Rational Herds and Endogenous Fluctuations in Consumer Sentiment

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Abstract

This thesis models *endogenous* fluctuations in consumer sentiment and sentiment driven endogenous business cycles in the context of a general equilibrium model with rational herds and social learning in consumer durable goods sector about a hidden state of nature. I want to understand the theoretical conditions under which one gets endogenous and asymmetric cyles in this class of models. My results show evidence of nonlinear and asymmetric theoretical impulse response functions, which I compare with stylized facts about asymmetric causal effects of consumer sentiments on business cycles from local projections. The learning dynamics are very sensitive to the variances specified for signal distributions, specially the variance for observation noise in social learning and also to how difficult it is to distinguish the two unobservable states. For some values, convergence of public beliefs to the truth is very slow and one gets rational herds on the wrong action and for others, one gets endogenous cycles in public beliefs and investor mass, leading to fast convergence to truth. I do not aim to achieve precise quantitative goals with this model, but the results outline the ability of this model to generate endogenous cycles and provide a information and noisy learning based theory of asymmetric, endogenous reversals.

Keywords: Consumer Sentiment, Endogenous Business Cycles, Informational Cascades, Rational Herds, Bayesian Learning, Social Learning, Information Frictions in Macroeconomics, Resolution Method, Nonlinear Business Cycle Models, Asymmetric Business Cycles, Two Sector New Keynesian Models.
JEL classification:

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1. INTRODUCTION

There is a large and growing literature in macroeconomics which attributes business cycle fluctuations to expectations, especially in light of the Great Recession, which did not seem to be driven by extremely unfavorable fundamentals. Many economists now recognize an enlarged role for beliefs in the narrative of the business cycle (see for example Kozlowski et al. (2019), Gennaioli and Shleifer (2020)).

My focus is on *modeling consumer sentiments*, their interaction, joint determination and joint evolution with economic activity over the business cycle with a special focus on modeling *endogenous* cycles by understanding the *transition* from a regime with excessive output into recession with only one initial impulse. I want to understand the theoretical conditions in social learning models under which one can get endogenous, sharp and sudden reversals in consumer sentiment from a state of high optimism to high pessimism and associated endogenous reversals in business cycles. Are these conditions plausible and likely to be satisfied? Is the effect of consumer sentiment reversal temporary or sentiment produces persistent economic effects in this class of models? Are effects of consumer sentiment reversal on business cycle activity asymmetric across business cycle stages of boom and recession in the model as it is the case in stylized facts, which I document from data? Endogenous transition into recession can also be produced by real forces orthogonal to consumer sentiment, some of which are discussed below. Is a consumer sentiment driven business cycle reversal stronger than a reversal driven by non-sentimental forces in social learning model of the kind I work with?

In order to answer these theoretical questions, I build a social learning model in general equilibrium which features endogenous business cycle fluctuations, driven by endogenous sentiment fluctuations, which are disciplined by the rationality of Bayesian learning. This is contrast with some of the major theoretical literature on *sentiments*. For instance, Angeletos and La'O (2013) and Angeletos et al. (2018) model sentiments as exogenous shocks to self-fulfilling higher order beliefs in island economies and the same is true for the classic sunspots literature, featuring multiple equilibria, steady state indeterminacy and dynamic indeterminacy Farmer (2016), Benhabib and Farmer (1999).

Given any origin of excessive output, it is obvious at one extreme level that even in the absence of any explicit mechanism for reversal, the boundedness of sets from which inputs are chosen and hence the boundedness of a well behaved aggregate production function guarantees the existence of a time T^{Max} and output Y^{Max} such that there is a upper cap on expansion and its duration. However, this places only weak restrictions on the exact duration of over-expansion, onset, violence and duration of the subsequent recession as well as the cause/mechanism of business cycle reversal. At a more nontrivial level, there are mechanisms of mean reversion and slowdown in the real economy if it gets strained beyond a certain point in absence of change in fundamentals through variable capacity utilization and its convex adjustment costs for example. There will also be supply side, congestion effects in input-output networks and strains on the monetary flows within the economy in absence of continued monetary expansion. On the other hand, if the output over-expansion is fueled by excessive credit growth, as suggested by recent historical evidence Schularick and Taylor (2012), Mian et al. (2017)[1], then eventual recognition of tail risks and overheating in financial markets paves the way for a *Minsky Moment* Minsky (1977), Bordalo et al. (2018)[2], engendering a recession endogenously.

However, it is not clear whether these *real* constraints, orthogonal to beliefs, despite their obvious existence are *binding* in causing business cycle reversals. It could be that transition into recession is largely independent of consumer sentiments, which then overreact to initial recession and contribute toward its deepening. It may also be that anticipated reversal based on consumer information creates a switch in consumer confidence toward pessimism, which is the main autonomous and causal impetus behind transitions. Recent evidence suggests that sentiments are important causal drivers of business cycles in general (see for example, Benhabib and Spiegel (2018); Gillitzer and Prasad (2018), Lagerborg et al. (2018)) and perhaps even transitions in particular, even though the evidence is not clear on the latter question. Ahmed and Cassou (2016) have studied asymmetric effects of consumer sentiment across business cycles using local projections but they have no convincing identification scheme, making a causal interpretation unjustified.[3]

Even if the powerful role of consumer sentiment as a driver of business cycles in general and reversals in particular is accepted, it begs the question, "Why do the sentiments shift?" Of course, they are responding to some information, even if the information is noisy and orthogonal to fundamentals. This is because analysis of consumer expectations data does not suggest that beliefs are characterized by random behavior or "sunspot" like

[1] Mian et al. (2017) provide global and historical evidence that rising household debt predicts recessions and intensity of recessions is related to prior debt expansion in household sector.

[2] These researchers build a microfounded and behavioral model of expectations, which they call diagnostic expectations and credit cycles in which beliefs overreact to incoming news because of the representativeness heuristic. This creates excessive optimism when credit spreads are low, during booms and also exaggeration of subsequent reversal when good news inflow slows down, leading to endogenous cycles in absence of change in fundamentals.

[3] They find that positive shocks to consumer sentiment shocks during recession regimes have no significant effects, which implies that recoveries cannot be achieved, solely through better confidence.

fluctuations. There is a structure, pattern, regularity and relative coherence in the manner in which consumer expectations evolve over the business cycle, especially when one examines cross-sectional heterogeneity. Certain demographic groups have consistently more pessimistic and inaccurate expectations such as women, ethnic minorities, lower socio-economic groups and young people Madeira and Zafar (2015); Curtin (2019). There is also an average pessimism bias across all demographic groups because of asymmetric recall of negative news in the elicited expectations, relative to estimates of rational expectations Curtin (2019), Bhandari et al. (2019). The volatility of consumer sentiment over the business cycle also varies across groups with higher socio-economic groups showing more volatility Curtin (2019). Meanwhile, the time series co-movements across demographic groups is very high. Moreover, consumer sentiment indices regularly predict recessions, though not by a long horizon. These stylized facts are further discussed in stylized facts section 1. In fact, the forward looking, informative and leading indicator nature of consumer sentiment data is precisely the reason why the University of Michigan survey and many other such surveys have become globally popular among central banks and economic practitioners. This evidence suggests that while "autonomous" components of consumer sentiment such as those driven by instruments are needed for econometric identification of plausibly exogenous variation, there is also an important *systematic, endogenous* component to these sentiments which is responding to, predicting and causing significant developments in the real economy. The sentiments can influence search intensity in labor markets, consumer durable goods purchases and so on. In fact, there is evidence that household expectations are predictive of economic and financial behavior Armantier et al. (2015); Armona et al. (2018) and high volatility in consumer durable goods purchases over the business cycle has been often attributed in the literature to consumer sentiment Katona et al. (1960); Mishkin et al. (1978). This is why I use a social learning framework with endogenous beliefs to model sentiments.

Meanwhile, the motivation for examining asymmetric effects of consumer sentiment over the business cycle, which is a distinct question from modeling endogenous beliefs and endogenous business cycle reversals[4] is driven by policy implications. For instance, if excessive consumer optimism is a more powerful causal driver of business cycle volatility in the expansion regime, then the focus of policymakers should be on preventing excessive spending sprees in the durable goods and luxury goods sector by preventing sentiments from becoming too optimistic so that an output over expansion can be avoided. This will be the case if conditional on a given path of excessive output, the causal mech-

[4]This is because even if I can model endogenous business cycle reversals driven by consumer sentiment, the cycles may not be asymmetric.

anisms pinning down business cycle reversal are largely independent of consumer sentiments. On the other hand, it could also be that due to loss aversion in preferences, salience of threatening events Tversky and Kahneman (1979), disproportionate media coverage of negative news Nimark (2014) and high elasticity of consumer durable goods spending, the transition point and depth of recessions is largely driven by consumer pessimism. In this case, managing sentiments plays a more important role in smoothing downturns and enabling gradual adjustment back to balanced growth path. It could also be that sentiments are uniformly important over the business cycle stages. These are important questions to understand from a policy perspective in order to optimally design the policy for management of household expectations and communication with households over the business cycle. Thus, I provide new stylized facts regarding causal effects of consumer sentiments across business cycle regimes and also provide evidence of asymmetric effects of consumer sentiment from my theoretical model.

1.1. Contributions

I present a novel, elegant, disciplined and simple modeling approach for modeling consumer sentiments. I ask as well as answer new and important theoretical questions in this class of social learning models in general equilibrium. By modeling endogenous fluctuations in sentiments and endogenous business cycles as well as exploring the issue of asymmetries, I add something new to the existing theoretical literature on sentiments. I also document new stylized facts from data about the asymmetric causal effects of consumer sentiments on business cycles, which is also a supplementary contribution even though my thesis is primarily focused on theory.

My other contributions lie in the specific details of modeling strategy and model solution. My model features two stages in every time, which is the related to the work of Lucas Jr (1972) and more recently Angeletos and La'O (2013) and Angeletos et al. (2018), among other work. However, the specific stage structure assumed as well as timing and information assumptions across stages in the context of modeling consumer durable investment makes the model novel, especially in terms of its application. Moreover, the stage, timing and information assumptions imply a minor deviation from the solution concept of rational expectations. It follows from the assumption of rational expectations that agents know the structural parameters in model as well as how the parameters are mapped into policy rule solutions by a linear solver for example. However, information/timing assumptions across stages and other details about belief updating imply that despite these rational expectations about structural parameters and *equilibrium mappings*, agents do not have rational expectations about the evolution of public beliefs. Furthermore, the manner in which stage 1 expectations of stage 2 phenomenon are formed in model implies that the functional form for conditional expectation is known which is related to the idea of PEA or parametrized expectations Den Haan and Marcet (1990), Lorenzoni and Marcet (1999) but yet distinct from it. The nature of these contributions will be made further clear once I flesh them out more explicitly in the relevant parts of the thesis.

Lastly, my model only has a numerical/computational solution. In order to solve the model, I draw upon the work of Kozlowski et al. (2019) and Schaal and Taschereau-Dumouchel (2020), mainly by using the crucial principle of resolution method/martingale property of public beliefs, which is a property that beliefs often satisfy in social learning models Chamley (2004). Once I use this martingale property, the solution becomes tractable and then I make some contributions as well. It follows from martingale property that I am able to use only a linear solver to solve the model in each stage 2, despite the fact that the model and its impulses are nonlinear since public beliefs evolve over time in a nonlinear fashion. Thus, I get a sequence of linear rational expectations solvers, one for each stage 2, even though the model features an infinite horizon and forward looking equations in every stage 2. There is also a fixed point problem between stage 1 and stage 2, which is not uncommon for models with such a stage structure (see for example Angeletos et al. (2018)) and indeed many dynamic macroeconomic models regularly involve fixed point problems. However, the specific nature of fixed point problem which arises in my model is novel and I solve this problem numerically, which is also another contribution.

1.2. Relation with Literature

Some researchers have modeled beliefs as exogenous fluctuations in sentiments or sunspots in multiple equilibrium models Farmer (2016), Benhabib and Farmer (1999) or sunspots in unique equilibrium models arising in the presence of higher order uncertainty and self-fulfilling fluctuations Angeletos and La'O (2013), Angeletos et al. (2018), Huo and Takayama (2015). The exogenous nature of sentiments in these models has been somewhat disciplined by the modeling strategy of Benhabib et al. (2015) and Acharya et al. (2019), in which the rational expectations equilibrium mapping imposes some discipline on sentiments and their stochastic properties in the former case and *endogenous* information ties the sentiments to equilibrium outcomes in the latter case.Benhabib et al. (2015) develop a model in which self-fulfilling, stochastic equilibria can arise, when firms are trying to forecast demand in next stage in a noisy world and there can be strategic complementarity in their price setting decisions because of aggregate demand externalities. The rational expectations mapping implies in their model that the variance of the stochastic equilibria in their model is endogenously determined. Acharya et al. (2019) show that in the context of a beauty contest with strategic complementarity, one can get *persistent*, sentiment driven business cycle fluctuations when agents can observe *endogenous* information which in their context means a signal about the aggregate equilibrium behavior of other agents.

Other researchers have used learning models in which agents act as econometricians and are trying to learn about the mapping between endogenous variables and fundamentals Eusepi and Preston (2011), which can be a source of persistent, self-fulfilling fluctuations, extrapolative behavior and endogenous reversals as in Adam et al. (2017), Adam and Merkel (2019). This is based on theory of internal rationality Adam and Marcet (2011), in which depending on evolution of relative strengths of income and substitution effects, one gets positive feedback, extrapolation and booms in stock price cycles due to expected capital gains and when wealth effects dominate substitution effects, due to growing value of stocks, one eventually gets an endogenous reversal and bust in stock prices. Yet another strand of literature draws upon theory of robustness as developed by Sargent and Hansen (2016) and ambiguity aversion Gilboa and Schmeidler (2004) with multiple priors utility and max-min decision problems to model subjective beliefs, their *pessimism* and asymmetries of the transitions into recessions (see for example Baqaee (2019), Bhandari et al. (2019), Ilut and Schneider (2014)).

There also exists the *news* view or Pigou cycle approach Pigou (1927), advocated by Beaudry and Portier (2004), which views expectation driven business cycles as the outcome of incentives to learn about fundamentals of economy in medium or long term; errors of optimism and pessimism in these forecasts can create *endogenous* cycles, in which an expansion sows the seeds of a recession if the initial optimism is unwarranted. There exists a lively debate on the frontier between the advocates of the news approach who view innovations in consumer confidence as being mainly driven by the news component about TFP fundamentals Barsky and Sims (2012), Beaudry and Portier (2014)[5] and those who emphasize the importance of noise more strongly Forni et al. (2017). Even the

[5]Some researchers such as Acharya et al. (2019) and Benhabib and Spiegel (2018) have questioned the identifying assumption of Barsky and Sims (2012) which ensures that by construction, animal spirits have only temporary/non-persistent effects.

noise approach can be of two types: noise in the observation/news about medium term fundamentals such as TFP Forni et al. (2017), Blanchard et al. (2013) or the noise can reflect consumer confidence about the *short* run evolution of the economy in near future a'la Angeletos et al. (2018), Angeletos and Lian (2019) and Benhabib and Spiegel (2018).

In relation to the theoretical efforts in literature, at a general level, my approach can be seen as related to both the *news* view in which there are endogenous cycles and the *sentiments* view of Angeletos and La'O (2013), Angeletos et al. (2018), Benhabib et al. (2015) Acharya et al. (2019), Benhabib and Spiegel (2018), in which consumer confidence about the *short* horizon is central rather than news about TFP. The *news* view has the attractive property of creating endogenous cycles, which I favor for theoretical and empirical reasons discussed below. However, I also think that consumer confidence about short run evolution of economy is a more important driver of business cycles relative to news about TFP, as also recognized by the *sentiments* approach based on empirical evidence cited below and elsewhere, which can not yet produce endogenous cycles and nonlinearities.

I favor endogenous cycles because the impulse, propagation view of business cycles has its limitations and the traditional reasons for favoring business cycles driven by persistent, exogenous shocks over endogenous cycles are on somewhat shaky grounds Beaudry et al. (2020), Galizia (2018), Boldrin and Woodford (1990). The classical literature on endogenous cycles fell out of favor since the 1980's because the spectral properties of macroeconomic time series showed no peaks at business cycle frequencies, which was interpreted as evidence against recurrent and regular endogenous cycles. Beaudry et al. (2020) question this evidence based on post-war US macroeconomic data and also argue for the existence of stochastic, limit cycles rather than deterministic ones. Secondly, the classical models featuring endogenous cycles usually produced these dynamics through ad-hoc, rule of thumb behavior, which was not microfounded. However, Boldrin and Woodford (1990) provided examples of equilibrium models with optimizing agents also featuring endogenous cycles and chaos.

Meanwhile, I favor the importance of sentiments because of recent, causal evidence, suggesting that sentiments are not just leading indicators (see for example Curtin (2019)) but also causal. For instance, convincing *causal* evidence from IV methods Benhabib and Spiegel (2018); Gillitzer and Prasad (2018), proxy-SVAR methods Lagerborg et al. (2018)[6] and structural estimation Angeletos et al. (2018); Bhandari et al. (2019); Ilut and Schneider (2014) suggests that autonomous shifts in consumer sentiments and subjective be-

[6]They use school shootings data in US to proxy for sentiment and show that lower sentiments after such incidents is correlated with higher unemployment over next one year horizon.

lief biases explain persistent business cycle fluctuations. Often, as shown by Gillitzer and Prasad (2018), this sentiment is driven by payoff irrelevant and non-fundamental variables such as political partisanship, which affects vehicle purchase rates in future for Australian data in their paper.

More specifically, my theoretical approach is most closely related to the social learning and informational cascades literature Chamley (2004), Banerjee (1992), Bikhchandani et al. (1992), as well as the literature on informational and asymmetric cycles Zeira (1994), Caplin and Leahy (1994), Gale (1996), Veldkamp (2005), Fajgelbaum et al. (2017). Social learning framework has been applied in a general equilibrium, macroeconomic context by only three papers to the best of my knowledge: Loisel et al. (2009), Fajgelbaum et al. (2017) and Schaal and Taschereau-Dumouchel (2020). Even these three papers are more closely tied to the *news* view and focus on diffusion of new technologies and endogenous tech driven boom bust cycles. I build and extend upon the theoretical frameworks of this macro-social learning literature but apply it in the novel context of modeling endogenous fluctuations in consumer sentiments and business cycles. The most closely related papers in terms of the solution method used and mathematical structure of my learning equations and signal specification are Schaal and Taschereau-Dumouchel (2020) and Kozlowski et al. (2019); the former focuses on herding cycles in the context of diffusion of new technologies and the latter models nonparametric learning about tail risks in the financial sector, creating nonlinear learning dynamics; consumer sentiment plays no central role in these papers.

2. Stylized Facts 1: Consumer Sentiments and Business Cycles

The graphs and table below document some prominent stylized facts about consumer sentiments and business cycles. Figure 1 shows evolution of consumer sentiment and NBER recession windows, along with a disaggregation of sentiment across demographic groups such as gender, education, region, age group and income group. It is clear that consumer sentiments are leading indicators for business cycles and predict onset of recessions for example. Since these are mere correlations, the leading nature of sentiments could reflect either their autonomous, causal role as drivers of business cycles or could reflect locally transmitted consumer knowledge about future economic developments which is not yet reflected in statistics collected by federal agencies. In the latter case, economic activity may not be causally driven by consumer behavior. From cross-sectional evidence, it is clear that time series comovements across demographic groups and regions is very high even though there is variation across groups in terms of the level of sentiment. Another salient fact is that volatility of index of consumer sentiment is higher for top income groups relative to middle and bottom income group.

In figure 2, I provide evidence regarding the lead, lag relation between index of consumer sentiment (ICS) and GDP growth rate for period 1978-2020. The vertical axis on these graphs should be interpreted as measuring $Corr(x_{t+h}, y)$, where x is variable listed first in title. Hence, $Corr(x_{t+h}, y) \leq 0$ for $h \leq 0$ (negative lag on x axis) would imply that first variable leads the second at business cycle frequency since the data is quarterly. It is clear from figure 2 that consumer sentiments are leading indicators for GDP. In figure 3, I report the cross-correlation between an optimism measure regarding vehicle buying conditions from Michigan data and personal consumption expenditure (PCE) growth rate for motor vehicles. The significant leading indicator nature of optimism regarding vehicle buying conditions for future vehicle purchase rates confirms the link between sentiments and future economic behavior.

In table 1, I document some descriptive evidence from the University of Michigan survey for period 1960-2020 regarding asymmetric recall of negative news. In all cases, the net favorable percentage is negative so that consumers systematically over-report and over recall negative news heard in last six months regarding various economic topics such as business conditions, employment, consumer demand, inflation conditions and credit conditions.

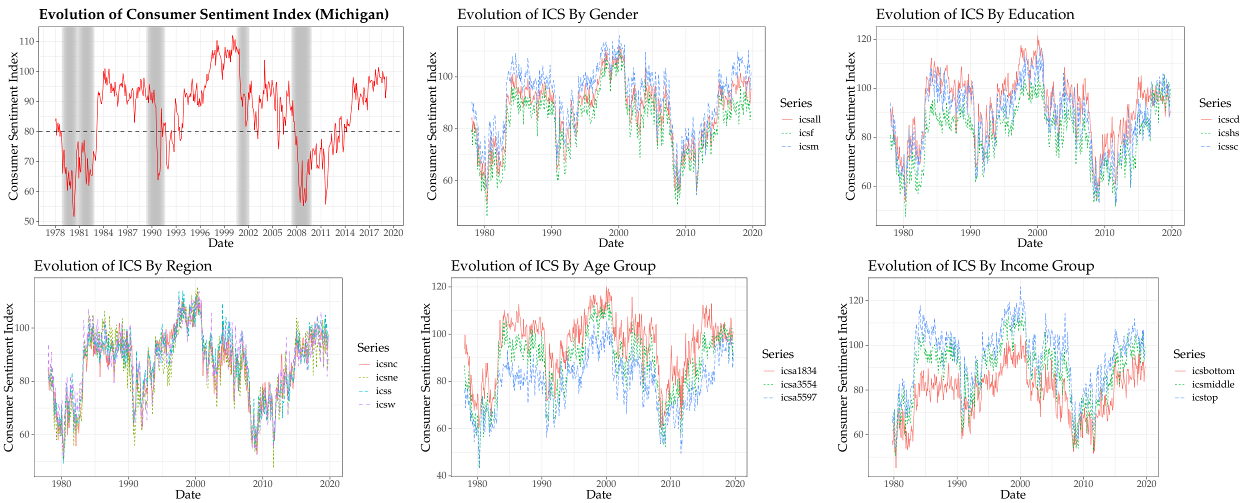


Figure 1: Age groups are 18-34, 35-54 and 55+. Income groups are bottom third, middle third and top third. Education groups are high school or less (hs), some college (sc) and college degree or more.

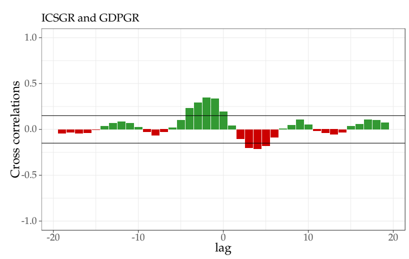


Figure 2: Cross Correlation between Quarterly ICS growth rate and GDP Growth Rate for 1978-2020.

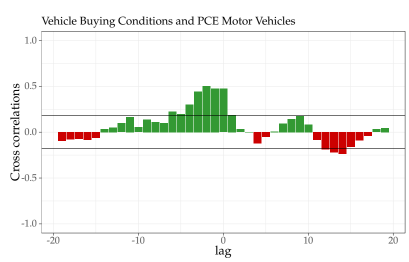


Figure 3: Cross Correlation between growth rate in optimism regarding vehicle buying conditions and PCE growth rate for motor vehicles for 1979-2020.

Table 1: Asymmetric Recall of Negative News

	(1) Favorable	(2) Unfavorable	(3) Net Favorable
News Heard About (% of respondents)			
Business Conditions	29.7	51.9	-22.2
Employment	11.5	20.8	-9.4
Government Elections	2.94	6.5	-3.5
Consumer Demand	2.35	3.33	-0.98
Inflation	1.07	4.26	-3.2
Credit Ease	2.2	2.6	-0.4
Stock Market	2.2	2.5	-0.3
Trade Deficit	0.5	1.84	-1.31

Note: Data is from University of Michigan Survey, 1960-1920.

3. Stylized Facts 2: Durable Goods and Business Cycles

The graphs and table below document some prominent evidence about durable goods and business cycles, mainly regarding the lead lag structure of durable goods spending with the cycle and relative volatility of durables over the business cycle. The cross-correlation graphs should be interpreted in the same way as before and figure 3 indicates that PCE durables growth rate is a leading indicator for GDP in post-war US data. Figure 5 (6) show that PCE growth rate for all consumption items (New Auto Vehicles) is a leading indicator for subsequent change in business inventories for the respective consumption category. Figure 7 documents the cyclical fluctuations of new and used vehicles along with NBER recession windows. Both types of vehicles display significant business cycle volatility but used vehicle market is even more volatile as is visible by the extreme fall in unit change of used vehicle sales from last year in the great recession.

Table 2 displays relative business cycle volatility statistics for various consumption categories relative to PCE, capacity utilization rates for various types of production relative to overall production, employment in various sectors, relative to all non-farm employment and CPI for various categories relative to overall urban inflation. The high relative volatility of durable sector as compared to non-durables across all these variables is clear. Another salient fact is that whenever data is available for used car inflation or used auto and trucks PCE growth rate for example, relative standard deviation is even higher than durables as a whole. This is also consistent with the evidence in Figure 7.

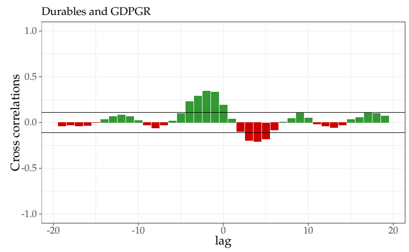


Figure 4: Cross Correlation between PCE durables growth rate and GDP Growth Rate for 1948-2020.

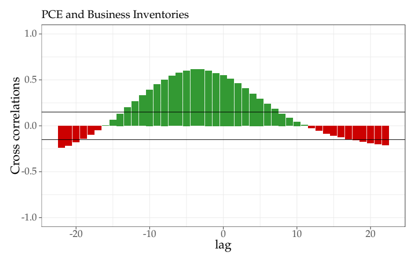


Figure 5: Cross Correlation between PCE Growth Rate and Change in Business Inventories for 1993-2020.

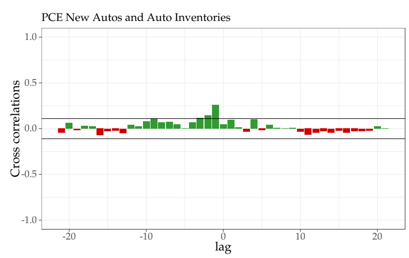


Figure 6: Cross Correlation between PCE New Autos Growth Rate and Change in Private Inventories for New Autos for 1948-2020.

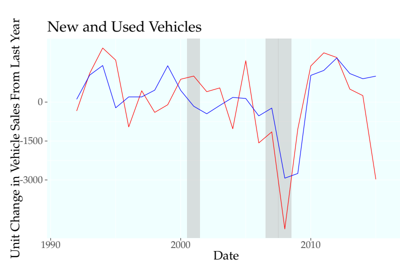


Figure 7: New (Blue) and Used (Red) Vehicles Over the Business Cycles with NBER Recessions in gray.

Table 2: Business Cycle Volatility Statistics

Parameter	(1) SD	(2) Relative SD
PCE		
GDPGR	3.46	1
PCEGR	1.98	0.57
Durables PCEGR	7.36	2.13
PCE Motor VehiclesGR	13.2	3.82
PCE Used Autos and TrucksGR	11.09	3.21
Non-durables PCEGR	1.8	0.52
Business Gross Fixed InvestmentGR	7.72	2.24
Capacity Utilization		
Overall	4.03	1
Durables	12.3	3.05
Non Durables	4.2	1.03
Manufacturing	4.62	1.15
Motor Vehicles	10.98	2.73
Employment		
All Employees NonFarm	2.93	1
Employees Manufacturing	5.40	1.84
Employees Durable Sector	7.83	2.67
Employees Non-Durable Sector	2.96	1.01
Employees Motor Vehicles Sector	7.83	2.67
CPI		
Inflation Urban Overall	2.8	1
Inflation Durables	3.3	1.2
Inflation Non Durables	3.62	1.3
Inflation New Vehicles	2.75	0.99
Inflation Used Cars	7.5	2.7

Note: Data is from FRED St Louis database. CPI data is for urban areas and time period is 1957-2020. Employment data is for 1940-2020. PCE and GDP data is for 1948-2020.

4. Stylized Facts 3: Local Projections

In this section, I document some stylized facts from the data regarding the impact of consumer sentiments on economic activity using an extant identification scheme, borrowed from Benhabib and Spiegel (2018). My main contribution to their empirical study is the use of local projection method a'la Jordà (2005) to also document the *asymmetric causal* effects (if any) of consumer sentiment on economic activity in future, in addition to some minor accretions to their empirical specifications such as also using data on durable good consumption by state, as a dependent variable.

4.1. Data

For my baseline estimation, I use almost identical data, identical variable definitions, identical instrumental variable and identical controls, as in Benhabib and Spiegel (2018) to facilitate comparison with their paper, but I use local projections to document asymmetries, unlike them. Quarterly sentiment data by states in US, available from 2005 through 2016 from the Surveys of Consumers, of Michigan (2020) is used. As also in Benhabib and Spiegel (2018), I use as base gauge of consumer sentiment, the answer to question BUS5, "Looking ahead, which would you say is more likely - that in the country as a whole we will have continuous good times during the next five years or so, or that we will have periods of widespread unemployment or depression, or what?". Respondents' answers are scored 1 to 5, with 1 representing the answer "good times", 2 representing "good with qualifications", 3 representing "pro-con", 4 representing "bad with qualifications" and 5 representing "bad times". There are also a modest number of responses characterised as "depends".

As my base measure of lagged sentiment, "GOOD", I consider the share of a state i at time $t-4$ whose respondents' answers were scored 1 or 2. My baseline dependent variable is $GGDP_{it}$ which is GDP growth rate at state level from $t-4$ to t, but I also use PCE (Personal Consumption Expenditures) growth rate as well as durable goods spending growth rate by state as dependent variables, unlike Benhabib and Spiegel (2018). I include other controls to condition on the characteristics of individual respondents. As my observation is at the state level, these are measured as state respondent averages, also at time $t-4$. My conditioning variables include income levels by state, "INCOME", which is calculated as the average of reported levels of respondent incomes within a state, "EDUC", which is the average of the highest year of education reported by respondents within a state, "INVEST", which is the share of state respondents who said that they hold investments and measure of output gap "YGAP". Data on GGDP, is obtained from Haver analytics, as is the measure of the national output gap, YGAP. Data on PCE and durable goods spending growth rate is obtained from FRED St Louis database, meanwhile data on other conditioning variables is accessed via the data appendix of Benhabib and Spiegel (2018).

4.2. Identification Scheme

On needs the maintained hypothesis for identification that states are sufficiently small that attitudes about the local economy will not distort the response about national economic conditions to prevent concerns about reverse causation. Given this assumption, cross-sectional treatment by using fixed effects should isolate the impact of differences in sentiment across states on future differences in state economic activity, in a within estimation. However, a potential problem is that household expectations about future national economic activity may be positively related to local experiences, raising the prospect of reverse causality. Thus, instrumental variables estimation is needed.

My instrument is borrowed from Benhabib and Spiegel (2018), who turn to political data as an instrument for local sentiment levels that vary systematically across states. The instrument is $Congpres_{it}$, which is the proportion of congress members in state who are from incumbent political party. There is a large literature that demonstrates a positive relationship between partisanship and economic assessments. A survey respondent who self-identifies as a member of one of the major political party is more optimistic about the national economic picture when the sitting national leader is from that same party. For instance, Gerber and Huber (2009) demonstrated that consumption changes following a political election are correlated with whether or not the election was won by the preferred political party of the respondent. They interpret this correlation as working through the sentiment channel. Political partisanship has also been used as an instrument for identifying a connection between sentiment and consumption. Mian et al. (2018) demonstrate that presidential elections are associated with changes in sentiment about the effectiveness of government policy in line with political partisanship but they do not find any statistically significant relationship between changes in the presidential party at the county level in the United States and changes in consumption. In contrast, Gillitzer and Prasad (2018) show for Australian survey data that higher sentiment - which is associated with having a member from your political party in office at the federal level - is associated with increased future vehicle purchase rates.

4.3. Local Projection Method

I use the local projection method with a logistic function approach to document the asymmetric effects of consumer sentiment across business cycle regimes of boom and recession. The shock variable in local projections is externally identified through a first stage regression which uses the $Congpres_{it}$ instrument, which is why I interpret the results as indicating causal effects.

In general, when we have a SVAR with n variables contained in Y_t, it can be written in matrix form as follows:

$$B_0 Y_t = \alpha + B(L) Y_t + \epsilon_t$$

The residuals ϵ_t are assumed to be white noise with zero mean. The structural shocks are contemporaneously uncorrelated and the variables have a contemporaneous effect on each other, which is captured by the square matrix B_0. However, estimating this SVAR without further assumptions is not possible because of the simultaneous identification problem. Traditionally, this identification problem was solved by orthogonalizing the reduced form residuals into orthogonal shocks using for example the Cholesky decomposition or other methods to recover the structural shocks. In his pioneering paper, Jordà (2005) proposed an alternative approach to estimate impulse responses. His first step consists of ordinary least squares (OLS) regressions for each forecast horizon:

$$y_{t+h} = \alpha_h + B_1^h y_{t-1} + ...B_p^h y_{t-p} + u_{t+h}^h. \tag{1}$$

for $h = 0, 1, .., H-1$. where α_h is a vector of constants, and B_i^h are parameter matrices. The vector elements u_{t+h}^h are autocorrelated and/or heteroscedastic disturbances. The collection of all regressions of Eq. (1) are called LPs (local projections). The slope matrix B_h can be interpreted as the response of y_{t+h} to a reduced form shock in t. Structural impulse responses are then estimated by the following:

$IR\widehat{(t, h, d_i)} = \hat{B}_1^h d_i$ where $d_i = B_0^{-1}$. As in the SVAR approach, the shock matrix d_i must be identified from a linear VAR. Given the serial correlation of u_{t+h}^h, Jordà (2005) proposed to estimate robust standard errors using the approach by Newey and West (1987).

A great advantage of LPs is their easy extension to nonlinear frameworks. The simplest approach to separate data into two regimes is using a binary (dummy) variable. The drawback, however, is that it lowers the degrees of freedom. As a remedy, Auerbach and Gorodnichenko (2012) proposed computing state probabilities with a logistic function that allows using all observations for the estimation, which is what I will also use given the relatively small number of observations in my data, especially with biennial data. The logistic function is the following:

$F(z_t) = \frac{\exp(-\gamma z_t)}{1+\exp(-\gamma z_t)}$, where $Var(z_t) = 1$, $E(z_t) = 1$.

For example, if z_t corresponds to changes in the gross domestic product (GDP) at time t, an increase in z_t would lead to a decrease in $F(z_t)$. Values close to zero of $F(z_t)$ would thus indicate periods of economic expansion. Auerbach and Gorodnichenko (2013) proposed standardizing the cyclical components of the variable using HP filter to obtain the variable z_t. The observations for the two regimes are the product of the transition function and the endogenous variables:

Regime 1 (R1) : $y_{t-l}.(1 - F(z_{t-l}))$, for $l = 1, ..., p$, Regime 2 (R2) : $y_{t-l}.F(z_{t-l})$, for $l = 1, ..., p$.

