ECON 202 Intermediate Macroeconomics: Two Period Consumption Saving Lec 21-22

Sonan Memon
Lecturer, Institute of Business Administration, Karachi.

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Dynamic Consumption Saving

i. This lecture corresponds to sections 9.1-9.3 from chapter 9 of GLS(2020).

ii. In modern macroeconomics, we solve a dynamic, consumption saving problem for the household rather than relying on ad hoc behavioral rules.

iii. We will work with two period models for simplification, even though the standard framework in modern macro is an infinite horizon one.

iv. The key insights from two period models carry over to models with multiple future periods, including infinite horizon models.

Lifetime Budget Constraint

i. The representative household lives for 2 periods and faces a *sequence* of *flow* budget constraints, 1 for each period which is: $C_t + S_t \leq Y_t$ and $C_{t+1} + S_{t+1} \leq Y_{t+1} + (1 + r_{t+1})S_t$.

ii. We assume that the household cannot die in debt and so that $S_{t+1} = 0 \implies C_t + S_t = Y_t$ and $C_{t+1} = Y_{t+1} + (1 + r_{t+1})S_t$.

iii. Solving for S_t from 2nd equation in point 2 above and substituting for S_t in the other equation yields the lifetime budget constraint or intertemporal constraint: $C_t + \frac{C_{t+1}}{1+r_{t+1}} = Y_t + \frac{Y_{t+1}}{1+r_{t+1}}$

iv. The lifetime budget constraint posits that present value of lifetime consumption should equal present value of lifetime income.

Utility Function and Discount Factor

i. Lifetime utility U is a weighted sum of flow utilities from each period of life i.e $U = u(C_t) + \beta u(C_{t+1})$, where $\beta \in (0, 1)$.
ii. β is the utility discount factor and $\frac{1}{1+r_{t+1}}$ has a similar role, where the latter is the goods discount factor.
iii. The standard regularity assumptions on u are that $u'(.) > 0$ and $u''(.) < 0$, where the former means that marginal utility of consumption is positive and latter means that we have diminishing marginal utility of consumption.

Some Canonical Utility Functions

i. Some canonical utility functions are:
$u(C_t) = \theta C_t$, $\theta > 0$,
$u(C_t) = C_t - \frac{\theta}{2} C_t^2$, $\theta > 0$,
$u(C_t) = \ln(C_t)$ and $u(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$, $\sigma > 0$.

ii. The first function above is linear, which means that $u'(.) > 0$, but this function does not satisfy $u''(.) < 0$ (check!)

iii. The second one is quadratic so that it satisfies $u''(.) < 0$, but it does not always satisfy $u'(.) > 0$ since there is a satiation point about which utility is decreasing in consumption.

iv. The third one is the log utility function, which satisfies both regularity assumptions: $u'(.) > 0$ and $u''(.) < 0$.

Canonical CRRA Function

i. $u(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$, $\sigma > 0$.

ii. This function is called the canonical CRRA (constant relative risk aversion) utility function.

iii. σ is coefficient of risk aversion and $\frac{1}{\sigma}$ governs the sensitivity of consumption growth to interest rates over time. A higher σ means more consumption smoothing/less consumption sensitivity over time.

iv. It follows from L'Hopital's Rule that $\lim_{\sigma \to 1} \frac{C_t^{1-\sigma}}{1-\sigma} = \ln(C_t)$ so that log utility is just a special instance of CRRA when $\sigma \to 1$.

L'Hopital's Rule

i. **Theorem: L'Hopital's Rule**
If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\pm\infty$ or if $\lim_{x\to a} \frac{f(x)}{g(x)}$ is equal to some other indeterminate form, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

ii. Examples of other indeterminate forms include $1^\infty, 0^0, \infty^0, \infty - \infty$ and $0 \times \infty$.

CRRA Function and Log Utility

i. **Claim:** $\lim_{\sigma \to 1} \frac{C_t^{1-\sigma}}{1-\sigma} = \ln(C_t)$

Proof:

Apply the rule on CRRA as $\sigma \to 1$ to get

$\lim_{\sigma \to 1} \frac{C_t^{1-\sigma}}{1-\sigma} = \lim_{\sigma \to 1} \frac{\exp\{\ln\left(C_t^{1-\sigma}\right)\}}{1-\sigma} = \lim_{\sigma \to 1} \frac{\exp\{(1-\sigma)\ln(C_t)\}}{1-\sigma}$

Now apply L'Hopital's rule (differentiate wrt σ)

$\lim_{\sigma \to 1} \frac{\exp\{(1-\sigma)\ln(C_t)\} \times -\ln(C_t)}{-1} = \lim_{\sigma \to 1} \frac{C_t^{1-\sigma}}{1-\sigma}$

Thus, $\lim_{\sigma \to 1} \frac{C_t^{1-\sigma}}{1-\sigma} = \lim_{\sigma \to 1} \frac{\exp\{(1-1)\ln(C_t)\} \times -\ln(C_t)}{-1} = 1 \times \frac{-\ln(C_t)}{-1} = \ln(C_t)$.

Optimization Problem

i. $\max\limits_{C_t, C_{t+1}} U = u(C_t) + \beta u(C_{t+1})$ subject to
$C_t + \frac{C_{t+1}}{1+r_{t+1}} = Y_t + \frac{Y_{t+1}}{1+r_{t+1}}$.

ii. Can solve the above problem using Lagrangian or by converting it into an unconstrained problem via substitution of constraint in objective function.

iii. We will solve it using Lagrangian; the book uses the substitution method.

iv. $\mathcal{L} = u(C_t) + \beta u(C_{t+1}) + \lambda \left(Y_t + \frac{Y_{t+1}}{1+r_{t+1}} - C_t - \frac{C_{t+1}}{1+r_{t+1}} \right)$.

v. The FOC's are: $\mathcal{L}_{C_t} = u'(C_t) = \lambda$,
$\mathcal{L}_{C_{t+1}} = \beta u'(C_{t+1}) = \frac{\lambda}{1+r_{t+1}}$.

