraint. In the context of prophet inequalities, such models have been studied extensively. Here one may think of models where, e.g., the seller can accept up to k offers, see, e.g., the work of Hajiaghayi et al. [21]. Other types of combinatorial restrictions one may consider are matroid constraints, as in the work of Kleinberg and Weinberg [26], or matchings constraints, see, e.g., the recent work of Ezra et al. [18] (and references therein). For a general combinatorial framework for prophet inequalities, we refer the reader to the work of Dütting et al. [15].

1.3 Technical preliminaries

A nonnegative probability distribution is described by a probability measure $\mathbb{P} : \mathcal{B} \to [0, 1]$, with $\int_x 1 \, \mathrm{d}\mathbb{P}(x)$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}_{\geq 0}$. We will interchangeably use the terms probability distribution and probability measure. For a random variable X distributed according to $\mathbb{P}$, denoted by $X \sim \mathbb{P}$, and function $g : \mathbb{R} \to \mathbb{R}$, we write

$$\mathbb{E}[g(X)] = \mathbb{E}_{\mathbb{P}}[g(X)] = \int_{\mathbb{R}_{\geq 0}} g(x) \, \mathrm{d}\mathbb{P}(x).$$

We always assume that the integral on the right hand side is well-defined. We denote the *mean* of X by $\mu = \mathbb{E}[X]$, and its *variance* by $\sigma^2 = \mathbb{E}[(X - \mathbb{E}[X])^2]$. The *mean absolute deviation (MAD)* of X is given by $d = \mathbb{E}\big[|X - \mathbb{E}[X]|\big]$, and can be seen as an L_1-norm dispersion measure. We say that the support of X is included in the interval $[a, b] \subseteq \mathbb{R}_{\geq 0}$ if $\int_a^b 1 \, \mathrm{d}\mathbb{P}(x) = 1$. In this work, for given $\mu, \sigma^2, L \geq 0$, we write $\mathcal{P}(\mu)$ for the class of all nonnegative distributions with mean μ; $\mathcal{P}(\mu, \sigma^2)$ for the class of all nonnegative distributions with mean μ and variance σ^2; and $\mathcal{P}(\mu, \sigma^2, L)$ for the set of all distributions with mean μ, variance σ^2 and support a subset of the interval $[0, L]$. We write $\mathcal{P}_2(\cdot) \subset \mathcal{P}(\cdot)$ for the set of all discrete distributions supported on two points (called two-point

distributions) satisfying the given information $(\cdot)$. For example, $\mathcal{P}_2(\mu)$ is the set of all nonnegative two-point distributions with mean μ. Analogous definitions holds for the case where the variance σ^2 is replaced by the mean absolute deviation $d \geq 0$.

We implicitly only consider combinations of the above parameters for which the resulting ambiguity sets are non-empty. In particular, for given $\mu, \sigma^2, L \geq 0$ it holds that $\mathcal{P}(\mu, \sigma^2, L)$ is non-empty if and only if

$$\sigma^2 \leq \mu(L - \mu) \tag{1.10}$$

and $\mathcal{P}(\mu, d, L)$ is non-empty if and only if

$$d \leq \frac{2\mu(L - \mu)}{L} \tag{1.11}$$

We give a proof of the non-emptiness conditions (1.10) and (1.11) in Appendix B. The same conditions hold true for the non-emptiness of $\mathcal{P}_2(\mu, \sigma^2, L)$ and $\mathcal{P}_2(\mu, d, L)$, respectively.

1.4 Outline

We continue in Section 2 with a proof of Theorem 1.1 showing the optimality of the robust thresholds in (1.3). We then provide a complete analysis of the two-point, mean-variance-support setting in Section 3, where we in particular proof Theorems 1.2 and 1.3. In Section 4 we then explain primal-dual moment bound problems, and provide our results for the general mean-variance-support setting (Theorem 4.1), as well as the mean-MAD(-support) setting (Theorem 4.2).

2 Optimality of robust thresholds

We first formally define (randomized) stopping rules in Section 2.1, and afterwards give the proof of Theorem 1.1 in Section 2.2.

2.1 Stopping rules

A (randomized) stopping rule τ is a collection of functions $(\tau_i)_{i=1,\ldots,n}$, where τ_i is a (Bernoulli) random variable that takes as inputs $v_1, \ldots, v_i$ and the limited information $I = (I_1, \ldots, I_n)$. Based on this input, the function τ_i selects value v_i with probability $r_i(v_1, \ldots, v_i, I)$. To be precise, we have

$$\tau_i(v_1, \ldots, v_i, I) = \begin{cases} 1 & \text{with prob. } r_i(v_1, \ldots, v_i, I) \\ 0 & \text{with prob. } 1 - r_i(v_1, \ldots, v_i, I) \end{cases} \tag{2.1}$$

for $i = 1, \ldots, n$. This means that the probability we select offer i is given by

$$\mathbb{P}[\tau = i \mid v_1, \ldots, v_i] = r_i(v_1, \ldots, v_i, I) \prod_{j=1}^{i-1} (1 - r_j(v_1, \ldots, v_j, I)). \tag{2.2}$$

We use i^* to denote the random variable representing the stopping moment, i.e., the first i for which $\tau_i(v_1, \ldots, v_i, I) = 1$ in a fixed tuple $(v_1, \ldots, v_n)$, which is distributed according to (2.2). We then write

$$X_\tau = \mathbb{E}[v_{i^*}] = \sum_{i=1}^{n} \mathbb{P}[\tau = i \mid v_1, \ldots, v_i] \cdot v_i$$

for the expected value of the offer that the stopping rule τ selects, on the fixed tuple $(v_1, \ldots, v_n)$. The expected payoff of the seller, where v_i is a realization of a random variable $X_i \sim \mathbb{P}_i$ for $i = 1, \ldots, n$, is then given by

$$\mathbb{E}_{\mathbb{P}_1,\ldots,\mathbb{P}_n}[X_\tau]. \tag{2.3}$$

2.2 Proof of Theorem 1.1

We will use (backwards) induction. It is not hard to see that the claim is true for $n = 1$, because if there is only one offer then it is clearly optimal to simply accept it always. Now suppose the claim is true for all instances with $n-1$ offers for some $n-1 \in \mathbb{N}$. We prove the thresholds in (1.3) are also optimal for instances with n offers.

Let $\tau = (\tau_1, \ldots, \tau_n)$ be a stopping rule that has higher expected robust payoff than the optimal strategy $\tau^* = (\tau_1^*, \ldots, \tau_n^*)$ given by the threshold-based strategy in (1.3). That is, assume that

$$\mathbb{E}_{\mathbb{P}_1,\ldots,\mathbb{P}_n}[X_\tau] > \mathbb{E}_{\mathbb{P}_1,\ldots,\mathbb{P}_n}[X_{\tau^*}]. \tag{2.4}$$

We will first argue (Lemma 2.1), using the recursion hypothesis, that it must be the case that τ and τ^* coincide for $i = 2, \ldots, n$, i.e., that τ is of the form $(\tau_1, \tau_{-1}^*) = (\tau_1, \tau_2^* \ldots, \tau_n^*)$. Secondly, we will argue (Lemma 2.2) that then also $\tau_1 = \tau_1^*$ must hold, which gives a contradiction.

We first introduce some additional notation. For a stopping rule σ with fixed value v_1 and $\sigma_1(v_1, I) = 0$, we write $\sigma_{-1} = (\sigma_2, \ldots, \sigma_n)$ for the stopping rule induced by fixing v_1 and $\sigma_1(v_1, I) = 0$ in $\sigma_2, \ldots, \sigma_n$ (we slightly abuse notation and call the resulting stopping rules $\sigma_2, \ldots, \sigma_n$ again; it will always be clear from context whether v_1 and $\sigma_1(v_1, I)$ are fixed or not). We remark that in, for the stopping rule τ^*, fixing v_1 does not change $\tau_2^*, \ldots, \tau_n^*$ as these are independent of v_1, which can be seen right away from (1.3). Note that σ_{-1} is a well-defined stopping rule on an instance with $n-1$ offers. Furthermore, because of the independence of the random variables, it holds that for $i = 2, \ldots, n$, v_i is still distributed according to $X_i \sim \mathbb{P}_i$.

Lemma 2.1. *Let τ be any stopping rule, and let τ^* be the stopping rule described by the thresholds in (1.3). Let $(\tau_1, \tau_{-1}^*) = (\tau_1, \tau_2^* \ldots, \tau_n^*)$. Then*

$$\min_{\mathbb{P}_1,\ldots,\mathbb{P}_n} \mathbb{E}[X_\tau] \leq \min_{\mathbb{P}_1,\ldots,\mathbb{P}_n} \mathbb{E}[X_{(\tau_1,\tau_{-1}^*)}]. \tag{2.5}$$

Proof. For any fixed realization v_1, it holds that

$$\mathbb{E}[X_\tau \mid v_1] = r_1(v_1, I) \cdot v_1 + (1 - r_1(v_1, I)) \cdot \mathbb{E}[X_\tau \mid v_1, \tau_1(v_1, I) = 0]. \tag{2.6}$$

Note that $\mathbb{E}[X_\tau \mid v_1, \tau_1(v_1, I) = 0]$ is the expected payoff of the seller on an instance with $n-1$ offers. The stopping rule on this instance is induced by the functions $(\tau_2, \ldots, \tau_n)$ in which v_1 is fixed, as well as the outcome $\tau_1(v_1, I) = 0$. The induction hypothesis then tells us that

$$\min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}[X_{\tau_{-1}} \mid v_1] \leq \min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}[X_{\tau_{-1}^*} \mid v_1]. \tag{2.7}$$

It follows we may extend (2.7), using (2.6), to

$$\min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}[X_\tau \mid v_1] \leq \min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}[X_{(\tau_1,\tau_{-1}^*)} \mid v_1]. \tag{2.8}$$

Here we use the calculus fact $\min_x f(x) \leq \min_x g(x) \Rightarrow \min_x \alpha \cdot u + (1-\alpha) f(x) \leq \min_x \alpha \cdot u + (1-\alpha) g(x)$ if α, u do not depend on x. We will next show how (2.5) can be derived from (2.8). Fix any $\epsilon > 0$ and let $(\mathbb{M}_2, \ldots, \mathbb{M}_n) \in \prod_{i=2}^n \mathcal{P}(I_i)$ be such that

$$\mathbb{E}_{\mathbb{M}_2,\ldots,\mathbb{M}_n}[X_\tau \mid v_1] \leq \min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}_{\mathbb{P}_2,\ldots,\mathbb{P}_n}[X_{(\tau_1,\tau^*_{-1})} \mid v_1] + \epsilon. \tag{2.9}$$

