ECON 202 Intermediate Macroeconomics: Solow Model Lec 15-19

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Solow Model

i. This lecture corresponds to chapter 5 from your main textbook.

ii. The Solow Model is a classic model for understanding long run growth and cross-country income differences. It was developed by Robert Solow in Solow (1956), work for which he also won a Nobel Prize.

iii. The theoretical framework makes powerful predictions that line up well with the data.

iv. We do not explicitly model the microeconomic underpinnings of the model.

v. The key equations of the model are an aggregate production function, a consumption/saving function, and an accumulation equation for physical capital.

Aggregate Production Function

i. $Y_t = A_t F\left(K_t, N_t\right)$

ii. Work with $A_t = A$.

iii. $F_K > 0,\ F_N > 0,\ F_{KK} < 0,\ F_{NN} < 0,\ F_{KN} > 0.$

iv. $F(\rho K_t, \rho N_t) = \rho F(K_t, N_t)$ (constant returns to scale assumption) here.

v. Both labor and capital goods are essential for production i.e $F(0, N_t) = F(K_t, 0) = 0.$

vi. Cobb Doughlas Function satisfies all of the above assumptions $F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}$ with $\alpha \in (0, 1)$.

Verification of Regularity Assumptions

i. Assumption 1: If $F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}$ and $\alpha \in (0, 1)$,
$F_K = \alpha K_t^{\alpha-1} N_t^{1-\alpha} > 0$,
$F_N = (1-\alpha) N_t^{-\alpha} K_t^{\alpha} > 0$,
$F_{NN} = -(1-\alpha)(\alpha) N_t^{-\alpha-1} K_t^{\alpha} < 0$ and
$F_{KK} = \alpha(\alpha-1) K_t^{\alpha-2} N_t^{1-\alpha} < 0$.

ii. Assumption 2: $F(\rho K_t, \rho N_t) = (\rho K_t)^{\alpha} (\rho N_t)^{1-\alpha}$
$= \rho^{\alpha-\alpha+1} K_t^{\alpha} N_t^{1-\alpha} = \rho F(K_t, N_t)$, as required.

iii. Assumption 3: $F(0, N_t) = 0^{\alpha} N_t^{1-\alpha} = 0$ and
$F(K_t, 0) = K_t^{\alpha} 0^{1-\alpha} = 0$.

Firm Problem

i. The problem of the firm is to choose capital and labor so as to maximize profit: i.e $\max_{K_t, N_t} \Pi_t = AF(K_t, N_t) - w_t N_t - R_t K_t$.

ii. The FOC's are: $w_t = AF_N(K_t, N_t)$ and $R_t = AF_K(K_t, N_t)$, i.e the price of each factor of production should equal its own marginal product (just take the first derivatives wrt choice variables).

iii. The FOC's are sufficient due to concavity of objective function.

Household Problem

i. There exists a **representative household** in the economy. This household is endowed with time N_t and an initial stock of capital K_t.

ii. Household earns income from supplying capital and labor to the firm $w_t N_t + R_t K_t$ and also gets share of profits from firm: Π_t.

iii. The household budget constraint is: $C_t + I_t \leq w_t N_t + R_t K_t + \Pi_t$.

Other Equilibrium Equations

i. Homogeneity with degree of 1, when combined with Euler's theorem here implies that $Y_t = AF_N(K_t, N_t)N_t + AF_K(K_t, N_t)K_t$.

ii. When the expression in last bullet point is combined with firm optimization conditions $w_t = AF_N(K_t, N_t)$ and $R_t = AF_K(K_t, N_t)$, we get $Y_t = w_t N_t + R_t K_t$.

iii. Thus, $\Pi_t = 0$ since $P_t = 1$, which reflects perfect competition assumption.

iv. Using equality instead of inequality in household budget constraint then gives $Y_t = C_t + I_t$, which is the goods market clearing condition.

Consumption, Investment and Capital Accumulation

i. The defining characteristic of Solow Model is that it assumes an exogenous and constant rate of savings s, out of income so that $I_t = sY_t$.

ii. Combining the investment rule with goods market clearing condition gives that $C_t = (1 - s)Y_t$.

iii. Note that capital is a stock variable and labor is a flow variable so that the law of motion for capital is $K_{t+1} = (1 - \delta)K_t + I_t$.

