How Many Different Ways Can the Tower of Hanoi with 3 Pegs Be Solved?

Word count: 3446

Table of Contents

Introduction

The original Tower of Hanoi is a popular puzzle involving three pegs and a tower of *n* disks stacked on one of the pegs in increasing radius top to bottom. While the exact origin of the puzzle is unknown, it was believed to be invented by French mathematician Édouard Lucas in 1883. (De Brabandere, 2017) The objective of the puzzle is to transfer the entire stack of disks onto a different peg in the same descending order of radius size by adhering to the following rules:

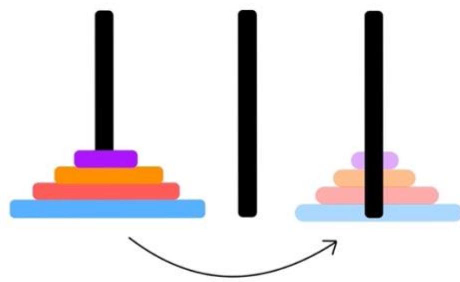

Figure 1: Objective of Tower of Hanoi

1. Only one disk (top disk on a peg) may be moved at a time
2. A larger disk cannot be stacked on top of a smaller one at any point in the game.

The puzzle is complete when all the disks have been transferred and stacked in descending order of radius on a different peg.

I will define the smallest disk as D_1, second smallest as D_2, third smallest as D_3, et cetera. D_n will then represent the largest disk in a game with n disks.

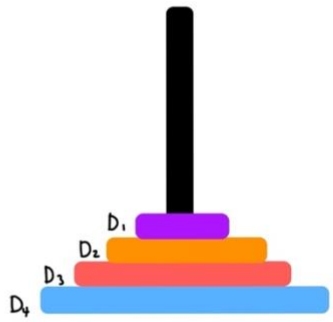

Figure 2: notation for disks

The Tower of Hanoi puzzle has been immensely popular for deriving a recursive formula for the minimum number of steps to solve the puzzle with n disks. (Plusadmin, 2006) However, I focused on a unique aspect of the game. After learning that the puzzle is used by psychologists to assess patients' cognitive abilities through tracking the sequence of their moves on the Hanoi Graph (Freiberger, 2016), I began to wonder, "how can the Hanoi Graph be used to determine the number of ways that the Tower of Hanoi with 3 pegs can be solved?" Indeed, this has been an original inquiry unanswered by the internet, so I decided to explore it for my Extended Essay in hopes of deriving a formula that determines the number of paths for the Hanoi graph with any number of disks.

For clarification, I will only be considering the original Tower of Hanoi game with 3 pegs with no variations. In this essay, I will answer my research question via constructing and analyzing the Hanoi Graph, a graphical representation of the relationship between all the possible configurations connected through legal moves within the Tower of Hanoi game. A configuration refers to a distinct state in the game every time a disk is moved, and a perfect configuration is when all the disks are stacked in increasing order of radius on one peg.

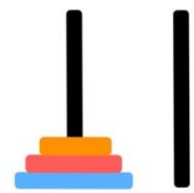

Figure 3: example of configuration *Figure 4: example of perfect configuration*

A "way" of solving, as mentioned in my question, "how can the Hanoi Graph be used to determine the number of ways that the Tower of Hanoi with 3 pegs can be solved?", is defined as a series of configurations one takes while solving the puzzle to get from the starting configuration to destination configuration. The following shows the most efficient method (Lewis, n.d.), which demonstrates one way of solving the Tower of Hanoi:

Figure 5: starting configuration in Tower of Hanoi game with 3 disks

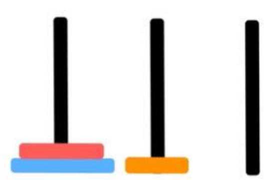

Figure 6: D_1 is moved

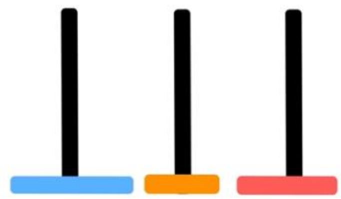

Figure 7: D_2 is moved

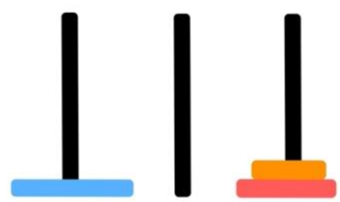

Figure 8: D_1 moved on top of D_2 to make room for D_3

Figure 9: D_3 is moved, establishing the foundation for final stack

Figure 10: D_1 moves off D_2, allowing D_2 to move

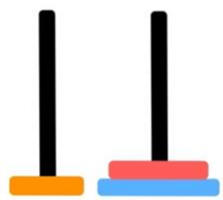

Figure 11: D_2 moves on top of D_3

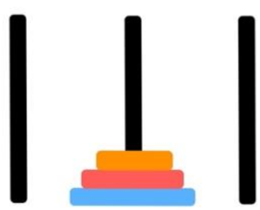

Figure 12: D_2 moves on top of D_3

In this essay, I will use the recursive property of the Hanoi graph to determine the relationship between number of paths in the graphs with one disk, two disks, and three disks, then derive a recursive formula to represent the number of paths for the Hanoi Graph with n disks.

What is recursion?

Recursion is a method of problem solving in which the solution of a larger problem depends on solutions smaller instances of the same problem. Eventually, the process leads to a base case which had been previously provided, and thus the problem can be solved. (CalcWorkshop, 2021) For example, if Bob needs to walk up 10 floors to get to his apartment, Bob first needs to walk up one staircase from the 9th floor, which would first require him to walk one staircase up from the 8th floor, which would require him to walk from the 7th and so on, eventually leading to the base case that he needs walk up floor one.

A recursive relation generates all terms beyond the first term in relation to previous terms, a_{n-1}. (CalcWorkshop, 2021) The first term or base case, a_1, must be defined along with the relation itself. The following is an example of a recursive relation:

$$a_1 = 1$$
$$a_n = a_{n-1} + 1$$

This relation would the produce the following sequence: 1, 2, 3, 4, 5…

The Hanoi Graph

Graph Theory

A graph, in terms of graph theory, is a non-linear data structure consisting of nodes and links between nodes. Each node is called a vertex, and each link is called an edge. A path is a sequence of edges joining a sequence of distinct vertices. (Carlson, 2020)

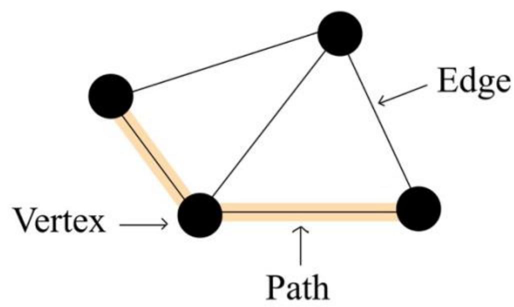

Figure 13: graph showing edge, vertex, and path

The Hanoi Graph is an example of an undirected graph, defined as a finite set of vertices connected with a finite set of bidirectional edges. A directed graph is one where the edges between two vertices are one-way and indicated with a particular direction. Therefore, paths that pass through these edges must pertain to its direction (Nykamp, n.d.)

Constructing the Hanoi Graph

A graph can be constructed of all the possible configurations in the Tower of Hanoi puzzle which visually represent the relationship from one configuration to another through adjacent steps. (Freiberger, 2012) Constructing such a graph benefits this investigation as it shows a compilation of all the possible configurations and their relationship.

A variable H_n will be used to represent the Hanoi graph with n disks. The notation for vertex $(c_1\ c_2\ c_3\ \ldots\ c_n)$will be used to represent the position of each of the disks, where c_1 represents the position of D_1 , c_2 represents the position of D_2 , c_3 represents the position of D_3 and so on. Pegs are numbered from 1 to 3 starting from left to right. For example, (211) means that D_1 is on the peg 2, while D_2 and D_3 are both on the first peg, stacked on top of one another.