Now fix some arbitrary $\mathbb{G}_1 \in \mathcal{P}(I_1)$. Integrating out (2.9) with respect to v_1 (using the measure $\mathbb{G}_1$) gives

$$\begin{aligned}
\mathbb{E}_{\mathbb{G}_1,\mathbb{M}_2,\ldots,\mathbb{M}_n}[X_\tau] &\leq \int_{v_1} \min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}_{\mathbb{P}_2,\ldots,\mathbb{P}_n}[X_{(\tau_1,\tau^*_{-1})} \mid v_1] + \epsilon \, \mathrm{d}\mathbb{G}_1(x) \\
&\leq \min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \int_{v_1} \mathbb{E}_{\mathbb{P}_2,\ldots,\mathbb{P}_n}[X_{(\tau_1,\tau^*_{-1})} \mid v_1] \, \mathrm{d}\mathbb{G}_1(x) + \epsilon \\
&= \min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}_{\mathbb{G}_1,\mathbb{P}_2,\ldots,\mathbb{P}_n}[X_{(\tau_1,\tau^*_{-1})}] + \epsilon,
\end{aligned} \tag{2.10}$$

where the second inequality is standard (and where we implicitly use the independence of the random variables $X_1, \ldots, X_n$). That is, we use that for any function g (for which the following expression is well-defined) it holds that $\int_t \min_x g(x,t) dt \leq \min_x \int_t g(x,t) dt$. Taking the minimum on the left with respect to $\mathbb{M}_2, \ldots, \mathbb{M}_n$ in (2.10) then gives

$$\min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}_{\mathbb{G}_1,\mathbb{P}_2,\ldots,\mathbb{P}_n}[X_\tau] \leq \mathbb{E}_{\mathbb{G}_1,\mathbb{M}_2,\ldots,\mathbb{M}_n}[X_\tau] \leq \min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}_{\mathbb{G}_1,\mathbb{P}_2,\ldots,\mathbb{P}_n}[X_{(\tau_1,\tau^*_{-1})}] + \epsilon \tag{2.11}$$

Next, taking the minimum with respect to $\mathbb{G}_1$ in (2.11), which was chosen arbitrarily (and recalling that ϵ was also chosen arbitrarily) then yields (2.5). □

Knowing Lemma 2.1 we continue by showing in Lemma 2.2 that it must then also hold that $\tau_1 = \tau_1^*$. This gives a contradiction with (2.4) and then completes the proof of Theorem 1.1.

Lemma 2.2. *Let τ_1 be any stopping rule for the first offer, and let τ^* be the stopping rule described by the thresholds in (1.3). Let $(\tau_1, \tau^*_{-1}) = (\tau_1, \tau_2^* \ldots, \tau_n^*)$. Then*

$$\min_{\mathbb{P}_1,\ldots,\mathbb{P}_n} \mathbb{E}[X_{(\tau_1,\tau^*_{-1})}] \leq \min_{\mathbb{P}_1,\ldots,\mathbb{P}_n} \mathbb{E}[X_{\tau^*}]. \tag{2.12}$$

Proof. We have (explanation is given afterwards)

$$\begin{aligned}
\min_{\mathbb{P}_1,\ldots,\mathbb{P}_n} \mathbb{E}[X_{(\tau_1,\tau^*_{-1})}] &= \min_{\mathbb{P}_1} \left(\min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}[X_{(\tau_1,\tau^*_{-1})}] \right) \\
&= \min_{\mathbb{P}_1} \left(\min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} r_1(v_1, I) \cdot v_1 + (1 - r_1(v_1, I)) \cdot \mathbb{E}[X_{(\tau_1,\tau^*_{-1})} \mid \tau_1(v_1, I) = 0] \right) \\
&= \min_{\mathbb{P}_1} \left(\min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} r_1(v_1, I) \cdot v_1 + (1 - r_1(v_1, I)) \cdot \mathbb{E}[X_{\tau^*_{-1}}] \right) \\
&= \min_{\mathbb{P}_1} \left(r_1(v_1, I) \cdot v_1 + (1 - r_1(v_1, I)) \cdot \min_{\mathbb{P}_2,\ldots,\mathbb{P}_n} \mathbb{E}[X_{\tau^*_{-1}}] \right) \\
&= \min_{\mathbb{P}_1} r_1(v_1, I) \cdot v_1 + (1 - r_1(v_1, I)) \cdot T(1).
\end{aligned} \tag{2.13}$$

The first and second equality are true by definition. The third equality is a bit more subtle. Here we use the fact that the functions $\tau_2^*, \ldots, \tau_n^*$ are independent of v_1 by definition of (1.3), so once

v_1 is rejected, we are simply left with an instance on $n-1$ offers on which the seller applies the stopping rule $\tau^*_{-1} = (\tau^*_2, \ldots, \tau^*_n)$. In the fourth equality, we use the fact that v_1 is independent of the distributions $\mathbb{P}_2, \ldots, \mathbb{P}_n$, and so we can bring the minimum inside. The final equality applies the definition of $T(1)$.

It is then not hard to see that, independent of what distribution $\mathbb{P}_1$ is, the expression

$$r_1(v_1, I) \cdot v_1 + (1 - r_1(v_1, I)) \cdot T(1)$$

in (2.13) is maximized by the setting

$$r_1(v_1, I) = \begin{cases} 1 & \text{if } v_1 \geq T(1) \\ 0 & \text{if } v_1 < T(1) \end{cases}.$$

This is precisely the function τ^*_1, which in turn yields (2.12). □

3 Ambiguity set $\mathcal{P}_2(\mu, \sigma^2, L)$

In this section we will give the proofs of Theorems 1.2 and 1.3 for $\mathbb{P} \in \mathcal{P}_2(\mu, \sigma^2, L)$, the ambiguity set central is this paper that contains all two-point distributions with a given mean μ, variance σ^2 and support contained in $[0, L]$. We shall start with the proof of Theorem 1.2 and then build on it in order to prove Theorem 1.3. But before doing so, we consider in Section 3.1 the ambiguity sets $\mathcal{P}(\mu)$ and $\mathcal{P}(\mu, \sigma^2)$ that give degenerate results, in the sense that nature chooses worst-case two-point distributions that make an expected payoff of μ best possible. These degenerate results provide motivation to consider two-point distributions, for a slightly richer information structure that includes the support, hence $\mathcal{P}_2(\mu, \sigma^2, L)$ as discussed in Section 3.2 and further.

3.1 Two (degenerate) ambiguity sets

First suppose that we are only given the (common) mean μ of every distribution, i.e., $\mathcal{P}_i = \mathcal{P}(\mu)$ for all $i = 1, \ldots, n$. In this case, (1.3) yields $T(i) = \mu$ for every $i = 0, \ldots, n-1$. To see this, notice that the seller can guarantee an (expected) payoff of μ by simply always selecting the first offer. On the other hand, if nature chooses identical deterministic distributions (i.e., point mass of 1 on μ) for all values, then clearly μ is also the best possible. It follows that the expected robust payoff of the seller is $r^* = \mu$ in this case. Notice that the reasoning above also remains valid if, in addition to the mean μ, we are also given an upper bound L on the support of the unknown distributions.

Next, assume that in addition we also know the common variance σ^2 of the distributions, i.e, we consider the ambiguity set $\mathcal{P}(\mu, \sigma^2)$ of all nonnegative distributions with mean μ and variance σ^2. Can we do better than an expected robust payoff of μ? In this case, nature can no longer simply choose deterministic distributions (point mass of 1 on μ), but we will argue that it can mimic such distributions. Also note that, if we assume the variance is *at most* σ^2, which is common in some works (e.g., [9]), we can right away argue that we cannot do better, as we can then again consider the setting where nature chooses deterministic distributions with point mass at μ. On the other hand, all our results seem to carry over to the setting in which we assume that the variance is *at least* σ^2, but we leave this to the interested reader. It follows directly from (1.3) that $T(n-1) = \mu$. We continue next with showing that also $T(n-2) = \mu$, which then also implies that $T(i) = \mu$ for every i. We will show that this result can be established already with discrete two-point distributions. A two-point distribution $\mathbb{P} \in \mathcal{P}_2(\mu, \sigma^2)$ can be written as

$$\mathbb{P}(x) = \begin{cases} p & \text{for } x = a \\ 1 - p & \text{for } x = b \end{cases} \tag{3.1}$$

with $a \leq \mu \leq b$. The mean and variance constraint imply that (after some calculus)

$$a = \mu - \sqrt{\frac{1-p}{p}}\sigma \quad \text{and} \quad b = \mu + \sqrt{\frac{p}{1-p}}\sigma. \tag{3.2}$$

Because of the constraint $a \geq 0$, it follows that

$$\frac{1}{1+c^2} \leq p < 1,$$

where $c = \mu/\sigma$ is the inverse of the *coefficient of variation*. Using (3.2) and the first form of $T(i)$ in (1.3), the threshold $T(n-2)$ is then given by

$$\begin{aligned} T(n-2) &= \min_{\mathbb{P}\in\mathcal{P}(\mu,\sigma^2)} \int_0^\infty \max\{T(n-1), x\}\, \mathrm{d}\mathbb{P}(x) \\ &= \min_{1/(1+c^2)\leq p<1} p\mu + (1-p)b \\ &= \min_{1/(1+c^2)\leq p<1} p\mu + (1-p)\left(\mu + \sqrt{\frac{p}{1-p}}\sigma\right) \\ &= \min_{1/(1+c^2)\leq p<1} \mu + \sqrt{p(1-p)}\sigma. \end{aligned} \tag{3.3}$$

In order to minimize the last expression, it is not hard to see that we should take $p \to 1$, resulting in $T(n-2) = \mu$.Alternatively, if we did not have the constraint of the random variables being nonnegative, we could also have taken $p \to 0$ (which comes at the cost of a becoming negative). So what actually happens when $p \to 1$? In this case nature's two-point distribution assigns almost all its probability mass to $\mu - g(\epsilon)$, for some g with $g(\epsilon) \to 0$ as $\epsilon \to 0$, and its remaining mass to the number $b(\epsilon)$ with $b(\epsilon) \to \infty$ as $\epsilon \to 0$ (to account for the variance constraint). In other words, it assigns most of its mass to a number just below the mean μ, and its remaining mass to a very large number.

The construction above heavily exploits the fact that we can make the number b in the support of the two-point distribution $\mathbb{P}$ arbitrarily large. This motivates the introduction of an upper bound on the support of the unknown distributions, i.e., the study of the ambiguity set $\mathcal{P}(\mu, \sigma^2, L)$.

3.2 Proof of Theorem 1.2

As before we use that a two-point distribution $\mathbb{P} \in \mathcal{P}_2(\mu, \sigma^2, L)$ can be written as (3.1) and (3.2) but now the constraints $a \geq 0$ and $b \leq L$ imply that p is bounded by

$$\frac{1}{1+c^2} \leq p \leq \frac{1}{1+\frac{\sigma^2}{(L-\mu)^2}} \quad \left(\leq 1 - \frac{1}{1+c^2} \text{ as } L \leq 2\mu\right), \tag{3.4}$$

where, again, $c = \mu/\sigma$. The two extreme choices for p yield the distributions f^* and g^* that were used in Section 1.1. We remark that the interval of p's satisfying the bounds in (3.4) is non-empty as for any distribution supported on $[0, L]$ with mean μ and variance σ^2, it holds that $\sigma^2 \leq \mu(L-\mu)$, because of (1.10), which is equivalent to $1/(1+c^2) \leq 1/(1+(\sigma/(L-\mu))^2)$. Furthermore, the assumption $L \leq 2\mu$ implies that $\sigma \leq \mu$, which yields $c \geq 1$.