Structural nonlinear impulse responses are estimated using the following: $IR^{R1}\widehat{(t, h, di)} = B_{1,R1}^h d_i$, for $h = 0, ..., H-1$, $IR^{R2}\widehat{(t, h, di)} = B_{1,R2}^h d_i$ for $h = 0, ..., H-1$, where $B_{1,R1}^0 = I$

and $B^0_{1,R2} = I$. The coefficient matrices $B^h_{1,R1}$ and $B^h_{1,R2}$ are obtained from the following LPs:

$$
\begin{aligned}
y_{t+h} = \alpha_h + B^h_{1,R1}.y_{t-1}.(1 - F(z_{t-1}) + ... + B^h_{p,R1}y_{t-p}.(1 - F(z_{t-1}) \\
+ B^h_{1,R2}.y_{t-1}.(F(z_{t-1}) + ... + B^h_{p,R2}y_{t-p}.(F(z_{t-1}) + u^h_{t+h}
\end{aligned}
\tag{2}
$$

with $h = 0, ..., H - 1$.

LP's can be applied to panel data as well, which is what I exploit to estimate IRFS on my panel data. Estimating impulse responses using LP's on panel data was put forward by Auerbach and Gorodnichenko (2013) and Jordà et al. (2015), among others. In order to implement the local projections on my panel data, I use the *lpirfs* package in R, developed by Adammer (2019).

My panel data equation is:

$$
y_{i,t+h} = a_{i,h} + \mu_{t+h} + shock_{i,t}\beta_h + x_{i,t}\gamma_h + \epsilon_{i,t+h} \tag{3}
$$

for $h = 0, 1, ..., H - 1$, where $shock_{i,t}$ is identified through an IV approach by using a first stage regression in 2SLS and represents a plausibly exogenous, positive shock to consumer sentiment. $x_{i,t}$ includes INCOME, EDUC, INVEST, YGAP and z_t is YGAP, i.e the regimes are separated based on output gap measure. $a_{i,h}$ are state/cross-section fixed effects and μ_{t+h} are time fixed effects. Time can be quarterly, annual or biennial across specifications. $shock_{i,t}$ can also be first estimated by an instrumental variable approach (see, e.g., Jordà et al. (2019)), which is what I also do, using the instrument from Benhabib and Spiegel (2018). My first stage regression equation for identifying $shock_{i,t}$ is:

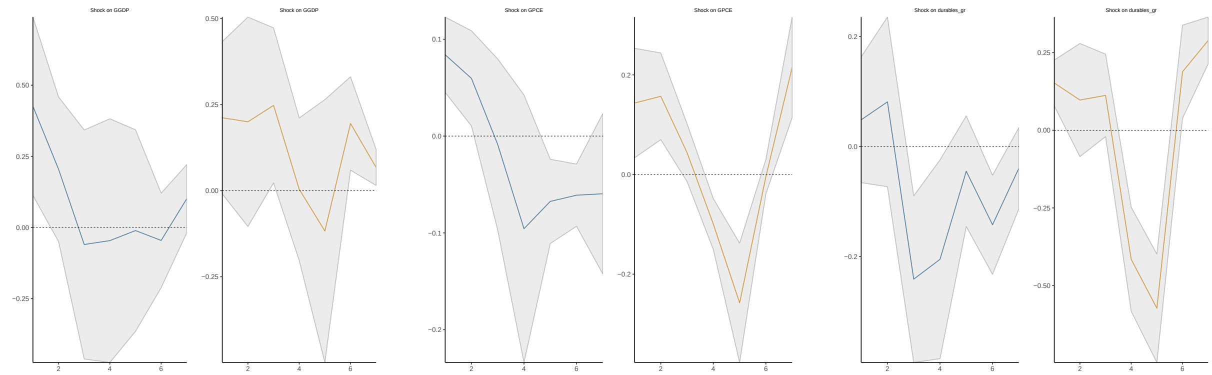
$$
CS_{i,t+h} = a_{i,h} + \mu_t + Congpres_{i,t}\beta_h + x_{i,t}\gamma_h + u_{i,t+h} \tag{4}
$$

where $CS_{i,t+h}$ is consumer sentiment measure, $Congpres_{i,t}$ is the instrument, discussed above and $shock_{i,t} = \widehat{CS_{i,t+h}}$ (predicted value) from this equation. $x_{i,t}$ includes other exogenous variables as before.

I also check robustness of results to alternative measures of consumer sentiment which include CSb5, which measures proportion of respondents who answer option 5 to the BUS question mentioned above, which is "bad times". Thus, CSb5 is a measure of consumer pessimism. I also have two more consumer optimism measures CSb1 and CSg1, where CSb1 is an even stronger measure of consumer optimism since it only includes respondents who answer 1 to the BUS5 question which is 'good times", whereas CSb1 is answer to question "BAGO" in survey which asks whether business conditions are better or worse than the previous year. I also do perturbations across different frequencies for data in order to address the problem of serial correlation in quarterly and annual data, which arises because instrument $Congpres_{it}$ changes its valye only biennially because of rules of political system. Thus, I report results for quarterly, annual and biennial specifications.

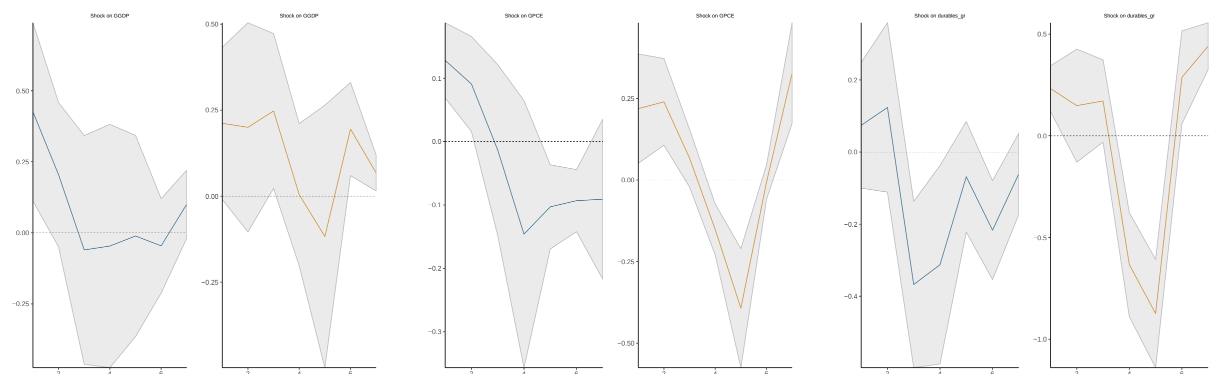
I use robust variance covariance matrix to account for heteroskedasticity in panel data. In order to separate the data into two regimes, I use value of $\gamma = 10$ as parameter in logistic function but my results are robust to alternative values of γ. Meanwhile, I use $\lambda = 6.25$ (1600) for annual/biennial (quarterly) data in order to HP filter output gap before using it for regime separation. I have 90% confidence intervals in the following impulse response graphs.

4.4. IRFS FROM LOCAL PROJECTIONS



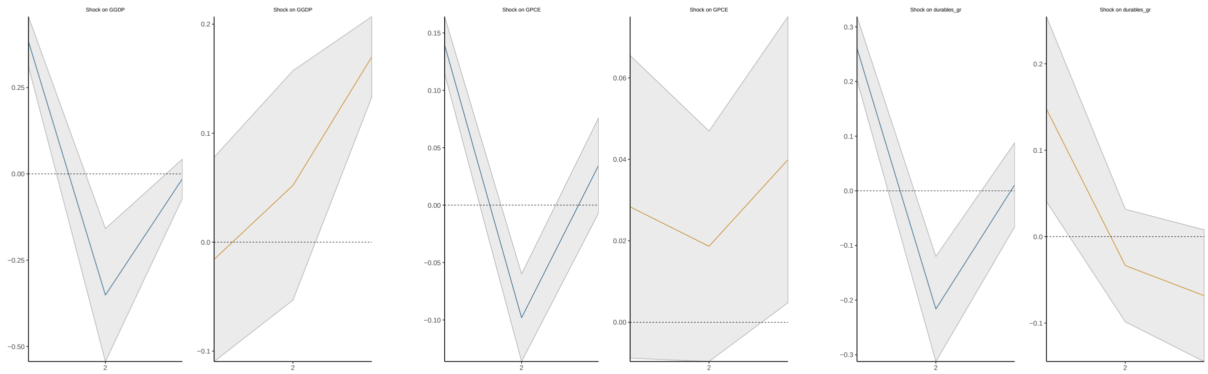
(a) (b) (c)

Figure 8: Impulse Response to a 1 sd CS Shock (Annual, CS)



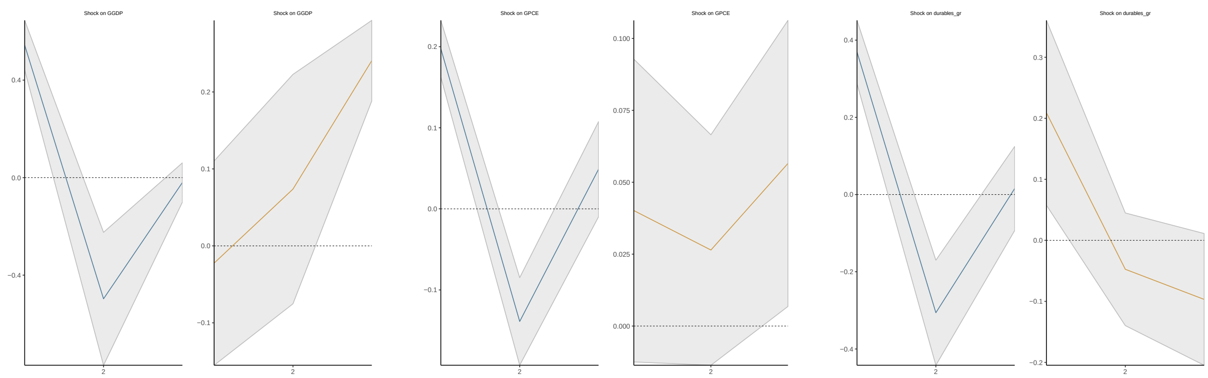
(a) (b) (c)

Figure 9: Impulse Response to a 1 sd CS Shock (Annual, CSg1 (Pessimism Shock))



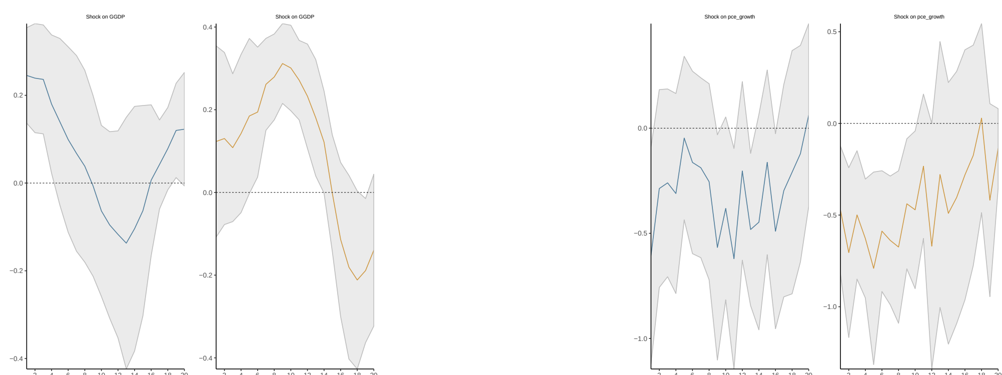
(a) (b) (c)

Figure 10: Impulse Response to a 1 sd CS Shock (Biennial, CS)



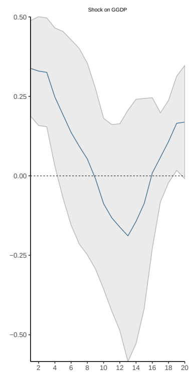
(a) (b) (c)

Figure 11: Impulse Response to a 1 sd CS Shock (Biennial, CSg1 (Pessimism Shock))

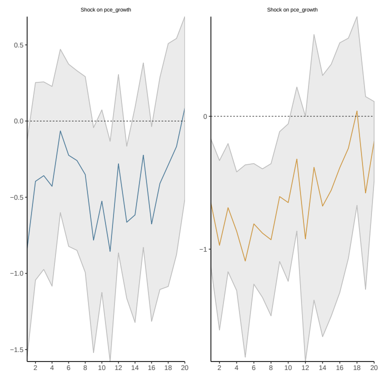


(a) (b)

Figure 12: Impulse Response to a 1 sd CS Shock (Quarterly, CS)



(a)



(b)

Figure 13: Impulse Response to a 1 sd CS Shock (Quarterly, CSg1 (Pessimism Shock))

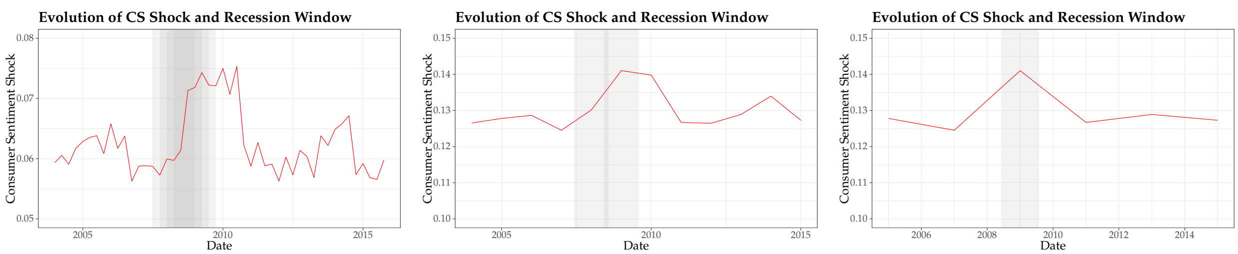


Figure 14: Identified Consumer Sentiment (CS) Shock Series and NBER Recession Window (Left is Quarterly Series, Middle is Annual and Right is Biennial).

4.5. Discussion of Stylized Facts

The graphs above display the impact of consumer sentiment on *GGDP* (left panel), *PCE* (middle panel) and durable goods spending (right most panel) for boom (left panel) and recession (right panel) regime. I omit the graphs corresponding to alternative definitions of consumer optimism CSb1 and CSb5 since they are quite similar to results for CS in the main text above. I include results for baseline measure "CS" for consumer optimism and consumer pessimism measure of "CSg1". However, I include graphs for all three frequencies and all dependent variables above. The omitted graphs are provided in appendix for completion and I also discuss the robustness of results to those omitted cases below.

The causal impact of a positive consumer sentiment shock on output is positive on impact in expansion regime for all perturbations. After impact, the effect dies down after some periods. For instance, in Figure 10, for quarterly data, the effect of consumer sentiment on GDP growth rate during boom is around 0.25% on impact. After about $6-8$ quarters, the effect dies down. The magnitude of the on-impact effect is slightly different across definitions of consumer sentiment and varies from 0.2 to 0.5%. As one would expect, a consumer pessimism shock ($CSg1$) produces a negative on impact effect of around 0.2% for quarterly data. For annual and biennial data, the magnitudes are similar to quarterly data and range around $0.2 - 0.5$%. For quarterly data, a consumer optimism shock during a recession regime produces no significant effect on impact, but it has a delayed, positive effect of around 0.3% after about 10 quarters, which dies down after about 16 quarters after shock. These effects are qualitatively similar across sentiment definitions and magnitudes range from around 0.3 to 0.5%. For biennial data, one also finds a delayed positive impact of around $0.15 - 0.3$% after $2 - 3$ periods during recession regime. Similarly, one finds only a delayed, positive effect for annual data.

The causal impact of a consumer optimism shock on growth rate of personal consumption expenditure (PCE) during boom regime and for annual/biennial frequency is positive on impact, in the range of $0.1 - 0.2$%, somewhat weaker than effects for GGDP. However, for quarterly data, one does not find any statistically significant causal evidence regarding impact of consumer optimism shock on PCE by state. For recession regime, consumer optimism shock has an impact effect of about $0.15 - 0.3$% on PCE in annual specifications. In the recession regime, the biennial specifications show no significant effects of consumer optimism on PCE. However, in the quarterly specifications, the impact effect of consumer optimism shock on PCE growth rate is around -0.5 to -1.5% during recession regime, which is strongly negative and a bit counter-intuitive.

Meanwhile for durable goods growth rate, I find significant on impact effects during boom regime for biennial data only, in the range of $0.25 - 0.4$%, stronger than magnitudes for PCE results but slightly weaker than GDP results. For annual specifications, I do not find any significant, positive effects of consumer optimism on durable goods growth rate. Data for durable goods was not available by state at quarterly level so I do not have any quarterly specifications in this case. For recession regime, I do find positive, significant, on-impact effects of an optimism shock on durables good growth rate for both annual and biennial specifications, in the range of $0.15 - 0.3$%.

Overall, for the most trusted specification with biennial data, due to the problem of serial correlation with other frequencies, I find intuitive, positive and statistically significant causal effects of consumer optimism on GDP growth rate, durable spending and PCE growth rate for boom regime, in line with results of Benhabib and Spiegel (2018) who did not report asymmetries. However, I find that these positive effects are delayed for some periods and quantitatively weaker during recession regime which is a novel form of causal asymmetry. This somewhat confirms the results of Ahmed and Cassou

(2016) who find no significant effect of sentiment during recessions, even though I do find significant impact of sentiment during recessions, despite the fact that this impact is delayed. Another noticeable asymmetry is that during recessions, I find no statistically significant effects of consumer optimism on PCE growth rates but only find significant effects for GDP and durable spending growth rate. These are novel results, regarding causal asymmetries in the effect of consumer optimism on economic activity.

The identified consumer sentiment shock series based on baseline measure of consumer sentiment CS is plotted for all three frequencies in figure 14. It is visible that consumer sentiment was at quite a low level at the beginning of 2007-2009 recession but it started improving significantly, rising to quite an optimistic level while the recession still persisted prior to recovery. This suggests that my identified consumer optimism shock series led the recovery i.e a rise in optimism predicted recovery. To the extent that the identification strategy identifies plausibly exogenous variation, these variations in consumer sentiment are orthogonal to economic fundamentals, suggesting an autonomous, causal role of consumer sentiments in generating a recovery from recession.

5. SOCIAL LEARNING MODELS

The general structure of social learning models with rational/Bayesian learning is the following: there exists either a sequence of countable agents or a sequence of continuum of agents born each period whose payoffs depend on some, true, hidden state of nature. Agents receive private information about the hidden state and the distribution of signals, conditional on true state satisfies the monotone likelihood ratio property, which ensures that signals have some precision. Agents can also observe the actions of past agents, with observation noise or without it. These actions of past agents are informative about the hidden state, since the structure of decisions, which maps private information into actions is known to everyone. Agents use Bayes' rule to update beliefs about true state of nature over time. Application of Bayes' rule generates the remarkable property of a martingale and martingale convergence theorem, which posits that expected value of change in beliefs is zero.

In models of social learning, a lot of private information can be hidden and small changes can lead to sudden and significant release of information through social learning which can lead to sharp switches in public beliefs over time Chamley (2004), a property which is particularly attractive for me, given that I want to understand endogenous reversals. Actions are more informative when they are drawn from continuous sets than when they are drawn from discrete sets. The weight of history, noisy observation and rational learning often slows down learning in these models.

"Rational herds" are defined in this set up as occurring if after some T, $\forall t \geq T$, all agents herd on one action, even though in principle they could get draws of signals such that they will break away from the herd, but that event does not realize. Social learning may be very slow during herds but it does exist. Cascades on the other hand are defined by the condition that all agents herd on one action, regardless of private signal received, which means actions are uninformative about hidden state and social learning completely stops. For generic, private signal distributions such as those with unbounded support, cascades are nongeneric but herds are generic i,e they eventually always occur, with probability one Chamley (2004). Cascades are nevertheless good stylized descriptions of public belief evolution because of slow rate of convergence to truth in these models, owing to observation noise and weight of history with social learning.

5.1. Social Learning with Dynamic Fundamental and Discrete Signal

In this section, I perform simulations from a social learning model with stochastic fundamental, based on theory of Moscarini et al. (1998) in order to illustrate some general properties of social learning models before I use this class of models in general equilibrium. The model is based on Moscarini et al. (1998) but I perform some novel numerical analysis on it.

The model features sequential, observational learning among a countable number of individuals, each of whom take one of two actions a_0 and a_1, where the payoff of actions depends on a binary, hidden state. Information available to each individual n includes private signal $\sigma^n \in \{\sigma^0, \sigma^1\}$ and public history of actions, $\mathcal{H}^n = \{a_1, a_2, ..., a_{n-1}\}$. The signal is informative in the sense that probability of observing a signal indicating a particular state, conditional on the fact that the true state is indeed that one is $\alpha > \frac{1}{2}$. Unlike the classical model of herds Bikhchandani et al. (1992) with fixed state, after each individual decision, the state switches with probability ϵ, i.e ϵ is the off diagonal term in the 2 by 2 stochastic matrix for this markov chain. This modifies the informativeness of the action taken by last individual, since the state might have switched in between. Given this model, Moscarini et al. (1998) derive the evolution of public beliefs q^n about the state over time and other theoretical results about nature of informational cascades in this environment.

In the graphs below, ϵ is the probability of switching from state 1 to state 2 or vice versa. The transition matrix is symmetric. The dynamics are qualitatively and quantitatively different across the different values of ϵ. As also guaranteed by the theoretical results in Moscarini et al. (1998), my simulations confirm that the cascades don't arise for $\epsilon \in [0.25, 0.75]$ and when they do arise, they are temporary. The results also confirm that some values of ϵ, the cascades are also alternating between high and low ones. For a prior probability of switching of 0.5, public beliefs stay constant at 0.5, since prior public beliefs q^0 have been specified at 0.5. The upper and lower blue lines demarcate the region for high and low cascades. All of these cycles arise endogenously in an observational learning context with bayesian learning without the need for exogenous impulses to create cycles.

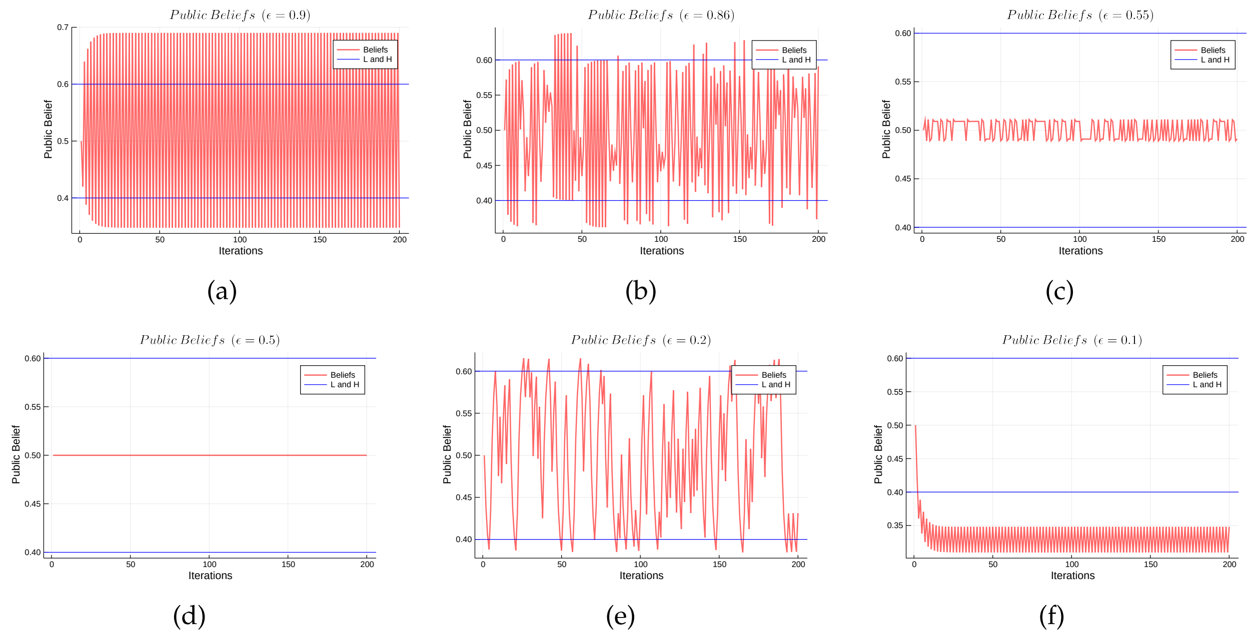
The public belief dynamics are governed by the following equations because of Bayesian updating Moscarini et al. (1998). When not in a cascade, then:

$$q^{n+1} := \left\{ \begin{array}{ll} f^0(q^n) = \frac{(1-\epsilon)(1-\alpha)q^n + \epsilon\alpha(1-q^n)}{(1-\alpha)q^n + \alpha(1-q^n)}, & \text{if } a^n = a_0 \\ f^1(q^n) = \frac{(1-\epsilon)\alpha q^n + \epsilon(1-\alpha)(1-q^n)}{\alpha q^n + (1-\alpha)(1-q^n)}, & \text{if } a^n = a_1 \end{array} \right\}$$

If one has entered an informational cascade, then the dynamics of public beliefs are governed by the following uninformative cascade equation:

$$q^{n+1} := \varphi(q^n) = (1-\epsilon)q^n + \epsilon(1-q^n)$$

The simulations of public beliefs for $T = 200$, $q^0 = 0.5$ and $\epsilon \in \{0.1, 0.2, 0.5, 0.55, 0.86, 0.9\}$ are displayed below:



6. Social Learning in General Equilibrium

My economy is populated by a representative household with a continuum of durable good shoppers who act in stage 1 and one non-durable good shopper, who makes all pertinent decisions in stage 2. Household has preferences, which are additively separable in consumption of durable, nondurable goods and leisure. There also exist two types of firms in two sectors (durable good and non-durable good sector) and a monetary authority in the model.

z is the hidden state which durable good shoppers are trying to learn about. If z is high, durable good shoppers are more likely to invest in durables such as vehicles or household appliances. The hidden state is a modeling device and is trying to capture an aggregate state of consumer confidence or aggregate state of economy, which consumers are trying to learn about. When they receive private signals, suggesting that z is high or when their social learning process through noisy observation of past durable good investment decisions by other consumers suggests that z is high, they are likely to spend more on durable goods. I do not provide explicit microfoundations for this behavior but external habits of the keeping up with Joneses variety or the informative value of other consumers' decisions in predicting future durable prices or future economic growth can all rationalize herding behavior by consumers in durable good sector.

In the initial steady state, $z = \bar{z}$, which can be interpreted as the level of optimism or consumer sentiment, associated with the balanced, growth path. This state can change to a new level, z^n, where $z^n \in \{z_L, z_H\}$, where $z_L < \bar{z} < z_H$. With probability λ, the state matures and its true value is revealed, after which all consumer uncertainty about the state of the economy is resolved and consumers learn about for instance, whether they have indeed entered a recession or not. Before the state matures, consumers are trying to learn if the aggregate state of consumer confidence is high or low, based on private signals and social learning, so that there exists some subjective uncertainty. Prior to revelation of true state, if economic activity enters a downturn, driven by the endogenous pessimism of consumers due to learning dynamics, one would interpret this as a consumer sentiment driven recession, where falling consumer sentiment is causing a recession in an autonomous fashion. Any fluctuations caused after revelation of true state can be thought of as an exogenous cause of fluctuations.

Payoff to durable good shopper h from investing in vehicles is the following, where i_{jt} is one when the vehicle is purchased and zero otherwise. c is cost of investment and R_t is related to hidden state and more explicitly defined below. In general equilibrium, $c = P_{d,t}$ and this is determined by the pricing decisions of the durable good sector firms in stage 2, given realized consumer demand for durable goods, which is predetermined from stage 1.

$y_{h,t} = i_{ht} \left(R_t - c\right)$

Each durable good shopper gets an additive, private signal s_h, which is centered on the true fundamental shock, conditional on the true shock being drawn by nature. The distribution for signal satisfies the MLRP (monotone likelihood ratio property), which guarantees that a high signal is more likely to be drawn from a high realization of $z + \eta$.

$s_h = z + \eta + v_h$, where v_h is iid and $v_h \sim F^v$

In addition to private signals about aggregate fundamental, consumers also observe the endogenous, aggregate durable goods purchase rates over time, which partially aggregates private information about fundamentals since individual investment decisions are mappings from private signals about fundamentals. After the vehicle investment decisions have been taken, consumers observe the mass of shoppers who purchased a durable good, with some additive noise.

$d_t = \int_0^1 i_{ht}\, dh + \epsilon_t$, where $\epsilon_t \sim F^\epsilon$ and is iid.

d_t is an endogenous signal, which is informative about the state z to some extent, but its information content is highly nonlinear and varies over time, as a function of history of shocks. Individual consumer h entertains a joint, probability distribution over the fundamental z and common noise shock η, based on its private information set $\mathcal{I}_{ht}$, which is:

$\pi_{ht}(\tilde{z},\tilde{\eta}) = Pr\left(z=\tilde{z},\tilde{\eta}\in[\tilde{\eta},\tilde{\eta}+d\tilde{\eta}]|\mathcal{I}_{ht}\right)$

The joint probability, with respect to public information is:

$\pi_{t}(\tilde{z},\tilde{\eta}) = Pr\left(z=\tilde{z},\tilde{\eta}\in[\tilde{\eta},\tilde{\eta}+d\tilde{\eta}]|\mathcal{I}_{t}\right)$

Public information consist of noisy observations of past measure of durable shoppers who carried out vehicle purchases, d_t, which is: $\mathcal{I}_t = \{d_{t-1}, d_{t-2},,,,d_0\}$ and private information is $\mathcal{I}_{ht} = \mathcal{I}_t \cup \{s_h\}$.

$\pi_{ht}(\tilde{z},\tilde{\eta}) = \frac{\pi_t(\tilde{z},\tilde{\eta})f_{\tilde{z}+\tilde{\eta}}(s_h)}{\int \pi_t(z,\eta)f_{z+\eta}(s_h)d(z,\eta)}$

$d_t^e = \int_0^{1-\mu} i_{ht}^*\left(\pi_t,\Omega_t,s_{ht}\right)dh$, where Ω_t consists of aggregate state variables of economy.

Publically observed mass of durable good investors with noise is: $d_t = d_t^e + \epsilon_t$, where $\epsilon \sim iid\, F$.

The private probability that the state is high is: $p_{ht} = \int \pi_{ht}(z_H,\eta)d\eta$. The public probability that the state is high is: $p_t = \int \pi_t(z_H,\eta)d\eta$.

Based on Bayes' rule and the given information structure, the joint probability is updated over time as:

$\pi_{t+1}(\tilde{z},\tilde{\eta}) = \frac{\pi_t(\tilde{z},\tilde{\eta})f^{\epsilon}((d_t - d^e(\pi_t,\Omega_t,\tilde{z},\tilde{\eta})))}{\int \pi_t(z,\eta)f^{\epsilon}((d_t-d^e(\pi_t,\Omega_t,z,\eta)))d(z,\eta)}$

Timing:

Stage 1: Each member of the continuum of durable good shoppers in household makes a static, binary decision about whether to invest in durable good or not, based on private signal observed about an aggregate state and aggregate mass of durable good consumers in last period, observed with noise in stage 1. However, in the presence of general equilibrium effects, the household must now forecast the price of durable goods, which will be set in stage 2 by durable sector firms and is unknown as of stage 1. This will require them to form static, stage 1 expectations of endogenous stage 2 variables.

Stage 2: Subsequently, mass of durable goods purchasing household members gets realized and public beliefs are updated, as shown in belief updating equations. The household in stage 2 must live with the implications of this predetermined and irreversible durable good decision for their budget and make other consumption (non-durable good) and labor supply decisions. Firms in the economy take these sources of demand as given and set prices, subject to a standard Calvo price friction. The monetary authority sets the policy rate based on taylor rule and markets clear. The market clearing values of all endogenous variables in stage 2 are observed by everyone and continue to be in the information set of all agents, moving forward.

Consumers have the following preferences and constraints:

6.1. Preferences and Constraints

The preferences for consumption (non-durables) and hours are separable and are of standard CRRA type with relative risk aversion of σ and Frisch elasticity of η. Preferences for durable good are also separable from the other two objects of utility. Purchasing one unit of durable good in stage 1 of time t gives durable good shopper h utility of $\frac{\tilde{Z}_t}{\left((1-\delta)D_{h,t-1}\right)^{\gamma}}$. Since social learning models become intractable with dynamic payoffs, I assume a Markovian and backward looking structure in preferences for durable goods, where only the undepreciated level of durable good stock from last period matters for current decision in stage 1. This means that the durable good purchase decision is not forward looking and dynamic, but it is static. Nevertheless, there is some history dependence because the evolution of durable good stock over time, reflecting past durable good purchase decisions affects current preference for buying an additional vehicle, only through past durable good stock level. Without this history dependence, preference for buying an additional vehicle today is going to be independent of past durable good stock, which is inconsistent with the sort of behavior one observes in vehicle purchases.

$\tilde{Z}_t$ is the consumer sentiment term, which is modeled as observationally equivalent to a preference shock. However, the log of this object does not follow any known stochastic process such as AR-1, unlike a standard preference shock; the value of hidden state associated with this term is Z, which is fixed but due to learning dynamics, the beliefs about the value of this hidden state are dynamic and agents receive signals about the log of Z, so that $\mathbb{E}_t\tilde{Z} := \tilde{Z}_t$ is not fixed over time. This time varying expectation of the fixed hidden state is the *consumer sentiment* term, which enters the utility function. The reason for modeling consumer sentiment as a pure, preference term is that I am trying to model pure consumer sentiments rather than beliefs which reflect information about TFP for example. For example, political partisanship can create a rise in sentiment when incumbent is in line with one's partisanship as shown in Gillitzer and Prasad (2018) for example; the effects of these kind of payoff irrelevant sentiment shocks will be analogous to pure preference shocks. However, I have endogenous beliefs about this hidden preference state, which evolve in response to noisy information over time, implying that consumer sentiment or consumer beliefs are endogenous and are disciplined by the rationality of bayesian learning.

$U(C_t(h), N_t(h)) = \left(\frac{C_t(h)^{1-\sigma}}{1-\sigma} - \psi \frac{N_t(h)^{1+\eta}}{1+\eta}\right) \tilde{Z}_t + \mathbb{1}_{I_t(h)} \frac{\tilde{Z}_t}{\left((1-\delta)D_{h,t-1}\right)^{\gamma}}$

If $\gamma = 0$, past durable stock has no impact on current preference for purchasing/replacing the old vehicle, for instance. If $\gamma > 1$, then for $(1-\delta)D_{h,t-1} > 1$, higher durable stock from past reduces preference for new purchase today. The extent of diminishing marginal utility in consumption of old vehicles is parametrized by γ, where a higher value of γ means low rate of diminishing marginal utility, so that having high durable stock from past reduces current preference for buying more very strongly. γ as well as the depreciation rate δ must be disciplined from some microeconomic/behavioral evidence regarding preferences for cars or other vehicles and their depreciation rates, since my focus in on the auto industry (Pending). I define $\tilde{Z_{Dh,t}} := \frac{\tilde{Z}_t}{\left((1-\delta)D_{h,t-1}\right)^{\gamma}}$.