Equilibrium

In sum, the Solow Model is completely characterized by the following 6 equations, holding simultaneously in equilibrium:

i. $Y_t = AF(K_t, N_t)$ (production function definition).

ii. $Y_t = C_t + I_t$ (goods market clearing condition).

iii. $K_{t+1} = (1 - \delta)K_t + I_t$ (law of motion for capital).

iv. $I_t = sY_t$ (investment/savings rule).

v. $w_t = AF_N(K_t, N_t)$ (firm optimality condition for labor choice).

vi. $R_t = AF_K(K_t, N_t)$ (firm optimality condition for capital choice).

Derivation of Central Equation

i. Combine the production function equation with investment rule to get $I_t = sAF(K_t, N_t)$.

ii. Then combine the expression derived in bullet point one above with law of motion for capital to get
$K_{t+1} = sAF(K_t, N_t) + (1-\delta)K_t$.

iii. Note that $k_t := \frac{K_t}{N_t}$ is capital per hours worked.

iv. Homogeneity (of degree 1) of production function implies that
$f(k_t) := F(k_t, 1) = F\left(\frac{K_t}{N_t}, \frac{N_t}{N_t}\right) = \left(\frac{1}{N_t}\right)^1 F(K_t, N_t)$

Derivation of Central Equation

i. Dividing everything in $K_{t+1} = sAF(K_t, N_t) + (1-\delta)K_t$ by N_t (labor hours), we get $\frac{K_{t+1}}{N_t} = (1-\delta)k_t + sA\frac{F(K_t,N_t)}{N_t}$.

ii. $f(k_t) = \frac{F(K_t,N_t)}{N_t} \implies \frac{K_{t+1}}{N_t} = (1-\delta)k_t + sAf(k_t)$.

iii. If we assume no population growth or no growth in labor hours over time, $N_t = N_{t+1}, \forall t \implies \frac{K_{t+1}}{N_t} = \frac{K_{t+1}}{N_{t+1}} = k_{t+1}$.

iv. Combining result in bullet point 3 above with equation in bullet point 2, we get $k_{t+1} = sAf(k_t) + (1-\delta)k_t$, the central equation of Solow Model.

Central Equation of Solow Model

$$k_{t+1} = \Phi(k_t) = sAf(k_t) + (1-\delta)k_t$$

Note that the equation above is expressed in **intensive** form, as should be evident from use of lower case f and lower case k_t.

Derivation of Firm Optimality Conditions in Intensive Form

i. $F_K(K_t, N_t) = F_K\left(\frac{K_t}{N_t}, \frac{N_t}{N_t}\right) = F_K(k_t, 1) = f'(k_t)$.

ii. Euler's theorem implies that
$F(K_t, N_t) = F_K(K_t, N_t)K_t + F_N(K_t, N_t)N_t$
$= f'(k_t)K_t + F_N(K_t, N_t)N_t$ (from first bullet above)
Dividing throughout by N_t
$\implies f(k_t) = f'(k_t)k_t + F_N(K_t, N_t)$
$\implies F_N(K_t, N_t) = f(k_t) - f'(k_t)k_t$

iii. Thus, the firm optimality conditions become $R_t = Af'(k_t)$.
$w_t = Af(k_t) - Af'(k_t)k_t$.

All Equations in Intensive Form

i. $y_t = Af(k_t)$ (production function definition).

ii. $y_t = c_t + i_t$ (goods market clearing condition).

iii. $k_{t+1} = (1-\delta)k_t + i_t$ (law of motion for capital).

iv. $i_t = sy_t$ (investment/savings rule).

v. $w_t = Af(k_t) - k_t Af'(k_t)$ (firm optimality condition for labor choice).

vi. $R_t = Af'(k_t)$ (firm optimality condition for capital choice).

INADA CONDITIONS

i. If we differentiate the central equation of Solow Model with respect to k_t, we get $\frac{dk_{t+1}}{dk_t} = \Phi'(k_t) = sAf'(k_t) + (1-\delta)$.

ii. $\lim_{k_t \to 0} f'(k_t) = \infty$ and
$\lim_{k_t \to \infty} f'(k_t) = 0$ are called **Inada Conditions**.

iii. The Inada conditions imply that as k_t approaches infinity, the slope $\Phi'(k_t)$ approaches $1-\delta$ and as $k_t \to 0$, $\Phi'(k_t) \to \infty$.

iv. Inada Conditions, along with diminishing marginal product of capital actually ensure that a unique and stable interior steady state exists. The rigorous proof of this fact is beyond the scope of this course.