Again we have $T(n) = 0$ and $T(n-1) = \mu$. Using the same reasoning as in (3.3), we now find the following minimization problem for p:

$$\begin{aligned} T(n-2) = \ & \min \ \mu + \sqrt{p(1-p)}\sigma \\ & \text{s.t.} \ \frac{1}{1+c^2} \leq \ p \ < \frac{1}{1+\frac{\sigma^2}{(L-\mu)^2}}. \end{aligned} \tag{3.5}$$

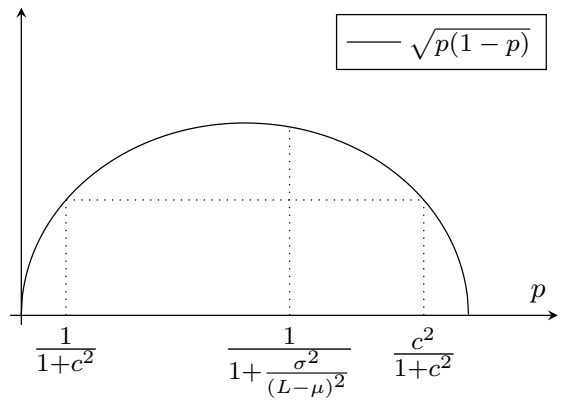

Figure 2: Sketch of where minimum of $h(p) = \sqrt{p(1-p)}$ is attained for $T(n-2)$.

By symmetry of the function $h(p) = \sqrt{p(1-p)}$ around $p = 1/2$, i.e., $h(1/(1+c^2)) = h(1 - 1/(1+c^2))$, it then follows that the minimum of the function $h(p)$ is attained at $p = 1/(1+c^2)$. Note that here we also use the fact that $1/(1+c^2) \leq 1/2$ as $c \geq 1$. An illustration of the situation is given in Figure 2. We can now determine the optimal robust threshold $T(n-2)$ by taking $p = 1/(1+c^2)$ in (3.5). We find

$$T(n-2) = \left(1 + \frac{1}{1+c^2}\right)\mu.$$

The worst-case distribution $\mathbb{P}_{n-1}$ attaining the minimum is given by

$$\mathbb{P}_{n-1}(x) = \begin{cases} \dfrac{1}{1+c^2} & \text{for } x = a = 0 \\ 1 - \dfrac{1}{1+c^2} & \text{for } x = b = \left(1 + \dfrac{1}{c^2}\right)\mu \end{cases}. \tag{3.6}$$

The point $b = \left(1 + 1/c^2\right)\mu = \mu + \sigma^2/\mu$ plays a special role, as it is the smallest possible rightmost point on which a distribution $\mathbb{P} \in \mathcal{P}_2(\mu, \sigma^2, L)$ is supported. The following lemma will also play an important role (also later on).

Lemma 3.1. *The sequence $(T(i))_{i=0,\ldots,n}$ described by (1.3) is nonincreasing for any collection of ambiguity sets $\mathcal{P}(I_i)$. Furthermore, suppose that every ambiguity set contains the two-point distribution in (3.6). Then $\mu = T(n-1) \leq \cdots \leq T(1) \leq T(0) \leq \mu + \sigma^2/\mu$.*

Proof. The fact that the sequence $(T(i))_{i=0,\ldots,n}$ is nonincreasing follows directly from the definition

$$T(i) = \min_{\mathbb{P}_{i+1} \in \mathcal{P}(I_{i+1})} \int_0^\infty \max\{T(i+1), x\} \, \mathrm{d}\mathbb{P}_{i+1}(x),$$

as $\max\{T(i+1), x\} \geq T(i+1)$. In order to prove the second claim, it suffices to consider the case $i = 0$, because we have just shown that the sequence $(T(i))_{i=0,\ldots,n}$ is nonincreasing. Suppose that nature chooses the two-point distribution (3.6) for every offer i. Then clearly the seller can never obtain an expected robust payoff of more than $\mu + \sigma^2/\mu$ as the value v_i of every offer is at most $\mu + \sigma^2/\mu$ (i.e., this is the largest point in the support of $\mathbb{P}$). □

From Lemma 3.1, we therefore can infer that for every $\mathbb{P}$ of the form (3.1) and every $i = 0, \ldots, n-1$, it holds that

$$a \leq \mu \leq T(i) \leq \mu + \sigma^2/\mu \leq b.$$

This means the integral in (1.3) is equal to

$$\begin{aligned} T(i) &= \min_{\mathbb{P}_{i+1}\in\mathcal{P}_2(\mu,\sigma^2,L)} \int_0^\infty \max\{T(i+1),x\}\,\mathrm{d}\mathbb{P}_{i+1}(x) \\ &= \min_{\frac{1}{1+c^2}\le\, p\, <\frac{1}{1+\frac{\sigma^2}{(L-\mu)^2}}} T(i+1)p + (\mu+\sqrt{p/(1-p)}\sigma)(1-p) \\ &= \min_{\frac{1}{1+c^2}\le\, p\, <\frac{1}{1+\frac{\sigma^2}{(L-\mu)^2}}} \mu + (T(i+1)-\mu)p + \sqrt{p(1-p)}\sigma \end{aligned} \tag{3.7}$$

for $i = 0,\dots,n-1$. In order to prove that the minimum in (3.7) remains to be attained at $p = 1/(1+c^2)$, we will use the following proposition.

Proposition 3.2. *Let $K_0, K_1 \ge 0$ and $p \in (0,1)$. Assume that $0 < y \le 1/2$ and z is such that $y \le z \le 1-y$. Let*

$$u(p) = K_0 p + \sqrt{p(1-p)}K_1.$$

Then $u(y) \le \min\{u(z), u(1-y)\}$ and the minimum of u in the range $(y, 1-y)$ is attained at y.

Proof. We write $u(p) = u_1(p) + u_2(p)$ with $u_1(p) = K_0 p$ and $u_2(p) = \sqrt{p(1-p)}K_1$. Note that $u_2(y) = u_2(1-y)$ by symmetry of u_2 around $p = 1/2$. Furthermore, $u_1(y) \le u_2(1-y)$ as $y \le 1-y$ by assumption of $0 \le y \le 1/2$. It follows that $u(y) \le u(1-y)$.

Secondly, the function $u(p)$ has at most one point in the interval $(0,1)$ where its derivative vanishes for any choice of K_0 and K_1. In combination with the fact that $u(p)$ has a positive derivative for any $0 < p < 1/2$, and $u(y) \le u(1-y)$, it then follows that for every point $y \le z \le 1-y$ satisfies $u(y) \le u(z)$ and that the minimum of u in $(y, 1-y)$ must be attained at y. □

The claim that $p = 1/(1+c^2)$ is where the minimum of (3.7) is attained now follows from applying Proposition 3.2 with $y = 1/(1+c^2) \le 1/2$, $K_0 = T(i+1) - \mu \ge 0$ and $K_1 = \sigma \ge 0$. We emphasize that this holds for any i. Plugging $p = 1/(1+c^2)$ into the expression (3.7) gives

$$T(i) = \mu + (T(i+1)-\mu)\frac{1}{1+c^2} + \frac{c}{1+c^2}\sigma = \mu + \frac{1}{1+c^2}T(i+1). \tag{3.8}$$

Here, and in subsequent sections, we will repeatedly make use of backwards recursions of this type. We now give a general description and solution of such recursions. The proof of Proposition 3.3 is left to the reader.

Proposition 3.3. *Let $\alpha, \beta, \gamma_0 \in \mathbb{R}$, and consider the backwards recursion $t(i) = \alpha + \beta \cdot t(i+1)$ for $i = 0,\dots,m$ with $t(m) = \gamma_0$. The solution to this recursion is given by*

$$t(i) = \left(\alpha \sum_{k=0}^{m-1-i} \beta^k\right) + \gamma_0 \beta^{m-i} \tag{3.9}$$

for $i = 0,\dots,m$. We use the convention that $\sum_{k=0}^{-1} \beta^k = 0$.

Applying Proposition 3.3 with $\alpha = \gamma_0 = \mu$ and $\beta = 1/1(1+c^2)$ then yields

$$T(i) = \mu \sum_{k=0}^{n-1-i} \left(\frac{1}{1+c^2}\right)^k = \mu \sum_{k=0}^{n-1-i} \left(\frac{\sigma^2}{\sigma^2+\mu^2}\right)^k.$$

Using the identity $\sum_{k=0}^{j} x^j = (1-x^{j+1})/(1-x)$, in combination with some calculus, then yields the identity as stated in Theorem 1.2. This completes the proof of Theorem 1.2.

3.3 Proof of Theorem 1.3

The setup for the proof of Theorem 1.3 is similar to that of the proof of Theorem 1.2. We again have $T(n) = 0$ and $T(n-1) = \mu$. The difference occurs when we consider the minimization problem

$$\begin{aligned} T(i) = \ & \min \ \mu + (T(i+1) - \mu)p + \sqrt{p(1-p)}\sigma. \\ & \text{s.t.} \ \frac{1}{1+c^2} \leq \ p \ < \frac{1}{1+\frac{\sigma^2}{(L-\mu)^2}} \end{aligned} \tag{3.10}$$

for $i = 0, \ldots, n-2$.

Because of the assumption $L > 2\mu$, it already holds for $i = n-2$ that the minimum is not attained at the leftmost point $p = 1/(1+c^2)$, but at the rightmost point $p = 1/(1+\sigma^2/(L-\mu)^2)$ instead. For certain parameter combinations the minimum is attained at both points simultaneously. To see this, one has to distinguish two cases: either $c \geq 1$, in which case the condition $L > 2\mu$ yields the claim, or $c < 1$, in which case $1/(1+c^2) \geq 1/2$ and so $h(1/(1+c^2)) \geq h(1/(1+\sigma^2/(L-\mu)^2))$ as $h(p) = \sqrt{p(1-p)}$ is decreasing on $[1/2, 1]$. These cases are illustrated in Figure 3.