$U(C_t, N_t) := \int_0^1 U(C_t(h), N_t(h))dh = \tilde{Z}_t \int_0^1 \frac{C_t(h)^{1-\sigma}}{1-\sigma} dh - \tilde{Z}_t \psi \int_0^1 \frac{N_t(h)^{1+\eta}}{1+\eta} dh + \int_0^1 \mathbb{1}_{I_t(h)} \tilde{Z_{Dh,t}}$

$C_t(h) = C_t$, $N_t(h) = N_t$ and $D_{h,t-1} = D_{t-1}$, in equilibrium since all households are identical. Also, note that $I_t = \int_0^1 I_t(h)$. Then,

$U(C_t, N_t) := \int_0^1 U(C_t(h), N_t(h)) = \tilde{Z}_t \frac{C_t^{1-\sigma}}{1-\sigma} - \tilde{Z}_t \psi \frac{N_t^{1+\eta}}{1+\eta} + \tilde{Z_{D,t}} I_t$, where $\tilde{Z_{D,t}} := \frac{\tilde{Z}_t}{((1-\delta)D_{t-1})^{\gamma}}$.

The household is subject to the following nominal and real budget constraints:

$P_{C,t}C_t(h) + P_{D,t}I_t(h) + Q_t B_t \leq W_t N_t(h) + B_{t-1} + \Pi_t(h)$

$C_t(h) + P_{d,t}I_t(h) + \frac{Q_t B_t}{P_t} \leq w_t N_t(h) + \frac{B_{t-1}}{P_t} + \tilde{\Pi_t}(h)$

$P_{d,t} := \frac{P_{D,t}}{P_{C,t}}$

$C_t(h) + P_{d,t}I_t(h) + \frac{B_t}{P_t} \leq w_t N_t(h) + \frac{1+i_{t-1}}{1+\pi_t} \frac{B_{t-1}}{P_{t-1}} + \tilde{\Pi_t}(h)$

$I_t = \int_0^1 I_t(h)dh$

$D_t = (1-\delta)D_{t-1} + I_t$

6.2. Static Durable Good Purchase Decision

6.2.1. *Durable Good Purchase Condition*

Each durable good shopper $h \in [0, 1]$ invests in durable good in period t and in stage 1, by solving the following *static* problem, in which shopper h invests in one unit of durable good if and only if the following condition is true. The condition compares the utility value from buying the vehicle, with the price of one unit of vehicle, but the price is determined in general equilibrium during stage 2, so that it must be forecasted as of stage 1. This investment can be thought of as either purchasing a new vehicle/appliance or replacing/refurbishing an old one.

$\mathbb{E}_{ht}\left[Z\tilde{D_{h,t}}\right] \geq \mathbb{E}_{ht}\left[P_{dt}\right]$

I assume that signals are received about $z := log(\tilde{Z}_t)$ for convenience. In log deviations, the purchase condition becomes:

$\mathbb{E}_{ht}\left[z - \gamma(1-\delta)d_{h,t-1}\right] \geq \mathbb{E}_{ht}\left[\hat{P_{dt}}\right]$

Since $\mathbb{E}_{ht}\left[z\right] = p_{ht}z_h + (1-p_{ht})z_l$ based on private beliefs p_{ht},

$\implies p_{ht}z_h + (1-p_{ht})z_l \geq \mathbb{E}_{ht}\left(\hat{P_{d,t}}\right) + \gamma(1-\delta)d_{h,t-1}$

6.2.2. *Stage 1 Expectations of Stage 2 Policy Rules*

The form of model to be solved using Sims' method during stage 2 is Sims (2002) is $\Gamma_0 x_t = \Gamma_1 x_{t-1} + C + \psi e_t + \Pi\eta_t$, where e_t is an exogenous, vector stochastic process, η_t are time t prediction errors satisfying $\mathbb{E}_t\eta_{t+1} = 0$ and x_t contains the endogenous variables. Given the solution for this model, $\Theta_C = AC$, where Θ_C is vector of constants which result as an output of model in policy rules. C is the vector of constants, which was specified as an input to the model and A is a non-stochastic, known matrix, representing the mapping between input constants and output constants. This mapping will depend on model structure such as model equations and parametrization. Θ_C^k refers to the component of constant part of solution output which corresponds to policy rule for variable k.

Then, the policy rule to be predicted is $\hat{P_{d,t}} = \left(G_{1,pd}\hat{x_{t-1}}\right) + \left(\Theta_C^{pd}\right) + \psi_{pd}e_t$, where $\hat{x_{t-1}}$ contain model variables in log deviations. Below, I use the fact that $\mathbb{E}_{ht}\left(e_t\right) = 0$, since the stochastic processes are mean zero and their realizations occur in stage 2, which implies that their stage 1 expectations will be zero. The policy rule coefficients can be expressed as $G_{1,pd} = \overline{G_{1,pd}}\, cons$, where $\overline{G_{1,pd}}$ are mappings from *cons* to policy rule coefficients and are row vectors. These mappings are nonstochastic and known to agents as of stage 1 but *cons* contains p_{t+1}, which is not known as of stage 1, so that an expectation must be formed about $G_{1,pd}$, the actual policy coefficients, by multiplying the known mapping with the stage 1 expectation of *cons*. The durable good purchase condition then becomes:

$p_{ht}z_h + (1-p_{ht})z_l \geq \mathbb{E}_{ht}\left(\overline{G_{1,pd}}\, cons\, \hat{x_{t-1}}\right) + \mathbb{E}_{ht}\left(\Theta_C^{pd}\right) + \gamma(1-\delta)d_{h,t}$, where $cons := -\frac{\psi_c}{\sigma}\left(\lambda\left(p_{t+1}(z_H) + (1-p_{t+1})z_L\right) + (1-\lambda)\,\bar{z}\right)$.

Given the definition of C, which contains all zeros apart from i_y position[7], at which it contains $cons := -\frac{\psi_c}{\sigma}\left(\lambda\left(p_{t+1}(z_H)+(1-p_{t+1})z_L\right)+(1-\lambda)\,\bar{z}\right)$, $\mathbb{E}_{ht}\left(\Theta_C^{pd}\right)$ becomes:[8]

$\mathbb{E}_{ht}\left(\Theta_C^{pd}\right) = a_C^{pd}\mathbb{E}_{ht}\,cons.$

The mapping a_C^{pd} is fixed for given matrices Γ_0, Γ_1 and given structure of C, i.e it contains some constant $cons$ at i_y position and zeros everywhere else.

Then, it can be shown that[9]:

$\mathbb{E}_{ht}\left(\Theta_C^{pd}\right) = \tilde{a}_{pd}\mathbb{E}_{ht}\left(p_{t+1}z_h+(1-p_{t+1})z_l\right)+\hat{a}_{pd}\bar{z}-\tilde{a}_{pd}\bar{z}$

,where $\tilde{a}_{pd} := -a_C^{pd}\lambda\frac{\psi_c}{\sigma}$ and $\hat{a}_{pd} := -a_C^{pd}\frac{\psi_c}{\sigma}$.

Notice that if $t=1$, so that $t-1=0$, where 0 indicates the nonstochastic steady state, then $k_{t-1}=k^{ss}$, where k^{ss} is the steady state value of variable where $k \in \hat{x_{t-1}}$. $k^{ss}=0$, $\forall k$, since variables represent log deviations from steady state and these are zero in steady state, by definition. Thus, for $t=1$ and given the structure of policy rules for the linear system based on Sims' solver, the policy rule apart from constant term $G_{1,pd}x_{t-1}$ becomes the following:

$$G_{1,pd}^{t}x_0 = 0 \tag{5}$$

For $\forall t \geq 2$,

$$G_{1,pd}^{t}x_{t-1} = G_{1,pd}^{t}\left(G_1^{t-1}x_{t-2}+\Theta_C^{t-1}\right) \tag{6}$$

The policy rule coefficients $G_{1,pd}^t$, G_1^t and Θ_C^t are indexed by time t because they depend on the current, fixed, updated value of public beliefs p_t, which is used to solve the system and these beliefs evolve over time.

6.2.3. *Belief Updating Equations*

Using Bayes' rule and given the assumed signal structure, one gets the following equation for updating of public beliefs Schaal and Taschereau-Dumouchel (2020).

$\pi_{t+1}(\tilde{z},\tilde{\eta}) = \frac{\pi_t(\tilde{z},\tilde{\eta})f^{\epsilon}((d_t-d^e(\pi_t,\Omega_t,\tilde{z},\tilde{\eta})))}{\int \pi_t(z,\eta)f^{\epsilon}((d_t-d^e(\pi_t,\Omega_t,z,\eta)))d(z,\eta)}$

[7] i_y position means the position at which the Euler equation shows up in the model system. In my case, i_y = 2 and this system will be fully derived later.

[8] I formulate the general equilibrium model in Sims' form later and derive that matrix C indeed has this form.

[9] Derivation is provided in appendix.

$$\pi_{t+1}(\tilde{z},\tilde{\eta}) = \frac{\pi_t(\tilde{z},\tilde{\eta}) f^{\epsilon}((d_t - d^e(\pi_t,\Omega_t,\tilde{z},\tilde{\eta})))}{\int \pi_t(z_h,\eta) f^{\epsilon}((d_t - d^e(\pi_t,\Omega_t,z_h,\eta)))d\eta + \int \pi_t(z_l,\eta) f^{\epsilon}((d_t - d^e(\pi_t,\Omega_t,z_l,\eta)))d\eta} \tag{7}$$

The posterior is updated using Bayes' rule i.e $f^{\eta}_{t+1}(\eta|d_t) = \frac{f_t(d_t|\eta) f^{\eta}_t(\eta)}{f_t(d_t)}$, for $t \geq 0$. At $t = 0$, we have the prior. Ignoring the marginal distribution, as is standard practice in bayesian econometrics and statistics due to intractability of the marginal, we get:

$f^{\eta}_{t+1}(\eta|d_t) \propto f_t(d_t|\eta) f^{\eta}_t(\eta)$, for $t \geq 0$.

For agents with private signals, information is updated about η by also using the private information, in addition to using the current, publicly updated posterior about η, which leads to further updating of posterior:

$f^{\eta}_{sh,t+1} := f^{\eta}_{t+1}(\eta|s_h,d_t) = \frac{f_{t+1}(s_h|\eta) f^{\eta}_{t+1}(\eta|d_t)}{f_{t+1}(s_h)}$, for $t \geq 0$.

$f^{\eta}_{t+1}(\eta|s_h,d_t) \propto f(s_h|\eta) f^{\eta}_{t+1}(\eta|d_t)$, for $t \geq 0$.

Given that p_{t+1}, which is the public belief that state is high is $\int \pi_{t+1}(z_H,\eta)d\eta$ and $\int \pi_t(z_h,\eta)d\eta = p_t f^{\eta}_t(\eta)$ and $\int \pi_t(z_l,\eta)d\eta = (1-p_t) f^{\eta}_t(\eta)$, it follows that

$p_{t+1} = \frac{\int \pi_t(z_h,\eta) f^{\epsilon}((d_t - d^e(\pi_t,\Omega_t,z_h,\eta)))d\eta}{\int \pi_t(z_h,\eta) f^{\epsilon}((d_t - d^e(\pi_t,\Omega_t,z_h,\eta)))d\eta + \int \pi_t(z_l,\eta) f^{\epsilon}((d_t - d^e(\pi_t,\Omega_t,z_l,\eta)))d\eta}$

It can then also be be shown that[10],

$$\mathbb{E}_{ht} p_{t+1} = \mathbb{E}_{ht}\left(\frac{1}{1 + \frac{(1-p_t)\int f^{\eta}_t(\eta) f^{\epsilon}\left(\left(d_t - F^{bar}_{z_l+\eta}(\hat{s}_t)\right)\right)d\eta}{p_t \int f^{\eta}_t(\eta) f^{\epsilon}\left(\left(d_t - F^{bar}_{z_h+\eta}(\hat{s}_t)\right)\right)d\eta}}\right).$$

I define

$$\Gamma(\hat{s}_t, s_h, p_t) := \mathbb{E}_{ht}\left(\frac{1}{1 + \frac{(1-p_t)\int f^{\eta}_t(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(d_t - F^{bar}_{z_l+\eta}(\hat{s}_t)(1-\mu)\right)\right)d\eta}{p_t \int f^{\eta}_t(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(d_t - F^{bar}_{z_h+\eta}(\hat{s}_t)(1-\mu)\right)\right)d\eta}}\right) \tag{8}$$

where $\hat{s}_t := \hat{s}_t(\pi_t, \Omega_t)$ is the cut off function.

The private beliefs are updated from public beliefs using private signals during stage 1 in the following manner:

[10]Derivations are provided in appendix.

$$\Psi(s_h, p_t) := p_{ht} = \frac{1}{1 + \frac{(1-p_t)\int f^{\eta}_{sh,t}(\eta) f_{z_l+\eta}(s_h) d\eta}{p_t \int f^{\eta}_{sh,t}(\eta) f_{z_h+\eta}(s_h) d\eta}} \tag{9}$$

6.2.4. Evaluating the Gamma Function

The Gamma function is evaluated by durable good shoppers in the following manner, which represents their stage 1 expectation of updated public beliefs p_{t+1} in stage 2, which will be updated after observing d_t at that stage. These updated public beliefs enter the Sims' solver in stage 2. Durable good shoppers know the updating equation which will be used in stage 2; the updating equation is a nonlinear mapping from the random variables η and ϵ to p_{t+1}, given that p_t is known. The durable good shoppers will have to integrate out this updating equation with respect to the two densities f^{ϵ} and f^{η} for random variables for noise terms, the realizations of which are yet to occur in stage 2. I use the general rule from probability theory Grinstead and Snell (2012) that $\mathbb{E}(G(x,y)) = \int\int G(x,y) f(x) h(y) dx\, dy$, where x and y are mutually independent random variables with densities $f(x)$ and $h(y)$; $G(.)$ is some absolutely continuous function, from $\mathbb{R}^2$ to $\mathbb{R}^1$. Since the posterior for f^{η} is updated over time, I use the current, most updated, posterior at time t in order to integrate out with respect to η. The gamma function can then be expressed as the following, where $\overline{F} = 1 - F$ and F is the CDF of the signal distribution.

$$\Gamma(\hat{s_t}, s_h, p_t) = \Psi(p_t, s_h) \int\int \left(\frac{1}{1 + \frac{(1-p_t)\int f^{\eta}_t(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(d_{ht} - \overline{F}_{z_l+\eta}(\hat{s_t})(1-\mu)\right)\right) d\eta}{p_t \int f^{\eta}_t(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(d_{ht} - \overline{F}_{z_h+\eta}(\hat{s_t})(1-\mu)\right)\right) d\eta}} \right) f^{\epsilon}(\epsilon) f^{\eta}_{t,s_h}(\eta) d\epsilon\, d\eta\, +$$

$$(1 - \Psi(p_t, s_h)) \int\int \left(\frac{1}{1 + \frac{(1-p_t)\int f^{\eta}_t(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(d_{lt} - \overline{F}_{z_l+\eta}(\hat{s_t})(1-\mu)\right)\right) d\eta}{p_t \int f^{\eta}_t(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(d_{lt} - \overline{F}_{z_h+\eta}(\hat{s_t})(1-\mu)\right)\right) d\eta}} \right) f^{\epsilon}(\epsilon) f^{\eta}_{t,s_h}(\eta) d\epsilon\, d\eta$$

,where $\tilde{d_{kt}} := (1-\mu)\overline{F}_{z_k+\eta}(\hat{s})$, for $k \in \{l, h\}$.

and $d_{kt} := \tilde{d_{kt}} + \mu\epsilon_t$.

Since I have a parametric, functional form for the static, stage 1 expectation of stage 2 beliefs p_{t+1}, the gamma function is related to the PEA or parametrized expectations approach to solving DSGE models Den Haan and Marcet (1990), Christiano and Fisher (2000). However, in my model, the functional form for Γ is known since the law of motion for beliefs is public knowledge, unlike the case of PEA in which one has to use function approximation methods such as spectral methods using chebyshev polynomials to approximate the unknown conditional expectation function. Another difference is that the expectation to be evaluated here is much simpler since it is static and not dynamic as in the use of PEA. On the other hand, the actual updating of beliefs in stage 2 from previous public beliefs p_t to new beliefs p_{t+1} is governed by the following function f, where $\hat{s}$ is determined from the most recent/previous stage 1. The following equation is valid for the case when the true state is low or equal to z_l and shocks other than noise shock such as ϵ have been set equal to prior value of zero[11], since I am doing impulse response analysis and this equation operates in stage 2 so that shock realization for ϵ in stage 2 has to be used, which is zero rather than stage 1 shock expectation.

$$p_{t+1} = f(\hat{s}, p_t) = \frac{1}{1+\frac{(1-p_t)\int f_t^{\eta}(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(\tilde{d_{lt}} - \overline{F}_{z_l+\eta}(\hat{s_t})(1-\mu)\right)\right) d\eta}{p_t \int f_t^{\eta}(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(\tilde{d_{lt}} - \overline{F}_{z_h+\eta}(\hat{s_t})(1-\mu)\right)\right) d\eta}}$$

Thus, the two functions are distinct. $\mathbb{E}_{ht}[p_{t+1}] = \Gamma(\hat{s_t}, s_h, p_t) \neq p_{t+1} = f(\hat{s}_t, p_t), \forall s_h$[12]. If one were to define the aggregate stage 1 expectation across durable shoppers as $\overline{\mathbb{E}_t[p_{t+1}]} := \int \Gamma(\hat{s_t}, s_h, p_t) f(s_h) ds_h$, it is still true that $\overline{\mathbb{E}_t[p_{t+1}]} \neq p_{t+1} = f(\hat{s}_t, p_t)$. This implies a deviation from RE which arises because the assumptions about timing, information and stage structure imply that the static, private, stage 1 expectation of durable shoppers about updated public beliefs in stage 2 will be formed based on the Γ function and not f function, which governs the actual updating of public beliefs from p_t to p_{t+1}. Thus, durable shoppers have rational expectations about structural parameters and equilibrium mappings but not public beliefs in next stage.

6.2.5. *Cutoff Functions*

Given an estimate of the cut off function's form, predetermined, public beliefs p_t and state variables Ω_t, the right hand side of the durable good purchase condition can then be approximated by the durable good shoppers,

The condition for purchasing durable good stock was:

$$p_{ht} z_h + (1-p_{ht}) z_l \geq \mathbb{E}_{ht}\left(G_{1,pd}\, cons\, \hat{x_{t-1}}\right) + \mathbb{E}_{ht}\left(\Theta_C^{pd}\right) + \gamma(1-\delta)\hat{D_{t-1}}$$

Then, using the previously derived expressions of $\mathbb{E}_{ht}\left[\Theta_C^{pd}\right]$ and $\mathbb{E}_{ht}\left[G_{1,pd}\hat{x_{t-1}}\right]$, the condition becomes:

$$p_{ht} z_h + (1-p_{ht}) z_l \geq \left(\overline{G_{1,pd}}\hat{x_{t-1}}\right)\mathbb{E}_{ht}\, cons + a_C^{pd}\mathbb{E}_{ht}\, cons + \gamma(1-\delta)\hat{D_{t-1}}$$

[11] This is why $\tilde{d_{lt}}$ shows up in this equation rather than d_{lt}.

[12] Every durable good shopper along the continuum has a distinct, private signal which means that every shopper has a distinct private expectation. Even if one were to consider the private expectation of the shopper for whom $\hat{s} = s_h$, $.\Gamma(\hat{s_t}, \hat{s_t}, p_t) \neq f(\hat{s}_t, p_t)$.

,where $a_C^{pd}\mathbb{E}_{ht}\, cons := \left(\tilde{a}_{pd}\mathbb{E}_{ht}\left(p_{t+1}z_h + (1-p_{t+1})z_l\right) + \hat{a}_{pd}\bar{z} - \tilde{a}_{pd}\bar{z}\right)$, as also derived in last section.

I define the cutoff function as the following:

$$c\left(p_t, s_h, \Omega_t\right) := \left(\overline{G_{1,pd}} x_{\hat{t-1}}\right)\mathbb{E}_{ht}\, cons + + \left(\tilde{a}_{pd}\mathbb{E}_{ht}\left(p_{t+1}z_h + (1-p_{t+1})z_l\right) + \hat{a}_{pd}\bar{z} - \tilde{a}_{pd}\bar{z}\right) + \gamma(1-\delta)D_{\hat{t-1}} \quad (10)$$

The gamma function can then be used to evaluate $\mathbb{E}_{ht}\,[p_{t+1}]$ and hence $\mathbb{E}_{ht}\,[cons]$ in the above expression and therefore, the cutoff function can be computed or approximated in any stage 1 by the durable good shoppers.

Using the definition of cutoff function and previously defined Ψ function for private beliefs, the durable good purchase condition can be expressed compactly as:

$$\Psi(s_h, p_t)\left(z_h - z_l\right) + z_l \geq c\left(p_t, s_h, \Omega_t\right)$$
$$\iff \Psi(s_h, p_t) \geq \frac{c(p_t, s_h, \Omega_t) - z_l}{z_h - z_l}$$
$$\iff \frac{1}{1 + \frac{(1-p_t)\int f^{\eta}_{sh,t}(\eta) f_{z_l+\eta}(s_h)d\eta}{p_t \int f^{\eta}_{sh,t}(\eta) f_{z_h+\eta}(s_h)d\eta}} \geq \tilde{c}\left(p_t, s_h, \Omega_t\right), \text{ where } \tilde{c}\left(p_t, s_h, \Omega_t\right) := \frac{c(p_t, s_h, \Omega_t) - z_l}{z_h - z_l}.$$
$$\iff 1 \geq \tilde{c}\left(p_t, s_h, \Omega_t\right)\left(1 + \frac{(1-p_t)\int f^{\eta}_{sh,t}(\eta) f_{z_l+\eta}(s_h)d\eta}{p_t \int f^{\eta}_{sh,t}(\eta) f_{z_h+\eta}(s_h)d\eta}\right)$$

For $t = 1$, $f_t^{\eta}(\eta)$ is normally distributed with mean zero and variance σ_η, which is the prior distribution. However, the posterior is not necessarily Gaussian in subsequent periods after Bayesian updating. For $t = 0$, $\int f_t^{\eta}(\eta) f^s_{z_k+\eta}(s_h)d\eta = pdf^{\tau_k}(s_h)$[13], where $\tau_k \sim \mathcal{N}(z_k, \sigma_s + \sigma_\eta)$ because of the convolution property of two Gaussians Rice (2006). In terms of standard normal density ϕ, $pdf^{\tau_k}(s_h) = \phi\left(\frac{s_h - z_k}{\sqrt{\sigma_\eta + \sigma_s}}\right)$, for $k \in \{l, h\}$ and the following analytical simplifications can then be obtained. The condition for purchasing durables now becomes:

$$1 \geq \tilde{c}\left(p_t, s_h, \Omega_t\right)\left(1 + \frac{(1-p_t)\phi\left(\frac{s_h - z_l}{\sqrt{\sigma_\eta^2 + \sigma_s^2}}\right)}{p_t \phi\left(\frac{s_h - z_h}{\sqrt{\sigma_\eta^2 + \sigma_s^2}}\right)}\right)$$

At the point of indifference between investing and not investing:

[13] The notation means density function of random variable τ_k, evaluated at s_h.

$$\left(\frac{1-\tilde{c}(p_t,s_h,\Omega_t)}{\tilde{c}(p_t,s_h,\Omega_t)}\right)\frac{p_t}{1-p_t}=\left(\frac{\phi\left(\frac{s_h-z_l}{\sqrt{\sigma_\eta^2+\sigma_s^2}}\right)}{\phi\left(\frac{s_h-z_h}{\sqrt{\sigma_\eta^2+\sigma_s^2}}\right)}\right)$$

One can solve the above equation for the value of cut off $\hat{s}$ for signal, at which this equality holds. The cut off $\hat{s}$ is implicitly defined by the value which leads to the following equality or the root of the G function, given Ω_t and π_t (holding as close as possible, in numerical terms if one uses numerical techniques for finding roots). The G function's form is derived in appendix.

$G\left(p_t,\hat{s}\left(p_t,\Omega_t\right),\Omega_t\right)\approx 0$

where,

$$G\left(p_t,\hat{s}\left(p_t,\Omega_t\right),\Omega_t\right):=\left\{\log\left(\left(\frac{1-\tilde{c}(p_t,\hat{s},\Omega_t)}{\tilde{c}(p_t,\hat{s},\Omega_t)}\right)\frac{p_t}{1-p_t}\right)-\frac{1}{2\left(\sigma_\eta^2+\sigma_s^2\right)}\left(z_h^2-z_l^2\right)\right\}\frac{\left(\sigma_\eta^2+\sigma_s^2\right)}{z_l-z_h}-\hat{s}\tag{11}$$

For $t>1$, the convolution property does not necessarily hold and the integral of product of two densities can no longer be equated with a convoluted Gaussian distribution. In this case, the H function for which one must find the root is derived as the following. The indifference condition is:

$$\left(\frac{1-\tilde{c}(p_t,s_h,\Omega_t)}{\tilde{c}(p_t,s_h,\Omega_t)}\right)\frac{p_t}{1-p_t}=\left(\frac{\int f^\eta_{sh,t}(\eta)f_{z_l+\eta}(s_h)d\eta}{\int f^\eta_{sh,t}(\eta)f_{z_h+\eta}(s_h)d\eta}\right)$$

$$H\left(\hat{s},p_t,\Omega_t\right):=\left(\frac{1-\tilde{c}(p_t,\hat{s},\Omega_t)}{\tilde{c}(p_t,\hat{s},\Omega_t)}\right)\frac{p_t}{1-p_t}-\left(\frac{\int f^\eta_{\hat{s},t}(\eta)f_{z_l+\eta}(\hat{s})d\eta}{\int f^\eta_{\hat{s},t}(\eta)f_{z_h+\eta}(\hat{s})d\eta}\right)$$

One must find $\hat{s}\left(p_t,\Omega_t\right)$: such that $H\left(\hat{s},p_t,\Omega_t\right)\approx 0$. In this case, one has to use numerical integration to evaluate the second expression in G function above and not just the cutoff function, as in the G function case so that the answer will be less accurate or less exact.

6.2.6. Numerical Solution

I use fixed point iteration method in order to numerically solve for the signal cutoff in model, i.e the root of the function H or G defined above, depending on the value of t. For any function $F(x)$, for which I have to solve the equation $F(x)=0$, fixed point iteration method posits another function $P(y):=y-F(y)$. Using fixed point iteration, one tries to find a point y^*, such that $P(y^*)=y^*$, i.e a fixed point of the function P. This implies that $y^*-F(y^*)=y^*$, which then implies that $F(y^*)=0$. In other words, a fixed point of the function P is the root of function F.

I evaluate the integrals in all these expressions using numerical integration. For the baseline, I just use the simpler composite Simpson's rule or Newton cote formulas. I check for robustness of results to different number of grid points for integration N and more accurate methods for numerical integration such as Gauss Hermite integration since I am working with gaussian densities. I use end points, b and a, based on $+$ and -3 standard deviations of the mean of the relevant densities. I use the following composite simpson's rule to evaluate the integrals:

Theorem 1 ***Composite Simpson's Rule:***
Let $f \in C^4[a,b]$, let n be even, $h := \frac{b-a}{n}$, and $x_j := a + jh$, then the composite simpson's rule for n equally spaced subintervals Ökten (2019) is:
$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^4(\zeta)$, *for some* $\zeta \in (a,b)$.

7. HOUSEHOLDS

The Lagrangian for representative household's problem in stage 2, when durable good shoppers have already taken their now irreversible decisions is:

$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma}\tilde{Z}_t - \psi \tilde{Z}_t \frac{N_t^{1+\eta}}{1+\eta} + \lambda_t \left(W_t N_t + B_{t-1} + \Pi_t - P_{C,t}C_t - P_{D,t}I_{d,t} - Q_t B_t\right)\right)$

The FOC's with respect to nondurable consumption and labor supply are:

$C_t^{-\sigma}\tilde{Z}_t = \lambda_t P_{C,t}$

$\psi N_t^{\eta}\tilde{Z}_t = \lambda_t W_t$

The Euler equation for consumption can then be expressed as the following, as is standard practice for New Keynesian models. Nevertheless, all derivations are provided in appendix for completion.

$$Q_t = \beta \mathbb{E}_t \left(\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \left(\frac{P_{C,t}}{P_{C,t+1}}\right) \left(\frac{\tilde{Z_{t+1}}}{\tilde{Z}_t}\right) \right) \tag{12}$$

In equilibrium $B_t = 0$, so that from budget constraint, we have that:

$w_t \left(\frac{w_t C_t^{-\sigma}}{\psi}\right)^{\frac{1}{\eta}} + \frac{\Pi_t}{P_{C,t}} = C_t + P_{D,t}I_{d,t}$ which implies that nondurable consumption will be effected by the predetermined value of $I_{d,t}$ (investment level of durables from stage 1) because of the budget constraint and it acts as isomorphic to a negative wealth effect from stage 1 on stage 2, given that this predetermined quantity of durables must be purchased in stage 2 at the stage 2 market prices.

8. FIRMS

There is a continuum of monopolistically, competitive firms within each sector $\omega \in \{C, D\}$ and each of them produces a differentiated good, using the same technology:

$Y_t(i)^{\omega} = A_t(N_t(i)^{\omega})^{1-\alpha}$

A competitive, final good firm bundles these varieties using a Dixit Stiglitz aggregator

$Y_t^{\omega} = \left(\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} \, di\right)^{\frac{\epsilon}{\epsilon-1}}$

Firms set prices in a staggered fashion a'la Calvo (1983), i.e. firms can reoptimize prices with probability $1-\theta$ each period and therefore take into account that they may not be able to adjust prices in the next period. Firms set prices to maximize expected profits subject to the demand schedule.

The cost minimizing problems solved by firms in the two sectors are the following:

$Min_{\{N_t^D(j)\}} \, W_t N_t^D(j)$

subject to $A_t(N_{t,j}^D)^{1-\alpha} \geq I_t \left(\frac{P_{d,t}(j)}{P_{d,t}}\right)^{-\epsilon_d}$ (Demand Meeting Constraint)

$Min_{\{N_t^C(j)\}} \, W_t N_t^C(j)$

$A_t(N_{t,j}^C)^{1-\alpha} \geq C_t \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_c}$ (Demand Meeting Constraint)

The solution to these minimization problems are:

$\lambda_t^D = \frac{W_t^D}{A_t(1-\alpha)(N_{j,t}^D)^{-\alpha}}$ or

$\frac{W_t^D}{MPL^D} = \lambda_t^D$

$\frac{W_t^C}{MPL^C} = \lambda_t^C$

The expressions for profits in both sectors can be expressed as the following:

$\Pi_{j,t}^C = \frac{P_{j,t}^C \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_c} (Y_t^C)}{P_t^C} - mc_t^C(1-\alpha)\left(\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_c} (Y_t^C)\right)$

$\Pi_{j,t}^D = \frac{P_{j,t}^D Y_{j,t}^D}{P_t^D} - mc_t^D(1-\alpha)\left(D_t\left(\frac{P_{d,t}(j)}{P_{d,t}}\right)^{-\epsilon_d}\right)$, where mc_t^k is the real marginal cost in sector k.

Each firm in each sector solves the following profit maximizing problem, where $\Lambda_{t,t+1}$

is the stochastic, discount factor, subject to the respective demand constraints. In non-durable consumption sector (C),

$$Max_{P^c_{j,t}}\; E_t \left\{ \sum_{s=0}^{\infty} (\phi)^s \Lambda_{t,t+s} \left(\Pi^{c,N}_{j,t+s} \right) \right\} =$$

$$Max_{P^c_{j,t}}\;\; E_t \left\{ \sum_{s=0}^{\infty} (\phi)^s \Lambda_{t,t+s} \left(\left(P^{1-\epsilon_c}_{c,j,t} \right) (P_{c,t+s})^{\epsilon_c} (Y^c_t) - \lambda^c_t \left(\left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon_c} (Y^c_t) \right) \right) \right\}$$

$$P^c_{j,t} = \frac{\epsilon_c}{\epsilon_c - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left((\phi\beta)^s U'(C_{t+s}) mc^c_{t+s} P^{\epsilon_c}_{c,t+s} Y^c_{t+s} \right)}{\mathbb{E}_t \sum_{s=0}^{\infty} \left((\phi\beta)^s U'(C_{t+s}) P^{\epsilon_c - 1}_{c,t+s} Y^c_{t+s} \right)} \tag{13}$$

$\iff\; P^c_{j,t} = \frac{\epsilon_c}{\epsilon_c - 1} \frac{S_{1,t}}{S_{2,t}}$

$S_{1,t} := U'(C_t) mc^c_t P^{\epsilon_c}_{c,t} Y^c_t + \phi\beta \mathbb{E}_t S_{1,t+1}$

$s_{1,t} := \frac{S_{1,t}}{P^{\epsilon_c}_{c,t}} = C^{-\sigma}_t mc^c_t Y^c_t + \phi\beta \mathbb{E}_t s_{1,t+1} \left(1 + \pi^c_{t+1} \right)^{\epsilon_c}$

$S_{2,t} := U'(C_t) P^{\epsilon_c - 1}_{c,t} Y^c_t + \phi\beta \mathbb{E}_t S_{2,t+1}$

$s_{2,t} := \frac{S_{2,t}}{P^{\epsilon_c - 1}_{c,t}} = C^{-\sigma}_t Y^c_t + \phi\beta \mathbb{E}_t s_{2,t+1} \left(1 + \pi^c_{t+1} \right)^{\epsilon_c - 1}$

$\implies \frac{S_{1,t}}{S_{2,t}} = P_{c,t} \frac{s_{1,t}}{s_{2,t}}$

Therefore, the optimal reset price can be written as:

$P^c_{j,t} = \frac{\epsilon_c}{\epsilon_c - 1} P_{c,t} \frac{s_{1,t}}{s_{2,t}}$

Then, divide by $P_{c,t-1}$ to get an expression in terms of inflation,

$(1 + \pi^*_{c,t}) = \frac{\epsilon_c}{\epsilon_c - 1} (1 + \pi_{c,t}) \frac{s_{1,t}}{s_{2,t}}$

If $\phi_C = 0$

$P^c_{j,t} = \mathcal{M}^c \lambda^c_t$ where λ^c_t is the nominal marginal cost and $\mathcal{M}^c = \frac{\epsilon_c}{\epsilon_c - 1}$ is the markup.