What is a Steady State?

i. In a dynamic model, a **steady state** is a resting point x^* such that if the variable starts out at this value and no shock ever hits the economy, it will remain at the steady state forever.

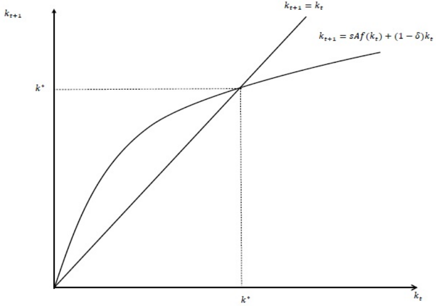
ii. The steady state is by definition, independent of time, so we remove the time subscripts and solve for the value x^* or sometimes denoted as x^{ss}.

iii. The steady state is a function of exogenous model parameters and will change if model parameters change.

iv. The steady state may or may not be stable (locally or globally) or there may or may not be monotonic convergence toward it.

Plot for Central Equation of Solow Model

Figure 5.1: Plot of Central Equation of Solow Model



Stability of Steady State and Monotonic Convergence

i. Given the Inada assumptions, there is an unique interior steady state which is also locally and globally stable.

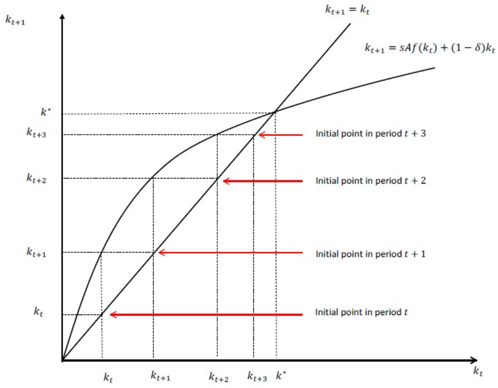
ii. k^* is globally stable if for *any* starting point k_0, the sequence $\{k_t\}_{t=0}^{\infty}$ converges to k^*.

iii. k^* is locally stable if $\{k_t\}_{t=0}^{\infty} \to k^*$ only if $k_0 \in (k^* - \epsilon, k^* + \epsilon)$ for some $\epsilon < \infty$.

iv. In this model, the convergence to steady state is also *monotonic* which means the sequence $\{k_t\}_{t=0}^{\infty}$ is either strictly decreasing or strictly increasing.

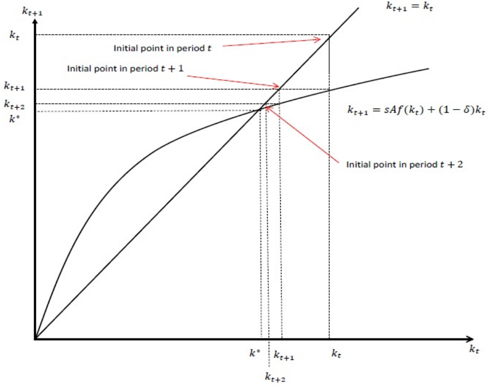
Convergence to Steady State for $k_t < k^*$

Figure 5.2: Convergence to Steady State from $k_t < k^*$



Convergence to Steady State for $k_t > k^*$

Figure 5.3: Convergence to Steady State from $k_t > k^*$



Alternative Version of Central Equation

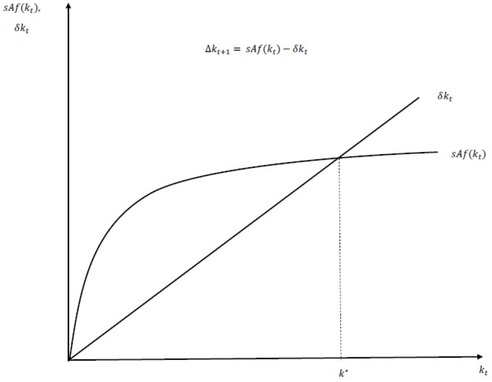
i. Subtracting k_t from both sides of the original central equation of Solow Model in intensive form and defining $\Delta k_{t+1} = k_{t+1} - k_t$, we get $\Delta k_{t+1} = sAf(k_t) - \delta k_t$

ii. The equation says that the net increase in capital to worker is equal to the investment in capital less the depreciation.

iii. δk_t is the break even level of investment.