The Hanoi graph H_3 will be constructed as follows. The game starts with all disks on peg 1, which is represented as (111). To begin the game, D_1 must be moved. Since there are two open pegs to choose from, here are two possible moves leading to two configurations (211) and (311) that can proceed from here:

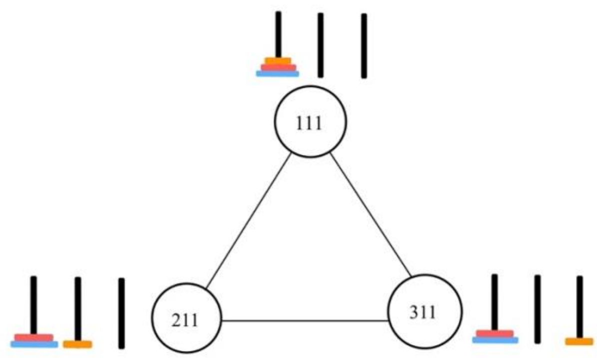

Figure 14: portion of Hanoi Graph showing movement of D_1 to

A relationship appears between the three vertices: they are interconnected and undirected through one movement represented by a single edge. Subsequently, there is only one possible outcome for moving D_2 for either of the configurations, that is to place it on the only other available peg. From there on, D_1 has the liberty to make two distinct legal moves once more.

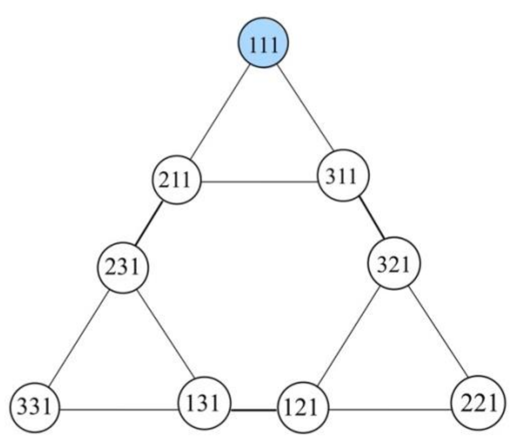

Figure 15: portion of Hanoi Graph showing movement of D_1 and D_2

The same undirected relationship and visual connection between all configurations we had noticed earlier also applies in the scope of the larger graph shown in Figure 15. Any configuration within the game can be reached from another configuration, for example (121) to (211), by following the path from (121) → (211).