Similarly, in durable good sector,

$Max_{P_{d,t}(j)}\, \mathbb{E}_t \left\{ \sum_{s=0}^{\infty} (\phi)^s \Lambda_{t,t+s} \left(\Pi^{D,N}_{j,t+s} \right) \right\}$ leads to the following condition, where I have used that $U'(D_{t+s}) = \frac{\tilde{Z}}{((1-\delta) D_{t+s-1})^{\gamma}}$,

$$P^d_{j,t} = \frac{\epsilon_d}{\epsilon_d - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \left((\phi\beta)^s \frac{\tilde{Z}}{((1-\delta) D_{t+s-1})^{\gamma}} mc^d_{t+s} P^{\epsilon_d}_{d,t+s} Y^d_{t+s} \right)}{\mathbb{E}_t \sum_{s=0}^{\infty} \left((\phi\beta)^s \frac{\tilde{Z}}{((1-\delta) D_{t+s-1})^{\gamma}} P^{\epsilon_d - 1}_{d,t+s} Y^d_{t+s} \right)} \tag{14}$$

$\implies P^d_{j,t} = \frac{\epsilon_d}{\epsilon_d - 1} \frac{S_{3,t}}{S_{4,t}}$

$S_{3,t} := \frac{\tilde{Z}}{((1-\delta)D_{t-1})^{\gamma}} mc_t^d P_{d,t}^{\epsilon_d} Y_t^d + \phi_D \beta \mathbb{E}_t S_{3,t+1}$

$s_{3,t} := \frac{S_{3,t}}{P_{d,t}^{\epsilon_d}} = \mathbb{E}_t \left[\frac{\tilde{Z}}{((1-\delta)D_{t-1})^{\gamma}}\right] mc_t^d Y_t^d + \phi_D \beta \mathbb{E}_t s_{3,t+1} \left(1+\pi_{t+1}^d\right)^{\epsilon_d}$

$S_{4,t} := \frac{\tilde{Z}}{((1-\delta)D_{t-1})^{\gamma}} P_{d,t}^{\epsilon_d - 1} Y_t^d + \phi_D \beta \mathbb{E}_t S_{4,t+1}(1+\pi_{c,t}^*)$

$s_{4,t} := \frac{S_{4,t}}{P_{d,t}^{\epsilon_d - 1}} = \mathbb{E}_t \left[\frac{\tilde{Z}}{((1-\delta)D_{t-1})^{\gamma}}\right] Y_t^d + \phi_D \beta \mathbb{E}_t s_{4,t+1} \left(1+\pi_{t+1}^d\right)^{\epsilon_d - 1}$

$s_{4,t} := \frac{S_{4,t}}{P_{c,t}^{\epsilon_d - 1}}$

As before, an equation in terms of inflation can be derived as the following:

$(1+\pi_{d,t}^*) = \frac{\epsilon_d}{\epsilon_d - 1}(1+\pi_{d,t})\frac{s_{3,t}}{s_{4,t}}$

If $\phi_D = 0$,

$P_{j,t}^d = \frac{\epsilon_d}{\epsilon_d - 1}\lambda_t^d$

$P_{j,t}^d = \mathcal{M}^d \lambda_t^d$ where λ_t^d is the nominal marginal cost.

The price dispersion terms are defined as:

$\Phi_c := \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_c} dj$

$\Phi_d := \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_d} dj$

Then, the expressions for aggregate production in each sector can be expressed as the following, which accounts for the distortion created because of the price rigidity:

$$Y_t^C = \frac{N_{t,C}^{1-\alpha} A_t}{\Phi_c} \tag{15}$$

$$Y_t^D = \frac{N_{t,D}^{1-\alpha} A_t}{\Phi_d} \tag{16}$$

The market clearing conditions in production sector and labor markets are the following:

$Y_t = \frac{P_{c,t}}{P_t} Y_t^c + \frac{P_{d,t}}{P_t} Y_t^d = \frac{P_{c,t}}{P_t} C_t + \frac{P_{d,t}}{P_t} I_t^d$

$N_t^D = \int_0^1 N_t(j)^C dj$

$N_t^C = \int_0^1 N_t(j)^D dj$

$N_t = N_t^D + N_t^C$ (Labor Supply = Labor Demand Equation)

The aggregate price index in the nondurable consumption sector can be expressed as the following:

$$P_{C,t} = \left[(1-\phi_C) \left(P^*_{C,t} \right)^{1-\epsilon_c} + \phi_C P^{1-\epsilon_c}_{C,t-1} \right]^{\frac{1}{1-\epsilon_c}} \tag{17}$$

The dynamics of the price dispersion are governed by the following equation:

$$\Phi_{c,t} = (1-\phi_C)(1+\pi^*_{c,t-1})^{-\epsilon_c}(1+\pi_{c,t-1})^{\epsilon_c} + \phi_C(1+\pi_{c,t-1})^{\epsilon_c}\Phi_{c,t-1} \tag{18}$$

Similarly in the durable good sector,

$$P_{D,t} = \left[(1-\phi_D) \left(P^*_{D,t} \right)^{1-\epsilon_d} + \phi_D P^{1-\epsilon_d}_{D,t-1} \right]^{\frac{1}{1-\epsilon_d}}$$

$$\Phi_{D,t} = (1-\phi_D)(1+\pi^*_{D,t-1})^{-\epsilon_d}(1+\pi_{D,t-1})^{\epsilon_d} + \phi_D(1+\pi_{D,t-1})^{\epsilon_d}\Phi_{D,t-1}$$

I define the aggregate inflation by aggregating inflation in the two sectors such that $\pi_t := \pi^{\psi_D}_{D,t}\pi^{1-\psi_D}_{C,t}$, where ψ_D is steady state aggregate output share of durable goods sector. I have omitted many of the derivations for equations in this section because they are fairly standard but all of them are provided in appendix.

9. Monetary Policy

I suppose that monetary policy follows a simple interest rate rule of the following kind:

$i_t = \varrho + \varrho_i i_{t-1} + (1-\varrho_i)(\phi_\pi \pi_t + \phi_y \hat{y}_t) + v_t$, where $\hat{y}_t = y_t - y$ is log deviation of output from its steady state value. $v_t = \varrho_v v_{t-1} + \epsilon^v_t$, where $\epsilon^v_t \sim N(0, \sigma_v)$ and is iid over time. This is the AR(1) monetary policy shock and ϱ_i is degree of interest rate smoothing; $\varrho_i \in [0,1]$.

The collection of all nonlinear equilibrium conditions is the following:

10. Equilibrium Conditions

$$Q_t = \beta \mathbb{E}_t \left(\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{P_{C,t}}{P_{C,t+1}} \right) \left(\frac{\tilde{Z}_{t+1}}{\tilde{Z}_t} \right) \right) \tag{19}$$

$$\frac{W_t}{P_{C,t}} = \psi N^\eta_t C^\sigma_t \tag{20}$$

$$mc_t^{\omega} = \frac{W_t}{MPL^{\omega}},\ \omega \in \{C, D\}. \tag{21}$$

$$Y_t = \frac{P_{C,t}}{P_t} Y_t^c + \frac{P_{D,t}}{P_t} Y_t^d = \frac{P_{C,t}}{P_t} C_t + \frac{P_{D,t}}{P_t} I_t^d \tag{22}$$

$$Y_t^{\omega} = \frac{A_t N_t^{1-\alpha}}{\Phi_{\omega,t}},\ \omega \in \{C, D\}. \tag{23}$$

$$\Phi_{\omega,t} = (1-\phi_{\omega})(1+\pi_{\omega,t}^*)^{-\epsilon_{\omega}}(1+\pi_{\omega,t})^{\epsilon_{\omega}} + \phi_{\omega}(1+\pi_{\omega,t})^{\epsilon_{\omega}}\Phi_{\omega,t-1},\ \omega \in \{C, D\}. \tag{24}$$

$$(1+\pi_{\omega,t+1})^{1-\epsilon_{\omega}} = (1-\phi_{\omega})(1+\pi_{\omega,t}^*)^{1-\epsilon_{\omega}} + \phi_{\omega},\ \omega \in \{C, D\}. \tag{25}$$

$$(1+\pi_{c,t}^*) = \frac{\epsilon_c}{\epsilon_c - 1}(1+\pi_{c,t})\frac{s_{1,t}}{s_{2,t}} \tag{26}$$

$$(1+\pi_{d,t}^*) = \frac{\epsilon_d}{\epsilon_d - 1}(1+\pi_{d,t})\frac{s_{3,t}}{s_{4,t}} \tag{27}$$

$$s_{1,t} := \frac{S_{1,t}}{P_{c,t}^{\epsilon_c}} = C_t^{-\sigma} mc_t^c Y_t^c + \phi_C \beta \mathbb{E}_t s_{1,t+1}\left(1+\pi_{t+1}^c\right)^{\epsilon_c} \tag{28}$$

$$s_{2,t} := \frac{S_{2,t}}{P_{c,t}^{\epsilon_c - 1}} = C_t^{-\sigma} Y_t^c + \phi_C \beta \mathbb{E}_t s_{2,t+1}\left(1+\pi_{t+1}^c\right)^{\epsilon_c - 1} \tag{29}$$

$$s_{3,t} = \mathbb{E}_t\left[\frac{\tilde{Z}}{((1-\delta)D_{t-1})^{\gamma}}\right] mc_t^d Y_t^d + \phi_D \beta \mathbb{E}_t s_{3,t+1}\left(1+\pi_{t+1}^d\right)^{\epsilon_d} \tag{30}$$

$$s_{4,t} = \mathbb{E}_t\left[\frac{\tilde{Z}}{((1-\delta)D_{t-1})^{\gamma}}\right] Y_t^d + \phi_D \beta \mathbb{E}_t s_{4,t+1}\left(1+\pi_{t+1}^d\right)^{\epsilon_d - 1} \tag{31}$$

$$P_{\omega,t} = \left[(1-\phi_{\omega})\left(P_{\omega,t}^*\right)^{1-\epsilon_{\omega}} + \phi_{\omega} P_{\omega,t-1}^{1-\epsilon_{\omega}}\right]^{\frac{1}{1-\epsilon_{\omega}}},\omega \in \{C, D\}. \tag{32}$$

$$\pi_t = \pi_{D,t}^{\psi_D}\pi_{C,t}^{1-\psi_D} \tag{33}$$

11. LOG LINEARIZATION

I use the following theorem Mertens (2015) for log linearization:

Theorem 2 *For any multivariable function $z = f(x_1, x_2, ..., x_n)$, around $(x_1^0, x_2^0, ..., x_n^0)$ (non-stochastic steady state) and given that $z_0 := f(x_1^0, x_2^0, ..., x_n^0)$ and $\hat{z} := \log\left(\frac{z}{z_0}\right) \approx \frac{z-z_0}{z_0}$, $\hat{x}_i := \log\left(\frac{x_i}{x_i^0}\right)$,*

$$\hat{z} \quad = \quad \log\left(\frac{z}{z_0}\right) \quad = \quad \sum_i \zeta f_i.\hat{x}_i,$$

where $\zeta f_i := \frac{f_i(x_1^0,...,x_n^0).x_i^0}{z_0}$ and f_i is the partial derivative of f with respect to i^{th} argument.

Applying the previously stated theorem, the log linearization of $w_t = \psi N_t^\eta C_t^\sigma$ is the following:

$\hat{w}_t = \zeta f_N.\hat{n} + \zeta f_C.\hat{c}$, $\zeta f_N = \frac{\psi\eta N^{\eta-1}C^\sigma N}{w}$, $\zeta f_C = \frac{\psi\sigma C^{\sigma-1}N^\eta C}{w}$

$\hat{w}_t = \frac{\psi\eta N^\eta C^\sigma}{w}\hat{n} + \frac{\psi\sigma C^\sigma N^\eta}{w}.\hat{c}$

Since $w = \psi N^\eta C^\sigma$, then $\hat{w}_t = \eta\hat{n}_t + \sigma\hat{c}_t$ in terms of log deviations or $w_t = \eta n_t + \sigma c_t$ in only log terms.

Similarly, the log linearization of the following equation $Y_t = \frac{P_{c,t}}{P_t}Y_t^c + \frac{P_{d,t}}{P_t}Y_t^d = \frac{P_{c,t}}{P_t}C_t + \frac{P_{d,t}}{P_t}I_t^d$ is:

$\frac{\frac{P_c}{P}C}{Y}\hat{c} + \frac{\frac{P_d}{P}I^d}{Y}\hat{i^d} = \hat{y}$

$\frac{\frac{P_c}{P}Y^c}{\frac{P_c}{P}Y^c+\frac{P_d}{P}Y^d}\hat{y}_c + \frac{\frac{P_d}{P}Y^d}{\frac{P_c}{P}Y^c+\frac{P_d}{P}Y^d}\hat{y^d} = \hat{y}$

$\frac{P_cY^c}{P_cY^c+P_dY^d}\hat{y}_c + \frac{P_dY^d}{P_cY^c+P_dY^d}\hat{y}_d = \hat{y}$

$(1-\psi_d)\hat{y}_c + \psi_d\hat{y}_d = \hat{y}$

Next I log linearize the Euler equation $Q_t = \beta\mathbb{E}_t\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{P_{C,t}}{P_{C,t+1}}\right)\left(\frac{\tilde{Z_{t+1}}}{\tilde{Z}_t}\right)\right)$.

As also in Galí (2015), I define $i_t = -\log(Q_t)$, $\varrho = -\beta$. This implies that $\beta = \exp(-\varrho)$, and $\exp i_t = \frac{1}{Q_t}$.

Then, the Euler equation can be written as:

$1 = \exp(-\varrho)\mathbb{E}_t\left(\exp\log\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{P_{C,t}}{P_{C,t+1}}\right)\left(\frac{\tilde{Z_{t+1}}}{\tilde{Z}_t}\right)\right)\right)\exp(i_t)$

The steady state interest rate is $i = \varrho + \sigma\gamma$ where γ is growth rate of non-durable consumption and since I assume a zero inflation steady state, it follows that the Euler equation can be log linearized as the following, where lower case letters represent log deviations from steady state.

$$1 = \mathbb{E}_t\left(\exp\left(-\sigma\Delta c_{t+1} - \pi_{c,t+1} + \Delta\log \tilde{Z_{t+1}} + i_t - \varrho\right)\right)$$
$$exp(.) \approx 1 + (i_t - i) - \sigma(\Delta c_{t+1} - \gamma) - (\pi^c_{t+1} - \pi) + \Delta\log\tilde{Z_{t+1}}$$
$$\implies 1 = \mathbb{E}_t\left(1 + i_t + \Delta\log\tilde{Z_{t+1}} - \sigma\Delta c_{t+1} - \pi^c_{t+1} - \varrho\right)$$
$$c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi^c_{t+1}\} + \mathbb{E}_t\Delta\log\tilde{Z_{t+1}} - \varrho\right)$$
$$c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi^c_{t+1}\} + \mathbb{E}_t\Delta\tilde{z_{t+1}} - \varrho\right), \text{ where } \tilde{z}_t := \log\tilde{Z}_t.$$

$\mathbb{E}_t\tilde{z}_t$ is my definition of current *consumer sentiment*, reflecting beliefs about the hidden state. I normalize $\mathbb{E}_t\tilde{z_{t+1}} := 0, \forall t$, since current beliefs about the sentiment of future selves is a higher order object and not amenable to convenient empirical measurement as well as theoretical conceptualization. $\mathbb{E}_t\left(z^N\right) := (p_t(z_H) + (1-p_t)z_L)$ and $\mathbb{E}_t\tilde{z}_t = \lambda\mathbb{E}_t\left(z^N\right) + (1-\lambda)\,\bar{z}, \forall t$. This is because current consumer sentiment is defined as the expected value of hidden state tomorrow because current consumer sentiment is a forward looking object and reflects beliefs about future. With probability λ, the state may mature tomorrow and if it does so, it has an expected value of $(p_t(z_H) + (1-p_t)z_L)$, given current public beliefs. With probability $1-\lambda$, the state does not mature and stays at its prior value of $\bar{z}$. Thus, for $\forall t$ when the true state has not been revealed yet,
$\mathbb{E}_t\Delta\log\tilde{Z_{t+1}} = \mathbb{E}_t\tilde{z_{t+1}} - \tilde{z}_t \; = 0 - (\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\,\bar{z})$.

The Euler equation thus becomes the following for time t when the state has not been revealed yet:

$$c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{c,t+1}\} - \left(\underbrace{\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\,\bar{z}}_{\text{Consumer Sentiment Term}}\right) - \varrho\right)$$

However, once the initial true state, whether high or low, has been revealed, all uncertainty has been resolved and after that point, $\forall t \geq t_{revealed}$, where $z_{revealed} \in \{z_L, z_H\}$:

$$c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{c,t+1}\} - \left(\underbrace{z_{revealed}}_{\text{Consumer Sentiment Term}}\right) - \varrho\right)$$

The log linearization of marginal costs equation $mc^\omega_t = \frac{W_t}{MPL^\omega}$ for each sector $\omega \in \{C, D\}$ is $\hat{mc}^\omega_t = \hat{w}_t + \alpha\hat{n}^\omega_t$.

The log linearization of the price dispersion equation, given that steady state price dispersion Φ_ω is 1 is:

$$\hat{\Phi_{w,t}} = -\epsilon_\omega(1-\phi_\omega)\hat{\pi^*_{\omega,t}} + \epsilon_\omega\hat{\pi_{\omega,t}} + \phi_\omega\hat{\Phi_{w,t-1}}, \forall\omega \in \{C, D\}. \tag{34}$$

The log linear approximation of $(1+\pi_{\omega,t})^{1-\epsilon_\omega} = (1-\phi_\omega)(1+\pi^*_{\omega,t})^{1-\epsilon_\omega} + \phi_\omega$ is :

$(1-\epsilon_\omega)\hat{\pi_{\omega,t}} = (1-\phi_\omega)(1-\epsilon_\omega)\hat{\pi^*_{\omega,t}}$

$\implies \hat{\pi_{\omega,t}} = (1-\phi_\omega)\hat{\pi^*_{\omega,t}}, \forall\omega \in \{C, D\}.$

Using the above expression in the log linear expression for dispersion, we get:

$\hat{\Phi_{w,t}} = -\epsilon_\omega(1-\phi_\omega)\hat{\pi^*_{\omega,t}} + \epsilon_\omega(1-\phi_\omega)\hat{\pi^*_{\omega,t}} + \phi_\omega\hat{\Phi_{w,t-1}} \implies \hat{\Phi_{w,t}} = \phi_\omega\hat{\Phi_{w,t-1}}, \forall\omega.$

The log linearization of $(1+\pi^*_{c,t}) = \frac{\epsilon_c}{\epsilon_c-1}(1+\pi_{c,t})\frac{s_{1,t}}{s_{2,t}}$ is $\hat{\pi^*_{c,t}} = \hat{\pi_{c,t}} + \hat{s_{1,t}} - \hat{s_{2,t}}$.

Similarly, the log linearization of $(1+\pi^*_{d,t}) = \frac{\epsilon_d}{\epsilon_d-1}(1+\pi_{d,t})\frac{s_{3,t}}{s_{4,t}}$ is $\hat{\pi^*_{d,t}} = \hat{\pi_{d,t}} + \hat{s_{3,t}} - \hat{s_{4,t}}$.

It can be shown that the log linear version of the auxilary variable $s_{1,t} = C_t^{-\sigma} mc_t^c Y_t^c + \phi\beta\mathbb{E}_t s_{1,t+1}\left(1+\pi_{c,t+1}\right)^{\epsilon_c}$ is the following:

$\hat{s_{1,t}} = \frac{C^{1-\sigma}mc^c\hat{mc_t}}{s_1} + \frac{(1-\sigma)C^{1-\sigma}mc^c\hat{c_{1,t}}}{s_1} + \phi\beta\mathbb{E}_t\hat{s_{1,t+1}} + \phi\beta\epsilon_c\mathbb{E}_t\hat{\pi_{c,t+1}}$

Solving the equation of $s_{1,t}$ in steady state implies that $s_1 = \frac{C^{1-\sigma}mc^c}{1-\phi\beta}$, which when substituted in the expression above yields the following expression:

$\hat{s_{1,t}} = \hat{mc_t}(1-\phi\beta) + (1-\sigma)(1-\phi\beta)\hat{c_{1,t}} + \phi\beta\mathbb{E}_t\hat{s_{1,t+1}} + \phi\beta\epsilon_c\mathbb{E}_t\hat{\pi_{c,t+1}}$

Similarly, The log linearization of $s_{2,t} = C_t^{-\sigma}Y_t^c + \phi\beta\mathbb{E}_t s_{2,t+1}\left(1+\pi_{c,t+1}\right)^{\epsilon_c-1}$ is the following:

$\hat{s_{2,t}} = (1-\sigma)(1-\phi\beta)\hat{c_t} + \phi\beta\mathbb{E}_t\hat{s_{2,t+1}} + \phi\beta(\epsilon_c-1)\mathbb{E}_t\hat{\pi^c_{t+1}}$

$\implies \hat{s_{1,t}} - \hat{s_{2,t}} = (1-\phi\beta)\hat{mc^c_t} + \phi\beta\mathbb{E}_t\pi^c_{t+1} + \phi\beta\mathbb{E}_t\left(\hat{s_{1,t+1}} - \hat{s_{2,t+1}}\right)$

Then, it can also be derived that:

$\frac{\hat{\pi_{c,t}}\phi}{1-\phi} = (1-\phi\beta)\hat{mc^c_t} + \phi\beta\mathbb{E}_t\pi^c_{t+1} + \phi\beta\mathbb{E}_t\left(\frac{\hat{\pi_{c,t+1}}\phi}{1-\phi}\right)$

$\implies \hat{\pi_{c,t}} = \frac{(1-\phi\beta)(1-\phi)}{1-\phi}\hat{mc^c_t} + \beta\mathbb{E}_t\left(\hat{\pi_{c,t+1}}\right)$

$\implies \hat{\pi_{c,t}} = \lambda_c\hat{mc^c_t} + \beta\mathbb{E}_t\left(\hat{\pi_{c,t+1}}\right)$, where $\lambda_c := \frac{(1-\phi_C\beta)(1-\phi_C)\Theta}{1-\phi}$, where $\Theta_C := \frac{1-\alpha}{1-\alpha+\alpha\epsilon_c}$

Similarly, we have a NKPC in the durable sector,

$\hat{\pi_{d,t}} = \lambda_d\hat{mc^d_t} + \beta\mathbb{E}_t\left(\hat{\pi_{d,t+1}}\right)$, where $\lambda_d := \frac{(1-\phi_D\beta)(1-\phi_D)\Theta}{1-\phi}$, where $\Theta_D := \frac{1-\alpha}{1-\alpha+\alpha\epsilon_d}$

The average log price markup in nondurable sector $\mu_{c,t} = p_{c,t} - \psi_{c,t}$ can be expressed as the following Galí (2015):

$\mu_{c,t} = -(w_t - p_{c,t}) + (-\alpha n_{c,t} + log(1-\alpha))$

$= -(\sigma c_t + \psi n_t) - \alpha n_{c,t} + log(1-\alpha)$ (Using Labor Supply Condition)

It can then be shown that:

$\hat{\mu_{c,t}} = \left(-\frac{\sigma}{\psi_c} + \frac{(-\psi(1-\zeta_d)-\alpha)(1-\alpha)}{\psi_c}\right)\hat{y_t} + \left(\frac{\sigma\psi_d}{\psi_c} - \frac{\psi_d(-\psi(1-\zeta_d)-\alpha)(1-\alpha)}{\psi_c} - \frac{\psi\zeta_d}{1-\alpha}\right)\hat{I_{d,t}}$

$\iff \hat{\mu_{c,t}} = \mu_y^c \hat{y}_t + \mu_{Id}^c \hat{I_{d,t}}$, where $\mu_y^c := \left(-\frac{\sigma}{\psi_c} + \frac{(-\psi(1-\zeta_d)-\alpha)(1-\alpha)}{\psi_c}\right)$ and $\mu_{Id}^c := \left(\frac{\sigma\psi_d}{\psi_c} - \frac{\psi_d(-\psi(1-\zeta_d)-\alpha)(1-\alpha)}{\psi_c} - \frac{\psi\zeta_d}{1-\alpha}\right)$.

Then, the NKPC in terms of log deviation of output from steady state can be expressed as the following. All derivations are provided in appendix but omitted here since this is fairly standard textbook material.

$\hat{\pi_{c,t}} = \kappa_1 (\hat{y}_t) + \beta\mathbb{E}_t (\hat{\pi_{c,t+1}}) + \kappa_2 (\hat{I_{d,t}})$, where $\kappa_1 := -\lambda_c\mu_y^c$ and $\kappa_2 := -\lambda_c\mu_{Id}^c$.

Similarly, average log price markup in durable sector $\mu_{d,t} = p_{d,t} - \psi_{d,t}$ can be expressed as the following:

$\mu_{d,t} = -(w_t - p_{d,t}) + (-\alpha n_{d,t} + log(1-\alpha))$

It can then be shown that:

$\mu_{d,t} = \left(-\frac{\sigma}{\psi_c} - \frac{\psi(1-\zeta_d)(1-\alpha)}{\psi_c}\right)\hat{y}_t + \left(\frac{\sigma\psi_d}{\psi_c} - \frac{\psi\zeta_d}{1-\alpha} - \frac{\alpha}{1-\alpha} + \frac{\psi(1-\zeta_d)\psi_d(1-\alpha)}{\psi_c}\right)\hat{I_{d,t}} - \hat{p_{c,t}} + \hat{p_{d,t}}$

$\iff \hat{\mu_{Id,t}} = \mu_y^{Id}\hat{y}_t + \mu_{Id}^{Id}\hat{I_{d,t}} - \hat{p_{c,t}} + \hat{p_{d,t}}$

Then, the NKPC in durable good sector is:

$\hat{\pi_{d,t}} = \kappa_1^{Id} (\hat{y}_t) + \beta\mathbb{E}_t (\hat{\pi_{c,t+1}}) + \kappa_2^{Id} (\hat{I_{d,t}}) - \lambda_d(\hat{p_{d,t}} - \hat{p_{c,t}})$, where $\kappa_1^{Id} := -\lambda_d\mu_y^{Id}$, $\kappa_2^{Id} := -\lambda_d\mu_{Id}^{Id}$ and $\hat{p_{d,t}} - \hat{p_{c,t}}$ is log deviation of the relative price between durable goods sector and consumption sector.

The log linear version of the aggregate inflation, which I defined as $\pi_t := \pi_{D,t}^{\psi_D}\pi_{C,t}^{1-\psi_D}$ is $\hat{\pi}_t = \psi_D(\hat{\pi_{d,t}}) + (1-\psi_D)\hat{\pi_{c,t}}$, where ψ_D is steady state output share of durable sector in aggregate production. This is abuse of notion but inflation π_t should not be confused with joint probability of the two hidden states, as in learning block of model.

12. Log Linear System of Equations

We know that:

$$\hat{c}_t = \mathbb{E}_t\{\hat{c_{t+1}}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{c,t+1}\} - \left(\underbrace{\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\bar{z}}_{\text{Consumer Sentiment Term}}\right) - \varrho\right)$$

For all $t \geq t_{revealed}$, where $z_{revealed} \in \{z_H, z_L\}$,

$$\hat{c}_t = \mathbb{E}_t\{\hat{c_{t+1}}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{c,t+1}\} - \left(\underbrace{z_{revealed}}_{\text{Consumer Sentiment Term}}\right) - \varrho\right)$$

Since $\hat{y}_t = \psi_c\hat{c}_t + (1-\psi_c)\hat{I_{d,t}}$, the Euler equation for consumption can be expressed as[14]:

$$\hat{y_t} = \mathbb{E}_t\{\hat{y_{t+1}}\} - \frac{\psi_c}{\sigma}\left(\hat{i_t} - \left(\underbrace{\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\,\bar{z}}_{\text{Consumer Sentiment Term}}\right) - \mathbb{E}_t\{\hat{\pi_{c,t+1}}\} - \hat{\Xi}_t\right)$$

,where $\hat{\Xi}_t := -\left(\frac{\sigma\psi_d}{\psi_c}\right)\mathbb{E}_t\{\Delta \hat{I_{d,t+1}}\}$, where $\mathbb{E}_t\{\Delta \hat{I_{d,t+1}}\} := \mathbb{E}_t\{\hat{I_{d,t+1}}\} - \hat{I_{d,t}}$.

The law of motion for durable goods' evolution is $D_t = (1-\delta)D_{t-1} + I_{d,t}$, which when log linearized becomes $\hat{D_t} = (1-\delta)\hat{D_{t-1}} + \frac{I}{D}\hat{I_{d,t}}$.

$\frac{I}{D}$ is the steady state proportion of investment of durable investment to durable stock. One must also log linearize $I_{d,t} = (1-\mu)\overline{F}\left(\hat{s}(p_t, \mathbf{x_{t-1}})\right) + \mu\epsilon_t$ around current beliefs p_t and current and known values of past endogenous variables, $\mathbf{x}$, where $\overline{F} := 1 - F$ and F is relevant CDF of private, signal distribution. In the next period, beliefs will be updated again and endogenous variables will evolve over time to new values and then, one must log linearize $I_{d,t}$ around those beliefs and values of endogenous variables. After log linearization, one gets the following expression, where $f(.)$ is the density function, corresponding to CDF F for the distribution of private signal s.

$$\hat{I_{d,t}} = (1-\mu)\left(\left(-f(\hat{s})\frac{\partial\hat{s}(.)}{\partial p_t}\right)P\hat{p_t} + \left(-f(\hat{s})\frac{\partial\hat{s}(.)}{\partial\hat{\mathbf{x_{t-1}}}}\right).\mathbf{X}.\hat{\mathbf{x_{t-1}}}\right) \tag{35}$$

Since beliefs are being held fixed moving forward because of the martingale property and/or resolution method Kozlowski et al. (2019), $\hat{p_t} = 0$, the above equation can be more compactly written as the following, where $a_{6,j} = -(1-\mu)f(\hat{s})\frac{\partial\hat{s}(.)}{\partial j_t}$ are the components of the vector $\mathbf{a_6}$, corresponding to variable j in $\hat{\mathbf{x_{t-1}}}$.

$$\hat{I_{d,t}} = \mathbf{a_6}.\mathbf{X}.\hat{\mathbf{x_{t-1}}} \tag{36}$$

The partial derivatives in equation 35 have to computed through numerical differentiation since $\hat{s}$ is being solved through a nonlinear solver and has no analytical expression. For baseline, I will use three point, midpoint formula for numerical differentiation by solving for $\hat{s}$ using nonlinear solver at different values in vicinity of current value of variables in $\hat{\mathbf{x_{t-1}}}$, in addition to actual, current values of $\hat{\mathbf{x_{t-1}}}$ (these current values correspond to x_0 in the formula below), to compute the difference quotients, needed for numerical differentiation. $\mathbf{X}$ contains the levels of variables in $\hat{\mathbf{x_{t-1}}}$ rather than log deviations. The three point, midpoint formula Ökten (2019) evaluates $f'(x_0)$ as the following:

[14] See Appendix for derivation.

Theorem 3
$f'(x_0) = \frac{f(x_0+h)-f(x_0-h)}{2h} - \frac{h^2}{6} f^3(\zeta_1)$, *where* $\zeta_1 \in (x_0 - h, x_0 + h)$ *and* $f \in C^3[x_0 - h, x_0 + h]$.

I specified the taylor rule as $\hat{i_t} = \varrho + \varrho_i \hat{i_{t-1}} + (1-\varrho_i)\left(\phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t\right) + \epsilon_{i,t}$ and I use $\varrho_i = 0.8$ for baseline, which is the typical value of this parameter, found in empirical evidence Clarida et al. (1999), reflecting significant inertia in policy adjustment.

Thus, the entire log linear system is the following, where ϵ_{y_t} and $\epsilon_{i,t}$ are the IID demand and monetary policy shocks respectively.

$$\hat{y_t} = \mathbb{E}_t\{\hat{y_{t+1}}\} - \frac{\psi_c}{\sigma}\left(\hat{i_t} - \left(\underbrace{\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\,\bar{z}}_{\text{Consumer Sentiment Term}}\right) - \mathbb{E}_t\{\hat{\pi_{c,t+1}}\} - \hat{\Xi}_t\right) + \epsilon_{y_t} \tag{37}$$

$$\hat{\pi_{c,t}} = \kappa_1^c \hat{y_t} + \beta \mathbb{E}_t\left(\hat{\pi_{c,t+1}}\right) + \kappa_2^c \hat{I_{d,t}} \tag{38}$$

$$\hat{\pi_{d,t}} = \kappa_1^{Id} \hat{y_t} + \beta \mathbb{E}_t\left(\hat{\pi_{d,t+1}}\right) + \kappa_2^{Id} \hat{I_{d,t}} - \lambda_d(\hat{p_{d,t}} - \hat{p_{c,t}}) \tag{39}$$

$$\hat{i_t} = \varrho_i \hat{i_{t-1}} + (1-\varrho_i)(\phi_\pi \hat{\pi}_t + \phi_y \hat{y_t}) + \epsilon_{i,t} \tag{40}$$

$$\hat{\pi}_t = \psi_C \hat{\pi_{c,t}} + \psi_D \hat{\pi_{d,t}} \tag{41}$$

$$\hat{\pi_{\omega,t}} = \hat{p_{\omega,t}} - \hat{p_{\omega,t-1}},\ for\ \omega \in \{C, D\}. \tag{42}$$

$$\hat{D_t} = (1-\delta)\hat{D_{t-1}} + \frac{I}{D}\hat{I_{d,t}} \tag{43}$$

$$\hat{I_{d,t}} = (\mathbf{a_6})\,.\mathbf{X}.\hat{\mathbf{x_{t-1}}} \tag{44}$$

12.1. SYSTEM IN SIMS' FORM

The general form of a linear rational expectations model to be solved by using the Sims method Sims (2002) is:

$\Gamma_0 \mathbf{\hat{x}_t} = \Gamma_1 \mathbf{x_{t}\hat{}_{-1}} + C + \psi e_t + \Pi \eta_t$, where e_t is an exogenous, vector stochastic process, η_t contains time t prediction errors satisfying $\mathbb{E}_t \eta_{t+1} = \mathbf{0}$ and $\mathbf{\hat{x}_t}$ contains the log deviations of endogenous variables in system. Prediction error for any variable y is $\eta_{t+1}^{y} := y_t - \mathbb{E}_{t-1} y_t$. This method allows for the possibility that Γ_0 can be singular and applies the generalized Schur decomposition to the pair of matrices $\{\Gamma_0, \Gamma_1\}$. Below, I define the relevant matrices needed in order to express equations 37 to 44 above in this Sims' formulation.

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\psi_c}{\sigma} & 1 & \frac{\psi_c}{\sigma} & 0 & -\psi_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & \lambda_d & -\lambda_d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{x_t} = \begin{bmatrix} \hat{i}_t \\ \zeta_t^{y} \\ \zeta_t^{\pi_c} \\ \zeta_t^{\pi_d} \\ \zeta_t^{I_d} \\ \hat{p_{c,t}} \\ \hat{p_{d,t}} \\ \hat{D}_t \\ \hat{y}_t \\ \hat{\pi}_t \\ \hat{\pi_{c,t}} \\ \hat{I_{d,t}} \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} \varrho_i & \phi_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_\pi & 0 & 0 \\ 0 & 1 & 0 & 0 & -\psi_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\kappa_1^c & 1 & 0 & -\kappa_2^c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\kappa_1^{Id} & 0 & 1 & -\kappa_2^{Id} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_C & \psi_D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{I}{D} & 0 & 0 & (1-\delta) & 0 & 0 & 0 & 0 \\ a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} & a_{6,8} & a_{6,9} & a_{6,10} & a_{6,11} & a_{6,12} \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi \eta_t = \begin{bmatrix} \phi_y & 0 & 0 & 0 \\ 1 & 0 & 0 & -\psi_d \\ -\kappa_1^c & 1 & 0 & -\kappa_2^c \\ -\kappa_1^{Id} & 0 & 1 & -\kappa_2^{Id} \\ 0 & \psi_C & \psi_D & 0 \\ 0 & 0 & 0 & \frac{I}{D} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_t^{y} \\ \eta_t^{\pi_c} \\ \eta_t^{\pi_d} \\ \eta_t^{I_d} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \\ -\frac{\psi_c}{\sigma}\left(\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\bar{z}\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\psi e_t = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{i,t} \\ \epsilon_{y,t} \end{bmatrix}$$

13. Model Solution

I provide a more detailed pseudo code, describing the computational implementation/algorithm of the solution method in appendix. Here, I describe the main contributions and principles used to solve the model.