Plot of Alternative Version of Central Equation of Solow Model

Figure 5.5: Alternative Plot of Central Equation of Solow Model



Algebra of Steady State with Cobb Doughlas Production Function

i. First, one can solve for steady state capital to labor ration through the central equation of Solow Model: $k^* = sA(k^*)^\alpha + (1-\delta)k^*$.

ii. Solve for k^* to get $k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$.

iii. The rest of model equations can then use this k^* to solve for other endogenous variables in steady state: $y^* = A(k^*)^\alpha$, $c^* = (1-s)A(k^*)^\alpha$, $i^* = sA(k^*)^\alpha$, $R^* = \alpha A(k^*)^{\alpha-1}$ and $w^* = (1-\alpha)A(k^*)^\alpha$.

Comparative Steady State Analysis

i. Since the steady state is a function of model parameters, it will change or shift to a new level when model parameters change.

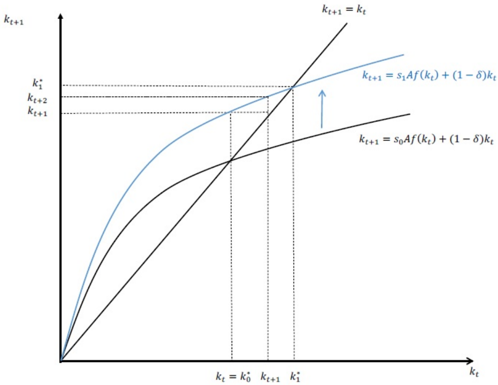
ii. For instance, it is clear from the solution for k^* that steady state capital to worker is increasing in A, s and α but decreasing in δ.

iii. This is similar in spirit to **comparative statics**, in which we shift the demand or supply curves for example to look at new competitive equilibria for prices and quantities.

iv. However, this is comparison of two steady states and not **equilibria**, since the equilibria are dynamic here.

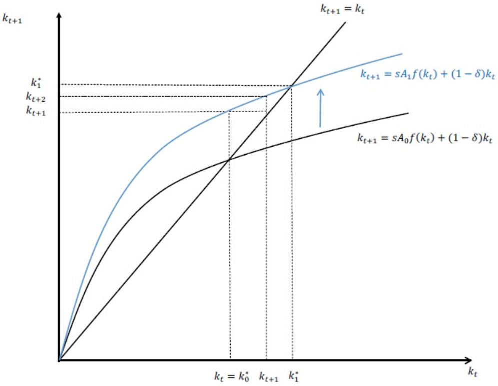
Change in Steady State Due to Change in s

Figure 5.6: Exogenous Increase in s, $s_1 > s_0$



Change in Steady State Due to Change in A

Figure 5.9: Exogenous Increase in A, $A_1 > A_0$



Transitional Dynamics or Dynamic Responses

i. In macroeconomics, an almost universal tool is called **impulse response functions** (IRF's), which show the *dynamic* response of a variable to a shock or perturbation to model economy.

ii. Impulse responses can be plotted for either a temporary change in some shock variable or parameter or a permanent change.

iii. For instance, in response to a permanent change in A, the steady state changes but dynamic responses show us the intermediate, transitional changes, governing the process of convergence from old steady state to new steady state.

Transitional Dynamics or Dynamic Responses

i. If the shock is temporary, the steady state does not change and we will return to original steady state but we can nevertheless observe the transitional dynamics, in which the variable responds to shock initially and eventually returns to the same starting point.

ii. Dynamic responses give us rich information about model's prediction about the joint dynamics of various variables and can be compared with empirical IRF's from empirical or econometric models such as Vector Auto-Regressions (VAR).

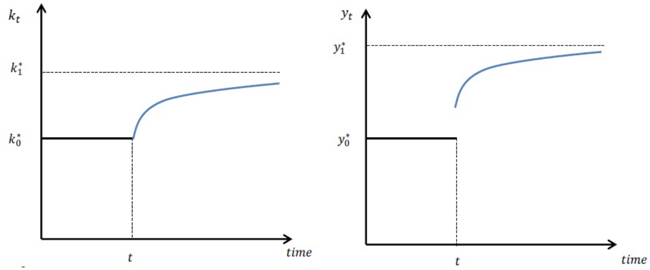
DYNAMIC RESPONSES WRT A

i. k_t is a **predetermined variable** so when A increases, k_t cannot respond immediately but since steady state capital to labor ratio has increased, in subsequent periods, capital accumulation will increase k_t until it reaches the new steady state.

ii. $y_t = Af(k_t)$ so that output increases in period t and beyond. The steady state output y^* is also increasing in A so that output eventually converges to higher steady state.