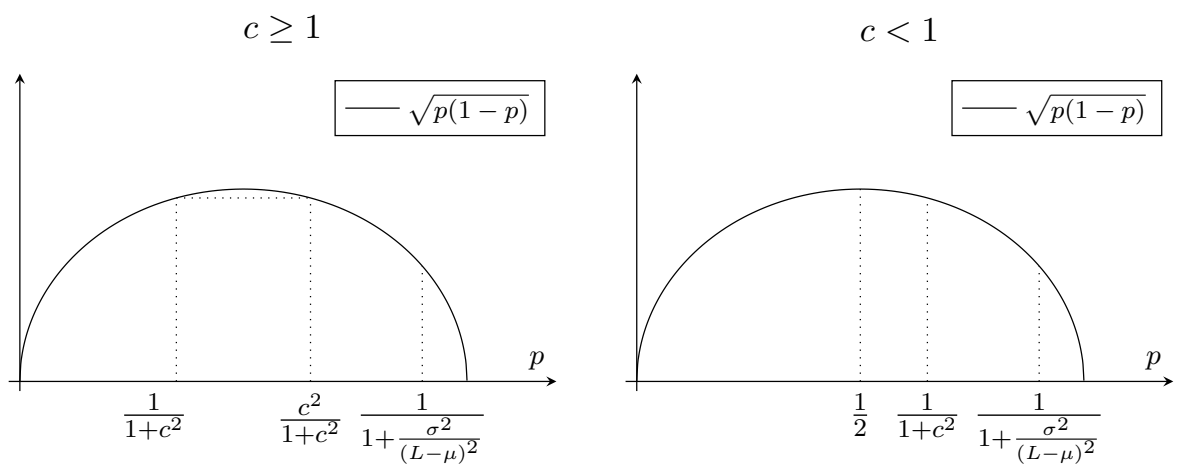

Figure 3: Sketch of where minimum of $h(p) = \sqrt{p(1-p)}$ is attained for $T(n-2)$ depending on whether $c \geq 1$ (left) or $c < 1$ (right).

The main question is now as follows: What happens for $T(i)$ for $i = 0, \ldots, n-3$? Is the minimum in (3.10) still attained at the rightmost point $p = 1/(1+\sigma^2/(L-\mu)^2)$ or will the minimum be attained at the leftmost point $p = 1/(1+c^2)$? The reason that we may focus on the extreme points of p is again because at every step of the backward recursion we are concerned with a function of the form $u(p)$ as in Lemma 3.2, which attains its minimum over an interval $[c, d] \subseteq [0, 1]$ at the boundary point c or d. It does not seem easy to give an explicit answer to this question in terms of the parameters μ, σ^2 and L right away. Nevertheless, we are able to describe the behavior of the minimum over the course of the backward recursion. We emphasize that this behavior relies on the fact that L is a constant independent of n.

We will show that there exists an $n_0 = n_0(\mu, \sigma^2, L) \in \{3, \ldots, n\}$ such that:

- For $n - n_0 < i \leq n-2$, the minimum in (3.10) will be attained at the rightmost point $p = 1/(1+\sigma^2/(L-\mu)^2)$.
- For $1 \leq i \leq n - n_0$, the minimum will be attained at the leftmost point $p = 1/(1+c^2)$.

Note that this means that if $n_0 = n$, then in fact the minimum is always attained at the rightmost point p. In other words, if n is large enough, there is a critical point $n - n_0$ at which the minimum switches from the rightmost point to the leftmost point, and after this, the leftmost point will continue to be the minimum for the remaining steps of the backwards recursion. We continue with the proof of this claim.

First of all, let n_0 be such that $n - n_0$ is the highest index i for which the minimum in (3.10) is attained at the leftmost point $p = 1/(1 + c^2)$ (or $n_0 = n$ otherwise). This means that for $n - n_0 + 1 \leq i \leq n - 2$ the minimum is attained at the rightmost point $p = 1/(1 + \sigma^2/(L - \mu)^2)$. Based on (3.10), the thresholds for these indices i are then given by the solution to the recursion

$$\begin{aligned} T(i) &= \mu + (T(i+1) - \mu)p + \sqrt{p(1-p)}\sigma \\ &= \mu + (T(i+1) - \mu)\frac{1}{1 + \frac{\sigma^2}{(L-\mu)^2}} + \sqrt{\frac{1}{1 + \frac{\sigma^2}{(L-\mu)^2}}\left(1 - \frac{1}{1 + \frac{\sigma^2}{(L-\mu)^2}}\right)}\sigma \\ &= \frac{L\sigma^2}{(L-\mu)^2 + \sigma^2} + \frac{(L-\mu)^2}{(L-\mu)^2 + \sigma^2}\, T(i+1). \end{aligned} \tag{3.11}$$

Using Proposition 3.3 with

$$\alpha = \frac{L\sigma^2}{(L-\mu)^2 + \sigma^2}, \quad \beta = \frac{(L-\mu)^2}{(L-\mu)^2 + \sigma^2}, \quad \text{and} \quad \gamma_0 = \mu$$

then yields for $n - n_0 + 1 \leq i \leq n - 1$ that

$$\begin{aligned} T(i) &= \frac{L\sigma^2}{(L-\mu)^2 + \sigma^2}\left[\sum_{k=0}^{n-2-i}\left(\frac{(L-\mu)^2}{(L-\mu)^2 + \sigma^2}\right)^k\right] + \mu\left(\frac{(L-\mu)^2}{(L-\mu)^2 + \sigma^2}\right)^{n-1-i} \\ &= L\left[1 - \left(1 - \frac{\mu}{L}\right)\left[\frac{(L-\mu)^2}{(L-\mu)^2 + \sigma^2}\right]^{n-1-i}\right]. \end{aligned} \tag{3.12}$$

Secondly, recall that $i = n - n_0$ is the first time the minimum is attained at the leftmost point $p = 1/(1+c^2)$. We claim that then in all the remaining steps $i = 1, \ldots, n - n_0 - 1$ of the backwards recursion, the minimum will also be attained at the leftmost point $p = 1/(1+c^2)$. To see this, note that the recursion, where $T(i)$ should be seen as a function of p, for $k < n - n_0$ reads

$$\begin{aligned} T(i) &= \mu + (T(i+1) - \mu)p + \sqrt{p(1-p)}\sigma \\ &= (T(i+1) - T(i+2))p + \left[\mu + (T(i+2) - \mu)p + \sqrt{p(1-p)}\sigma\right] \\ &=: g_1^i(p) + g_2^i(p) \end{aligned} \tag{3.13}$$

with $g_1^i(p) = (T(i+1) - T(i+2))p$ and $g_2^i(p) = \mu + (T(i+2) - \mu)p + \sqrt{p(1-p)}\sigma$. Now suppose that the minimum of $T(i)$ is attained at $p = 1/(1+c^2)$ for $i = \ell + 1, \ldots, n - n_0$. We claim that the same holds for $i = \ell$. To see this, note that $g_2^i(p)$ is precisely the expression we minimized for $i = \ell + 1$, so it follows that $g_2^i(1/(1+c^2)) \leq g_2^i(1/(1 + \sigma^2/(L-\mu)^2))$ as we assume the minimum is attained at $p = 1/(1+c^2)$ for $i = \ell + 1$. Furthermore, we know from Lemma 3.1 that the optimal robust thresholds in (1.3) form a non-increasing sequence. This means that $g_1^i(p)$ is an non-decreasing function in p, and therefore $g_1^i(1/(1+c^2)) \leq g_1^i(1/(1 + \sigma^2/(L-\mu)^2))$. Hence the function $T(\ell)$, as a function of p, is minimized at $p = 1/(1+c^2)$ as well.

A sketch of the behavior of the minimum is given in Figure 4. Roughly speaking, what happens is that the top of the function $T(i) = g_1^i(p) + g_2^i(p)$ moves to the right as i decreases (i.e., as we go

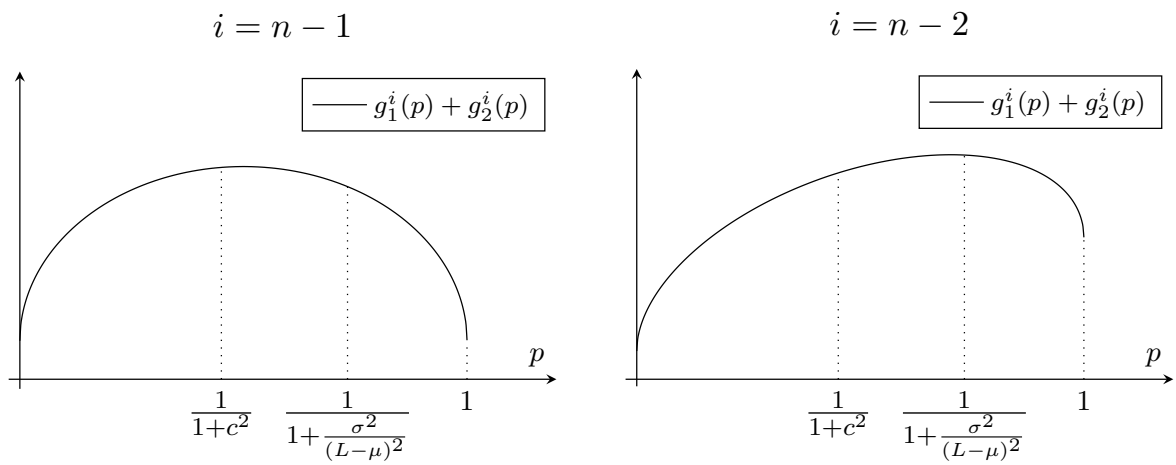

Figure 4: Behavior of minimum of $T(i) = g_1^i(p) + g_2^i(p)$ for $i = n-2$ (left) and $i = n-3$ (right). For $i = n-1$ the minimum is attained at the rightmost feasible $p = 1/(1+\sigma^2/(L-\mu)^2)$, but for $i = n-2$ it is attained at the leftmost feasible $p = 1/(1+c^2)$. These plots are made with $\mu = 1$, $\sigma^2 = 0.82\mu$ and $L = 2.5\mu$.

further into the backward recursion). This means that as soon as the minimum is attained on the left it will be attained on the left in the future as well.

Finally, we can again solve the recursion using Proposition 3.3 and obtain

$$\begin{aligned} T(i) &= \left(\sum_{k=0}^{n-n_0-i} \left(\frac{\sigma^2}{\mu^2+\sigma^2} \right)^i \right) \mu + \left(\frac{\sigma^2}{\mu^2+\sigma^2} \right)^{n-n_0+1-i} T(n-n_0+1) \\ &= \mu + \frac{\sigma^2}{\mu} \left[1 - \left(\frac{\sigma^2}{\mu^2+\sigma^2} \right)^{n-n_0-i} \right] + T(n-n_0+1) \left(\frac{\sigma^2}{\sigma^2+\mu^2} \right)^{n-n_0+1-i} \end{aligned} \tag{3.14}$$

for $i = 1, \ldots, n-n_0$, where $T(n-n_0+1)$ is given by (3.12).

In order to understand what happens when $n \to \infty$, we carry out a little thought experiment. Suppose that $n_0 = 1$, even if $n \to \infty$, meaning that the minimum is attained at $p = 1/(1+\sigma^2/(L-\mu)^2)$ for every step of the backwards recursion. This then means that $T(0) \to L$ as $n \to \infty$, as can be seen from (3.12). However, we know that nature can always guarantee that the seller gets at most $\mu + \sigma^2/\mu$ by Lemma 3.1. Note that in fact $\mu + \sigma^2/\mu \leq L$, as this is equivalent to $\sigma^2 \leq \mu(L-\mu)$ which is true for any distribution with mean μ, variance σ^2 and support range in $[0, L]$. This means that, when the strict inequality $\sigma^2 < \mu(L-\mu)$ holds, it must be the case that at some point $n - n_0$ in the recursion the minimum will be attained at the leftmost point $p = 1/(1+c^2)$ for the first time. This means we will always switch to the thresholds as in (3.14). These have the property that $T(0) \to \mu + \sigma^2/\mu$ as $n \to \infty$, regardless of $n_0 = n_0(\mu, \sigma^2, L)$, which itself is independent of n.