13.1. General Strategy

Public beliefs about the hidden state are updated at the beginning of every stage 2 after observing the mass of investing durable good shoppers with observation noise based on Bayesian updating and the corresponding law of motion for belief updating, derived in learning section. However, because of the martingale property of public beliefs, after updating public beliefs at the beginning of stage 2, they can be held fixed, moving forward at this updated value rather than considering beliefs as dynamic and changing over time in the model solution. In mathematical terms, the martingale property implies that joint probability of the two hidden states satisfies $\mathbb{E}_t[\pi_{t+k}(z,\eta)] = \pi_t(z,\eta), \forall k \geq 0$. This principle makes the solution tractable and was developed by Kozlowski et al. (2019) who refer to it as the resolution method; this principle was also used by Schaal and Taschereau-Dumouchel (2020) to solve a social learning model in general equilibrium.

In every stage 1, one has to calculate the signal cutoff by evaluating the root of cutoff function, which uses information from last stage 2. Once the signal cutoff is determined, one can determine the mass of investing agents, which will be observed publicly with observation noise in beginning of upcoming stage 2 by all agents. During stage 1, one also solves the fixed point problem between stage 1 and 2, before proceeding to stage 2; the nature of the fixed point problem is explained in next section. In addition, after solving the fixed point problem, one uses numerical differentiation[15] to evaluate the terms in log linear equation for $\hat{I_{d,t}}$, which will be used in the stage 2 solution.

In every stage 2, I use a linear solver or to be specific the Sims' solver in order to solve for policy rules, given the fixed, updated value of public beliefs after observing mass of investing agents with noise in beginning of stage 2. Public beliefs are held fixed at this updated value, moving forward because of the martingale property. Thus, I get a sequence of linear rational expectations solvers $\{G_1^t, \Theta_C^t, G_0^t\}_{t=0}^{M}$[16], one for each iteration/each stage 2, corresponding to the fixed/constant value of updated public beliefs at that particular stage 2. Since there is a sequence of Sims' solvers, one has to be keep track of the evolution of log deviation of endogenous variables over time, using appropriate policy rule coefficients at each step, as policy rule coefficients change after every stage 2. Keeping track of evolution of endogenous variables is important because after every iteration, one uses updated endogenous variables from last period to log linearize $\hat{I_{d,t}}$ around these new values from last period and the definition of impulse responses given below also requires keeping track of x_{t+h-1} at every horizon h for computation of IRFS.

13.2. Fixed Point Problem Between Stage 1 and Stage 2

There is a fixed point problem between stage 1 and 2 because when deciding whether to invest in durable good or not during stage 1, durable good shoppers have to forecast policy rule for $\hat{P_{d,t}}$ in upcoming stage 2, as also derived in earlier section. In order to forecast the policy rule, they must know the matrices Γ_0 and Γ_1, used to solve model in stage 2. In addition to some deep parameters, which are assumed to be known, the matrix Γ_1 contains 12 scalars $a_{6,k}$, $\forall k$ (row 6), contained in $\mathbf{a_6}$. Any given beliefs $\mathbf{a_6^{beliefs}}$ that durable shoppers have for this row, will be associated with given *mappings*[17] $\overline{G_{1,pd}}$ and $\overline{a_{pd}}$ for policy rule of $\hat{P_{d,t}}$ and hence, associated with a given stage 1 expectation of $\hat{P_{d,t}}$. This stage 1 expectation enters the cutoff function $\tilde{c}$, as derived earlier and hence determines the terms in $\mathbf{a_6}$ because these terms depend on signal cutoff and derivatives

[15] To be specific, I use Newton, three point formulas to evaluate numerical differentiation, as also explained before in text.

[16] For linear RE system in Sims' form $\Gamma_0 x_{t+1} = \Gamma_1 x_t + C + \psi e_{t+1}$, the form of solution is $x_t = G_1 x_{t-1} + \Theta_C + G_0 e_t$.

[17] In the cutoff function section, it was explained what these *mappings* mean.

of cutoff function i.e $a_{6,j} = -(1-\mu)f(\hat{s})\frac{\partial \hat{s}}{\partial j}, \forall j \in \mathbf{x_{t-1}}$. However, the coefficients in $\mathbf{a_6}$ enter equation 45 for $\hat{I_{d,t}}$, which is one of the equations in the stage 2 system, implying that any believed values for $\mathbf{a_6}^{\mathbf{beliefs}}$ will effect the actual values $\mathbf{a_6}$ that end up being used in the system during 2nd stage, which is the essence of fixed point problem. Hence, analagous to the notion of a rational, expectations equilibrium, one must solve this fixed point problem, so that equilibrium beliefs $\mathbf{a_6^{beliefs}}$ of shoppers in stage 1 end up producing a cutoff rule such that up to some numerical error or tolerance, the implications of these beliefs confirm with reality i.e the actual values of $\mathbf{a_6}$ or $\mathbf{a_6} \approx \mathbf{a_6^{beliefs}}$ in stage 2. This means that shoppers have rational expectations about the mappings $\overline{G_{1,pd}}$ and $\overline{a_{pd}}$. In the pseudo code section of appendix, I more fully describe the algorithm for solving this fixed point problem using fixed point iteration. More succinctly,

$\mathbf{a_{6,t}^{beliefs}} \implies \overline{\mathbf{G_{1,pd}}}(\mathbf{a_{6,t}^{beliefs}})$ and $\overline{a_{pd}}(\mathbf{a_{6,t}^{beliefs}}) \implies \tilde{c}(\mathbf{a_{6,t}^{beliefs}};) \implies \hat{s}((\mathbf{a_{6,t}^{beliefs}});) \implies \mathbf{a_{6,t+1}}(\mathbf{a_{6,t}^{beliefs}})$.

I must find $\mathbf{a_{6,*}^{beliefs}}$ such that the above fixed point iteration is solved i.e $\mathbf{a_{6,*}^{beliefs}} \approx \mathbf{a_{6,*}}(\mathbf{a_{6,*}^{beliefs}}) \approx \mathbf{a_{6,*}}$.

13.3. Defining Impulse Response Functions

In general, impulse response functions are defined as:

$$IRF(h) = \mathbb{E}_t\{x_{t+h}\} - \mathbb{E}_{t-1}\{x_{t+h}\} | shock_t = shock_0 \tag{45}$$

,where h is horizon and x_t is any endogenous or exogenous variable. In my model, time $t-1$ means steady state and time t means the first stage 2 just after the first stage 1, in which the economy has been perturbed or hit with a noise shock or any other shock equal to $shock_0$. In my model, the noise shock is a permanent shock and it has direct effects only during stage 1 when durable good shoppers use signals to update beliefs, but the implications of those effects for belief updating means that noise shock also has indirect effects for solution of endogenous variables during stage 2. Suppose that x_t contains the set of endogenous variables to be solved for using Sims solver in stage 2. Fix $h = 0$. I define the impulse response function for noise shocks in my model as the following:

$$IRF(h) := G_1^{t+h} x_{t+h-1} + \Theta_C^{t+h} - (G_1^{prior} x_{t+h-1} + \Theta_C^{prior}) \tag{46}$$

The policy rule matrices G_1^{t+h} and Θ_C^{t+h} are indexed by $t+h$ since value of updated public belief in period $t+h$ which is p_{h+1} is used to solve the stage 2 system and this level of belief determines the value of these matrices. G_1^{prior} and Θ_C^{prior} are policy rule coefficients, corresponding to the value of public beliefs equal to prior of p_0. Thus, I define the IRF as measuring the difference in policy rules when one uses the current public beliefs to determine the solution, relative to when one uses prior public belief to solve for policy rule coefficients. x_{t+h-1} is predetermined and known as of time $t+h$ and is *common* across the two terms. $x_{t+h-1} = G_1^{t+h-1} x_{t+h-2} + \Theta_C^{t+h-1}$ and this value is determined based on actual policy rule coefficients in period $t+h-1$, based on public beliefs in period $t+h-1$ rather than coefficients based on prior public beliefs. Of course for $h=0$, $G_1^{t+h-1} = G_1^{prior}$ and $\Theta_C^{t+h-1} = \Theta_C^{prior}$ but this is not true for $h>0$. For $h=0$, $x_{t+h-1} = 0$, since variables are expressed in log deviations and they are zero in steady state $\implies IRF(0) = \Theta_C^t - \Theta_C^{prior}$, where Θ_C^t corresponds to public beliefs p_1 and Θ_C^{prior} corresponds to prior belief p_0.

14. Calibration

The table below shows calibration of parameters used in the general equilibrium model. Some of these values have justification from microeconomic evidence such as the one for Frisch elasticity Chetty et al. (2011) ($\eta = 2$) and others are standard values used in literature such as Taylor rule coefficients satisfying the Taylor principle, for e.g $\phi_\pi > 1$, based on the work of Clarida et al. (1999), for example. I use value of 2 for risk aversion parameter σ based on recent evidence Crump et al. (2019) based on FRBNY-SCE (survey of consumer expectations data), measuring subjective elasticity of intertemporal substitution (EIS) through micro, survey data; they find that $EIS \approx 0.5 \implies \sigma \approx 2$ for CRRA preferences. The Calvo friction parameters in both sectors are set equal to $\phi_C = \phi_D = 0.92$, implying a price duration of about one year, since I have a monthly interpretation of time in the model and $\phi_C = 0.92 \implies \frac{1}{1-\phi_C} = 12$ months.[18] I assume that elasticities of substitution across varieties in each sector ϵ_c and ϵ_d are equal to 6, in order to target a steady state markup of 20% in both sectors. Monacelli (2009) argue that annual durables' depreciation rate should be 4%. Since I have a monthly interpretation of the model, I set $\delta = 0.0033$ in my model, which implies an annual depreciation rate of 4%.

In order to calibrate p_0, the prior probability of high state, which corresponds to the

[18] The reason for assuming equal degree of price stickiness in both sectors is that Cantelmo and Melina (2018) estimate a 2 sector New Keynesian model and find that the degree of stickiness in durable sectors is approximately the same as in nondurable sector, unlike the traditional assumption in New Keynesian models that prices are fully flexible in durable sector.

strength of belief on part of consumers that they may transit into a boom state from the prior state $\bar{z}$, I use the University of Michigan consumer sentiment data. Since my focus is on the durable goods sector, I use data from the question regarding buying conditions for vehicles in the next year, to which people may answer, "Good Times", "Uncertain Times" and "Bad Times". I define

$p_0 := \frac{GoodTimes - UncertainTimes}{GoodTimes - UncertainTimes + BadTimes}$,

since the probability mass has to be divided between the two states z_H and z_L in my model and these correspond to either a state of extreme optimism or extreme pessimism. I adjust "Good Times" measure from the data downwards by subtracting Uncertain Times from it because I want to capture the subset of people who answer "Good Times" who are really optimistic, filtering out those who may be just mildly optimistic or believe that the economy will continue on the current balanced growth path.[19] The average value of this proportion is 0.63, for the Michigan data series, ranging from 1978-2020, which is the value of p_0 I use in calibration.

In the model, λ measures subjective uncertainty resolution about the future state of economy. A higher value of λ means that subjective uncertainty resolves at relatively short horizons. For example, suppose that people are trying to understand how the economy will evolve over the next 12 months, starting from the steady state or $t = 0$, where the time after next 12 months is indexed by $t + 12$ and time is measured in months. Under this interpretation, a higher value of λ means that consumers start to have a high degree of subjective certainty about where the economy is headed at $t + 12$ at $t + k$ where k is small. On the other hand, a lower value of λ would mean that subjective uncertainty about $t + 12$ is significantly low only at $t + k$, for k large or $k \approx 12$. In other words, when they approach $t + 12$ really closely, only then they are likely to become certain about whether the economy will transit into recession. In order to measure λ, I measure subjective uncertainty from FRBNY-SCE[20] data, which elicits subjective distributions regarding inflation expectations in the next 12 months by providing ten bins, representing different ranges in which inflation may lie in the future and the subjective probabilities across bins must sum to 1. I use data on inflation expectations because for this question, the survey elicits a subjective distribution, not just for next 12 months, but also for another horizon, further into future, for the 12 months, ranging from next 24 months to 36 months, allowing me to also measure reduction in subjective uncertainty at different horizons. In order

[19] Admittedly, this is an imprecise adjustment but the data do not allow me to get an even cleaner measure than this one.

[20] This is the Federal reserve bank of New York's survey of consumer expectations, running from 2013 to 2019.

to measure subjective uncertainty, I use the measure, proposed by Krüger and Pavlova (2019), which is simple to use, has desirable statistical interpretation[21] and overcomes the deficiencies of measures, based on interquartile range (IQR)[22] and measures based on fitted generalized beta distribution proposed by Engelberg et al. (2009)[23] in the context of such bin data. Krüger and Pavlova (2019) define the expected, ranked probability score ($ERPS$) as follows:

$$ERPS = \sum_{k=1}^{K} P_k(1 - P_k) \tag{47}$$

, where $P_k = \sum_{j=1} p_j$ is the cumulative subjective probabilities reported by respondents for first k bins and $K = 10$ is the total number of bins.

Based on FRBNY-SCE, I measure $ERPS_{24-36} - ERPS_{12}$ for inflation expectations in each respondent year and then take an average of this reduction across sample; $ERPS_{24-36}$ refers to subjective uncertainty at 24 to 36 month horizon regarding inflation expectations and $ERPS_{12}$ is the same quantity for 12 month horizon. I find that relative to the $ERPS_{24-36}$ level, which is 0.46 there is a 7.9% reduction in subjective uncertainty at the 12 month horizon, corresponding to $ERPS_{12}$, which is equal to 0.42. This is a significant reduction, but there remains substantial uncertainty about the next 12 months, which implies that λ is likely to be small. Based on this evidence, I want the number $(1-\lambda)^k$ to be quite high for k small, but I want $(1-\lambda)^k \approx 1$ or high for $k \approx 12$. This is because λ is the probability of success of a Bernoulli random variable and with independent Bernoulli trials and $(1-\lambda)^k = Pr\{\text{Uncertainty About } t+12 \text{ Has Not Resolved After } t+k \text{ Months}\}$[24]. As k approaches 12, this probability should converge to 1 and when $k \approx 1$, $(1-\lambda)^k \approx 1$. If I use the condition that $(1-\lambda)^1 \approx 0.9$, I get that $\lambda \approx 0.1$, and then, $(1-\lambda)^{12} = 0.9^{12} \approx 1$. This is why I use $\lambda = 0.1$ in calibration. I check for robustness of results to values of $\lambda \in [0.01, 0.2]$.

In order to calibrate ζ_d, the steady state proportion of labor used in durable sector[25], relative to aggregate labor in model, I use US data on employment in durable and non-durable sectors from FRED St Louis database for the period 1939-2020 and calculate aver-

[21] It can be interpreted as the generalized entropy function of the ranked probability score.

[22] Surveys like the SCE feature censoring in that participants do not specify the probability distribution within each bin. This property precludes the direct computation of a standard measure of spread, such as the standard deviation or interquartile range from the distribution.

[23] This method does not work well, when probability mass is concentrated on a few bins, such as only one or two, which happens quite frequently in this data.

[24] Pr refers to probability.

[25] When I solve the model with a moving steady state, this proportion is updated over time and the calibrated value here is the proportion used in first period.

age proportion of employees in durable sector, relative to durable and non-durable sector combined. The average proportion during this period is 0.38, so I use $\zeta_d = 0.38$ in my calibration. $\psi_D := \frac{P_d Y^d}{P_c Y^c + P_d Y^d}$ and in order to calibrate this steady state proportion of expenditure on durables, relative to durables and nondurables combined, I use data on PCE (personal consumption expenditures) from FRED St Louis database for US and for the time period 1947-2020, I measure the average proportion of PCE on durables, relative to durables and nondurables combined, which is equal to 0.3, so I use value of 0.3 for ψ_D in my calibration.

For variances of signal distributions, I report how my results vary, based on various values of σ_s, σ_ϵ and σ_η. The values of states z_H, z_L and $\bar{z}$ are theoretical objects and do not have any readily available empirical counterpart. Of course, it must be that $z_L < \bar{z} < z_H$. Admittedly, these state parameters have the least amount of discipline imposed on them, relative to all other parameters. Nevertheless, I do analyze how my results vary with different values for these parameters as well. In general, when the two states are very far apart or close, they are easier or difficult to distinguish respectively and learning dynamics can vary accordingly.

Table 3: GE Calibration

Parameter	(1) Baseline	(2) Alternatives
risk aversion σ	2	2
frisch elasticity η	2	2
taylor rule ϕ_π	1.5	1.5
taylor rule ϕ_y	0.125	0.125
interest rate smoothing ϱ_i	0.8	0.8
signal variance σ_ϵ	5	0.2-10
signal variance σ_s	15	2.5-15
signal variance σ_η	0.3	0.2-5
signal variance σ_u	2.5	2.5
returns to scale α	0	0
elasticity of substitution ϵ_c	6	6
elasticity of substitution ϵ_d	6	6
depreciation rate of durables δ	0.0033	0.0033
durable good marginal utility parameter γ	1.02	1.02
probability of uncertainty resolution λ	0.1	0.01-0.2
low state z_L	1.5	0-1.5
high state z_H	2	2
steady state value of state $\bar{z}$	1.75	1.75
prior probability of high state p_0	0.63	0.63
steady state output share of non-durables ψ_c	0.3	0.3
share of noisy durable shoppers μ	0.1	0.1
steady state employment share durable sector ζ_d	0.38	0.38
steady state inflation nondurables $\pi_{c,ss}$	0	0
steady state inflation durables $\pi_{d,ss}$	0	0
steady state durable good stock D_{ss}	2	2
interval size for numerical differentiation formula h	10^{-10}	10^{-10}
iterations/horizon of IRF M	50	50
number of gridpoints integration N	50	10-100

Note

15. Theoretical Impulse Response Functions

In the IRF (impulse response function) graphs[26], the absolute magnitudes of the log deviations from steady state have no interpretation but relative magnitudes, reflecting asymmetries for example do have an interpretation. Even these have a qualitative meaning and can reflect asymmetries if contractions are stronger than expansions for example but do not suggest a particular magnitude for the asymmetry. The impulses are plotted as a reaction to a permanent, false positive (henceforth FP) noise shock Schaal and Taschereau-Dumouchel (2020), where FP means that the true state is z_l or low but the permanent noise shock is equal to η_0, so that the private signal distribution is centered around $z_l + \eta_0$. This creates the possibility that durable shoppers may confound the high $z_l + \eta_0$ with high state or may believe that $z = z_h$ with high probability, especially if η_0 is an unlikely and large, positive draw from their current, most updated posterior about η. However, if as a result of Bayesian and social learning, shoppers eventually allocate a lot of probability mass to the interval $[\eta_0 - \epsilon, \eta_0 + \epsilon]$, for $\epsilon \approx 0$ in their updated posterior, then the confounding will cease to exist and public beliefs will display an endogenous and sudden crash/reversal toward zero, which means convergence to the truth since the true state is low.

The impulse responses are generated under the assumption that uncertainty is never resolved exogenously during the period when impulses are plotted or in other words, the Bernoulli random variable with probability of success equal to $\lambda = 0.1$ never realizes as success during this period. This assumption is made to accentuate the effects of endogenous learning dynamics and the associated endogenous fluctuations in economic activity. For example, if the economy is likely to transit into a recession for reasons orthogonal to consumer sentiment, then it will eventually become obvious to consumers that they have indeed entered into a recession, despite the fact that they might be in a rational herd phase with unjustifiably optimistic beliefs based on endogenous beliefs. However, any fluctuations in beliefs and economic activity attributable to such an uncertainty resolution, which can be likened to a consumer version of *Minsky Moment* are *exogenous* and this is why I do not simulate uncertainty resolution in impulses.

One must interpret the log deviation of mass of investing shoppers in durables ($\hat{I_{d,t}}$) with caution. The mass of investing shoppers and hence actual production of durables in stage 2 is predetermined from stage 1 and its evolution is plotted in the graph on mass of investors. One cannot hold this level of mass constant, moving *forward* in the forward

[26] The graphs are displayed in section 17. All omitted graphs can be found in appendix.

looking solution of the model at its stage 1 value, since this mass is a function of not just beliefs, which can be held fixed, due to martingale property/resolution method but also past values of endogenous variables through the cutoff function as derived in durable good purchase decision section; these other endogenous variables cannot be held fixed moving forward in model solution. Hence, $I_{d,t}$ must be treated as dynamic, which is why I log linearized the investor mass to get $\hat{I_{d,t}}$. However, once one has obtained model solution by incorporating $\hat{I_{d,t}}$ in stage 2 system, in the next stage 1, the mass of investing agents $I_{d,t}$ is determined by the solution to the cutoff problem, which uses information from previous stage 2 model solution. Hence, the evolution of level of mass $I_{d,t}$ over time is a distinct object from $\hat{I_{d,t}}$ and $\hat{I_{d,t}}$ does not have any clear interpretation; it has a fuzzy interpretation but essentially reflects the expectations of how this mass may evolve moving forward, in deviations from its current value. Nevertheless, $\hat{I_{d,t}}$ does effect the solution for output, nondurable consumption, inflation and other variables in the stage 2 solution and those do have clear interpretations as variables.

In particular, there is a negative correlation between the impulses in $\hat{I_{d,t}}$ and log deviation of non-durable consumption $\hat{c}_t$ in my results. This relationship reflects the negative wealth effect of higher stage 1 investment in durables on stage 2 non-durable consumption. In my model, there is no mechanism available to consumers for expanding their consumption possibilities of non-durables by compensating for the negative wealth effect of higher predetermined and irreversible durable investment apart from supplying more labor. In particular, there is no auto-credit in model so that higher durable investment in stage 1 always comes at the cost of reduced purchasing power for spending on non-durables to the extent that higher labor supply does not compensate for this effect. Having said this, a fall in $\hat{I_{d,t}}$ does not necessarily mean a fall in mass of durable investment, since $\hat{I_{d,t}} \neq I_{d,t}$, as discussed in last paragraph. Hence, in the forward looking solution of model when for example $\hat{I_{d,t}}$ falls but non-durable consumption expands, the best possible interpretation of this is that durable investment is expected to rise in log deviations from current investor mass and hence this *expected* negative wealth effect causes non-durable consumption to fall. Since higher labor supply entails a disutility, higher labor supply can compensate for this negative wealth effect only partially. Furthermore, it is perfectly possible for mass of investing agents $I_{d,t}$, $\hat{c}_t$ and $\hat{y}_t$ to comove and expand on impact for instance, as they do in some cases below, even though $Corr(\hat{c}_t, \hat{I_{d,t}}) < 0$ and $\hat{I_{d,t}}$ contracts.

For the baseline case, in response to FP noise shock of 2 standard deviations, with $\sigma_s = 15$ and $\sigma_\epsilon = 0.2$, the public beliefs display a rising degree of optimism, i.e the probability that state is high increases from prior of 0.63 to 1 in few periods; subsequently, the beliefs display a sudden, sharp and endogenous reversal after 5 periods as there is recognition of the truth. Driven by the optimism of beliefs, mass of investors in durables rises above 8% on impact, before there is a sharp reversal toward zero and nobody invests. Non-durable consumption, aggregate output, aggregate hours and nondurable inflation enter into temporary boom before reverting back to steady state. Meanwhile, aggregate inflation and durable inflation display a significant fall below steady state/contraction on impact. The interest rate is being set by central bank in response to deviations of aggregate output and aggregate inflation from steady state; the interest rate falls quite significantly on impact since $\phi_\pi > \phi_y$ and aggregate inflation is falling. Hence, one gets an endogenous cycle in evolution of public beliefs but no cycle in other macroeconomic variables for this case.

The comovements between aggregate quantities such as hours, non-durable consumption, output and nondurable inflation are positive, since they all expand in response to noise shock. However, both durable inflation and aggregate inflation comove negatively with these variables. Hence, in terms of the relationship between aggregate quantities and aggregate inflation, this permanent noise shock is similar to a positive supply shock. This is in contrast with the work of Lorenzoni (2009) for example who argued that noise shocks in an island type economy behave like demand shocks. Rousakis (2012) made a similar point in the context of a very different model and argued that expectational shocks can behave like supply shocks in the presence of endogenous response of monetary policy under some conditions[27]. Hence, my findings further bolster this extant theoretical result in the literature by establishing the same result but in the context of a completely different model.

Meanwhile, when one perturbs the economy with different noise shock size of 0.1 standard deviation, while keeping $\sigma_s = 15$ and $\sigma_\epsilon = 0.2$ fixed, the qualitative behavior of all variables is the same. The previously discussed results are robust to this smaller shock size. For a shock size, larger than 2 standard deviations, the results are observationally equivalent to a higher variance σ_η since shock size = $\eta_0 = 2\sigma_\eta$ for 2 sd shock. If one varies the variance of signal distribution σ_η to 5 from baseline value of $\sigma_\eta = 0.3$, one gets similar responses as in the case with $\sigma_\eta = 0.3$. This shows that results are robust to alternative shock sizes, both greater and lower than two standard deviations, given fixed values for

[27] In Rousakis (2012), expectational shocks behave like supply shocks if $\phi_y > \frac{1}{\eta}$ i.e when monetary policy reaction to output is greater than inverse Frisch elasticity. In my calibration, this condition is not satisfied but I still find that noise/sentiment shocks behave like supply shocks since my model is completely different.

other parameters. I omit the graphs for these two cases below because the results are very similar to the baseline case. Nevertheless, all omitted IRF graphs are provided in appendix.

On the other hand, for $\sigma_s = 2.5$, $\sigma_\epsilon = 0.2$, and 2 sd noise shock, the qualitative behavior of results displays significant changes relative to baseline. A lower value of σ_s means that signal distribution is more tightly centered around $z_l + \eta_0$ so that a more precise signal about the state is received by durable shoppers, which facilitates convergence toward truth. Public beliefs do not show any rising degree of optimism any more and converge quickly and monotonically to the truth from prior value of 0.63 toward 0 in about five periods. One gets a contraction in aggregate output, nondurable consumption, aggregate hours and nondurable inflation on impact. All these variables experience a subsequent recovery, which brings nondurable consumption and hours to a level above steady state but output continues to be in recession even though the recession is milder than before. As also in earlier cases, the comovement between aggregate quantities and aggregate inflation is negative, so that aggregate inflation increases on impact. The mass of investing durable shoppers rises significantly on impact, *herds* at that high value for several periods and sharply reverts back to zero after 50 periods. These results show that an endogenous crash in beliefs can generate a persistent recession/contraction in output and $\hat{I_{d,t}}$ where the latter has to be interpreted with caution. However, hours and non-durable consumption, despite an initial contraction are able to recover even above steady state values. This is mainly because of the wealth effect of $\hat{I_{d,t}}$ on $\hat{c}_t$, also explained above; since $\hat{I_{d,t}}$ is settling permanently on a value below steady state, $\hat{c}_t$ can settle on a value permanently above steady state.

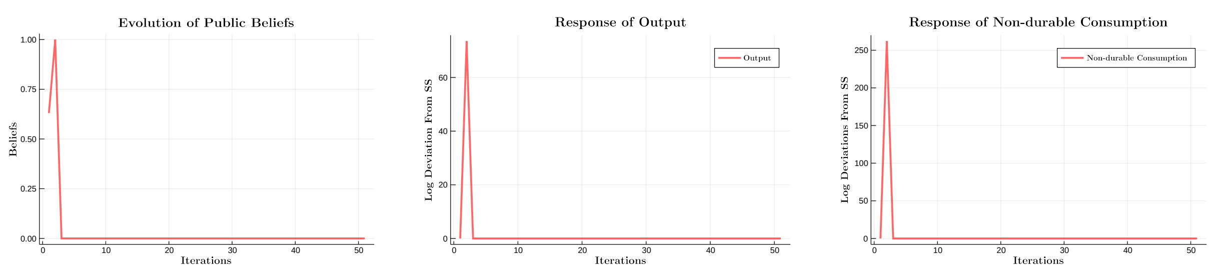
In perturbation corresponding to section 16.3, I perturb σ_ϵ from baseline value of 0.2 to 5, which makes the observation noise in the social learning process more significant, implying that durable shoppers will trust past observations of mass of durable investors less, making the bayesian learning and convergence to truth slower. In this case, beliefs keep rising slowly toward increasing degree of optimism for several periods and no convergence toward truth occurs even for very large number of iterations such as 1000 (not shown). This is an example of rational herding toward the wrong belief; public beliefs are not literally constant over time, which is why this is a rational herd and not informational cascade, since there is some positive learning. The impulse responses are qualitatively distinct, relative to previous cases. The mass of investors rises on impact but slowly falls toward a value of 4%, before sharply falling toward zero after period 50. Meanwhile, output, hours, nondurable consumption and nondurable inflation comove positively with each other and in this case also display endogenous cycles. The cycle in output is asymmetric and causes output to fall way below steady state after the relatively mild and brief over-expansion; eventually output reverts to steady state again. The cycles in other three endogenous variables have an endogenous reversal which leads these variables to stabilize at a value above steady state; reversal in these variables is not asymmetrically strong, unlike that of output. This is also explained by the wealth effect from $\hat{I_{d,t}}$ since $\hat{I_{d,t}}$ settles on a value, permanently below steady state and so consumption settles above steady state. Since public beliefs do not converge toward truth of zero, the endogenous reversals in economic activity are not driven by endogenous reversal in consumer sentiment. The reversals are driven by other general equilibrium effects, orthogonal to consumer sentiment such as the stabilizing response of central banks and other cross-equation restrictions in model, which cause the policy rule of output to react to changes in other variables of system, once the system is perturbed. These cross-equation responses are sufficiently strong so that output actually falls way below steady state after a brief expansion, which is a kind of overshooting of stabilizing responses.

In section 16.4, in addition to using $\sigma_{\epsilon} = 5$, I also change value of z_l from 1.5 to 1, which makes the two states easier to distinguish relative to previous case. The latter change facilitates learning about the hidden state and accelerates convergence toward truth. However, the noise in observation still dominates the results and I do not get any convergence to truth even after many periods. The only difference is that rather than a rational herd in which beliefs keep slowly rising toward higher values for many periods, beliefs now enter into a herding phase which more closely approximates a *cascade* in which beliefs stabilize at an almost constant value of 0.8 for many periods. In this case, beliefs are closer to truth after a large number of periods relative to previous case, even though still quite far from it. Output, nondurable consumption, hours and nondurable inflation display similar qualitative dynamics as in the previous case and expand initially followed by an endogenous reversal, which leaves output below initial steady state. All other variables also show almost identical qualitative behavior as in previous case.

In section 16.5, I further reduce the value of z_l to 0.5 which creates an endogenous reversal in public beliefs after period 75. The endogenous and sharp reversal of beliefs toward the truth i.e a recognition that the state is low produces a sharp contraction in output and other endogenous variables. The mass of investors in durables displays nonlinear and endogenous cycles; the impact effect is positive, followed by a reversal, expansion and another sharp crash after period 100. This volatility of durable spending over the business cycle is consistent with the stylized facts regarding durable good spending's volatility reported earlier. Prior to endogenous reversal in consumer sentiment, output expanded initially because of rising optimism and then cross-equation responses/GE effects caused output to contract below steady state before eventually reverting toward steady state. However, an endogenous crash of consumer confidence in period 75 creates a sharp and *persistent* recession, which is followed by only an almost negligible recovery despite the presence of a stabilizing central bank. This shows that endogenous consumer sentiment reversals can create strong and persistent recessions in my model, despite the presence of other stabilizing forces. Thus, endogenous cycles in beliefs and economic activity arise when two conditions are satisfied. Firstly, information is sufficiently noisy/imprecise i.e σ_s and σ_ϵ are sufficiently high so that that monotonic convergence toward zero does not occur and public beliefs herd toward unjustified optimism. Secondly, when the two hidden states ar also sufficiently easy to distinguish so that eventually the herding phase terminates, followed by a sharp reversal.