Dynamic Response of k_t and y_t when A increases

Figure 5.10: Dynamic Responses to Increase in A



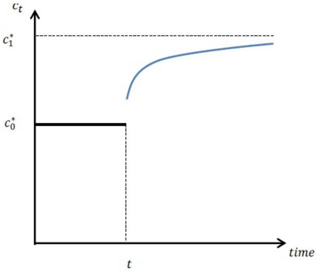
Dynamic Responses wrt A

i. $c_t = (1-s)y_t$, so given a fixed s, it just follows the changes in y_t by increasing on impact as well as afterwards.

ii. $i_t = sy_t$, so i_t also increases on impact and afterwards.

iii. A higher steady state capital to worker enables a higher steady to state consumption and saving simultaneously since it changes the size of "pie".

Dynamic Response of c_t and i_t wrt A



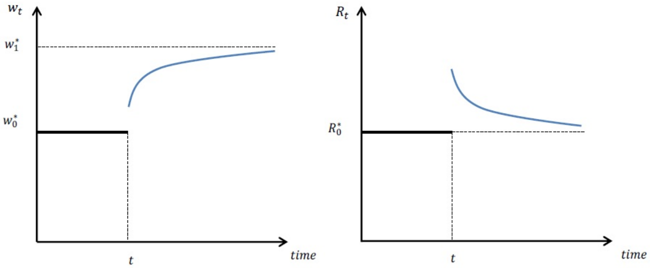
DYNAMIC RESPONSES WRT A

i. $w_t = A\left(f(k_t) - k_t f'(k_t)\right)$ which implies that w_t rises in period t when k_t has not changed yet but A has increased.

ii. In subsequent periods, k_t also rises. Since $f''(k_t) < 0$ ($F_{KK} < 0$) implies that $\frac{d(f(k_t) - k_t f'(k_t))}{dk_t} = -k_t f''(k_t) > 0$, higher k_t means wages continue to rise subsequently as well.

iii. $R_t = Af'(k_t)$, implying that R_t increases on impact but decreases after period t since $f''(k_t) < 0$ and steady state R^* is independent of A in this model.

Dynamic Response of w_t and R_t wrt A



Dynamic Responses wrt A

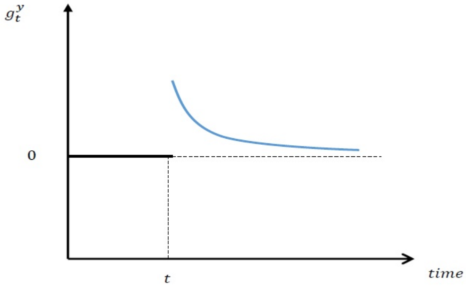
i. g_t^y increases right from the outset since y_t increases from period t.

ii. After period t, y_t continues to rise but at a decreasing rate since eventually it stabilizes at new steady state so that g_t^y falls after initial spike and equals zero eventually.

iii. This implies that even a permanent but one time increase in A will only lead to economic growth over the short run and ultimately we converge to new steady state with *no growth*.

Dynamic Response of g_t^y wrt A

Figure 5.11: Dynamic Path of Output Growth



Golden Rule of Saving

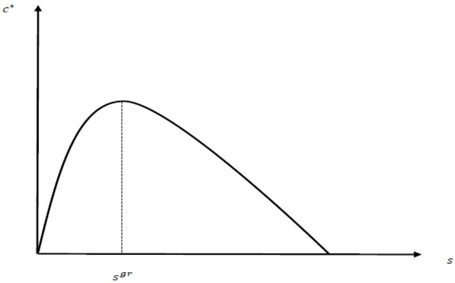
i. Observe the equation for steady state consumption in Solow Model $c^* = (1-s)Af(k^*)$.

ii. Increase in s increases k^* and hence $f(k^*)$ but also reduces $1-s$ so that change in s has an ambiguous effect on c^*.

iii. The golden rule of saving s^{gr} is the level of saving which maximizes steady state consumption and satisfies $Af'(k^*) = \delta$ (take FOC).

Golden Rule of Saving

Figure 5.12: s and c^*: The Golden Rule



Golden Rule of Saving and Dynamic Inefficiency

i. If $s < s_{GR}$, then increasing saving will increase new steady state consumption but will reduce consumption in the short run, below the old steady state level.