In order to compute the value of n_0 exactly, one has to compare the values of $T(i)$ obtained by either applying the recursion in (3.8) or (3.11) in every step i. The minimum of these two values determines the worst-case distribution (and the robust threshold). We know, by the analysis above, that in the first steps the minimum will be given by (3.8). The first step i for which the minimum is given by (3.11), determines the value of n_0.

3.4 An asymptotic result

Both in the case $L \leq 2\mu$ (Theorem 1.2) and in the case $L > 2\mu$ (Theorem 1.3) it follows that as $n \to \infty$, the expected robust payoff of the optimal stopping strategy of the gambler approaches $\mu + \sigma^2/\mu$. We argue that we can also obtain this payoff approximately (as n grows large) by means of simply setting a threshold of

$$T = \mu + \frac{\sigma^2}{\mu} \tag{3.15}$$

at every step of the game. That is, if one is only interested in obtaining the optimal payoff when n is large, and not so much about identifying the optimal strategy at every individual step of the game, then this can be done in a fairly simple manner. A slightly weaker version of this result for the more general set $\mathcal{P}(\mu, \sigma^2, L)$ is given in Section 4.3.

Theorem 3.4. *Consider the stopping rule τ that selects a value v_i if $v_i \geq T = \mu + \sigma^2/\mu$. Then*

$$\lim_{n\to\infty} r^*(\tau) = T = \mu + \frac{\sigma^2}{\mu}$$

in case nature is allowed to choose adversarial two-point distributions from $\mathcal{P}_2(\mu, \sigma^2, L)$.

Proof. The proof simply relies on showing that with high probability, as $n \to \infty$, there will be a value whose weight is at least T. Note that for any two-point distribution in $\mathcal{P}_2(\mu, \sigma^2, L)$ with positive probability on points a and b, it must be the case that $b \geq \mu + \sigma^2/\mu$. Furthermore, the probability mass assigned to the point b is always bounded away from zero (in particular by a constant depending on μ, σ^2 and L, but independent of n). This means that, as n grows large the probability that at least one of the values will be greater or equal than $\mu + \sigma^2/\mu$, approaches 1. In other words, no matter which adversarial distributions nature chooses, there will also be a constant probability mass assigned to the interval $[\mu + \sigma^2/\mu, L]$ for all of them, and, hence, with high probability one of the values will lie in that interval. Therefore, the probability that the gambler is able to select a value v_i with $v_i \geq T$ approaches 1. □

We remark that the analysis in the proof of Theorem 3.4 heavily relies on the fact that μ, σ^2 and L are independent of n.

4 Moment bound problems

In this section we present our results for the ambiguity sets $\mathcal{P}(\mu, \sigma^2, L)$, consisting of all distributions with mean μ, variance σ^2 and support upper bound L, and the set $\mathcal{P}(\mu, d, L)$, in which the variance is replaced by the mean absolute deviation (MAD). We start with the result for $\mathcal{P}(\mu, \sigma^2, L)$ as summarized in Theorem 4.1. We again refer the reader to Figure 1 for an illustration of the obtained robust thresholds.

Theorem 4.1 (Ambiguity set $\mathcal{P}(\mu, \sigma^2, L)$)**.** *Let $n \in \mathbb{N}$, $\mu, L > 0$ and $\sigma^2 \geq 0$. Let $\mathcal{P}(\mu, \sigma^2, L)$ be the set of all distributions with mean μ, variance σ^2 and whose support is a subset of the interval $[0, L]$. Let $\mathcal{P}(I_i) = \mathcal{P}(\mu, \sigma^2, L)$ for all $i = 1, \ldots, n$. Then the optimal robust threshold in (1.3) equals*

$$T(i) = \mu + \frac{\sigma^2}{\mu}\left[1 - \left(1 - \frac{\mu}{L}\right)^{n-1-i}\right] \tag{4.1}$$

for $i = 1, \ldots, n-1$, and $T(n) = 0$. Furthermore, as $n \to \infty$, the seller's expected robust payoff $r^* = T(0)$ approaches $\mu + \frac{\sigma^2}{\mu}$.

For the ambiguity set $\mathcal{P}(\mu, d, L)$, we obtain the result in Theorem 4.2. As was already discussion in Section 1.1, the obtained thresholds are independent of the support upper bound L, and in fact also holds for the ambiguity set $\mathcal{P}(\mu, d)$ in which no support upper bound is assumed.

Theorem 4.2 (Ambiguity set $\mathcal{P}(\mu, d, L)$)**.** *Let $n \in \mathbb{N}$, $\mu, d, L \geq 0$. Let $\mathcal{P}(\mu, d, L)$ be the set of all distributions with mean μ, mean absolute deviation d and whose support is a subset of the interval $[0, L]$. Let $\mathcal{P}(I_i) = \mathcal{P}(\mu, d, L)$ for all $i = 1, \ldots, n$. Then the optimal robust threshold in (1.3) equals*

$$T(i) = \frac{2\mu^2}{2\mu - d} - \left[\frac{2\mu^2}{2\mu - d} - \mu\right]\left(\frac{d}{2\mu}\right)^{n-1-i} \tag{4.2}$$

for $i = 1, \ldots, n-1$, and $T(n) = 0$. Furthermore, as $n \to \infty$, the seller's expected robust payoff $r^* = T(0)$ approaches $2\mu^2/(2\mu - d)$.

This section is structured as follows. We first illustrate the primal-dual method for moment bound problems in Section 4.1, and use it to prove a result of Cox [12] in this language. This result will the be applied in Section 4.2 for recursively determining the robust thresholds of Theorem 4.1. We provide the proof of Theorem 4.2, also using the primal-dual moment bound approach, in Section 4.4. An asymptotic result for the set $\mathcal{P}(\mu, \sigma^2, L)$, similar in spirit as Theorem 3.4, is given in Section 4.3.

4.1 Primal-dual method by example

For $\mathcal{P}(\mu, \sigma^2, L)$, based on the formulation (1.4), we need to solve the maximin problem

$$\max_{\mathbb{P} \in \mathcal{P}(\mu, \sigma^2, L)} \mathbb{E}_{\mathbb{P}}[\min\{\xi, X\}] \tag{4.3}$$

with $\xi = T(i+1)$ considered a constant. The problem can be viewed as a semi-infinite linear optimization problem (LP)

$$\begin{aligned} &\max_{\mathbb{P}} \int_{[0,L]} \min\{t, x\} \, \mathrm{d}\mathbb{P}(x) \\ &\text{s.t.} \int_{[0,L]} 1 \, \mathrm{d}\mathbb{P}(x) = 1, \ \int_{[0,L]} x \, \mathrm{d}\mathbb{P}(x) = \mu, \ \int_{[0,L]} x^2 \, \mathrm{d}\mathbb{P}(x) = \mu^2 + \sigma^2 \end{aligned} \tag{4.4}$$

with a finite number of constraints (three in this case), and an infinite number of variables (all distributions $\mathbb{P}$ in the ambiguity set $\mathcal{P}$). The Richter-Rogosinski Theorem (see, e.g., [33, 40]) states that there exists an extremal distribution for problem (4.4) with at most three support points. While finding these points in closed form is typically not possible for general semi-infinite problems, we show that this is possible for the problem at hand by resorting to the dual problem; see e.g. [24] and [31]. To explain this dual approach, we install a small intermezzo, and first cast (4.4) in the more general form of generalized moment bound problems:

$$\begin{aligned} \text{(P)} \quad p^* = \max_{\mathbb{P} \in \mathcal{P}(L)} \ & \mathbb{E}[h(X)] \\ \text{s.t.} \quad & \mathbb{E}[g_j(X)] = q_j \text{ for } j = 0, \ldots, m, \end{aligned}$$

where X is a random variable with nonnegative probability measure $\mathbb{P}$, and $h, g_0, \ldots, g_m$ are real-valued functions. We use the convention that the function g_0 always represents the constraint that $\mathbb{P}$ is a probability measure (which is essentially redundant as we already assume that $\mathcal{P}(L)$ is a

class of probability distributions). That is, we have $g_0 \equiv 1$, and $q_0 = 1$. The *dual program* of a moment bound problem of the form (P) is given by

$$\begin{aligned} \text{(D)} \quad d^* = \min_{\lambda_0,\ldots,\lambda_m} & \sum_{j=0}^{m} \lambda_j q_j \\ \text{s.t.} \quad & M(x) := \sum_{j=0}^{m} \lambda_j g_j(x) \geq h(x) \quad \forall x \in [0, L]. \end{aligned}$$

It is not hard to see that *weak duality* holds, i.e., $p^* \leq d^*$. Furthermore, under mild assumptions, also *strong duality* holds, i.e., $p^* = d^*$. Finally, it is known that the (extremal) distributions solving the primal program (P) are discrete distributions that are supported on at most $n+1$ points (we refer to [31] for an overview of these claims and further references).

The above primal-dual thinking provides a constructive method for solving a maximin problem by exploiting the dual problem formulation. The solution of the dual problem not only gives an upper bound for the primal problem (by weak duality), but also identifies a candidate worst-case distribution. By testing this candidate distribution in the primal problem, and showing that both problems give the same result (strong duality), one can thus solve the semi-infinite LP. We will now demonstrate this constructive method for $\mathcal{P}(\mu, \sigma^2, L)$ and then for $\mathcal{P}(\mu, d, L)$ in Section 4.4, both presenting semi-infinite LPs with three constraints. For $\mathcal{P}(\mu, \sigma^2, L)$ this rederives a result in [12], and for $\mathcal{P}(\mu, d, L)$ this gives a similar result but then for MAD instead of variance. Related results for mean-MAD ambiguity in the context of Chebyshev-like inequalities can be found [34].