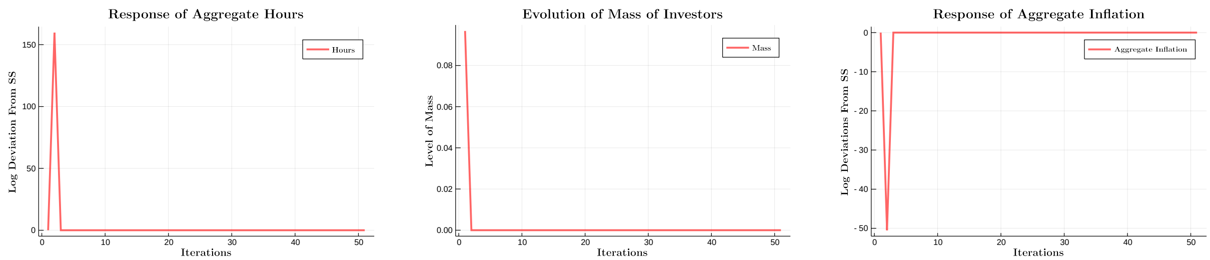
16. Theoretical IRF Results

16.1. Baseline FP, $\sigma_s = 15$, $\sigma_\epsilon = 0.2$, shock size = 2 sd



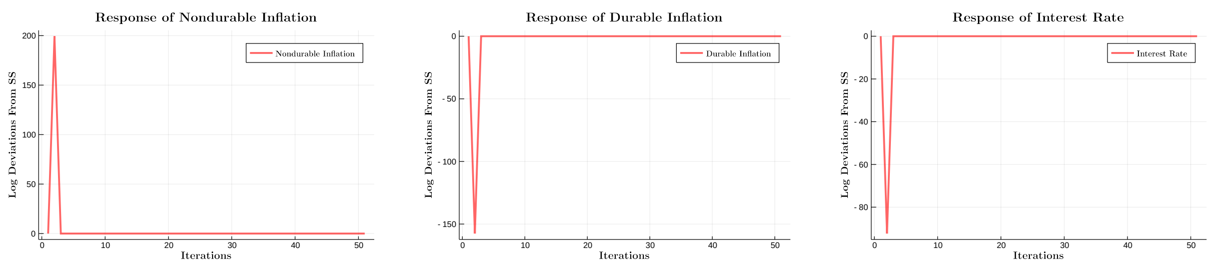
(g) (h) (i)

Figure 15: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a) (b) (c)

Figure 16: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



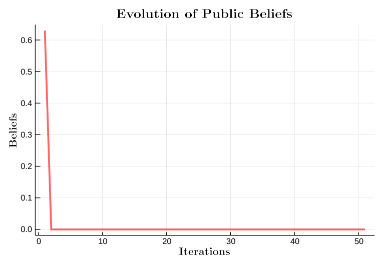
(a) (b) (c)

Figure 17: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

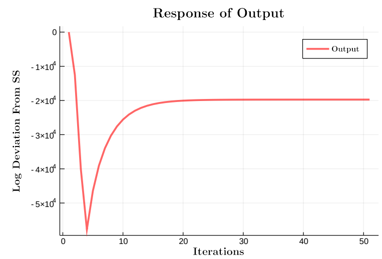
(a) (b)

Figure 18: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

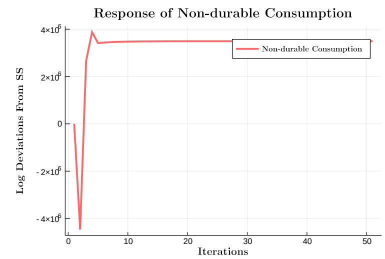
16.2. FP, $\sigma_s = 2.5$, $\sigma_\epsilon = 0.2$, shock size = 2 sd



(a)

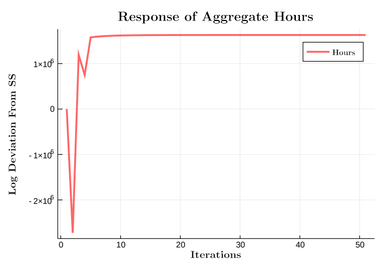


(b)

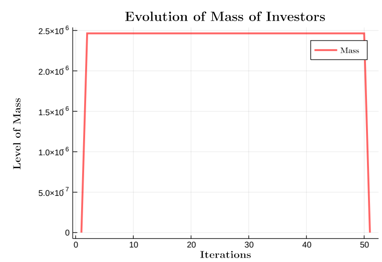


(c)

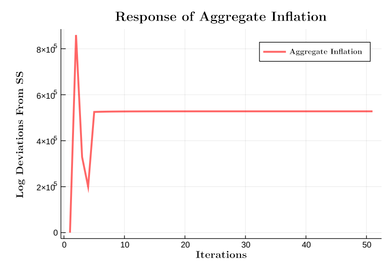
Figure 19: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 2.5$)



(a)

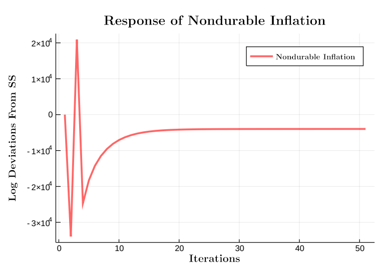


(b)

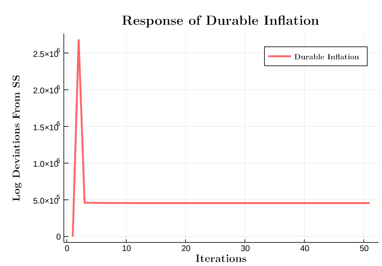


(c)

Figure 20: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 2.5$)



(a)



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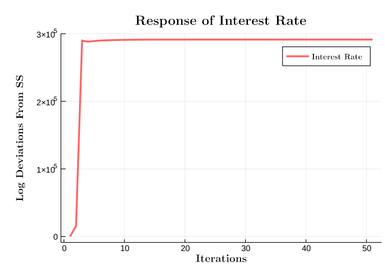


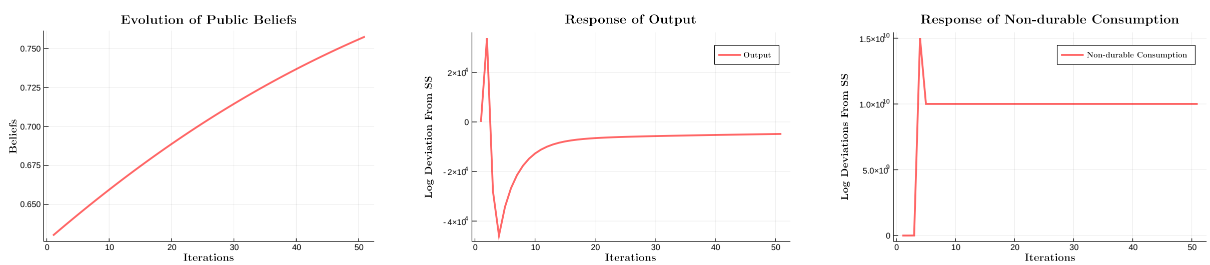
(c)

Figure 21: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 2.5$)

(a) (b)

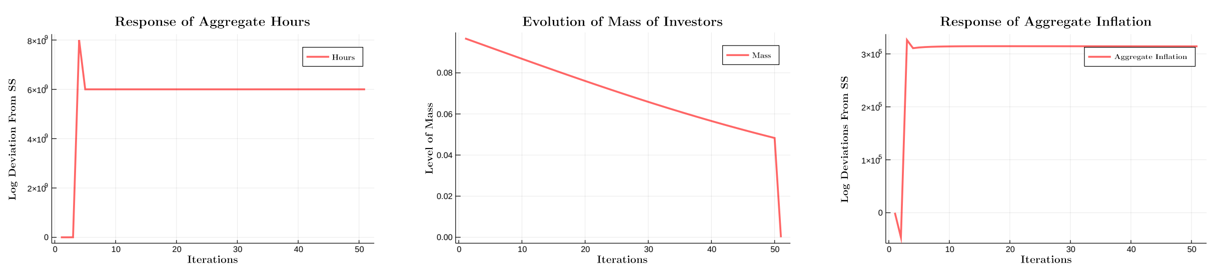
Figure 22: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 2.5$)

16.3. FP, $\sigma_s = 15$, $z_l = 1.5$, SHOCK SIZE = 2 SD, $\sigma_\epsilon = 5$



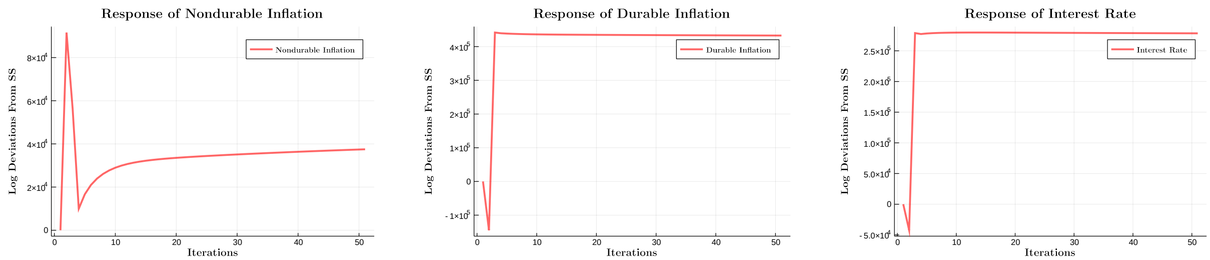
(a) (b) (c)

Figure 23: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



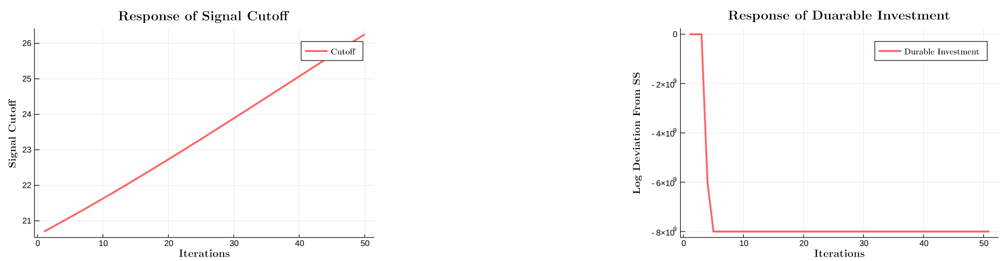
(a) (b) (c)

Figure 24: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a) (b) (c)

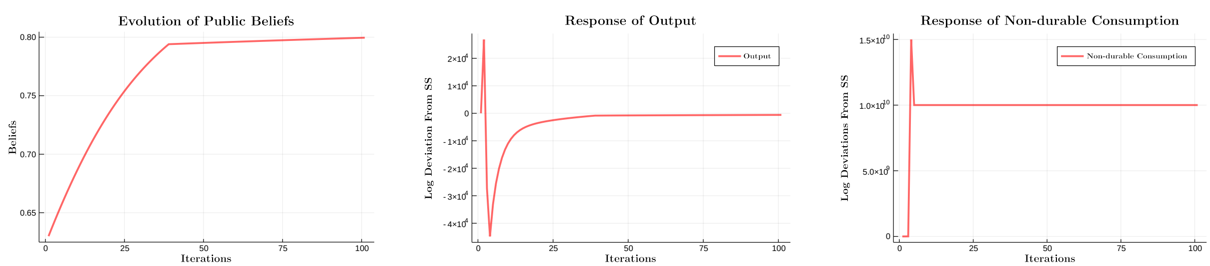
Figure 25: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a) (b)

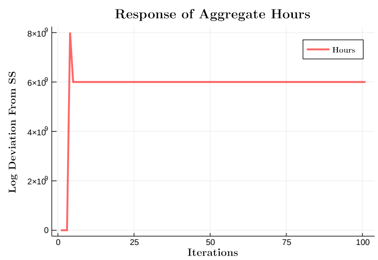
Figure 26: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

16.4. FP, $\sigma_s = 15$, $\sigma_\epsilon = 5$, ZL = 1, SHOCK SIZE = 2 SD

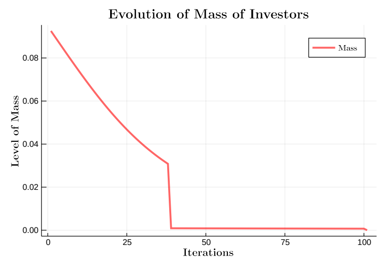


(a) (b) (c)

Figure 27: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a)



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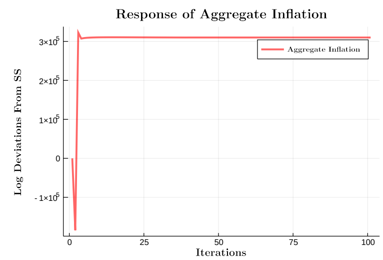
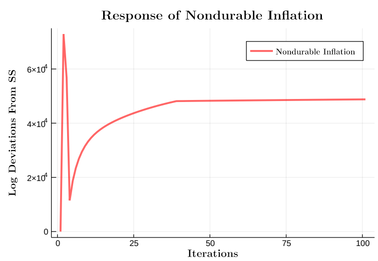
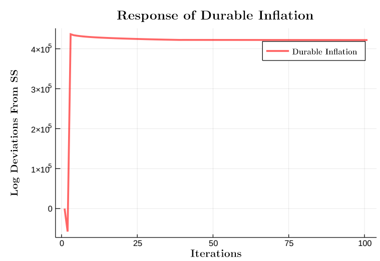


(c)

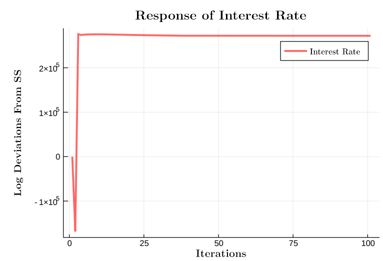
Figure 28: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a)

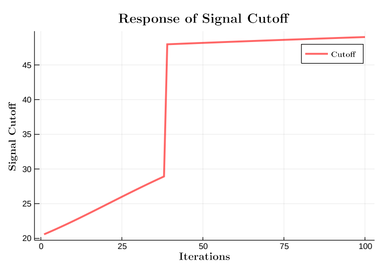


(b)

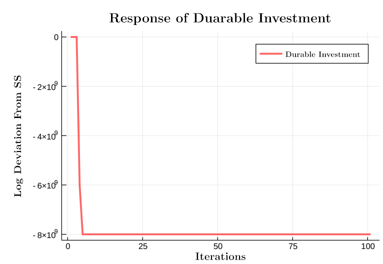


(c)

Figure 29: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



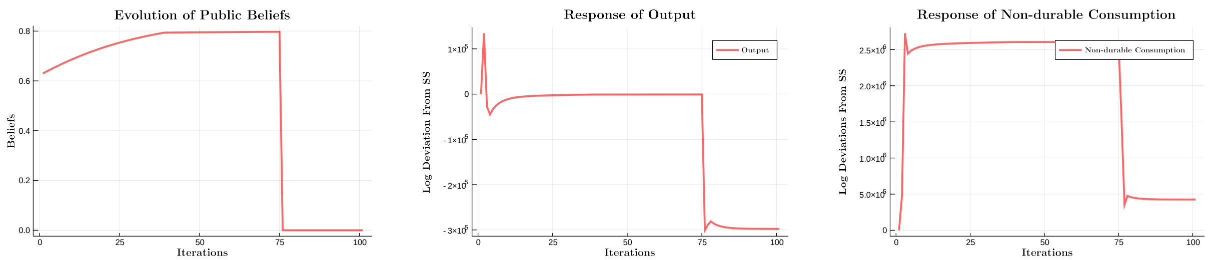
(a)



(b)

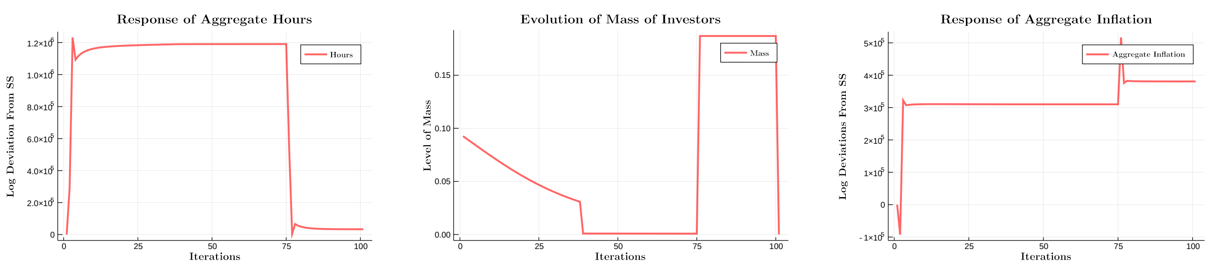
Figure 30: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

16.5. FP, $\sigma_s = 15$, $\sigma_\epsilon = 5$, ZL = 0.5, SHOCK SIZE = 2 SD, ENDOGENOUS REVERSAL



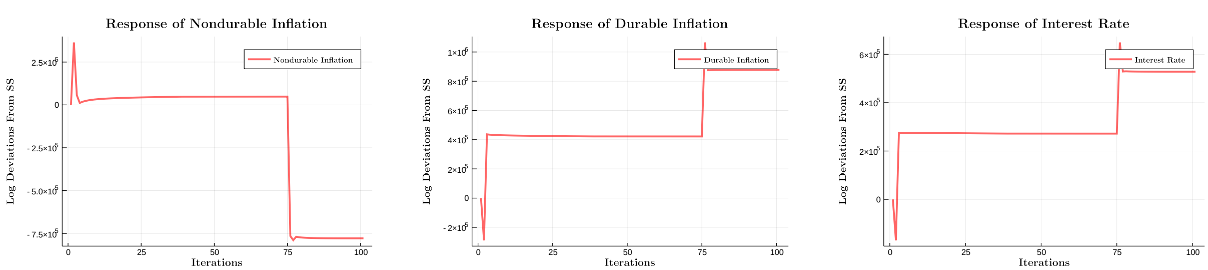
(a) (b) (c)

Figure 31: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a) (b) (c)

Figure 32: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a) (b) (c)

Figure 33: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

(a) (b)

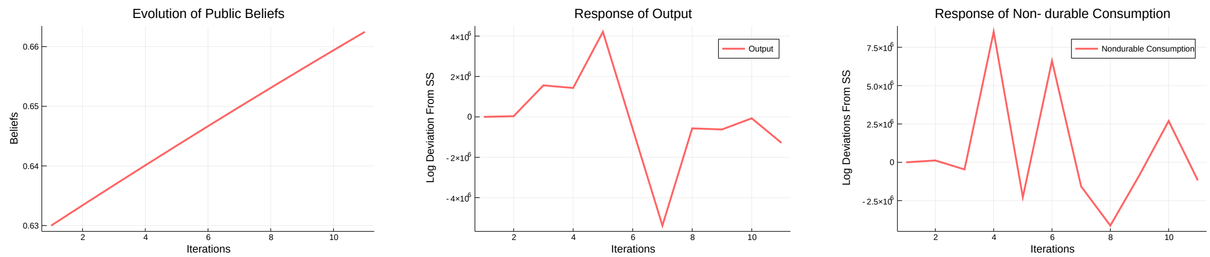
Figure 34: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

17. Moving Steady State

In this section, I present results from two different perturbations of parameters, when I solve the model with a moving steady state instead. This means that after every stage 2, I update the value of parameters ψ_c and ζ_d based on the new ratios of output produced in nondurable sector relative to aggregate output and employment share in durable sector, accessible after model solution in last stage 2. Therefore, the log linear system in every stage 2 log linearizes equations around the steady state implied by the previous stage 2 solution. Thus, a positive response on graph now means a deviation above the current steady state, rather than one fixed steady state, as before.

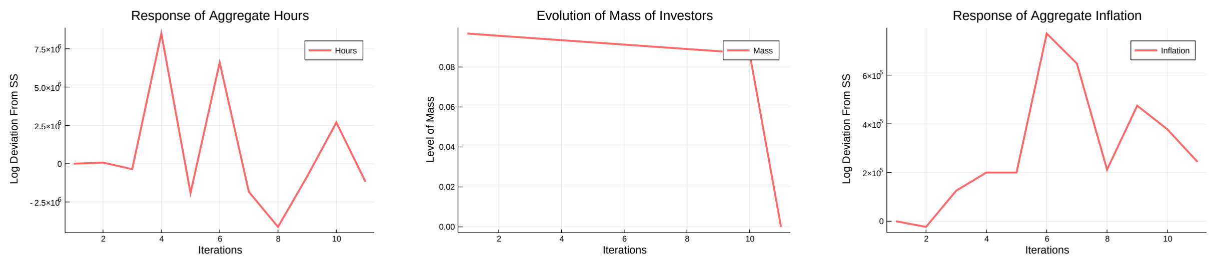
In both perturbations below, I get endogenous and asymmetric cycles in output and aggregate inflation, where recessions are stronger than expansions. In the second perturbation below, I get asymmetric recessions and endogenous cycles in output, aggregate inflation, aggregate hours and non-durable consumption, driven by endogenous crash in beliefs. As before, the positive supply shock interpretation of fluctuations caused by my noise shock remains valid.

17.1. FP, Moving SS, $\sigma_s = 15$, $z_l = 1.5$, shock size = 2 sd, $\sigma_\epsilon = 5$



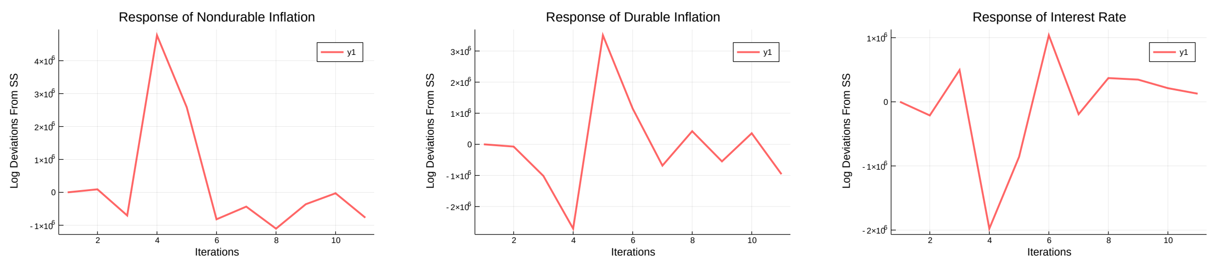
(a) (b) (c)

Figure 35: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



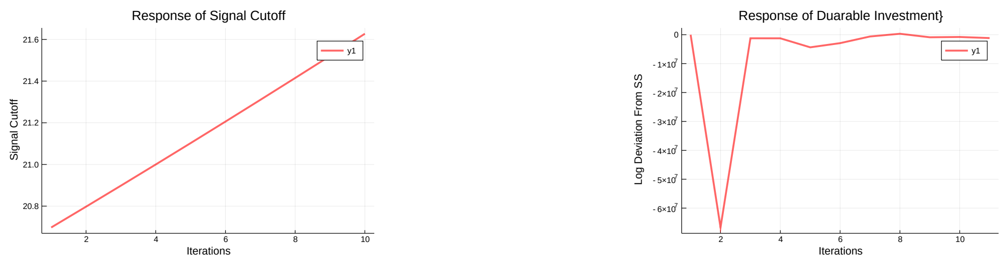
(a) (b) (c)

Figure 36: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a) (b) (c)

Figure 37: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



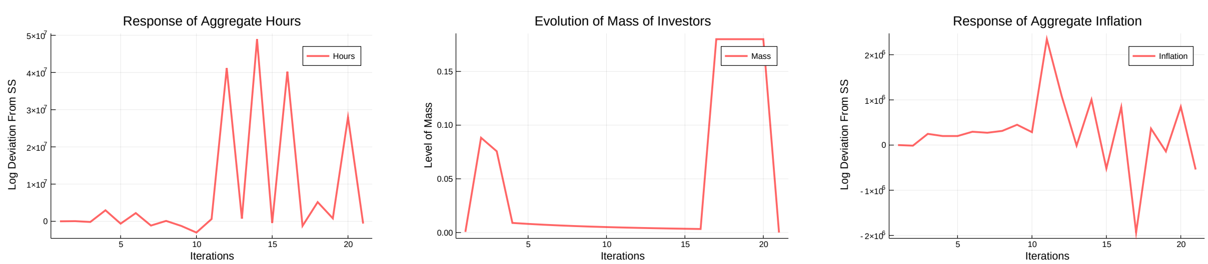
(a) (b)

Figure 38: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

17.2. FP, Moving SS, $\sigma_s = 15$, $z_l = 0.5$, shock size = 2 sd, $\sigma_\epsilon = 2.5$

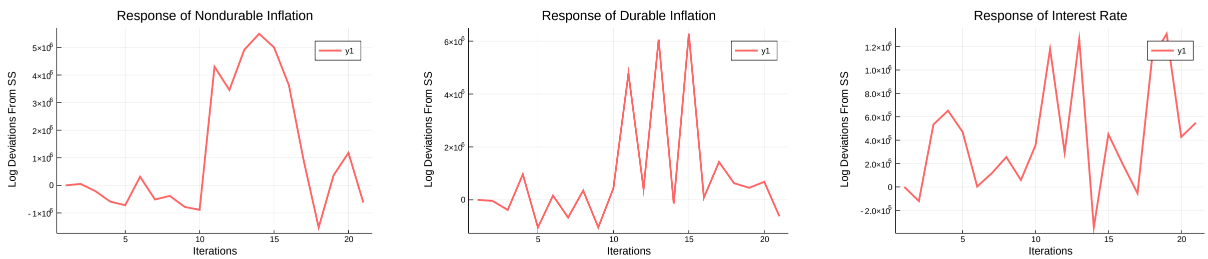
(a) (b) (c)

Figure 39: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



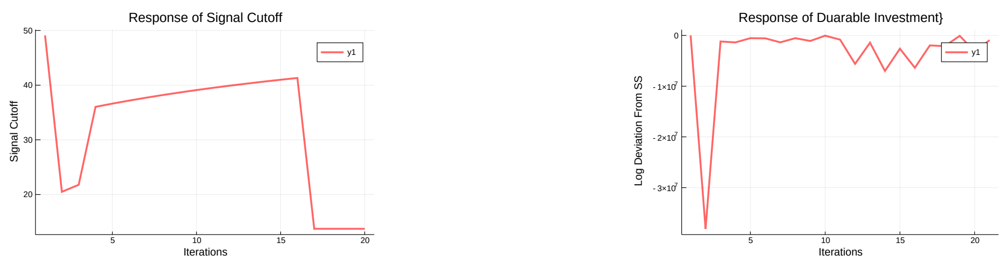
(a) (b) (c)

Figure 40: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a) (b) (c)

Figure 41: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



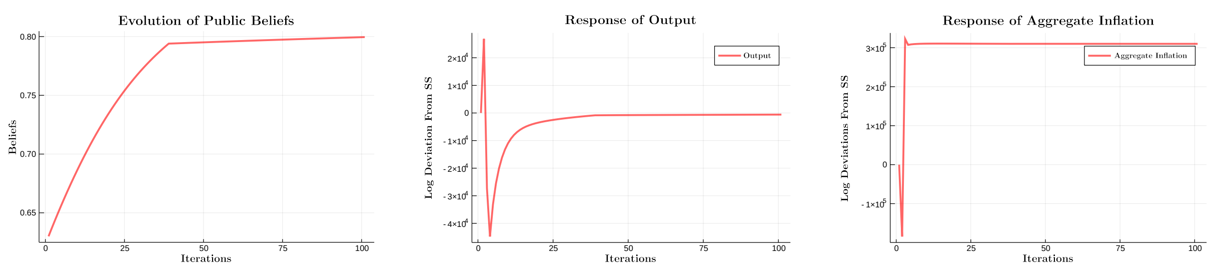
(a) (b)

Figure 42: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

18. SENTIMENT SHOCKS DURING BOOM/RECESSION

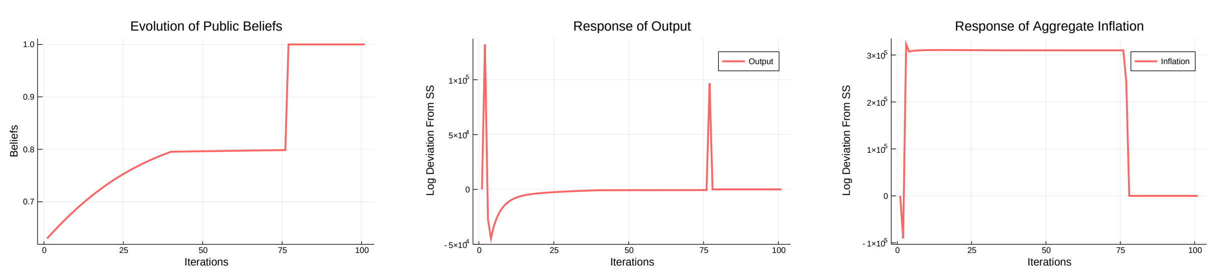
In order to connect the theoretical results with the stylized facts from local projections, I now present results in which I perturb the economy with an exogenous sentiment shock, either of an optimistic kind, in which beliefs sharply jump toward one or a pessimistic kind in which beliefs crash toward zero; these shocks can be interpreted as exogenous uncertainty resolution about the true state, which will be high in the case of an optimism shock and low in the case of a pessimism shock. I control the timing of these shocks so that I can perturb the economy with an optimism/ pessimism shock during a recession and boom/non-recession phase, which leads to four cases. These four cases along with the corresponding baseline impulses in absence of any sentiment shock are displayed below. These theoretical impulses speak with the local projection evidence presented earlier, which showed empirical IRF'S in the case of a consumer optimism/pessimism shock, instrumented by political partisanship variable for both an expansion/boom regime and recession regime. Consistent with the empirical evidence, my model can produce causal effects of optimism and pessimism shocks on economic activity in the right direction i.e optimism is expansionary and pessimism is contractionary, both during recession regime and expansion/non-regime.

An optimism shock during a boom/non-recession phase creates a significant expansion (contraction) in output (aggregate inflation) (Figure 52). Meanwhile, a pessimism shock during non-recession (Figure 53) phase creates a very sharp and persistent recession (expansion) in output (aggregate inflation). A pessimism shock during non-recession regime has more persistent and stronger effects on output relative to an optimism shock during non-recession regime; i.e optimism is good for the economy during a boom phase but pessimism is extremely deleterious in terms of creating a reversal during boom phase. Meanwhile, during recession phase, an optimism shock (Figure 55) significantly accelerates recovery toward steady state relative to no shock case (Figure 54). On the other hand, a pessimism shock during recession significantly slows down the recovery which would have occurred in absence of sentiment shock and makes the recession very persistent. It is also true that optimism and pessimism shocks during recession regime have weaker quantitative effects relative to the same shocks during boom/non-recession phase, which is also consistent with the stylized facts from local projections section.



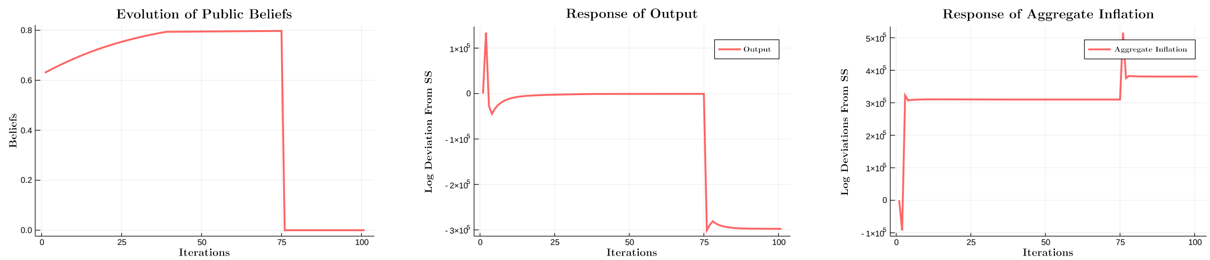
(a) (b) (c)

Figure 43: Impulse Response to a FP 2 sd Permanent Noise Shock, No Sentiment Shock



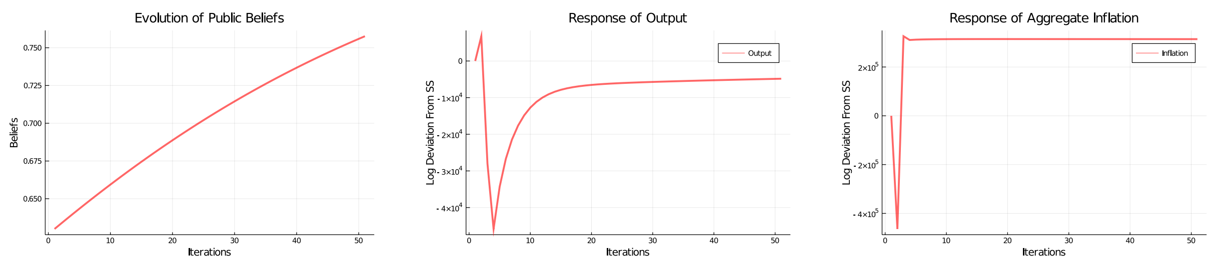
(a) (b) (c)

Figure 44: Impulse Response to a FP 2 sd Permanent Noise Shock, Optimism Shock During Boom/Non-Recession at Period 75



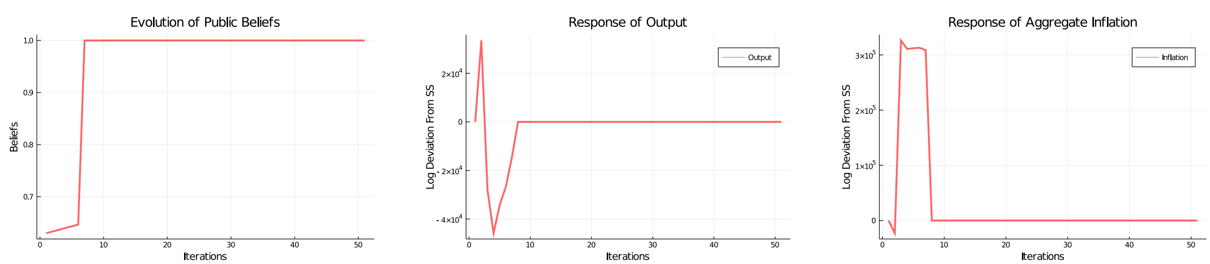
(a) (b) (c)

Figure 45: Impulse Response to a FP 2 sd Permanent Noise Shock, Pessimism Shock During Boom/Non-Recession at Period 75



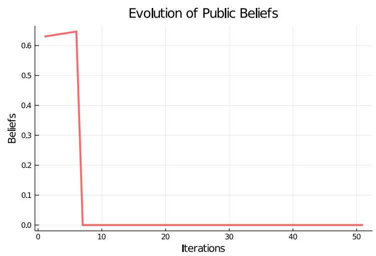
(a) (b) (c)

Figure 46: Impulse Response to a FP 2 sd Permanent Noise Shock, No Sentiment Shock

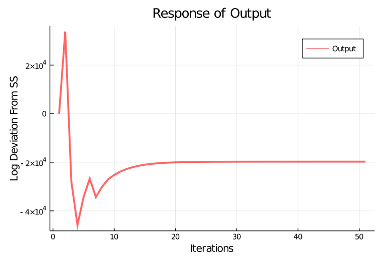


(a) (b) (c)

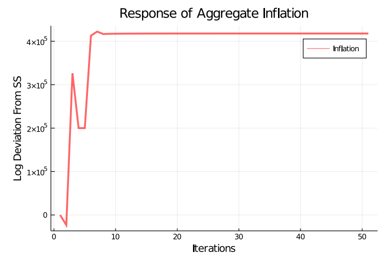
Figure 47: Impulse Response to a FP 2 sd Permanent Noise Shock and Optimism Shock During Recession at Period 5



(a)



(b)



(c)

Figure 48: Impulse Response to a FP 2 sd Permanent Noise Shock and Pessimism Shock During Recession at Period 5

19. Conclusion

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20. Appendix

20.1. Data and Variables

University of Michigan Survey

FRBNY-SCE

Haver Analytics

FRED Database St Louis

Data used from replication files of Benhabib and Spiegel (2018).

20.2. SUPPLEMENTARY DERIVATIONS

Derivation of Static Durable Pricing Equation:

The price charged by all durable good sector firms in stage 2, who take the aggregate demand for durables from stage 1 as given is going to be:

$P_{d,t} = \frac{\epsilon_d - 1}{\epsilon_d}\varphi_{t,d} = \mathcal{M}_d \varphi_{t,d}$, where $\varphi_{t,d}$ is nominal, marginal cost in durable goods sector and $\mathcal{M}_d$ is markup.

Nominal marginal cost is $\varphi_{t,d} = \frac{W_t}{A_t(1-\alpha)\left(N_{t,d}\right)^{-\alpha}}$.

$\implies P_{d,t} = \mathcal{M}_d \frac{W_t}{A_t(1-\alpha)\left(N_{t,d}\right)^{-\alpha}}$

Abstracting from TFP shocks and imposing $A_t = 1$ and expressing the previous pricing equation in log linear terms,

$\hat{P_{d,t}} = \hat{w}_t + \alpha \hat{n_t^d}$

Using the log linearized labor supply condition, the pricing equation is:

$\hat{P_{d,t}} = \hat{n_t^s} + \sigma \hat{c}_t + \alpha \hat{n_t^d}$

The durable good investment level cannot be considered as fixed, moving forward, since it is a function of past endogenous variables such as consumption and past endogenous state variables such as past durable good stock and these cannot be considered as fixed moving forward. Hence, $I_{d,t}$ is not held fixed but is expressed in log deviations $\hat{I_{d,t}}$ by log linearizing the goods market clearing condition:

$\hat{y}_t = \psi_c \hat{c}_t + (1-\psi_c)\hat{I_{d,t}}$

Similarly, the log linearization of $N_t^s = N_{t,d}^d + N_{t,c}^d$ is:

$\hat{n_t^s} = \zeta_d \hat{n_{t,d}^d} + (1-\zeta_d)\hat{n_{t,c}^d}$, where $\zeta_d := \frac{N_d^d}{N_d^d + N_c^d}$ is the steady state proportion of labor demanded in durable good sector. Also, note that $\hat{I_{d,t}} = (1-\alpha)\hat{n_{t,d}^d}$ from inverse production function.