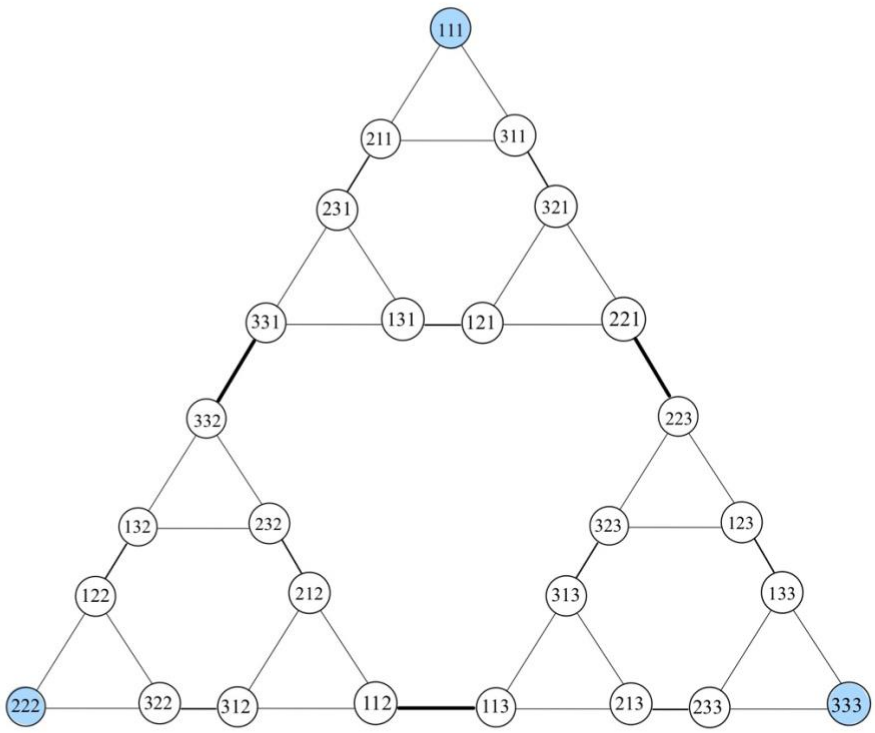

Figure 16: complete H_3 graph

Following through with all the available moves, the completed Hanoi graph for 3 disks is shown in *Figure 16.* Each of the vertices of the largest triangle highlighted in blue is a perfect configuration. If each edge is counted as one move, one can follow along the path and find all the possible paths. It appears that the complete triangle is essentially three smaller triangles joined through one connecting move in between. Moreover, *those* triangles are each made up of three even smaller triangles, which are each made of three vertices. Thus, the Hanoi tower possesses a recursive nature where the graphs are self-similar as the number of disks increase.

Furthermore, the most efficient solution which involves the minimum number of steps in solving the Tower of Hanoi for n disks (Lewis, n.d.), $2^n - 1$, can be visually deduced by counting the number of edges in the shortest path between the source and destination vertices (111) and (333).

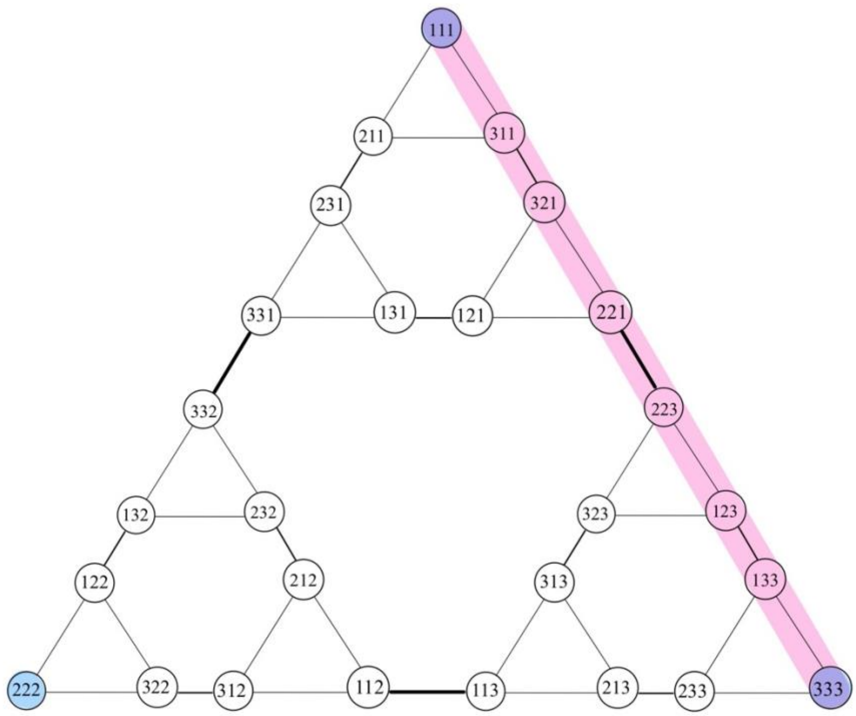

Figure 17: the most efficient solution presented on the H_3 graph

Properties

The following properties within the Hanoi Graph, which reveal the recursive nature upon which it is constructed, are useful in the process of deriving a formula to answer the research question