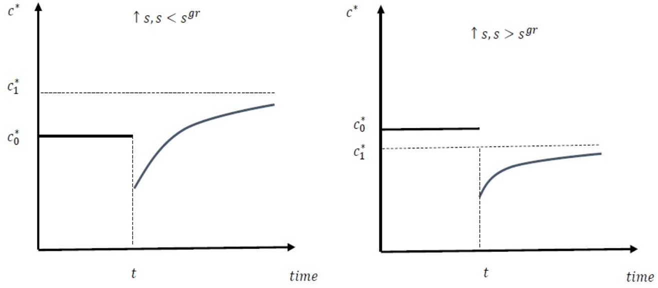
ii. If $s > s_{GR}$, then increasing saving will take the saving level even further away from golden rule level so that this will reduce consumption, both in short run and long run.

iii. If $s < s_{GR}$, then increasing or reducing saving does not have clear normative consequences for welfare due to impatience and possibly slow convergence to new steady state.

iv. If $s > s_{GR}$, we say that the economy is **dynamically inefficient** because *reducing* saving unambiguously increases welfare since consumption will increase in current and all future periods.

Dynamic Response of Consumption wrt s when $s > (<) s_{GR}$

Figure 5.14: Effects of $\uparrow s$ Above and Below the Golden Rule



Implications of Basic Solow Model

i. Sustained economic growth is not possible with mere capital accumulation over time and eventually one just converges to a steady state with no growth.

ii. Even if there is a permanent but one time increase in savings or A (TFP), we can only get positive growth rate for some time.

iii. Even repeated increases in s over time cannot generate sustained progress in growth rates since $s \leq 1$ (bounded above by 1).

iv. Sustained growth in living standards is only possible through sustained/repeated technical progress or improvement in total factor productivity or growth in A.

Homogeneous Functions

i. For any scalar ρ, a real valued function is homogeneous of degree k if $f(\rho x_1, ..., \rho x_n) = \rho^k f(x_1, ..., x_n), \forall \rho > 0$. For example, constant returns to scale production function is homogeneous with $k = 1$.

ii. If $z = f(\boldsymbol{x})$ is homogeneous function of degree k, where $\boldsymbol{x} \in \mathbb{R}^n$, then first order partial derivatives of f are homogeneous functions of degree $k - 1$.

iii. Consider Cobb Doughlas function $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$. $f(\rho x_1, \rho x_2) = (\rho x_1)^{\alpha} (\rho x_2)^{1-\alpha} = \rho^{\alpha} x_1^{\alpha} \rho^{1-\alpha} x_2^{1-\alpha} = \rho^{\alpha+1-\alpha} x_1^{\alpha} x_2^{1-\alpha} = \rho^1 f(x_1, x_2) \implies$ that this function is homogeneous of degree 1, which means constant returns to scale in the production function.

Back to main.

Euler's Theorem

i. **Euler's theorem:** Let $f(\boldsymbol{x})$ be a C^1, homogeneous function of degree k on $\mathbb{R}^{\boldsymbol{n}}_{+}$. Then $\forall \boldsymbol{x}$,
$x_1 \frac{\partial f}{\partial x_1}(\boldsymbol{x}) + x_2 \frac{\partial f}{\partial x_2}(\boldsymbol{x}) + \ldots x_n \frac{\partial f}{\partial x_n}(\boldsymbol{x}) = kf(\boldsymbol{x})$ or
$\boldsymbol{x}.\boldsymbol{\nabla} f(\boldsymbol{x}) = k.f(\boldsymbol{x})$, where $\boldsymbol{\nabla} f(\boldsymbol{x})$ is the gradient vector.

ii. Example: Consider $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.
$\boldsymbol{x}.\boldsymbol{\nabla} f(\boldsymbol{x}) = x_1.(\alpha x_1^{\alpha-1} x_2^{1-\alpha}) + x_2.\left((1-\alpha).x_2^{-\alpha} x_1^{\alpha}\right)$
$= x_1^{\alpha}.(\alpha x_2^{1-\alpha}) + \left((1-\alpha).x_2^{1-\alpha} x_1^{\alpha}\right)$
$= \alpha x_1^{\alpha} x_2^{1-\alpha} + x_2^{1-\alpha} x_1^{\alpha} - \alpha x_1^{\alpha} x_2^{1-\alpha} = x_2^{1-\alpha} x_1^{\alpha} = 1 * f(x_1, x_2)$
(Euler's theorem satisfied).

Back to main.

Thank you