$\implies \hat{P_{d,t}} = \hat{n_t^s} + \sigma \hat{c}_t + \alpha \hat{n_{t,d}} = \zeta_d \hat{n_{t,d}^d} + (1-\zeta_d)\hat{n_{t,c}^d} + \sigma \hat{c}_t + \frac{\alpha}{1-\alpha}\hat{I_{d,t}}$

$\hat{n_t^s} + \sigma \hat{c}_t + \alpha \hat{n_{t,d}} = \frac{\zeta_d}{1-\alpha}\hat{I_{d,t}} + \frac{1-\zeta_d}{1-\alpha}\hat{c}_t + \sigma \hat{c}_t + \frac{\alpha}{1-\alpha}\hat{I_{d,t}}$

$\hat{n_t^s} + \sigma \hat{c}_t + \alpha \hat{n_{t,d}} = \frac{\zeta_d + \alpha}{1-\alpha}\hat{I_{d,t}} + \frac{1-\zeta_d}{1-\alpha}\hat{c}_t + \sigma \hat{c}_t$

$\hat{n_t^s} + \sigma \hat{c}_t + \alpha \hat{n_{t,d}} = \frac{\zeta_d + \alpha}{1-\alpha}\hat{I_{d,t}} + \frac{(1-\zeta_d)\hat{c}_t + \sigma \hat{c}_t(1-\alpha)}{(1-\alpha)}$

$\hat{n_t^s} + \sigma \hat{c}_t + \alpha \hat{n_{t,d}} = \frac{\zeta_d + \alpha}{1-\alpha}\hat{I_{d,t}} + \frac{((1-\zeta_d) + \sigma(1-\alpha))\hat{c}_t}{(1-\alpha)}$

$\hat{n_t^s} + \sigma \hat{c}_t + \alpha \hat{n_{t,d}} = \mu_d \hat{I_{d,t}} + \mu_c \hat{c}_t$, where $\mu_d := \frac{\zeta_d + \alpha}{1-\alpha}$ and $\mu_c := \frac{((1-\zeta_d) + \sigma(1-\alpha))}{(1-\alpha)}$.

$= \mu_d \hat{I_{d,t}} + \mu_c \hat{c}_t$

$\implies \hat{P_{d,t}} = c_{1,d}\hat{I_{d,t}} + c_{2,c}\hat{c}_t$

,where $c_{1,d} := \frac{\zeta_d+\alpha}{1-\alpha}$

and $c_{2,c} := \frac{((1-\zeta_d)+\sigma(1-\alpha))}{(1-\alpha)}$.

Derivation Section 6.2.2

Since $\mathbb{E}_{ht}\left(\Theta_C^{pd}\right) = \left(a_C^{pd}\,\mathbb{E}_{ht}\left(-\frac{\psi_c}{\sigma}\left(\lambda\left(p_{t+1}(z_H) + (1-p_{t+1})z_L\right) + (1-\lambda)\,\bar{z}\right)\right)\right)$

Given $\hat{z_{t+1}} := p_{t+1}z_H + (1-p_{t+1})z_L$, algebraic manipulation implies that:

$\mathbb{E}_{ht}\left(\Theta_C^{pd}\right) = \tilde{a}^{pd}\mathbb{E}_{ht}\left(\hat{z_{t+1}}\right) + \hat{a}_{pd}\bar{z} - \tilde{a}_{pd}\bar{z}$, where $\tilde{a}^{pd} := -a_C^{pd}\lambda\frac{\psi_c}{\sigma}$ and $\hat{a}^{pd} := -a_C^{pd}\frac{\psi_c}{\sigma}$.

Derivation Section 6.2.3 Equation 8

bayes rule is:

signal structure blah blah blah

$\pi_{t+1}(\tilde{z},\tilde{\eta}) = \frac{\pi_t(\tilde{z},\tilde{\eta})f^{\epsilon}\left(\frac{1}{\mu}(d_t - d^e(\pi_t,\Omega_t,\tilde{z},\tilde{\eta})(1-\mu))\right)}{\int \pi_t(z,\eta)f^{\epsilon}\left(\frac{1}{\mu}(d_t - d^e(\pi_t,\Omega_t,z,\eta)(1-\mu))\right)d(z,\eta)}$

$$\pi_{t+1}(\tilde{z},\tilde{\eta}) = \frac{\pi_t(\tilde{z},\tilde{\eta})f^{\epsilon}\left(\frac{1}{\mu}(d_t - d^e(\pi_t,\Omega_t,\tilde{z},\tilde{\eta})(1-\mu))\right)}{\int \pi_t(z_h,\eta)f^{\epsilon}\left(\frac{1}{\mu}(d_t - d^e(\pi_t,\Omega_t,z_h,\eta)(1-\mu))\right)d\eta + \int \pi_t(z_l,\eta)f^{\epsilon}\left(\frac{1}{\mu}(d_t - d^e(\pi_t,\Omega_t,z_l,\eta)(1-\mu))\right)d\eta} \tag{48}$$

Derivation Section 6.2.3 Equation blah

$\mathbb{E}_{ht}\left(p_{t+1}\right) = \mathbb{E}_{ht}\left(\int \pi_{t+1}(z_H,\eta)d\eta\right)$

$$\iff \mathbb{E}_{ht}\left(p_{t+1}\right) = \mathbb{E}_{ht}\left(\frac{\int \pi_t(z_h,\eta)f^{\epsilon}\left(\frac{1}{\mu}(d_t - d^e(\pi_t,\Omega_t,z_h,\eta)(1-\mu))\right)d\eta}{\int \pi_t(z_h,\eta)f^{\epsilon}\left(\frac{1}{\mu}(d_t - d^e(\pi_t,\Omega_t,z_h,\eta)(1-\mu))\right)d\eta + \int \pi_t(z_l,\eta)f^{\epsilon}\left(\frac{1}{\mu}(d_t - d^e(\pi_t,\Omega_t,z_l,\eta)(1-\mu))\right)d\eta}\right)$$

$$\iff \mathbb{E}_{ht}\left(p_{t+1}\right) = \mathbb{E}_{ht}\left(\frac{1}{1+\frac{\int \pi_t(z_l,\eta)f^{\epsilon}\left(\frac{1}{\mu}\left(d_t - d^e(\pi_t,\Omega_t,z_l,\eta)(1-\mu)\right)\right)d\eta}{\int \pi_t(z_h,\eta)f^{\epsilon}\left(\frac{1}{\mu}\left(d_t - d^e(\pi_t,\Omega_t,z_h,\eta)(1-\mu)\right)\right)d\eta}}\right)$$

Given that $\int \pi_t(z_h,\eta)d\eta = p_t f_t^{\eta}(\eta)$ and $\int \pi_t(z_l,\eta)d\eta = (1-p_t)f_t^{\eta}(\eta)$,

$$\implies \mathbb{E}_{ht}\left(p_{t+1}\right) = \mathbb{E}_{ht}\left(\frac{1}{1+\frac{(1-p_t)\int f_t^{\eta}(\eta)f^{\epsilon}\left(\frac{1}{\mu}\left(d_t - d^e(\pi_t,\Omega_t,z_l,\eta)(1-\mu)\right)\right)d\eta}{p_t\int f_t^{\eta}(\eta)f^{\epsilon}\left(\frac{1}{\mu}\left(d_t - d^e(\pi_t,\Omega_t,z_h,\eta)(1-\mu)\right)\right)d\eta}}\right)$$

$$\mathbb{E}_{ht}p_{t+1} = \mathbb{E}_{ht}\left(\frac{1}{1+\frac{(1-p_t)\int f_t^{\eta}(\eta)f^{\epsilon}\left(\frac{1}{\mu}\left(d_t - F^{bar}_{z_l+\eta}(\hat{s_t})(1-\mu)\right)\right)d\eta}{p_t\int f_t^{\eta}(\eta)f^{\epsilon}\left(\frac{1}{\mu}\left(d_t - F^{bar}_{z_h+\eta}(\hat{s_t})(1-\mu)\right)\right)d\eta}}\right).$$

Derivation Section 6.2.3 Equation Blah

Derive the below:

$\Psi(s_h,p_t) := p_{ht} = \frac{1}{1+\frac{(1-p_t)\int f^{\eta}_{sh,t}(\eta)f_{z_l+\eta}(s_h)d\eta}{p_t\int f^{\eta}_{sh,t}(\eta)f_{z_h+\eta}(s_h)d\eta}}$

Derivation Section 6.2.4 Gamma Function Derivation

Derive the below explicitly

$$\Gamma(\hat{s_t}, s_h, p_t) = \Psi(p_t, s_h) \int\int \left(\frac{1}{1+\frac{(1-p_t)\int f_t^{\eta}(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(d_{ht}-F^{bar}_{z_l+\eta}(\hat{s_t})(1-\mu)\right)\right)d\eta}{p_t \int f_t^{\eta}(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(d_{ht}-F^{bar}_{z_h+\eta}(\hat{s_t})(1-\mu)\right)\right)d\eta}} \right) f^{\epsilon}(\epsilon) f^{\eta}_{t,s_h}(\eta) d\epsilon\, d\eta\, +$$

$$(1-\Psi(p_t, s_h)) \int\int \left(\frac{1}{1+\frac{(1-p_t)\int f_t^{\eta}(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(d_{lt}-F^{bar}_{z_l+\eta}(\hat{s_t})(1-\mu)\right)\right)d\eta}{p_t \int f_t^{\eta}(\eta) f^{\epsilon}\left(\frac{1}{\mu}\left(d_{lt}-F^{bar}_{z_h+\eta}(\hat{s_t})(1-\mu)\right)\right)d\eta}} \right) f^{\epsilon}(\epsilon) f^{\eta}_{t,s_h}(\eta) d\epsilon\, d\eta$$

,where $\tilde{d_{kt}} := (1-\mu) F_{z_k+\eta}(\hat{s})$, for $k \in \{l, h\}$.

and $d_{kt} := \tilde{d_{kt}} + \mu\epsilon_t$.

Derivation of G Function:

The indifference condition is:

$$\left(\frac{1-\tilde{c}(p_t,s_h,\Omega_t)}{\tilde{c}(p_t,s_h,\Omega_t)}\right)\frac{p_t}{1-p_t} = \left(\frac{\phi\left(\frac{s_h-z_l}{\sqrt{\sigma_\eta^2+\sigma_s^2}}\right)}{\phi\left(\frac{s_h-z_h}{\sqrt{\sigma_\eta^2+\sigma_s^2}}\right)} \right)$$

where $\tilde{c}\left(p_t, \hat{s}, \Omega_t\right) := \frac{\left(G_{1,pd}\hat{x_{t-1}}\right)+\tilde{a}_{pd}(\Gamma(\hat{s},p_t)z_h+(1-\Gamma(\hat{s},p_t))z_l)+\hat{a}_{pd}\bar{z}-\tilde{a}_{pd}\bar{z}+\tilde{\gamma}z_h-z_l}{z_h-z_l}$, and $\tilde{\gamma} := \gamma(1-\delta)D_{t-1}$.

$$\iff \left(\frac{1-\tilde{c}(p_t,s_h,\Omega_t)}{\tilde{c}(p_t,s_h,\Omega_t)}\right)\frac{p_t}{1-p_t} = \exp\left\{\frac{1}{2}\left(\frac{s_h-z_h}{\sqrt{\sigma_\eta^2+\sigma_s^2}}\right)^2 - \frac{1}{2}\left(\frac{s_h-z_l}{\sqrt{\sigma_\eta^2+\sigma_s^2}}\right)^2\right\}$$

One can solve the above equation for the value of cut off $\hat{s}$ for signal, at which this equality holds. The cut off $\hat{s}$ is implicitly defined by the value which leads to the following equality or the root of the G function, given Ω_t and π_t (holding as close as possible, in numerical terms if one uses numerical techniques for finding roots).

$$\log\left(\left(\frac{1-\tilde{c}(p_t,s_h,\Omega_t)}{\tilde{c}(p_t,s_h,\Omega_t)}\right)\frac{p_t}{1-p_t}\right) = \left\{\frac{1}{2\left(\sigma_\eta^2+\sigma_s^2\right)}\left((s_h-z_h)^2-(s_h-z_l)^2\right)\right\}$$

$$\log\left(\left(\frac{1-\tilde{c}(p_t,s_h,\Omega_t)}{\tilde{c}(p_t,s_h,\Omega_t)}\right)\frac{p_t}{1-p_t}\right) = \left\{\frac{1}{2\left(\sigma_\eta^2+\sigma_s^2\right)}\left(\left(-2s_hz_h+(z_h)^2\right)-\left(-2s_hz_l+(z_l)^2\right)\right)\right\}$$

$$\log\left(\left(\frac{1-\tilde{c}(p_t,s_h,\Omega_t)}{\tilde{c}(p_t,s_h,\Omega_t)}\right)\frac{p_t}{1-p_t}\right) = \left\{\frac{1}{2\left(\sigma_\eta^2+\sigma_s^2\right)}\left(2s_h\left(z_l-z_h\right)+z_h^2-z_l^2\right)\right\}$$

One must find the root of the following G function:

$$G\left(p_t, \hat{s}\left(p_t, \Omega_t\right), \Omega_t\right) := \left\{\log\left(\left(\frac{1-\tilde{c}(p_t,\hat{s},\Omega_t)}{\tilde{c}(p_t,\hat{s},\Omega_t)}\right)\frac{p_t}{1-p_t}\right) - \frac{1}{2\left(\sigma_\eta^2+\sigma_s^2\right)}\left(z_h^2-z_l^2\right)\right\}\frac{\left(\sigma_\eta^2+\sigma_s^2\right)}{z_l-z_h} - \hat{s} \tag{49}$$

Derivation of Euler Equation:

The Lagrangian for representative household's problem in stage 2, when durable good shoppers have already taken their decisions, which are now irreversible is:

$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} \tilde{Z}_t - \psi \tilde{Z}_t \frac{N_t^{1+\eta}}{1+\eta} + \lambda_t \left(W_t N_t + B_{t-1} + \Pi_t - P_{C,t} C_t - P_{D,t} I_{d,t} - Q_t B_t \right) \right)$

The FOC's with respect to nondurable consumption and labor supply are the following. These conditions are necessary and sufficient because of concavity of objective function and linearity of budget constraint:

$C_t^{-\sigma} \tilde{Z}_t = \lambda_t P_{C,t}$

$\psi N_t^{\eta} \tilde{Z}_t = \lambda_t W_t$

$\lambda_t Q_t = \beta \mathbb{E}_t \left(\lambda_{t+1} \right)$

$\frac{\psi N_t^{\eta}}{W_t} = \frac{C_t^{-\sigma}}{P_{C,t}}$

$\frac{\psi N_t^{\eta}}{w_t} = C_t^{-\sigma}$

$w_t = \psi N_t^{\eta} C_t^{\sigma}$

$\tilde{Z}_t \frac{C_t^{-\sigma}}{P_{C,t}} Q_t = \beta \mathbb{E}_t \left(\tilde{Z_{t+1}} \frac{C_{t+1}^{-\sigma}}{P_{C,t+1}} \right)$

$C_t^{-\sigma} Q_t = \beta \mathbb{E}_t \left(\frac{\tilde{Z_{t+1}}}{\tilde{Z}_t} \frac{C_{t+1}^{-\sigma} P_{C,t}}{P_{C,t+1}} \right)$

The Euler equation for consumption can then be expressed as the following:

$$Q_t = \beta \mathbb{E}_t \left(\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{P_{C,t}}{P_{C,t+1}} \right) \left(\frac{\tilde{Z_{t+1}}}{\tilde{Z}_t} \right) \right) \tag{50}$$

Derivation of Profit Expression:

$\Pi_{j,t}^C = \frac{P_{j,t}^C Y_{j,t}^C}{P_t^C} - \frac{W_t^C N_{j,t}^C}{P_t^C}$

$\Pi_{j,t}^C = \frac{P_{j,t}^C Y_{j,t}^C}{P_t^C} - \frac{MPL^C \lambda_t^C N_{j,t}^C}{P_t^C}$

$\Pi_{j,t}^C = \frac{P_{j,t}^C Y_{j,t}^C}{P_t^C} - \frac{\lambda_t^C (1-\alpha) Y_{j,t}^C}{P_t^C}$

$\Pi_{j,t}^C = \frac{P_{j,t}^C Y_{j,t}^C}{P_t^C} - mc_t^C (1-\alpha) Y_{j,t}^C$

$\Pi_{j,t}^C = \frac{P_{j,t}^C Y_{j,t}^C}{P_t^C} - mc_t^C (1-\alpha) Y_{j,t}^C$

$\Pi_{j,t}^D = \frac{P_{j,t}^D Y_{j,t}^D}{P_t^D} - mc_t^D (1-\alpha) Y_{j,t}^D$

$\Pi_{j,t}^C = \frac{P_{j,t}^C Y_{j,t}^C}{P_t^C} - mc_t^C (1-\alpha) \left(C_t \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_c} \right)$

$\Pi^{C}_{j,t} = \frac{P^{C}_{j,t} Y^{C}_{j,t}}{P^{C}_{t}} - mc^{C}_{t}(1-\alpha)\left(\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_c}\left(Y^{C}_{t}\right)\right)$

Derivation of Pricing FOC:

$Max_{P^{c}_{j,t}}\ E_t\left\{ \sum_{s=0}^{\infty}(\phi)^s \Lambda_{t,t+s}\left(\Pi^{c,N}_{j,t+s}\right) \right\} =$

$Max_{P^{c}_{j,t}}\ \ E_t\left\{ \sum_{s=0}^{\infty}(\phi)^s \Lambda_{t,t+s}\left(\left(P^{1-\epsilon_c}_{c,j,t}\right)(P_{c,t+s})^{\epsilon_c}\left(Y^{c}_{t}\right) - \lambda^{c}_{t}\left(\left(\frac{P_t(j)}{P_{t+s}}\right)^{-\epsilon_c}\left(Y^{c}_{t}\right)\right)\right) \right\}$

FOC:

$(1-\epsilon_c)P^{-\epsilon_c}_{c,j,t}\sum_{s=0}^{\infty}\left((\phi)^s \Lambda_{t,t+s}P^{\epsilon_c-1}_{c,t+s}Y^{c}_{t+s}\right) + \epsilon_c(P_t(j))^{-\epsilon_c-1}\sum_{s=0}^{\infty}\left((\phi)^s \Lambda_{t,t+s}\lambda^{c}_{t,t+s}P^{\epsilon_c}_{t+s}Y^{c}_{t+s}\right) = 0$

$(1-\epsilon_c)\sum_{s=0}^{\infty}\left((\phi)^s \Lambda_{t,t+s}P^{\epsilon_c-1}_{c,t+s}Y^{c}_{t+s}\right) + \epsilon_c(P_t(j))^{-1}\sum_{s=0}^{\infty}\left((\phi)^s \Lambda_{t,t+s}\lambda^{c}_{t+s}P^{\epsilon_c}_{t+s}Y^{c}_{t+s}\right) = 0$

$(1-\epsilon_c)\sum_{s=0}^{\infty}\left((\phi)^s \Lambda_{t,t+s}P^{\epsilon_c-1}_{c,t+s}Y^{c}_{t+s}\right) = -\epsilon_c(P_t(j))^{-1}\sum_{s=0}^{\infty}\left((\phi)^s \Lambda_{t,t+s}\lambda^{c}_{t}P^{\epsilon_c}_{t+s}Y^{c}_{t+s}\right)$

Using the fact that $U'(C_t) = P^{c}_{t}\lambda^{n}_{t}$ and multiplying and dividing the above expression by P^{c}_{t+s},

$$P^{c}_{j,t} = \frac{\epsilon_c}{\epsilon_c - 1}\frac{\mathbb{E}_t\sum_{s=0}^{\infty}\left((\phi\beta)^s U'(C_{t+s})mc^{c}_{t+s}P^{\epsilon_c}_{c,t+s}Y^{c}_{t+s}\right)}{\mathbb{E}_t\sum_{s=0}^{\infty}\left((\phi\beta)^s U'(C_{t+s})P^{\epsilon_c-1}_{c,t+s}Y^{c}_{t+s}\right)} \tag{51}$$

Derivation of Law of Motion of Price:

$P^{1-\epsilon_c}_{t,C} = \int_0^1 P^{1-\epsilon_c}_{j,t,C}dj$

Among the unit interval of firms, $(1-\phi)$ mass of firms re-optimize and set the optimal reset price $P^{*}_{C,t}$ and the rest of mass set their price equal to the last period's price. So that,

$P^{1-\epsilon_c}_{C,t} = \int_0^{1-\phi} P^{*1-\epsilon_c}_{C,t}dj + \int_{1-\phi}^{1} P_{t-1}(j)dj$

$P^{1-\epsilon_c}_{C,t} = (1-\phi)P^{*1-\epsilon_c}_{C,t} + \int_{1-\phi}^{1} P_{t-1}(j)^{1-\epsilon_c}dj$

$\int_{1-\phi}^{1} P_{t-1}(j)^{1-\epsilon_c}dj = \phi\int_0^1 P_{t-1}(j)^{1-\epsilon_c}dj = \phi P^{1-\epsilon_c}_{t-1}$ (follows from symmetry of all firms and appropriate law of large numbers.) Then, the aggregate price index can be written in the following form:

$P^{1-\epsilon_c}_{C,t} = (1-\phi)\left(P^{*}_{C,t}\right)^{1-\epsilon_c} + \phi P^{1-\epsilon_c}_{C,t-1}$

The aggregate price index in the nondurable consumption sector can be expressed as the following:

$$P_{C,t} = \left[(1-\phi_C)\left(P^{*}_{C,t}\right)^{1-\epsilon_c} + \phi_C P^{1-\epsilon_c}_{C,t-1}\right]^{\frac{1}{1-\epsilon_c}} \tag{52}$$

Divide both sides by $P_{C,t-1}^{1-\epsilon_c}$ and define reset inflation as $\pi^*_{c,t} := \frac{P^*_t}{P_{t-1}} - 1$

$(1+\pi_{c,t+1})^{1-\epsilon_c} = (1-\phi)(1+\pi^*_{c,t})^{1-\epsilon_c} + \phi$

$\Phi_{c,t} := \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_c} dj$

Again using the Calvo friction property, this can be decomposed as:

$\Phi_c = \int_0^{1-\phi} \left(\frac{P^*_{c,t}}{P_{c,t}}\right)^{-\epsilon_c} dj + \int_{1-\phi}^1 \left(\frac{P_{c,t-1}(j)}{P_{c,t}}\right)^{-\epsilon_c} dj$

Dividing and multiplying by appropriate powers of P_{t-1}:

$\Phi_{c,t} = \int_0^{1-\phi} \left(\frac{P^*_{c,t}}{P_{c,t-1}}\right)^{-\epsilon_c} \left(\frac{P_{c,t-1}}{P_{c,t}}\right)^{-\epsilon_c} dj + \int_{1-\phi}^1 \left(\frac{P_{c,t-1}(j)}{P_{c,t-1}}\right)^{-\epsilon_c} \left(\frac{P_{c,t-1}}{P_{c,t}}\right)^{-\epsilon_c} dj$

The dynamics of the price dispersion are governed by the following equation:

$$\Phi_{c,t} = (1-\phi_C)(1+\pi^*_{c,t-1})^{-\epsilon_c}(1+\pi_{c,t-1})^{\epsilon_c} + \phi_C(1+\pi_{c,t-1})^{\epsilon_c}\Phi_{c,t-1} \tag{53}$$

NKPC Derivation Section 12

It can be shown that the log linear version of the auxilary variable $s_{1,t} = C_t^{-\sigma} mc^c_t Y^c_t + \phi\beta\mathbb{E}_t s_{1,t+1}\left(1+\pi_{c,t+1}\right)^{\epsilon_c}$ is the following, where I have used the fact that $Y^c_t = C_t$.

$\hat{s_{1,t}} = \frac{C^{1-\sigma}mc^c\hat{mc_t}}{s_1} + \frac{(1-\sigma)C^{1-\sigma}mc^c\hat{c_{1,t}}}{s_1} + \phi\beta\mathbb{E}_t\hat{s_{1,t+1}} + \phi\beta\epsilon_c\mathbb{E}_t\hat{\pi_{c,t+1}}$

Solving the equation of $s_{1,t}$ in steady state implies that $s_1 = \frac{C^{1-\sigma}mc^c}{1-\phi\beta}$, which when substituted in the expression above yields the following expression:

$\hat{s_{1,t}} = \hat{mc_t}(1-\phi\beta) + (1-\sigma)(1-\phi\beta)\hat{c_{1,t}} + \phi\beta\mathbb{E}_t\hat{s_{1,t+1}} + \phi\beta\epsilon_c\mathbb{E}_t\hat{\pi_{c,t+1}}$

Similarly, The log linearization of $s_{2,t} = C_t^{-\sigma}Y^c_t + \phi\beta\mathbb{E}_t s_{2,t+1}\left(1+\pi_{c,t+1}\right)^{\epsilon_c-1}$ is the following, where I have used the fact that $s_2 = \frac{C^{1-\sigma}}{1-\phi\beta}$ and that $Y^c_t = C_t$.

$\hat{s_{2,t}} = (1-\sigma)(1-\phi\beta)\hat{c_t} + \phi\beta\mathbb{E}_t\hat{s_{2,t+1}} + \phi\beta(\epsilon_c-1)\mathbb{E}_t\pi^{\hat{c}}_{t+1}$

$\implies \hat{s_{1,t}} - \hat{s_{2,t}} = (1-\phi\beta)\hat{mc^c_t} + \phi\beta\mathbb{E}_t\pi^c_{t+1} + \phi\beta\mathbb{E}_t\left(\hat{s_{1,t+1}} - \hat{s_{2,t+1}}\right)$

From the log linear expression for reset price inflation above,

$\hat{\pi^*_{c,t}} - \hat{\pi_{c,t}} = \hat{s_{1,t}} - \hat{s_{2,t}}$

Also,

$\hat{\pi_{c,t}} = (1-\phi)\hat{\pi^*_{c,t}}$,

$\frac{\hat{\pi_{c,t}}}{1-\phi} - \hat{\pi_{c,t}} = \hat{s_{1,t}} - \hat{s_{2,t}}$

$\frac{\hat{\pi_{c,t}} - \hat{\pi_{c,t}}(1-\phi)}{1-\phi} = \hat{s_{1,t}} - \hat{s_{2,t}}$

$\frac{\hat{\pi_{c,t}}\phi}{1-\phi} = \hat{s_{1,t}} - \hat{s_{2,t}}$

$\frac{\hat{\pi_{c,t}}\phi}{1-\phi} = (1-\phi\beta)\hat{mc^c_t} + \phi\beta\mathbb{E}_t\pi^c_{t+1} + \phi\beta\mathbb{E}_t\left(\frac{\hat{\pi_{c,t+1}}\phi}{1-\phi}\right)$

$\hat{\pi_{c,t}} = \frac{(1-\phi\beta)(1-\phi)}{1-\phi}\hat{mc_t^c} + \beta(1-\phi)\mathbb{E}_t\left(\hat{\pi_{c,t+1}}\right) + \phi\beta\mathbb{E}_t\left(\hat{\pi_{c,t+1}}\right)$

$\hat{\pi_{c,t}} = \frac{(1-\phi\beta)(1-\phi)}{1-\phi}\hat{mc_t^c} + \beta\mathbb{E}_t\left(\hat{\pi_{c,t+1}}\right)$

$\hat{\pi_{c,t}} = \lambda_c\hat{mc_t^c} + \beta\mathbb{E}_t\left(\hat{\pi_{c,t+1}}\right)$, where $\lambda_c := \frac{(1-\phi_C\beta)(1-\phi_C)\Theta}{1-\phi}$, where $\Theta_C := \frac{1-\alpha}{1-\alpha+\alpha\epsilon_c}$

The average log price markup in nondurable sector $\mu_{c,t} = p_{c,t} - \psi_{c,t}$ can be expressed as the following Galí (2015):

$\mu_{c,t} = -(w_t - p_{c,t}) + (-\alpha n_{c,t} + log(1-\alpha))$

$= -(\sigma c_t + \psi n_t) - \alpha n_{c,t} + log(1-\alpha)$ (Using Labor Supply Condition)

$= -\sigma c_t + \psi n_t - \alpha n_{c,t} + log(1-\alpha)$

In log deviations from steady state (using $\alpha = 0$):

$\hat{\mu_{c,t}} = -\sigma\hat{c_t} - \psi\hat{n_t} - \alpha\hat{n_{c,t}}$

Using the fact that $\hat{n_t} = \zeta_d\hat{n_{d,t}} + (1-\zeta_d)\hat{n_{c,t}}$,

$\hat{\mu_{c,t}} = -\sigma\hat{c_t} - \psi\left(\zeta_d\hat{n_{d,t}} + (1-\zeta_d)\hat{n_{c,t}}\right) - \alpha\hat{n_{c,t}}$

$\hat{\mu_{c,t}} = -\sigma\hat{c_t} - \psi\zeta_d\hat{n_{d,t}} + \left(-\psi(1-\zeta_d) - \alpha\right)\hat{n_{c,t}}$

Since $\hat{c_t} = \frac{\hat{y_t} - \psi_d\hat{I_{d,t}}}{\psi_c}$,

$\hat{\mu_{c,t}} = -\sigma\left(\frac{\hat{y_t} - \psi_d\hat{I_{d,t}}}{\psi_c}\right) + \hat{n_{c,t}}\left(-\psi(1-\zeta_d) - \alpha\right) - \psi\zeta_d\hat{n_{d,t}}$

Using the inverse production functions, we have that $\hat{c_t} = (1-\alpha)\hat{n_{c,t}}$, so that $\hat{n_{c,t}} = \frac{\hat{c_t}}{1-\alpha}$ and $\hat{n_{d,t}} = \frac{\hat{I_{d,t}}}{1-\alpha}$. Then,

$\hat{\mu_{c,t}} = -\sigma\left(\frac{\hat{y_t} - \psi_d\hat{I_{d,t}}}{\psi_c}\right) + \frac{\hat{c_t}}{1-\alpha}\left(-\psi(1-\zeta_d) - \alpha\right) - \psi\zeta_d\frac{\hat{I_{d,t}}}{1-\alpha}$.

$\hat{\mu_{c,t}} = -\sigma\left(\frac{\hat{y_t} - \psi_d\hat{I_{d,t}}}{\psi_c}\right) + \frac{\frac{\hat{y_t} - \psi_d\hat{I_{d,t}}}{\psi_c}}{1-\alpha}\left(-\psi(1-\zeta_d) - \alpha\right) - \psi\zeta_d\frac{\hat{I_{d,t}}}{1-\alpha}$

It can then be shown that:

$\hat{\mu_{c,t}} = \left(-\frac{\sigma}{\psi_c} + \frac{(-\psi(1-\zeta_d)-\alpha)(1-\alpha)}{\psi_c}\right)\hat{y_t} + \left(\frac{\sigma\psi_d}{\psi_c} - \frac{\psi_d(-\psi(1-\zeta_d)-\alpha)(1-\alpha)}{\psi_c} - \frac{\psi\zeta_d}{1-\alpha}\right)\hat{I_{d,t}}$

$\iff \hat{\mu_{c,t}} = \mu_y^c\hat{y_t} + \mu_{Id}^c\hat{I_{d,t}}$, where $\mu_y^c := \left(-\frac{\sigma}{\psi_c} + \frac{(-\psi(1-\zeta_d)-\alpha)(1-\alpha)}{\psi_c}\right)$ and $\mu_{Id}^c := \left(\frac{\sigma\psi_d}{\psi_c} - \frac{\psi_d(-\psi(1-\zeta_d)-\alpha)(1-\alpha)}{\psi_c} - \frac{\psi\zeta_d}{1-\alpha}\right)$.

Under flexible prices, $\hat{\mu_{c,t}} = 0$ since markup is fixed and the corresponding natural output is:

$0 = \mu_y\hat{y_t^n} + \mu_{Id}\hat{I_{d,t}^n}$

$\hat{\mu_{c,t}} - 0 = \mu_y\left(\hat{y_t} - \hat{y_t^n}\right) + \mu_{Id}\left(\hat{I_{d,t}} - \hat{I_{d,t}^n}\right)$

Then, since $\hat{mc_t^c} = -\mu_{c,t}$,

$\hat{\pi_{c,t}} = -\lambda_c\hat{\mu_t^c} + \beta\mathbb{E}_t\left(\hat{\pi_{c,t+1}}\right)$

$\hat{\pi_{c,t}} = -\lambda_c \left(\mu_y^c \hat{y}_t + \mu_{Id}^c \hat{I_{d,t}} \right) + \beta \mathbb{E}_t \left(\hat{\pi_{c,t+1}} \right)$

$\hat{\pi_{c,t}} = \kappa_1 \left(\hat{y}_t \right) + \beta \mathbb{E}_t \left(\hat{\pi_{c,t+1}} \right) + \kappa_2 \left(\hat{I_{d,t}} \right)$, where $\kappa_1 := -\lambda_c \mu_y^c$ and $\kappa_2 := -\lambda_c \mu_{Id}^c$.

Derivation of NKPC Durable Sector:

Similarly, average log price markup in durable sector $\mu_{d,t} = p_{d,t} - \psi_{d,t}$ can be expressed as the following:

$\mu_{d,t} = -(w_t - p_{d,t}) + (-\alpha n_{d,t} + log(1-\alpha))$

$\alpha = 0$ means that $\mu_{d,t} = -(w_t - p_{d,t}) + (-\alpha n_{d,t})$

Using the inverse production function, $\mu_{d,t} = -(w_t - p_{d,t}) - \alpha \frac{\hat{I_{d,t}}}{1-\alpha}$.

Since $w_t = \sigma c_t + \psi n_t + p_{c,t}$

$\implies w_t - p_{d,t} = \sigma c_t + \psi n_t + p_{c,t} - p_{d,t}$,

In log deviations, $\hat{w}_t - \hat{p_{d,t}} = \sigma \hat{c}_t + \psi \hat{n}_t + \hat{p_{c,t}} - \hat{p_{d,t}}$.

$\iff \mu_{d,t} = -\sigma \hat{c}_t - \psi(\zeta_d \hat{n_{t,d}} + (1-\zeta)\hat{n_{t,c}}) - \hat{p_{c,t}} + \hat{p_{d,t}} - \alpha \frac{\hat{I_{d,t}}}{1-\alpha}$.

Again, using inverse production functions and log linear expression, connecting consumption with output,

$$\iff \mu_{d,t} = -\sigma \left(\frac{\hat{y}_t - \psi_d \hat{I_{d,t}}}{\psi_c} \right) - \psi \left(\zeta_d \frac{\hat{I_{d,t}}}{1-\alpha} + (1-\zeta_d) \frac{\left(\frac{\hat{y}_t - \psi_d \hat{I_{d,t}}}{\psi_c} \right)}{1-\alpha} \right) - \hat{p_{c,t}} + \hat{p_{d,t}} - \alpha \frac{\hat{I_{d,t}}}{1-\alpha}.$$

It can then be shown that:

$$\mu_{d,t} = \left(-\frac{\sigma}{\psi_c} - \frac{\psi(1-\zeta_d)(1-\alpha)}{\psi_c} \right) \hat{y}_t + \left(\frac{\sigma \psi_d}{\psi_c} - \frac{\psi \zeta_d}{1-\alpha} - \frac{\alpha}{1-\alpha} + \frac{\psi(1-\zeta_d)\psi_d(1-\alpha)}{\psi_c} \right) \hat{I_{d,t}} - \hat{p_{c,t}} + \hat{p_{d,t}}$$

$$\iff \hat{\mu_{Id,t}} = \mu_y^{Id} \hat{y}_t + \mu_{Id}^{Id} \hat{I_{d,t}} - \hat{p_{c,t}} + \hat{p_{d,t}}$$

Then, the NKPC in durable good sector is:

$\hat{\pi_{d,t}} = \kappa_1^{Id} \left(\hat{y}_t \right) + \beta \mathbb{E}_t \left(\hat{\pi_{c,t+1}} \right) + \kappa_2^{Id} \left(\hat{I_{d,t}} \right) - \lambda_d(\hat{p_{d,t}} - \hat{p_{c,t}})$, where $\kappa_1^{Id} := -\lambda_d \mu_y^{Id}$, $\kappa_2^{Id} := -\lambda_d \mu_{Id}^{Id}$ and $\hat{p_{d,t}} - \hat{p_{c,t}}$ is log deviation of the relative price between durable goods sector and consumption sector.