The dual problem of (4.4) is given by

$$\begin{aligned} \min_{\lambda_0,\lambda_1,\lambda_2} \quad & \lambda_0 + \mu\lambda_1 + (\mu^2 + \sigma^2)\lambda_2 \\ \text{s.t.} \quad & \lambda_0 + \lambda_1 x + \lambda_2 x^2 \geq \min\{x, \xi\} \quad \forall x \in [0, L]. \end{aligned}$$

The dual problem searches the tightest majorant $M(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2$, a candidate being the

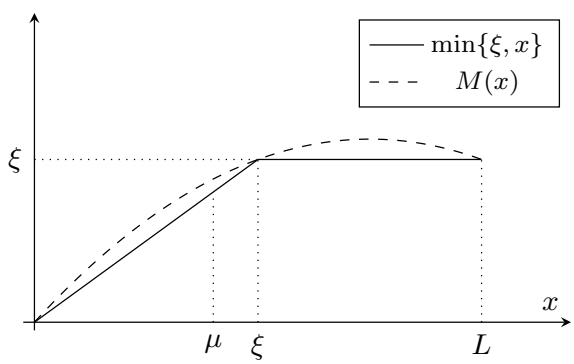

Figure 5: Majorant $M(x)$ for the function $\min\{\xi, x\}$ when $\mu - \frac{\sigma^2}{L-\mu} \leq \xi \leq \mu + \frac{\sigma^2}{\mu}$.

concave parabola that touches $\min\{\xi, x\}$ at $\{0, \xi, L\}$, with

$$\lambda_0 = 0, \quad \lambda_1 = \frac{L+\xi}{L}, \quad \lambda_2 = -\frac{1}{L} \tag{4.5}$$

and dual objective

$$\frac{\mu(L+\xi) - (\mu^2 + \sigma^2)}{L}. \tag{4.6}$$

For the primal problem, construct the distribution $\mathbb{P}$ on $\{0, \xi, L\}$ with probabilities

$$\mathbb{P}(0) = \frac{L\xi - (L+\xi)\mu + \mu^2 + \sigma^2}{L\xi}, \quad \mathbb{P}(\xi) = \frac{L\mu - (\mu^2 + \sigma^2)}{(L-\xi)\xi}, \quad \mathbb{P}(L) = \frac{\mu^2 + \sigma^2 - \xi\mu}{(L-\xi)L}. \tag{4.7}$$

This three-point distribution has primal objective value (4.6), which thus is the optimum. Note that this is only a distribution when

$$\mu - \frac{\sigma^2}{L-\mu} \leq \xi \leq \mu + \frac{\sigma^2}{\mu}. \tag{4.8}$$

For $\xi > \mu + \frac{\sigma^2}{\mu}$ consider the majorant $M(x) = x$ with dual objective μ. To find a distribution with a matching primal objective, consider all distributions on $[0, \xi]$ in $\mathcal{P}(\mu, \sigma^2, L)$, such as the three-point distribution on $\{0, \mu, \xi\}$. For $\xi < \mu - \frac{\sigma^2}{L-\mu}$ consider the majorant $M(x) = \xi$ with dual objective ξ. Distributions with a matching primal objective are all distributions in $\mathcal{P}(\mu, \sigma^2, L)$ on $[\xi, L]$, in $\mathcal{P}(\mu, \sigma^2, L)$, including the three-point distribution on $\{\xi, \mu, L\}$. Taken together, we obtain the following lemma:

Lemma 4.3 (Cox [12])**.** *Let $\mathcal{P}(\mu, \sigma^2, L)$ be the set of all distribution with mean μ, variance σ^2 and support contained in the interval $[0, L]$. Then*

$$\max_{\mathbb{P} \in \mathcal{P}(\mu,\sigma^2,L)} \int_{[0,L]} \min\{\xi, x\} \, \mathrm{d}\mathbb{P}(x) = \begin{cases} \xi & \text{for } 0 \leq \xi \leq \mu - \dfrac{\sigma^2}{L-\mu} \\ \dfrac{\mu(L+\xi) - (\mu^2+\sigma^2)}{L} & \text{for } \mu - \dfrac{\sigma^2}{L-\mu} \leq \xi \leq \mu + \dfrac{\sigma^2}{\mu} \\ \mu & \text{for } \mu + \dfrac{\sigma^2}{\mu} \leq \xi \leq L \end{cases}. \tag{4.9}$$

For the proof of Theorem 4.1 presented below, the max-min result in Lemma 4.3 presents the worst-case scenario that serves as input for finding the robust thresholds. As will become clear, nature chooses as worst-case distributions the three-point distributions in (4.7), which is close to a two-point distribution. That is, as $\xi \uparrow \mu + \frac{\sigma^2}{\mu}$, the distribution in (4.7) converges in probability to the distribution

$$\mathbb{P}(x) = \begin{cases} \dfrac{\sigma^2}{\mu^2+\sigma^2} & \text{for } x = 0 \\ \dfrac{\mu^2}{\mu^2+\sigma^2} & \text{for } x = \mu + \frac{\sigma^2}{\mu} \end{cases}. \tag{4.10}$$

Note that this is the same distribution as in (3.6).

4.2 Proof of Theorem 4.1

In this section we give the proof of Theorem 4.1 by combining Lemma 4.3, Theorem 1.1, and the next result. We will work with the second form of $T(i)$ as given in (1.3), i.e.,

$$T(i) = \mu + T(i+1) - \max_{\mathbb{P}_{i+1} \in \mathcal{P}(\mu,\sigma^2,L)} \int_0^\infty \min\{T(i+1), x\} \, \mathrm{d}\mathbb{P}_{i+1}(x) \tag{4.11}$$

for $i = 0, \ldots, n-1$, with $T(n) = 0$. Note that we have $\mathbb{E}[X_i] = \mu$ for every $i = 1, \ldots, n$.

In the first step of the backwards recursion, we obtain $T(n-1) = \mu$, which follows from the first case of the solution as given in (4.9) by considering $t = 0$. In the next step of the recursion, we therefore start in the second case of (4.9) with $t = \mu$. Remember that the second case was given by the optimal value

$$p^*(t) = \frac{\mu(L+t) - (\mu^2 + \sigma^2)}{L}$$

for $\mu - \sigma^2/(L-\mu) \leq t \leq \mu + \sigma^2/\mu$.

From Lemma 3.1 we know that $\mu = T(n-1) \leq \ldots T(1) \leq T(0) \leq \mu + \sigma^2/\mu$. Since $\mu + \sigma^2/\mu$ is precisely the upper bound of the range of the second case of (4.9), this means that $T(i)$, for $i = 0, \ldots, n-2$, is determined by the recursion

$$T(i) = \mu + T(i+1) - \frac{\mu(L + T(i+1)) - (\mu^2 + \sigma^2)}{L} = \frac{\mu^2 + \sigma^2}{L} + \left(1 - \frac{\mu}{L}\right) T(i+1),$$

i.e., a recursion based on the solution to the second case of (4.9). Using Proposition 3.3 with $\alpha = (\mu^2 + \sigma^2)/L$, $\beta = 1 - \mu/L$ and $\gamma_0 = \mu$, then gives

$$T(i) = \frac{\mu^2 + \sigma^2}{L} \left(\sum_{k=0}^{n-2-i} \left(1 - \frac{\mu}{L}\right)^k \right) + \mu \left(1 - \frac{\mu}{L}\right)^{n-1-i}$$

for $i = 0, \ldots, n$. Using the identity $\sum_{k=0}^{j} x^j = (1 - x^{j+1})/(1-x)$, in combination with some calculus, then yields the identity as stated in Theorem 4.1. This completes the proof of Theorem 4.1.

As already illustrated in Section 4.1, the worst-case distribution in every step of the recursion is given by the three-point distribution in (4.7) with $\xi = T(i+1)$. In Figure 6 we have given an overview of how the probability mass of this worst-case three-point distribution changes over time. It can be seen that the mass on L approaches 0.

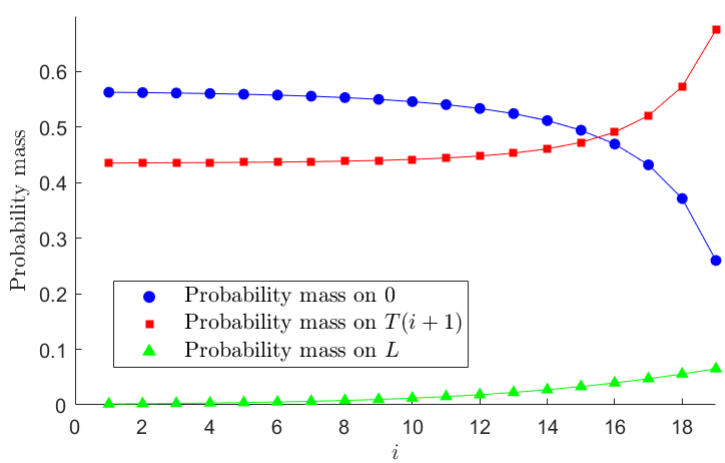

Figure 6: Sketch of probability mass on points $\{0, T(i+1), L\}$ for $i = 1, \ldots, n-1 = 19$, for worst-case three-point distribution (4.7) with $\xi = T(i+1)$. The chosen parameters are $\mu = 1$, $\sigma^2 = 1.3\mu$ and $L = 5\mu$.

4.3 An asymptotic result

As for the case of two-point distributions, we can also obtain the payoff of $\mu + \sigma^2/\mu$ approximately (as n grows large) by means of simply setting a threshold of

$$T = \mu + \frac{\sigma^2}{\mu} - \epsilon \tag{4.12}$$

at every step of the game, for any $\epsilon > 0$. That is, if one is only interested in obtaining the optimal payoff when n is large, and not so much about identifying the optimal strategy at every individual step of the game, then this can be done in a fairly simple manner. Theorem 4.4 below is a slightly weaker version of Theorem 3.4. We need to choose a slightly lower threshold than in Theorem 3.4, as nature can circumvent the reasoning in the proof of Theorem 3.4 by using three-point distributions.

Theorem 4.4. *Let $\epsilon > 0$. Consider the stopping rule τ that selects the first offer v_i for which $v_i \geq T = \mu + \sigma^2/\mu - \epsilon$. Then*

$$\lim_{n\to\infty} r^* \in \left[\mu + \frac{\sigma^2}{\mu} - \epsilon, \mu + \frac{\sigma^2}{\mu}\right]$$

if every offer is equipped with the ambiguity set $\mathcal{P}(\mu, \sigma^2, L)$.

Proof. The proof simply relies on showing that with high probability, as $n \to \infty$, there will be a value whose weight is at least T. We do this by showing that, for any $\mathbb{P} \in \mathcal{P}(\mu, \sigma^2, L)$, there is always a constant probability mass $q = q(\mu, \sigma^2, L, \epsilon) > 0$ assigned to the interval $[\mu + \sigma^2/\mu - \epsilon, L]$ (we emphasize that this is not true in case $\epsilon = 0$). This means that as n grows large, with high probability there will be at least one offer whose value is at least $\mu + \sigma^2/\mu - \epsilon$, and therefore the seller will always be able to obtain a payoff of at least $\mu + \sigma^2/\mu - \epsilon$ with high probability. We also know from Lemma 3.1 that nature can be make sure the seller never gets a robust expected payoff of more than $\mu + \sigma^2/\mu$, so this would then complete the claim.

It remains to show that for any $\mathbb{P} \in \mathcal{P}(\mu, \sigma^2, L)$ a constant probability mass $q = q(\mu, \sigma^2, L) > 0$ is assigned to the interval $[\mu + \sigma^2/\mu - \epsilon, L]$. This can be derived directly, e.g., from a result of De Schepper and Heijnen [14]. In particular, from [14] it follows that

$$\min_{\mathbb{P}\in\mathcal{P}(\mu,\sigma^2,L)} \mathbb{P}(X \geq \mu + \sigma^2/\mu - \epsilon) = \frac{\epsilon}{L(L - \mu - \sigma^2/\mu)} =: q(\mu, \sigma^2, L).$$

This completes the proof. □

We remark that the analysis in the proof of Theorem 4.4 heavily relies on the fact that μ, σ^2 and L are independent of n, otherwise the parameter q might also depend on n, i.e., $q = q(\mu(n), \sigma^2(n), L(n))$ which might result in the fact that $\lim_{n\to\infty} q(\mu(n), \sigma^2(n), L(n)) = 0$.

