Answers to theory questions LB

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1 Pros and cons of LB

1.1 Pros

- Explicit update rule, no system of equations to solve.
  - Done by doing a trick. Introduce a new distribution function and see that the sum over states give the same global quantities.
- Easy to parallelize.
  - Each cell only depend locally on the neighbouring cells in the streaming step and only on itself in the collision.
- Simulating mesoscale
  - Unlike NS, LB looks at distributions of "bunches" of particles. May catch behaviour too fine for NS to detect.
- Can handle multiphase flows relatively easy.
  - Usually done by modifying the collision operator.

1.2 Cons

- Explicit update rule
  - Need small timesteps.
  - Finite propagation speed of information.
- Only using a square grid can be limiting.

2 Convective-/viscous term

Consider the boltzmann equation,

$$\frac{\delta f}{\delta t} + \vec{v} \cdot \nabla f = -\Delta(f - f^{eq}), \tag{1}$$

and the BGK approximation

$$\Delta(f - f^{eq}) = -\frac{1}{\tau}(f - f^{eq}) \tag{2}$$

3 Linear LBM to non-linear NSE

In the LBM we have a linear update rule,

$$f_i(\vec{x}+\vec{c}_i dt, t+dt) = f_i(\vec{x}, t) - \frac{1}{\tau}(f_i(\vec{x}, t) - f_i^{eq}), \tag{3}$$

which gives us the updated distribution of particles with velocity $\vec{c}_i$ at position $\vec{x}+\vec{c}_i dt$ and time $t+dt$. $i = 0 : Q-1$ where Q is the size of our velocity space.

3.1 Heuristic argument

The NSE solves a global system on a macroscale. To get the macroscopic velocities from the LBM velocities we need to integrate the velocity density distribution over the velocity set. We hence introduce nonlinearity.

4 LBM memory usage

5 Conservation of mass and momentum in the collision rule

Collision rule:

$$f_i^*(\vec{x},\, t) = f_i(\vec{x},\, t) - \frac{1}{\tau}(f_i(\vec{x},\, t) - f_i^{eq}(\vec{x},\, t)) \tag{4}$$

Approximation to the equilibrium state which holds when the distribution is close to equilibrium:

$$f_i^{eq}(\vec{x},\, t) \approx w_i \rho \left[1 + \frac{\vec{c}_i \cdot \vec{u}}{c_s^2} + \frac{(\vec{c}_i \cdot \vec{u})^2}{c_s^4} - \frac{\vec{u} \cdot \vec{u}}{c_s^2}\right]. \tag{5}$$

Properties on w_i (there are more, but only these are needed here):

$$\sum_{i=0}^{Q-1} w_i c_i^\alpha c_i^\beta = c_s^2 \delta_{\alpha\beta}; \quad \alpha = x,\, y,\, z; \quad \beta = x,\, y,\, z. \tag{6}$$

$$\sum_{i=0}^{Q-1} w_i = 1, \tag{7}$$

$$\sum_{i=0}^{Q-1} w_i c_i^\alpha = 0, \quad \alpha = x,\, y,\, z, \tag{8}$$

5.1 Conservation of mass

Before the collision step we have the density

$$\rho = \sum_{i=0}^{Q-1} f_i(\vec{x}, t). \tag{9}$$

After the collision the density is

$$\rho^* = \sum_{i=0}^{Q-1} f_i^*(\vec{x}, t). \tag{10}$$

Lets first look at the sum over the equilibrium distribution:

$$\sum_{i=0}^{Q-1} f_i^{eq}(\vec{x}, t) = \sum_{i=0}^{Q-1} \left(w_i \rho \left[1 + \frac{\vec{c}_i \cdot \vec{u}}{c_s^2} + \frac{(\vec{c}_i \cdot \vec{u})^2}{2c_s^4} - \frac{\vec{u} \cdot \vec{u}}{2c_s^2} \right] \right). \tag{11}$$

By virtue of eq. 6, 7 and 8 eq. 11 becomes after some calculation

$$\sum_{i=0}^{Q-1} f_i^{eq}(\vec{x}, t) = \rho. \tag{12}$$

Hence the after collison density becomes

$$\rho^* = \rho, \tag{13}$$

and we have mass conservation.

5.2 conservation of momentum

Momentum before collision:

$$\rho \vec{u} = \sum_{i=0}^{Q-1} f_i(\vec{x}, t) \vec{c}_i. \tag{14}$$

After the collision the momentum is

$$\rho^* \vec{u}^* = \sum_{i=0}^{Q-1} f_i^*(\vec{x}, t) \vec{c}_i. \tag{15}$$

Lets first look at the sum over the equilibrium distribution:

$$\sum_{i=0}^{Q-1} w_i \vec{c}_i f_i^{eq}(\vec{x}, t) = \sum_{i=0}^{Q-1} \left(\vec{c}_i w_i \rho \left[1 + \frac{\vec{c}_i \cdot \vec{u}}{c_s^2} + \frac{(\vec{c}_i \cdot \vec{u})^2}{2c_s^4} - \frac{\vec{u} \cdot \vec{u}}{2c_s^2} \right] \right). \tag{16}$$

By using eq. 8 the first and last term inside the square bracket becomes zero. The second term inside the square bracket becomes

$$\sum_{i=0}^{Q-1} \left(\frac{w_i \vec{c}_i (\vec{c}_i \cdot \vec{u}_i)}{2c_s^2} \right) = \vec{u}. \tag{17}$$

By expanding the third term and using eq. 6 we get

$$\sum_{i=0}^{Q-1}\left(w_i\vec{c}_i\frac{(\vec{c}_i\cdot\vec{u}_i)^2}{2c_s^4}\right)=\sum_{i=0}^{Q-1}\left(w_i\vec{c}_i\frac{\vec{c}_i\cdot\vec{c}_i+2\vec{c}_i\cdot\vec{u}+\vec{u}\cdot\vec{u}}{2c_s^4}\right) \tag{18}$$

$$=\sum_{i=0}^{Q-1}\frac{w_i\vec{c}_i(\vec{c}_i\cdot\vec{c}_i)}{2c_s^4}+\frac{\vec{u}}{c_s^2}. \tag{19}$$

6 Conservation of mass and momentum in the streaming rule

6.1 Heuristic argument

6.1.1 Single cell

We have seen that the collision rule conserves mass in a cell. Assume now that the collision step has happened and we look at at the distributions streaming into a cell. The distribution in each direction streaming in is only dependant on the state after the previous collision in the neighbouring cells. That means that all the Q distributions being streamed into our cell are independent of each other, and hence the density in the current cell may be different than before the streaming.

6.1.2 Over neighbouring cells