Derivation of Euler Equation In Terms of Output

We know that:

$$\hat{c}_t = \mathbb{E}_t\{\hat{c_{t+1}}\} - \frac{1}{\sigma} \left(i_t - \mathbb{E}_t\{\pi_{t+1}^c\} - \left(\underbrace{\lambda \left(p_t(z_H) + (1-p_t) z_L \right) + (1-\lambda) \bar{z}}_{\text{Consumer Sentiment Term}} \right) - \varrho \right)$$

For all $t \geq t_{revealed}$, where $z_{revealed} \in \{z_H, z_L\}$,

$$\hat{c}_t = \mathbb{E}_t\{\hat{c_{t+1}}\} - \frac{1}{\sigma} \left(i_t - \mathbb{E}_t\{\pi_{t+1}^c\} - \left(\underbrace{z_{revealed}}_{\text{Consumer Sentiment Term}} \right) - \varrho \right)$$

Using the log linear expression for aggregate resource constraint $\hat{y}_t = \psi_c \hat{c}_t$ and for aggregate price level $\left(\hat{P}_t = \psi_d \hat{P_{d,t}} + (1-\psi_d) \hat{P_{c,t}} \right)$, I write the above Euler equation for non-durable goods in terms of aggregate output and non-durable good inflation:

$$\hat{y_t} = \mathbb{E}_t\{\hat{y_{t+1}}\} - \frac{\psi_c}{\sigma}\left(\hat{i_t} - \left(\underbrace{\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\,\bar{z}}_{\text{Consumer Sentiment Term}}\right) - \mathbb{E}_t\{\pi^{\hat{c}}_{t+1}\}\right)$$

Since $\hat{y_t} = \psi_c \hat{c}_t + (1-\psi_c)\hat{I_{d,t}}$, the Euler equation for consumption can be expressed as:

$$\frac{\hat{y_t} - (1-\psi_c)\hat{I_{d,t}}}{\psi_c} = \mathbb{E}_t\left\{\frac{\hat{y_{t+1}} - (1-\psi_c)\hat{I_{d,t+1}}}{\psi_c}\right\} - \frac{1}{\sigma}\left(\hat{i_t} - \mathbb{E}_t\{\pi^c_{t+1}\} - \left(\underbrace{\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\,\bar{z}}_{\text{Consumer Sentiment Term}}\right)\right)$$

Multiplying throughout by ψ_c,

$$\hat{y_t} - (1-\psi_c)\hat{I_{d,t}} = \mathbb{E}_t\left\{\hat{y_{t+1}} - (1-\psi_c)\hat{I_{d,t+1}} - \frac{\psi_c}{\sigma}\left(\hat{i_t} - \mathbb{E}_t\{\hat{\pi_{t+1}}\} - \left(\underbrace{\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\,\bar{z}}_{\text{Consumer Sentiment Term}}\right)\right)\right\}$$

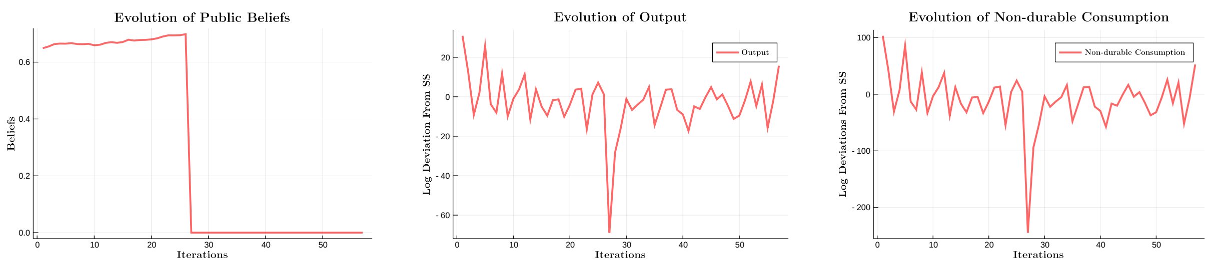
$$\hat{y_t} = \mathbb{E}_t\{\hat{y_{t+1}}\} - \frac{\psi_c}{\sigma}\left(\hat{i_t} - \left(\underbrace{\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\,\bar{z}}_{\text{Consumer Sentiment Term}}\right) - \mathbb{E}_t\{\hat{\pi_{c,t+1}}\} - \hat{\Xi}_t\right)$$

,where $\hat{\Xi}_t := -\left(\frac{\sigma\psi_d}{\psi_c}\right)\mathbb{E}_t\{\Delta \hat{I_{d,t+1}}\}$, where $\mathbb{E}_t\{\Delta \hat{I_{d,t+1}}\} := \mathbb{E}_t\{\hat{I_{d,t+1}}\} - \hat{I_{d,t}}$.

21. Simulation

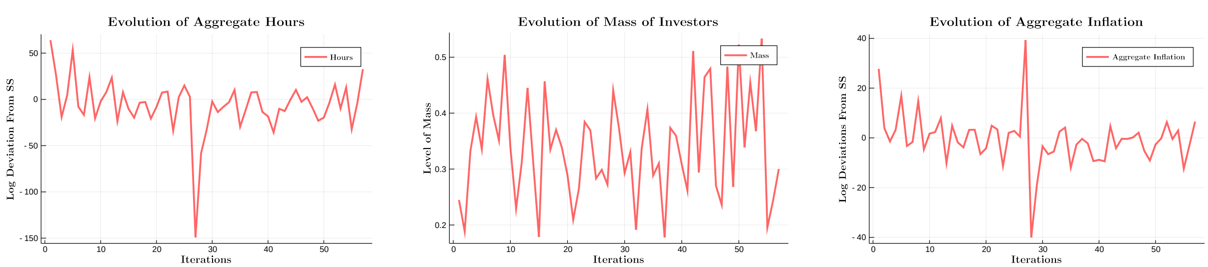
In this section, I provide results from simulation of the model, when not just uncertainty resolution but all other random variables have been drawn from their distributions rather than fixed at mean values fo zero; the iid demand shock and monetary policy shock have also been drawn from their distributions. All other parameters are at their baseline values. In the graphs below, at period 30, uncertainty about the true state is resolved exogenously and since the true state is low, beliefs crash toward certainty about the true state and this has sharp impacts on mass of investing agents as well as all other endogenous variables; it creates a sudden, asymmetric and brief recession [28]. However, due to the presence of general equilibrium effects arising from the presence of a active central bank, concerned about stabilization and simulated, IID demand/MP shocks, I also get a quick recovery.

Simulation: True State = Low, $\sigma_s = 15$, $\sigma_\epsilon = 5$, $z_l = 1.5$



(a) (b) (c)

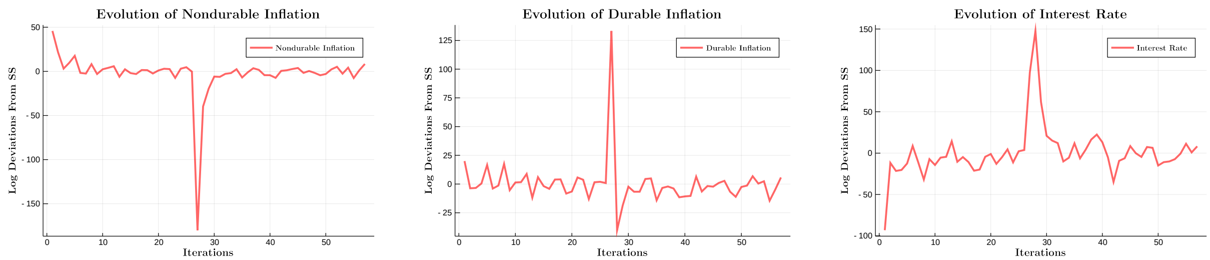
Figure 49: Simulation (True State = Low)



(a) (b) (c)

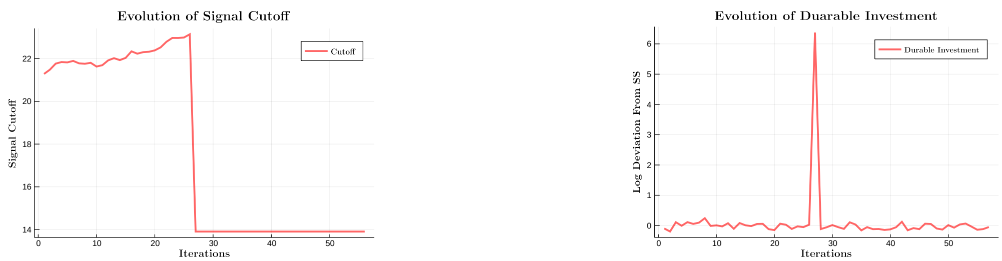
Figure 50: Simulation (True State = Low)

[28]Notice that asymmetry only has a qualitative interpretation so that the results should *not* be interpreted as implying that my model predicts that contractions are 6 to 7 times stronger than typical growth rate of economy



(a) (b) (c)

Figure 51: Simulation (True State = Low)



(a) (b)

Figure 52: Simulation (True State = Low)

21.1. PSEUDO CODE FOR GE MODEL

Define Parameters and Matrices. Define $TOL = 10^3$.
for j in 1:M-1

while $(diff > TOL)$ & $(k \leq maxiter)$

1. Define matrices Γ_0 and Γ_1 with initial guess for $a^l_{6,k}, \forall k$, and a^l, starting from $l = 1$.

2. Use these matrices to solve for policy rules using Gensys solver [29] and store the relevant coefficients and *mappings*[30] from policy rules.
3. Use these mappings as inputs in the RootSolver and solve for the cutoff.
4. Evaluate the derivatives needed to compute coefficients in the log linear expression for $\hat{I_{d,t}}$. Use numerical differentiation to compute the required coefficients[31].
5. Use these derivatives and cutoffs to evaluate constants $a^{l+1}_{6,k}$ and a^{l+1} and measure differences $d_{1,k} = |a^{l+1}_{6,k} - a^l_{6,k}|, \forall k$ and $d_2 = |a^{l+1} - a^l|$.

[29] I use the Julia package developed by Normann Rion Rion (2019) to solve the linear rational expectations model using Sims solver at each iteration.

[30] One example of a mapping is the scalar a^y_c, defined earlier.

[31] I use Newton, three point formulas to evaluate the RootSolver in the close neighborhood of current values of D_{t-1} and Ξ_t to calculate difference quotients, as also explained earlier in paper.

6. Check whether $diff := max\left(max\left(d_{1,k}, \forall k\right), d_2\right) \leq TOL$ at this $l+1$ step. If this is true, then while loop ends; otherwise it continues until *maxiter* is reached or $diff \leq TOl$.
$l = l + 1$
end **while** loop.

3. Use initial guess for cutoff $\hat{s}^t_{Guess}$ to initiate fixed point iteration process to solve for signal cutoff $\hat{s}^t$, which is the root of nonlinear function H/G for the system with scalars $a^*_{6,j}$ and a^*_2, the solutions to fixed point problem in while loop above.

2. Update guess for next step, equal to current solution $\hat{s}^{t+1}_{Guess} = \hat{s}^t$.

3. Use this solved cutoff $\hat{s}^t$ to calculate mass of investing agents for durable goods d_t by evaluating (1 - cdf of the signal distribution) evaluated at this cutoff, where the signal is centered around the true state $z_l + \eta_0$, known to the modeler but not agents.

4. Once the mass of investing agents is known, use the equation for updating of beliefs to update the public beliefs from p_t to p_{t+1}.

5. If doing a simulation rather than impulse response, then check whether the Bernoulli random variable, with probability of success equal to λ has realized as success or not. If it has done so, then replace the constant in Euler equation by $q_{t,t}z_t + q_{t,t'}z_{t'} + q_{t,t''}z_{t''}$, where the realized state is $z_t = z^N \in \{z_L, z_H\}$.

6. Use these updated beliefs to calculate $-\frac{\psi_c}{\sigma}\left(\lambda\left(p_t(z_H) + (1-p_t)z_L\right) + (1-\lambda)\,\bar{z}\right)$ and use this *constant* value in the C matrix of Sims' Solver. Use Gensys solver to solve for the policy rules.[32] If doing a simulation instead, then use $q_{t,t}z_t + q_{t,t'}z_{t'} + q_{t,t''}z_{t''}$ as constant if uncertainty has been resolved while using Gensys solver and continue to simulate the three state Markov Chain.

7. Extract relevant constants and mappings from solution of policy rules and store values of all endogenous variables at this iteration.

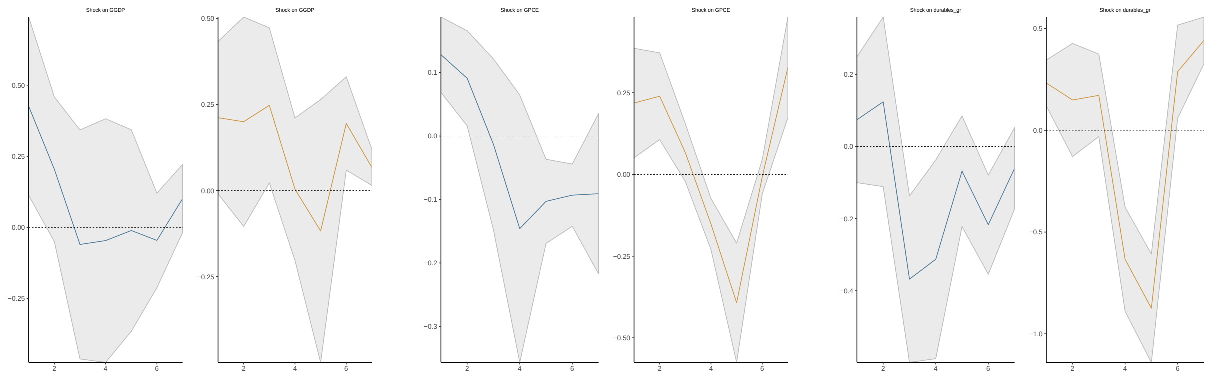
8. Compute $\mathbb{E}_t\{x_{t+j-1}\} = A^t\mathbb{E}_t x_{t+j-2} + \Theta^t_C$ and $\mathbb{E}_{t-1}\{x_{t+j-1}\} = A^{t-1}\mathbb{E}_{t-1}x_{t+j-2} + \Theta^{t-1}_C$, where $x_{t-1} = 0$ and x_t is value after *first* stage 2 system has been solved. Compute $IRF(j-1) = A^t\mathbb{E}_t x_{t+j-2} + \Theta^t_C - A^{t-1}\mathbb{E}_{t-1}x_{t+j-2} + \Theta^{t-1}_C$. $IRF(0)$ means the first impulse reaction in endogenous variables after first stage 2 system has been solved for $j = 1$.
end **for** loop.

Plot impulse responses for all variables.
Plot evolution of public beliefs and level of mass of durable good investing agents.
Perform Simulations.

[32] After beliefs are updated in beginning of stage 2, they are held constant, moving forward at this *updated* value and not treated as dynamic. This is because of the *martingale* property of beliefs Kozlowski et al. (2019) Schaal and Taschereau-Dumouchel (2020), which ensures that a law of iterated expectations type result holds, guaranteeing that conditional on time t information $\mathcal{I}_t$, $\mathbb{E}_t[\pi_{t+k}(z,\eta)] = \pi_t(z,\eta), \forall k \geq 1$.

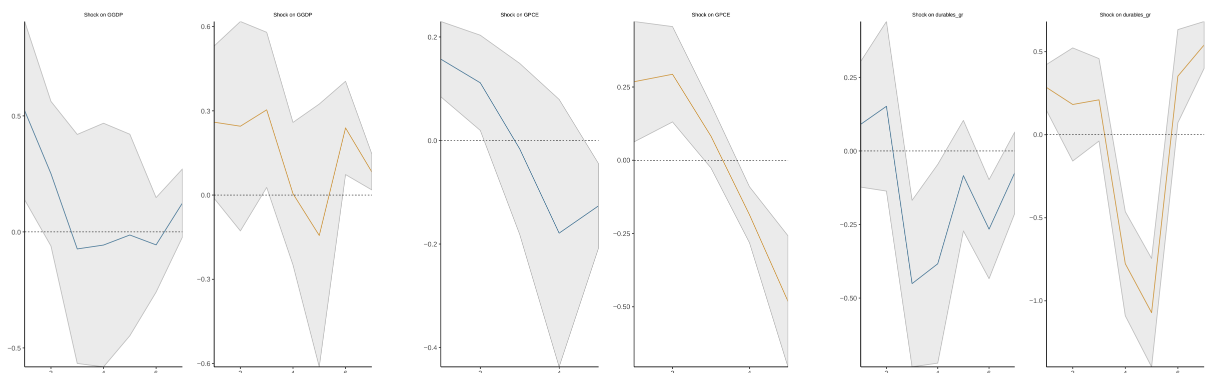
21.2. OMITTED IRF GRAPHS

Omitted Empirical IRFS (Local Projections):



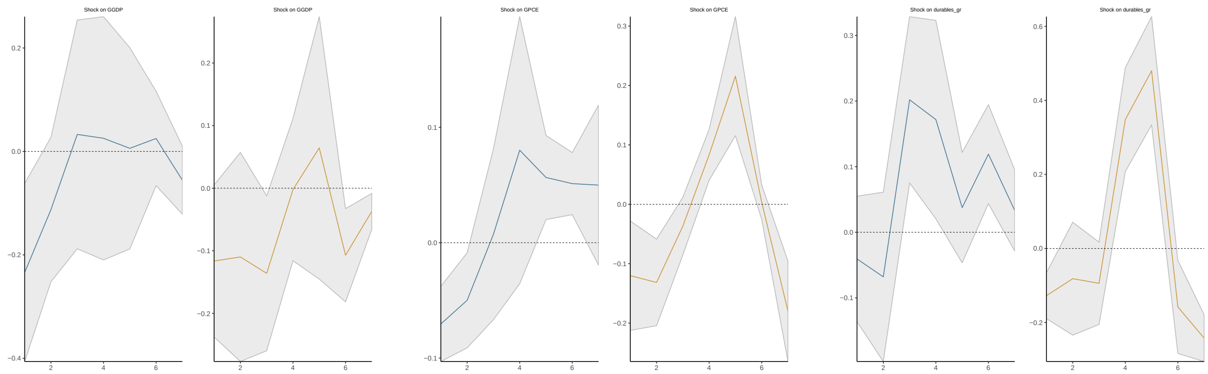
(a) (b) (c)

Figure 53: Impulse Response to a 1 sd CS Shock (Annual, CSg1)



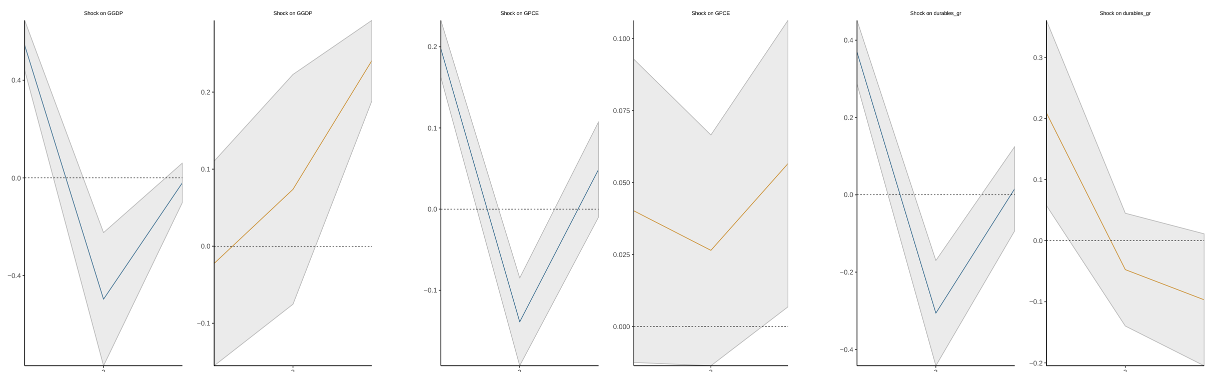
(a) (b) (c)

Figure 54: Impulse Response to a 1 sd CS Shock (Annual, CSb1)



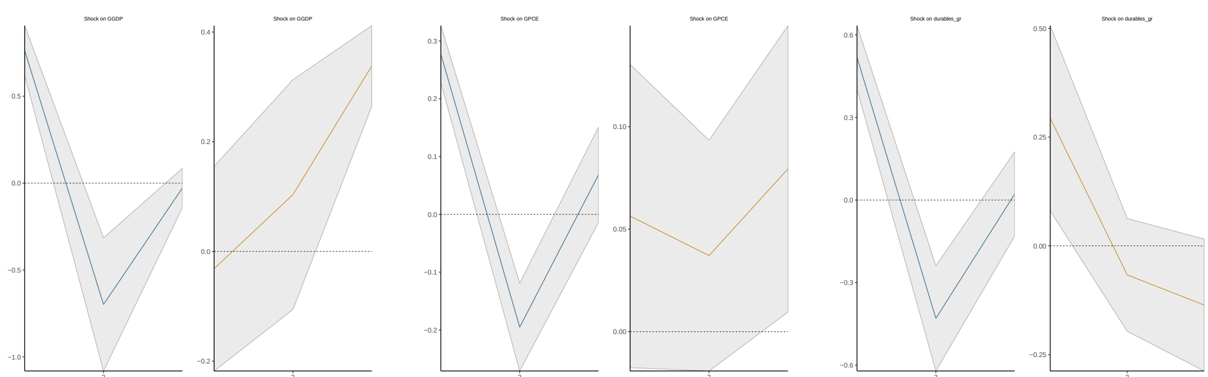
(a) (b) (c)

Figure 55: Impulse Response to a 1 sd CS Shock (Annual, CSb5)



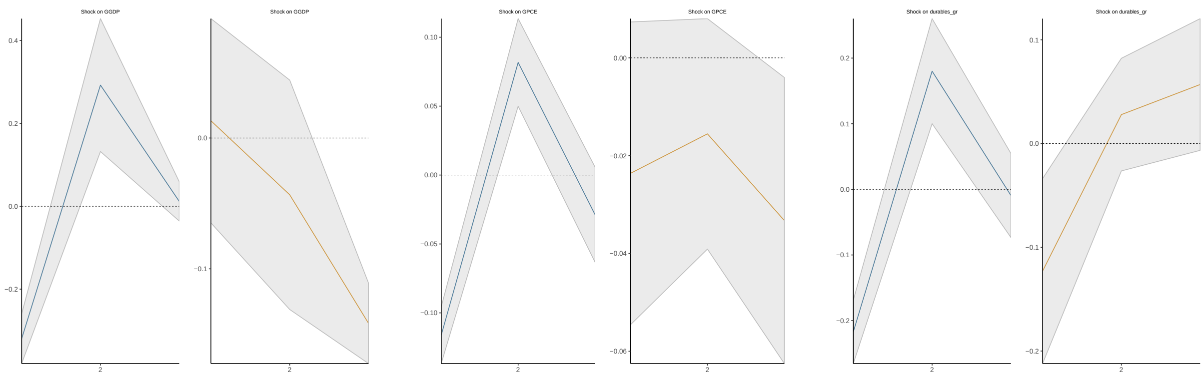
(a) (b) (c)

Figure 56: Impulse Response to a 1 sd CS Shock (Biennial, CSg1)



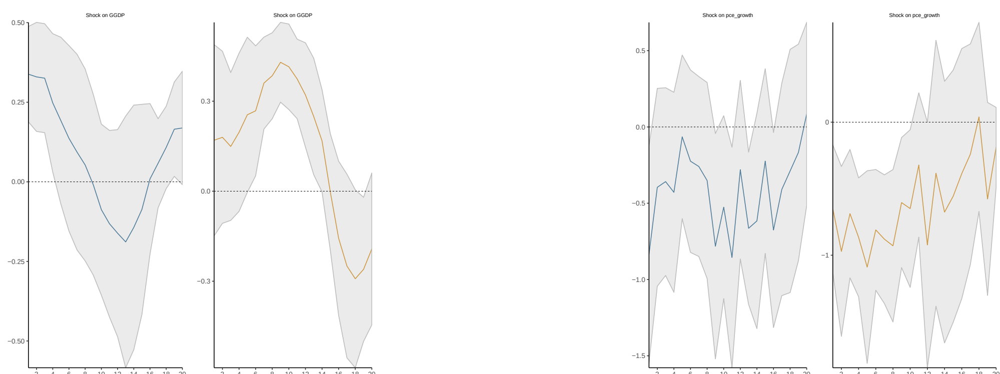
(a) (b) (c)

Figure 57: Impulse Response to a 1 sd CS Shock (Biennial, CSb1)



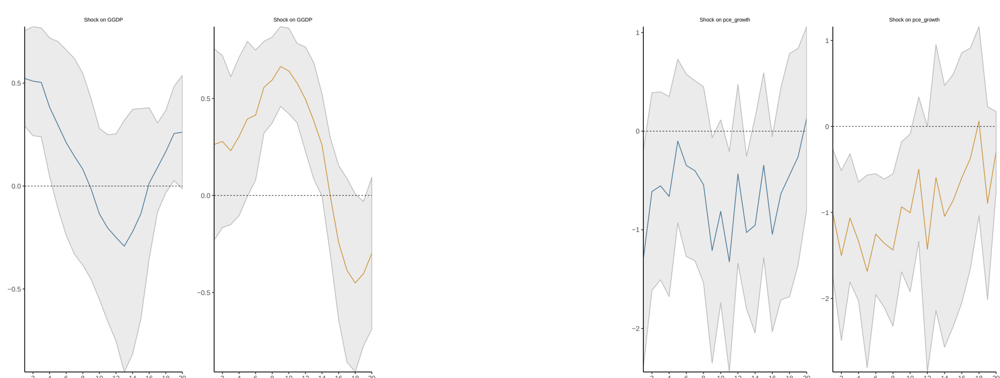
(a) (b) (c)

Figure 58: Impulse Response to a 1 sd CS Shock (Biennial, CSb5)



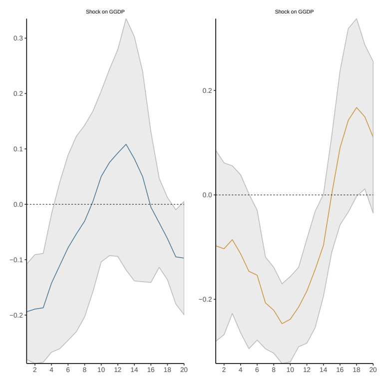
(a) (b)

Figure 59: Impulse Response to a 1 sd CS Shock (Quarterly, CSg1)

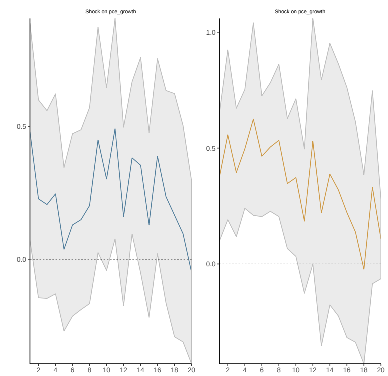


(a) (b)

Figure 60: Impulse Response to a 1 sd CS Shock (Quarterly, CSb1)



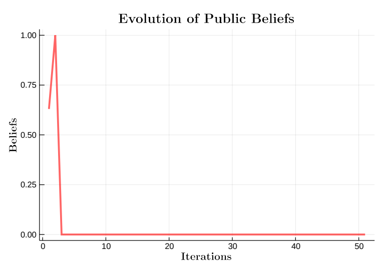
(a)



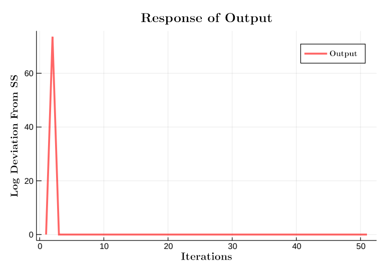
(b)

Figure 61: Impulse Response to a 1 sd CS Shock (Quarterly, CSb5)

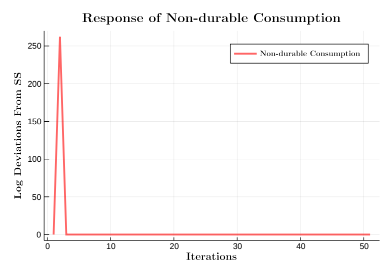
21.2.1. FP, $\sigma_s = 15$, $\sigma_\epsilon = 0.2$, shock size = 0.1 sd



(a)

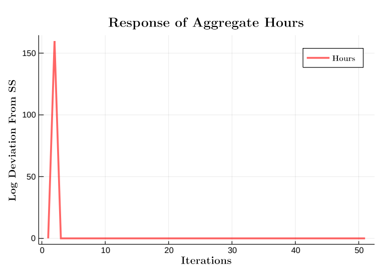


(b)

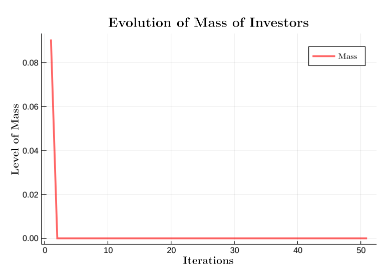


(c)

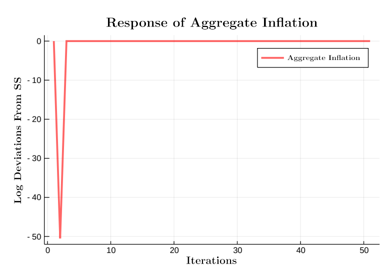
Figure 62: Impulse Response to a 0.1 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a)

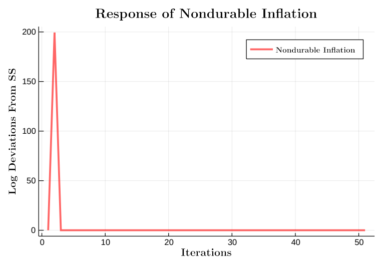


(b)

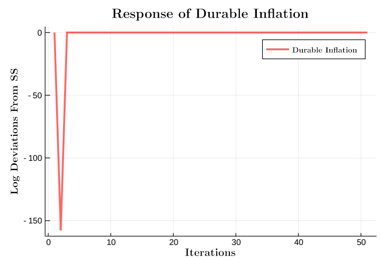


(c)

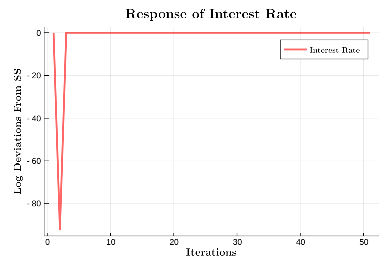
Figure 63: Impulse Response to a 0.1 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a)

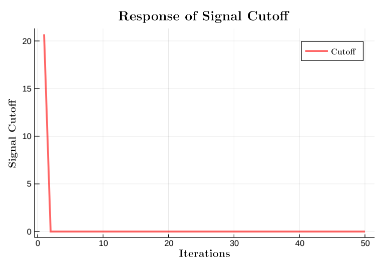


(b)

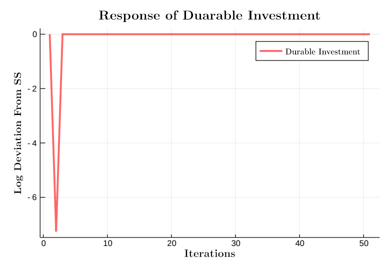


(c)

Figure 64: Impulse Response to a 0.1 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



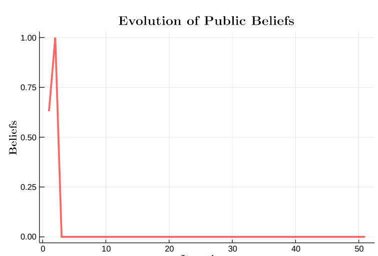
(a)



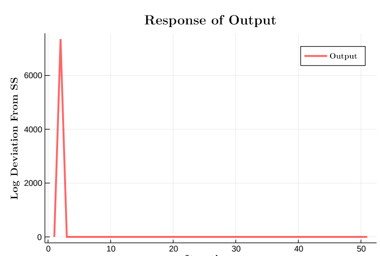
(b)

Figure 65: Impulse Response to a 0.1 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

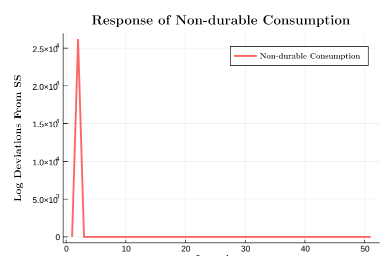
21.2.2. FP, $\sigma_s = 15$, $\sigma_\epsilon = 0.2$, $\sigma_\eta = 5$, shock size = 2 sd



(a)

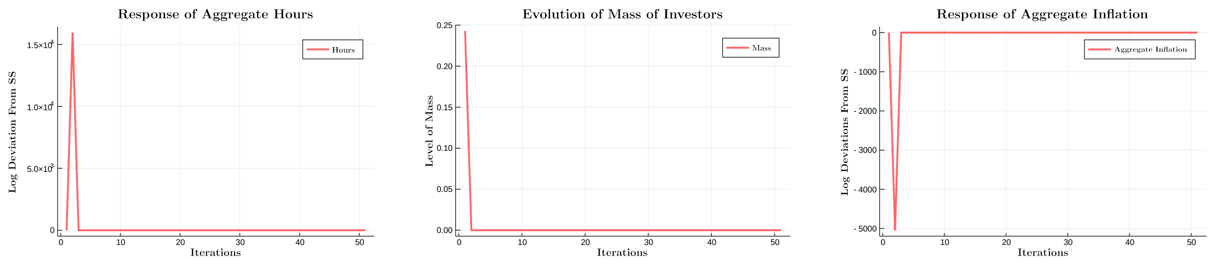


(b)



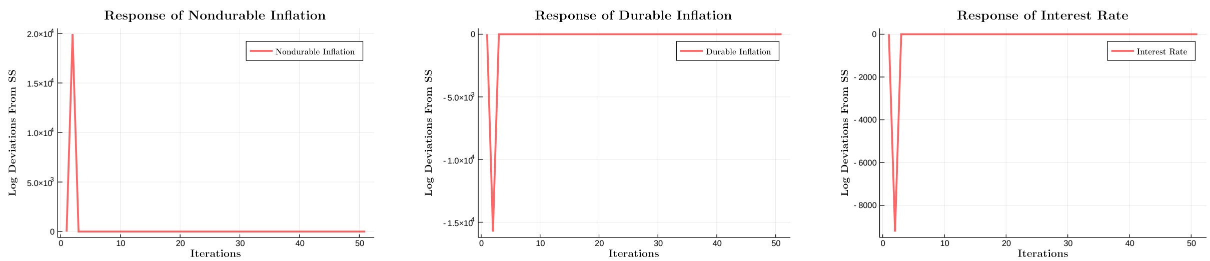
(c)

Figure 66: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a) (b) (c)

Figure 67: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)



(a) (b) (c)

Figure 68: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

(a) (b)

Figure 69: Impulse Response to a 2 sd Permanent Noise Shock (False Positive, $\sigma_s = 15$)

21.3. GitHub Link for Replication Data and Code

The Github link for replication data and code in R (for empirical part) and Julia (theoretical part) for all graphs and results is: https://github.com/sonanmemon/MPhil_Thesis_2020_Oxford. I use the Julia package developed by Normann Rion Rion (2019) to solve the linear rational expectations model at each iteration.