4.4 Proof of Theorem 4.2

In this section we switch attention to the ambiguity set $\mathcal{P}(\mu, d, L)$ with all probability distributions with mean μ, mean absolute deviation d, and support contained in the interval $[0, L]$. In order to prove Theorem 4.2, we rely on the following lemma, which can be seen as a mean-MAD-support analogue of the result of Cox [12]. A related result for mean-MAD ambiguity in the context of Chebyshev-like inequalities is given in [34].

Lemma 4.5. *Let $\mathcal{P}(\mu, d, L)$ be the set of all distribution with mean μ, MAD d and support contained in the interval $[0, L]$. Then*

$$\max_{\mathbb{P}\in\mathcal{P}(\mu,d,L)} \int_0^L \min\{\xi, x\}\, \mathrm{d}\mathbb{P}(x) = \begin{cases} \xi, & \xi \in [0, \xi_1], \\ \mu - \frac{d(L-\xi)}{2(L-\mu)}, & \xi \in [\xi_1, \mu], \\ \xi - \frac{d\xi}{2\mu}, & \xi \in [\mu, \xi_2], \\ \mu, & \xi \in [\xi_2, L], \end{cases} \tag{4.13}$$

with ξ_1 and ξ_2 given by

$$\xi_1 = \mu - \frac{d(L-\mu)}{2(L-\mu) - d}, \quad \xi_2 = \mu + \frac{d\mu}{2\mu - d}.$$

Proof. The dual of the primal moment bound problem in (4.13) is given by

$$\begin{aligned} \min_{\lambda_0,\lambda_1,\lambda_2} \quad & \lambda_0 + \mu\lambda_1 + d\lambda_2 \\ \text{s.t.} \quad & \lambda_0 + \lambda_1 x + \lambda_2|\mu - x| \geq \min\{x, \xi\} \quad \forall x \in [0, L]. \end{aligned}$$

We write $M(x) = \lambda_0 + \lambda_1 x + \lambda_2|\mu - x|$ and call $M(x)$ the majorant (as in majorizes $\min\{x, \xi\}$). The dual problem searches the tightest majorant $M(x) = \lambda_0 + \lambda_1 x + \lambda_2|\mu - x|$ that minimizes the objective value. In order to establish (4.13), we will construct for all regimes in (4.13) a primal-dual pair of feasible solutions with the same objective value, after which strong duality implies optimality.

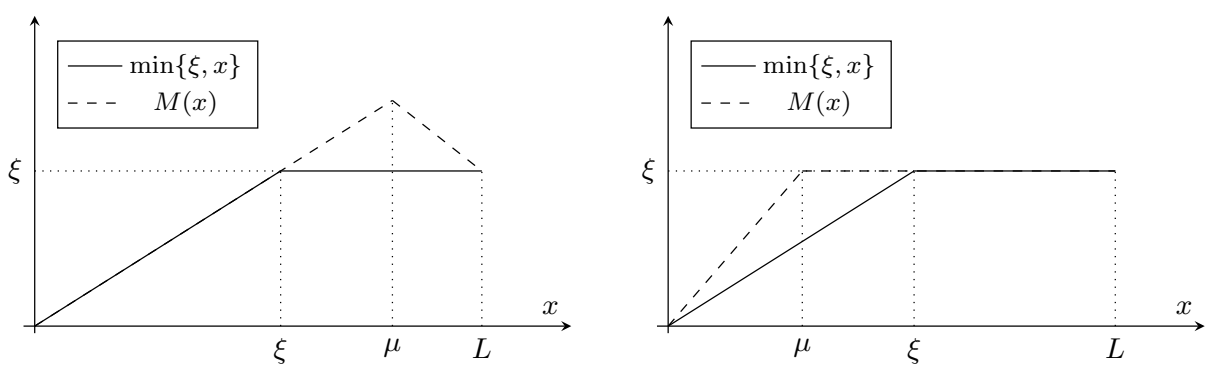

Figure 7: Majorants $M(x)$ for the function $\min(\xi, x)$ for $\mu > \xi$ (left) and $\mu < \xi$ (right).

The function $h(x) = \min(\xi, x)$ is piecewise concave, and candidate tightest majorants $M(x)$ are piecewise linear with a possible kink at $x = \mu$.

First, assume $\mu \geq \xi$ and consider the majorant in Figure 7 with

$$\lambda_0 = -\frac{\mu\xi - \mu L}{2(L-\mu)}, \quad \lambda_1 = -\frac{2\mu - L - \xi}{2(L-\mu)}, \quad \lambda_2 = -\frac{L-\xi}{2(L-\mu)} \tag{4.14}$$

and dual objective

$$\mu - \frac{d(L-\xi)}{2(L-\mu)}. \tag{4.15}$$

For the primal problem, we construct a distribution $\mathbb{P}$ on $\{0, \xi, L\}$ with probabilities

$$\mathbb{P}(0) = 1 - \frac{\mu}{\xi} + \frac{d(L-\xi)}{2\xi(L-\mu)}, \quad \mathbb{P}(\xi) = \frac{\mu}{\xi} - \frac{dL}{2\xi(L-\mu)}, \quad \mathbb{P}(L) = \frac{d}{2(L-\mu)}, \mu/\sigma \tag{4.16}$$

which is only a distribution when $\xi \geq \xi_1$ with

$$\xi_1 = \mu - \frac{dL}{2(L-\mu)}. \tag{4.17}$$

and when

$$d \leq \frac{2\mu(L-\mu)}{L}. \tag{4.18}$$

The latter condition is always satisfied in order to ensure that $\mathcal{P}(\mu, d, L)$ is nonempty. The three-point distribution in (4.16) has as objective value the expression in (4.15), which thus is the optimum.

Secondly, let $\xi \leq \xi_1$ and consider the majorant with $\lambda_0 = \xi, \lambda_1 = 0, \lambda_2 = 0$ and hence the dual solution with objective value ξ. For the primal problem, consider *any* distribution supported on $\{\xi, \mu, L\}$. All such distributions give ξ as primal objective. It is not hard to see that such a distribution always exists.

Thirdly, assume $\mu \leq \xi$ and consider the majorant with $M(0) = 0$, $M(\mu) = M(L) = \xi$, see Figure 7, which results in

$$\lambda_0 = \frac{\xi}{2} \quad \lambda_1 = \frac{\xi}{2\mu}, \quad \lambda_2 = -\frac{\xi}{2\mu},$$

and dual objective value

$$\lambda_0 + \lambda_1\mu + \lambda_2 d = \xi - \frac{d\xi}{2\mu}. \tag{4.19}$$

Consider here the two-point distribution $\mathbb{P}$ with support $\{0, \xi_2\}$ and

$$\mathbb{P}(0) = \frac{d}{2\mu}, \quad \mathbb{P}(\xi_2) = 1 - \frac{d}{2\mu}$$

For any $\mu \leq \xi \leq \xi_2$ the above distribution yields a primal objective of $\xi(1 - d/2\mu)$ which equal the dual objective in (4.19). This leaves the final case $\xi > \xi_2$. Consider the majorant with $\lambda_1 = 1$ and hence dual objective μ. One may consider any primal distribution $\mathbb{P}$ supported on $\{0, \mu, \xi\}$ (of which there always exists at least one). For such a distribution the primal objective is precisely the expectation of $\mathbb{P}$, which is μ. This concludes the proof. □

We continue with the proof of Theorem 4.2.

Proof of Theorem 4.2. We roughly follow the proof of Theorem 4.1. We have $T(n) = 0$ as always. Note that if nature always plays the two-point distribution

$$\mathbb{P}(x) = \begin{cases} \dfrac{d}{2\mu} & \text{for } x = 0 \\ 1 - \dfrac{d}{2\mu} & \text{for } x = \mu + \dfrac{d\mu}{2\mu - d} \end{cases} \tag{4.20}$$

then the seller can never obtain a payoff of more than $\mu + d\mu/(2\mu - d)$. This implies that $\mu \leq T(n-1) \leq T(n-2) \leq \cdots \leq T(1) \leq T(0) \leq \mu + d\mu/(2\mu - d)$. Looking at Lemma 4.5, this means we are always in the third regime for $i = n-1, \ldots, 0$.

We then have

$$\begin{aligned} T(i) &= \mu + T(i+1) - \max_{\mathbb{P}_{i+1} \in \mathcal{P}(\mu,\sigma^2,L)} \int_0^\infty \min\{T(i+1), x\} \, \mathrm{d}\mathbb{P}_{i+1}(x) \\ &= \mu + T(i+1) - \left[T(i) - \frac{dT(i)}{2\mu} \right] = \mu + \frac{d}{2\mu} T(i). \end{aligned}$$

Invoking Proposition 3.3 with $\alpha = \gamma_0 = \mu$ and $\beta = \frac{d}{2\mu}$ then gives

$$T(i) = \frac{2\mu^2}{2\mu - d} - \left[\frac{2\mu^2}{2\mu - d} - \mu \right] \left(\frac{d}{2\mu} \right)^{n-1-i},$$

which converges to $2\mu^2/(2\mu - d)$ as $n \to \infty$. □

References

[1] Amine Allouah, Achraf Bahamou, and Omar Besbes. Pricing with samples. *Operations Research*, 2022.

[2] Amine Allouah and Omar Besbes. Prior-independent optimal auctions. *Management Science*, 66(10):4417–4432, 2020.

[3] Jerry Anunrojwong, Santiago Balseiro, and Omar Besbes. On the robustness of second-price auctions in prior-independent mechanism design. *arXiv preprint arXiv:2204.10478*, 2022.

[4] Pablo Azar, Silvio Micali, Constantinos Daskalakis, and Seth Matthew Weinberg. Optimal and efficient parametric auctions. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 596–604. SIAM, 2013.

[5] Pablo D. Azar, Robert Kleinberg, and S. Matthew Weinberg. Prophet inequalities with limited information. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1358–1377, 2014.

[6] Pablo Daniel Azar and Silvio Micali. Parametric digital auctions. In *Proceedings of the Fourth Conference on Innovations in Theoretical Computer Science*, pages 231–232, 2013.

[7] Dirk Bergemann and Karl H Schlag. Pricing without priors. *Journal of the European Economic Association*, 6(2-3):560–569, 2008.

[8] Dirk Bergemann and Karl H. Schlag. Robust monopoly pricing. *Journal of Economic Theory*, 146(6):2527–2543, 2011.

[9] Vinicius Carrasco, Vitor Farinha Luz, Nenad Kos, Matthias Messner, Paulo Monteiro, and Humberto Moreira. Optimal selling mechanisms under moment conditions. *Journal of Economic Theory*, 177:245–279, 2018.

