Answers to theory questions LB

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1 Pros and cons of LB

1.1 Pros

- Explicit update rule, no system of equations to solve.
  - Done by doing a trick. Introduce a new distribution function and see that the sum over states give the same global quantities.
- Easy to parallelize.
  - Each cell only depend locally on the neighbouring cells in the streaming step and only on itself in the collision.
- Simulating mesoscale
  - Unlike NS, LB looks at distributions of "bunches" of particles. May catch behaviour too fine for NS to detect.
- Can handle multiphase flows relatively easy.
  - Usually done by modifying the collision operator.

1.2 Cons

- Explicit update rule
  - Need small timesteps.
  - Finite propagation speed of information.
- Only using a square grid can be limiting.

2 Convective-/viscous term

Consider the boltzmann equation,

$$\frac{\delta f}{\delta t} + \vec{v} \cdot \nabla f = -\Delta(f - f^{eq}), \tag{1}$$

and the BGK approximation

$$\Delta(f - f^{eq}) = -\frac{1}{\tau}(f - f^{eq}) \tag{2}$$

TODO: Find right phrasing/argument

Collision term: viscous/diffusion term
$\vec{v} \cdot \nabla f$: convective term

3 Linear LBM to non-linear NSE

In the LBM we have a linear update rule,

$$f_i(\vec{x} + \vec{c}_i dt, t + dt) = f_i(\vec{x}, t) - \frac{1}{\tau}(f_i(\vec{x}, t) - f_i^{eq}), \tag{3}$$

which gives us the updated distribution of particles with velocity $\vec{c}_i$ at position $\vec{x} + \vec{c}_i dt$ and time $t + dt$. $i = 0 : Q - 1$ where Q is the size of our velocity space.

3.1 Heuristic argument

The NSE solves a global system on a macroscale. To get the macroscopic velocities from the LBM velocities we need to integrate the velocity density distribution over the velocity set. We hence introduce nonlinearity.

4 LBM memory usage

TODO:

5 Conservation of mass and momentum in the collision rule

Collision rule:

$$f_i^*(\vec{x}, t) = f_i(\vec{x}, t) - \frac{1}{\tau}(f_i(\vec{x}, t) - f_i^{eq}(\vec{x}, t)) \tag{4}$$

Approximation to the equilibrium state which holds when the distribution is close to equilibrium:

$$f_i^{eq}(\vec{x}, t) \approx w_i \rho \left[1 + \frac{\vec{c}_i \cdot \vec{u}}{c_s^2} + \frac{(\vec{c}_i \cdot \vec{u})^2}{c_s^4} - \frac{\vec{u} \cdot \vec{u}}{c_s^2}\right]. \tag{5}$$

Properties on w_i (there are more, but only these are needed here):

$$\sum_{i=0}^{Q-1} w_i c_i^\alpha c_i^\beta = c_s^2 \delta_{\alpha\beta}; \quad \alpha = x, y, z; \quad \beta = x, y, z. \tag{6}$$

$$\sum_{i=0}^{Q-1} w_i = 1, \tag{7}$$

$$\sum_{i=0}^{Q-1} w_i c_i^\alpha = 0, \quad \alpha = x, y, z, \tag{8}$$

5.1 Conservation of mass

Before the collision step we have the density

$$\rho = \sum_{i=0}^{Q-1} f_i(\vec{x},\, t). \tag{9}$$

After the collision the density is

$$\rho^* = \sum_{i=0}^{Q-1} f_i^*(\vec{x},\, t). \tag{10}$$

Lets first look at the sum over the equilibrium distribution:

$$\sum_{i=0}^{Q-1} f_i^{eq}(\vec{x},\, t) = \sum_{i=0}^{Q-1} \left(w_i \rho \left[1 + \frac{\vec{c}_i \cdot \vec{u}}{c_s^2} + \frac{(\vec{c}_i \cdot \vec{u})^2}{2c_s^4} - \frac{\vec{u} \cdot \vec{u}}{2c_s^2} \right] \right). \tag{11}$$

By virtue of eq. 6, 7 and 8 eq. 11 becomes after some calculation

$$\sum_{i=0}^{Q-1} f_i^{eq}(\vec{x},\, t) = \rho. \tag{12}$$

Hence the after collison density becomes

$$\rho^* = \rho, \tag{13}$$

and we have mass conservation.

5.2 conservation of momentum

Momentum before collision:

$$\rho\vec{u} = \sum_{i=0}^{Q-1} f_i(\vec{x},\, t)\vec{c}_i. \tag{14}$$

After the collision the momentum is

$$\rho^*\vec{u}^* = \sum_{i=0}^{Q-1} f_i^*(\vec{x},\, t)\vec{c}_i. \tag{15}$$

Lets first look at the sum over the equilibrium distribution:

$$\sum_{i=0}^{Q-1} w_i \vec{c}_i f_i^{eq}(\vec{x},\, t) = \sum_{i=0}^{Q-1} \left(\vec{c}_i w_i \rho \left[1 + \frac{\vec{c}_i \cdot \vec{u}}{c_s^2} + \frac{(\vec{c}_i \cdot \vec{u})^2}{2c_s^4} - \frac{\vec{u} \cdot \vec{u}}{2c_s^2} \right] \right). \tag{16}$$

By using eq. 8 the first and last term inside the square bracket becomes zero. The second term inside the square bracket becomes

$$\sum_{i=0}^{Q-1} \left(\frac{w_i \vec{c}_i (\vec{c}_i \cdot \vec{u}_i)}{2c_s^2} \right) = \vec{u}. \tag{17}$$

By expanding the third term and using eq. 6 we get

$$\sum_{i=0}^{Q-1}\left(w_i\vec{c}_i\frac{(\vec{c}_i\cdot\vec{u}_i)^2}{2c_s^4}\right)=\sum_{i=0}^{Q-1}\left(w_i\vec{c}_i\frac{\vec{c}_i\cdot\vec{c}_i+2\vec{c}_i\cdot\vec{u}+\vec{u}\cdot\vec{u}}{2c_s^4}\right) \tag{18}$$

$$=\sum_{i=0}^{Q-1}\frac{w_i\vec{c}_i(\vec{c}_i\cdot\vec{c}_i)}{2c_s^4}+\frac{\vec{u}}{c_s^2}. \tag{19}$$

TODO: Finish computation

6 Conservation of mass and momentum in the streaming rule

TODO: Consider momentum

6.1 Heuristic argument

6.1.1 Single cell

We have seen that the collision rule conserves mass in a cell. Assume now that the collision step has happened and we look at at the distributions streaming into a cell. The distribution in each direction streaming in is only dependant on the state after the previous collision in the neighbouring cells. That means that all the Q distributions being streamed into our cell are independent of each other, and hence the density in the current cell may be different than before the streaming.

6.1.2 Over neighbouring cells

The same argument as before may be applied to a collection of cells which are not the entire domain. If we have a subset of cells there is always some that stream out of the subset and get distributions streamed in from the outside. Since the distributions streamed in are independent of each other the mass in this subset of cells is not necessarily conserved between timesteps.