Euler Equation

i. Equating λ from the first two FOC's, stated in last bullet point on previous slide, we get $u'(C_t) = \beta(1 + r_{t+1})u'(C_{t+1})$, the **Consumption Euler Equation**.

ii. **Interpretation:** If I give you one more unit of endowment, you can either consume it today, which gives you marginal utility of $u'(C_t)$, the LHS.

iii. Alternatively, you could save this one unit to get $(1 + r_{t+1})$ tomorrow, which when ultimately consumed gives $u'(C_{t+1})$, but this also has to be discounted by β (RHS).

iv. Euler equation says that $LHS = RHS$ at an optimum, dynamic consumption path.

Euler Equation for Log Utility and CRRA Utility

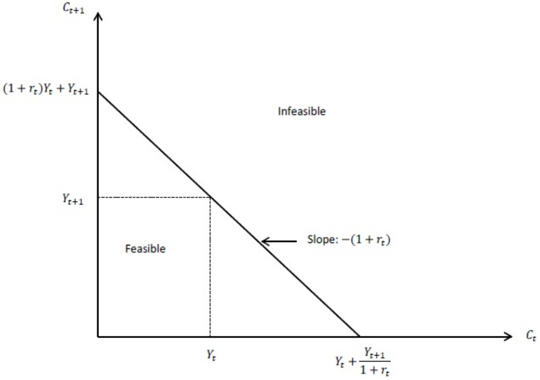
i. $\frac{C_{t+1}}{C_t} = \beta(1 + r_{t+1})$ is the Euler equation for log utility.

ii. $C_t^{-\sigma} = \beta(1 + r_{t+1})C_{t+1}^{-\sigma}$ is the Euler equation for iso-elastic/CRRA utility function. Notice that when $\sigma = 1$, we get the Euler equation for log utility.

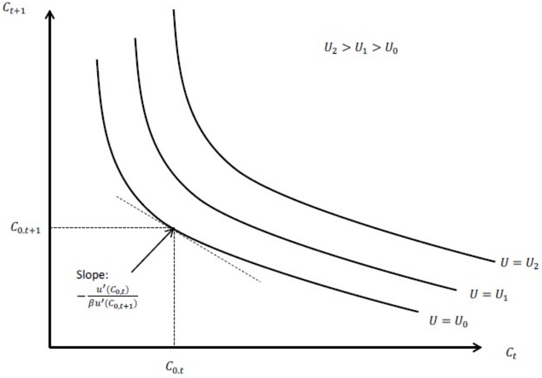
iii. Take logs and rearrange the Euler equation for CRRA utility to get $\Delta \ln(C_{t+1}) = \ln(C_{t+1}) - \ln(C_t) \approx \frac{1}{\sigma}\ln(\beta) + \frac{1}{\sigma}r_{t+1}$.

iv. The equation above suggests that growth in consumption over time depends positively on β (patience), r_{t+1} and inversely on σ, which measures consumption smoothing motive.

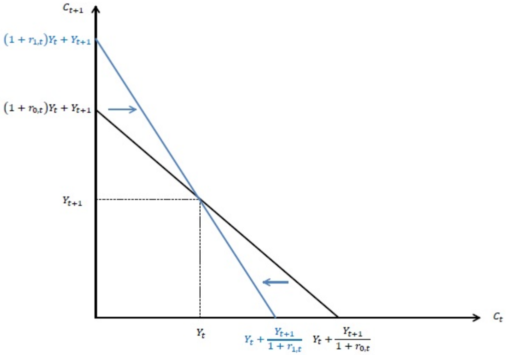
Intertemporal Budget Constraint



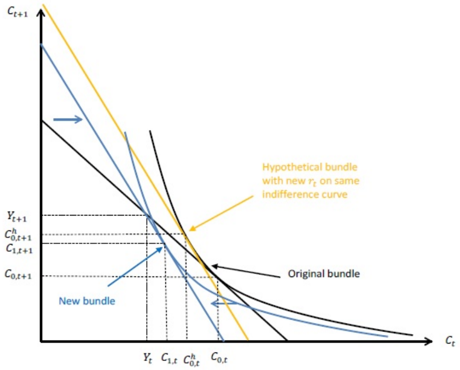
Indifference Curves and Optimal Solution



Rotation in Intertemporal Constraint due to higher r_{t+1}



Income and Substitution Effects for Borrower



Income and Substitution Effects

Table 9.1: Income and Substitution Effects of Higher r_t

	Substitution Effect	Income Effect	Total Effect
C_t			
Borrower	-	-	-
Saver	-	+	?
C_{t+1}			
Borrower	+	-	?
Saver	+	+	+

Consumption Function

i. The optimization problem for consumption saving can be thought of as producing a consumption function $C_t = C(Y_t, Y_{t+1}, r_{t+1})$.

ii. If we assume that substitution effects dominate income effects, then $\frac{\partial C}{\partial r_{t+1}} < 0$.

iii. The Euler equation for log utility from a previous slide can be solved for C_{t+1} and when we substitute for C_{t+1} in lifetime budget constraint, we get $C_t = \frac{1}{1+\beta}\left(Y_t + \frac{Y_{t+1}}{1+r_{t+1}}\right)$.

iv. Thinking of current consumption as proportional to the present discounted value of the stream of income is an insightful way to think about consumption-saving behavior.

Thank you