[10] Ethan Che. Robust reserve pricing in auctions under mean constraints. *Available at SSRN 3488222*, 2019.

[11] Jose Correa, Paul Dütting, Felix Fischer, and Kevin Schewior. Prophet inequalities for independent and identically distributed random variables from an unknown distribution. *Mathematics of Operations Research (forthcoming)*, 2021.

[12] S.H. Cox. Bounds on expected values of insurance payments and option prices. *Transactions of the Society of Actuaries*, 43:231–260, 1991.

[13] Constantinos Daskalakis and George Pierrakos. Simple, optimal and efficient auctions. In *International Workshop on Internet and Network Economics*, pages 109–121. Springer, 2011.

[14] Ann De Schepper and Bart Heijnen. General restrictions on tail probabilities. *Journal of Computational and Applied Mathematics*, 64(1-2):177–188, 1995.

[15] Paul Dutting, Michal Feldman, Thomas Kesselheim, and Brendan Lucier. Prophet inequalities made easy: Stochastic optimization by pricing nonstochastic inputs. *SIAM Journal on Computing*, 49(3):540–582, 2020.

[16] Paul Dütting and Thomas Kesselheim. Posted pricing and prophet inequalities with inaccurate priors. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, pages 111–129, 2019.

[17] Laurent El Ghaoui and Arnab Nilim. Robust solutions to Markov decision problems with uncertain transition matrices. *Operations Research*, 53(5):780–798, 2005.

[18] Tomer Ezra, Michal Feldman, Nick Gravin, and Zhihao Gavin Tang. Online stochastic max-weight matching: Prophet inequality for vertex and edge arrival models. In *Proceedings of the 21st ACM Conference on Economics and Computation*, pages 769–787, 2020.

[19] Thomas S. Ferguson. Optimal stopping and applications, 2006.

[20] Yiannis Giannakopoulos, Diogo Pocas, and Alexandros Tsigonias-Dimitriadis. Robust revenue maximization under minimal statistical information. In *International Conference on Web and Internet Economics*, pages 177–190. Springer, 2020.

[21] Mohammad Taghi Hajiaghayi, Robert Kleinberg, and Tuomas Sandholm. Automated online mechanism design and prophet inequalities. In *AAAI*, volume 7, pages 58–65, 2007.

[22] Wei He and Jiangtao Li. Correlation-robust auction design. *Journal of Economic Theory (forthcoming)*, page 105403, 2022.

[23] Nicole Immorlica, Sahil Singla, and Bo Waggoner. Prophet inequalities with linear correlations and augmentations. In *Proceedings of the 21st ACM Conference on Economics and Computation*, pages 159–185, 2020.

[24] K. Isii. On sharpness of Tchebycheff-type inequalities. *Annals of the Institute of Statistical Mathematics*, 14(1):185–197, 1962.

[25] Garud N. Iyengar. Robust dynamic programming. *Mathematics of Operations Research*, 30(2):257–280, 2005.

[26] Robert Kleinberg and Seth Matthew Weinberg. Matroid prophet inequalities. In *Proceedings of the Forty-Fourth Annual ACM Symposium on Theory of Computing*, pages 123–136, 2012.

[27] Cağıl Kocyiğit, Garud N. Iyengar, Daniel Kuhn, and Wolfram Wiesemann. Distributionally robust mechanism design. *Management Science*, 66(1):159–189, 2020.

[28] Ulrich Krengel and Louis Sucheston. Semiamarts and finite values. *Bulletin of the American Mathematical Society*, 83(4):745 – 747, 1977.

[29] Brendan Lucier. An economic view of prophet inequalities. *SIGecom Exchanges*, 16(1):24–47, sep 2017.

[30] Paul Milgrom. *Interdependence of Types and Values*, pages 157–207. Churchill Lectures in Economics. Cambridge University Press, 2004.

[31] Ioana Popescu. A semidefinite programming approach to optimal-moment bounds for convex classes of distributions. *Mathematics of Operations Research*, 30(3):632–657, 2005.

[32] Frank Riedel. Optimal stopping with multiple priors. *Econometrica*, 77(3):857–908, 2009.

[33] Werner Wolfgang Rogosinski. Moments of non-negative mass. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 245(1240):1–27, 1958.

[34] Ernst Roos, Ruud Brekelmans, Wouter van Eekelen, Dick den Hertog, and Johan SH van Leeuwaarden. Tight tail probability bounds for distribution-free decision making. *European Journal of Operational Research*, 2021.

[35] Aviad Rubinstein, Jack Z. Wang, and Seth Matthew Weinberg. Optimal single-choice prophet inequalities from samples. In *11th Innovations in Theoretical Computer Science Conference*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.

[36] Ester Samuel-Cahn. Comparison of threshold stop rules and maximum for independent non-negative random variables. *The Annals of Probability*, pages 1213–1216, 1984.

[37] Herbert Scarf. A min-max solution of an inventory problem. *Studies in the mathematical theory of inventory and production*, 1958.

[38] Karl H. Schlag and Andriy Zapechelnyuk. Value of information when searching for a secretary. Technical report, Mimeo, 2016.

[39] Karl H. Schlag and Andriy Zapechelnyuk. Robust sequential search. *Theoretical Economics*, 16(4):1431–1470, 2021.

[40] Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczyński. *Lectures on Stochastic Programming: Modeling and Theory*. SIAM, 2009.

[41] Alex Suzdaltsev. Distributionally robust pricing in independent private value auctions. *arXiv preprint arXiv:2008.01618*, 2020.

[42] Robert Wilson. *Game-theoretic analyses of trading processes*, pages 33–70. Econometric Society Monographs. Cambridge University Press, 1987.

[43] Alexander Wolitzky. Mechanism design with maxmin agents: Theory and an application to bilateral trade. *Theoretical Economics*, 11(3):971–1004, 2016.

A Regarding correlations

In this section we argue that if correlations are allowed between the random variables $X_1, \ldots, X_n$, then the maximin problem in (1.2) is typically not very interesting. For example, suppose that $I_i = I_j$, so that $\mathcal{P} = \mathcal{P}(I_i) = \mathcal{P}(I_j)$, for all $i, j = 1, \ldots, n$ (as we do for the results in Section 3).

If nature chooses a common distribution $\mathbb{P}$ from $\mathcal{P}$ for all offers, whose mean μ is as small as possible, and fully correlates the random variables $X_1, \ldots, X_n$ (meaning that all variables always

have their realized value in common), then it is clear that in expectation the seller can obtain a payoff of at most μ regardless of its chosen stopping rule. On the other hand, always simply selecting the first offer will definitely guarantee a payoff of μ independent of which (possibly correlated) distributions nature chooses. This then yields the solution to (1.2) with $r^* = \mu$.

We remark that more might be possible in case one, in some way, restricts the amount of correlation that is allowed between the random variables. In the context of prophet inequalities, we refer the interested reader to the work of Immorlica et al. [23], and references therein, for possible models along this line.

B Non-emptyness conditions for mean-dispersion ambiguity sets

The following lemma provides a formal proof of the claims made in Section 1.3 regarding the non-emptiness of the ambiguity sets $\mathcal{P}(\mu, \sigma^2, L)$ and $\mathcal{P}(\mu, d, L)$.

Lemma B.1. *Let μ, σ, d, $L \geq 0$. The ambiguity set $\mathcal{P}(\mu, \sigma^2, L)$, consisting of all distributions with mean μ, variance σ^2 and support contained in $[0, L]$, is nonempty if and only if*

$$\sigma^2 \leq \mu(L - \mu).$$

The set $\mathcal{P}(\mu, d, L)$, consisting of all distributions with mean μ, mean absolute deviation d and support contained in $[0, L]$, is non-empty if and only if

$$d \leq \frac{2\mu(L - \mu)}{L}.$$

Proof. We first consider the mean-variance-support ambiguity set $\mathcal{P}(\mu, \sigma^2, L)$. Consider the two-point distribution

$$\mathbb{P}(x) = \begin{cases} \dfrac{\sigma^2}{\mu^2 + \sigma^2} & \text{for } x = 0 \\ \dfrac{\mu^2}{\mu^2 + \sigma^2} & \text{for } x = \mu + \dfrac{\sigma^2}{\mu} \end{cases}. \tag{B.1}$$

First suppose that $\sigma^2 \leq \mu(L - \mu)$. This condition is equivalent to $\mu + \sigma^2/\mu \leq L$, which means that $\mathbb{P} \in \mathcal{P}(\mu, \sigma^2, L)$, and, hence, $\mathcal{P}(\mu, \sigma^2, L)$ is non-empty. For the converse we will use a standard result of which we next include a proof for completeness. Suppose that $\mathcal{P}(\mu, \sigma^2, L)$ is non-empty and consider any random variable $X \sim \mathbb{P}'$ where $\mathbb{P}' \in \mathcal{P}(\mu, \sigma^2, L)$. If $L = 1$, we have

$$\sigma^2 = \mathbb{E}[(X - \mu)^2] = \mathbb{E}[X^2] - \mu^2 \leq \mathbb{E}[X] - \mu^2 = \mu(1 - \mu) = \mu(L - \mu)$$

where in the inequality we use the fact that X is supported on $[0, 1]$. For general values of L, the claim follows by considering the rescaled random variable $Y = X/L$.

We continue with the mean-MAD-support ambiguity set $\mathcal{P}(\mu, d, L)$. Consider the two-point distribution

$$\mathbb{P}(x) = \begin{cases} \dfrac{d}{2\mu} & \text{for } x = 0 \\ 1 - \dfrac{d}{2\mu} & \text{for } x = \dfrac{2\mu^2}{2\mu - d} \end{cases}. \tag{B.2}$$

First suppose that $d \leq 2\mu(L-\mu)/L$. This condition is equivalent to $2\mu^2/(2\mu - d) \leq L$ and so $\mathbb{P} \in \mathcal{P}(\mu, d, L)$. Conversely, suppose that $\mathcal{P}(\mu, d, L)$ is non-empty and let $\mathbb{P}' \in \mathcal{P}(\mu, d, L)$. If we put all the probability mass in $[0, \mu)$ on its conditional mean in that interval, and, similarly, all the probability mass in $[\mu, L]$ on its conditional mean in that interval, then the resulting two-point distribution $\mathbb{P}''$, supported on, say, $\{x, y\}$ with $x \leq y$, still has mean μ and MAD d (because of the linear nature of the MAD), and is therefore contained in $\mathcal{P}(\mu, d, L)$. It is not hard to see that the distribution $\mathbb{P}$ in (B.2) is the two-point distribution with mean μ and MAD d for which both points in the support are chosen as small as possible (among all choices for two-point distributions with mean μ and MAD d). This means that

$$\frac{2\mu^2}{2\mu - d} \leq y \leq L$$

where the second inequality holds as $\mathbb{P}'' \in \mathcal{P}(\mu, d, L)$. Again, the resulting inequality is equivalent to $d \leq 2\mu(L-\mu)/L$, which completes the proof. $\square$