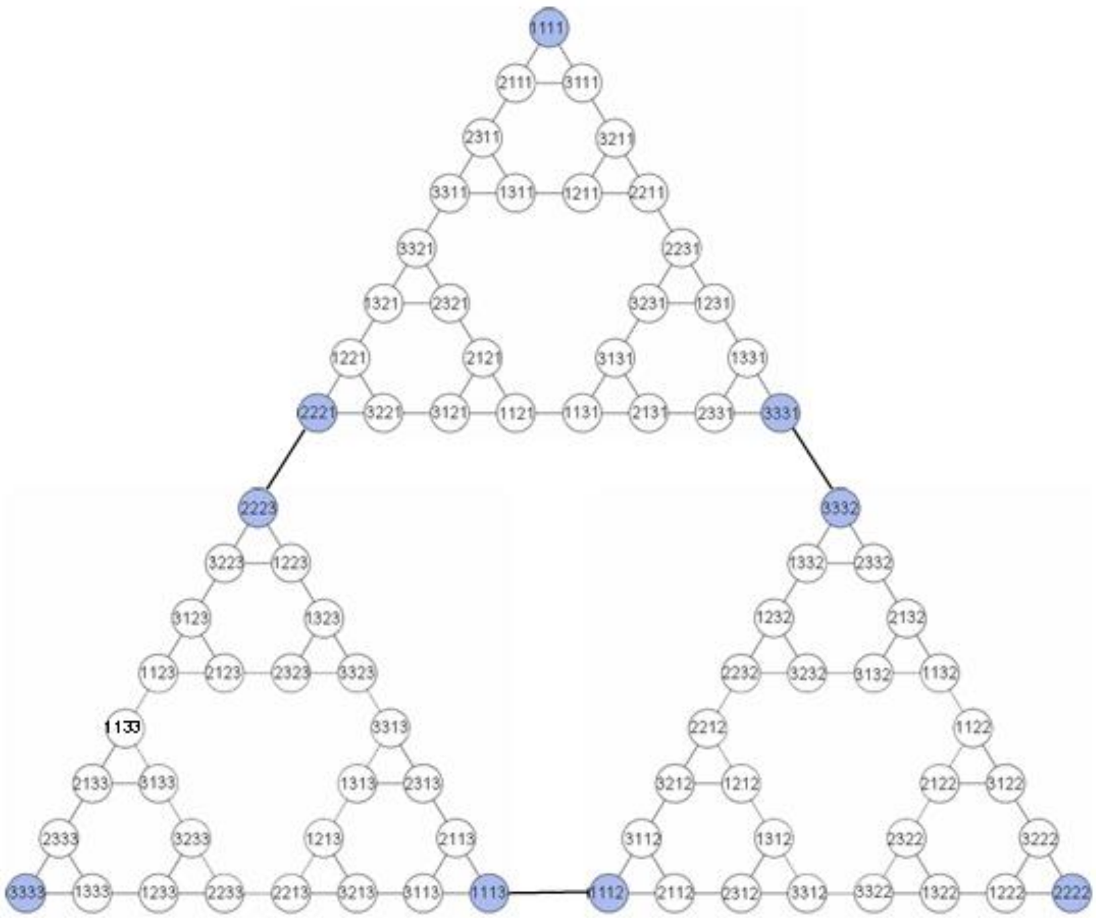

Figure 18: H_4 graph (Freiberger, 2012)

This is the Hanoi graph with 4 disks, H_4, which consists of three of our previous H_3 graphs, each connected to one another through one edge between the vertices that represents the perfect states of each H_3 triangle. The connection between

$$(2221) \rightarrow (2223)$$
$$(1113) \rightarrow (1112)$$
$$(3331) \rightarrow (3332)$$

represents possible places where movement D_4 disk after the completion of H_3. The D_n piece remains dormant while the D_{n-1} stack is being made. Thus, I extrapolate a property that the Hanoi graph H_{n-1} for n disks is identical to the graph of H_n stack for n-1 disks. Consequently, the possible paths are identical as well.

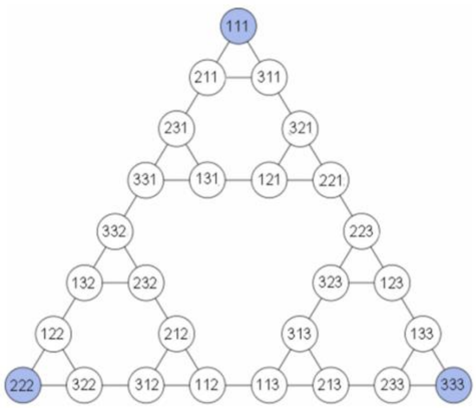

Figure 19: H_3 graph (Freiberger, 2012)

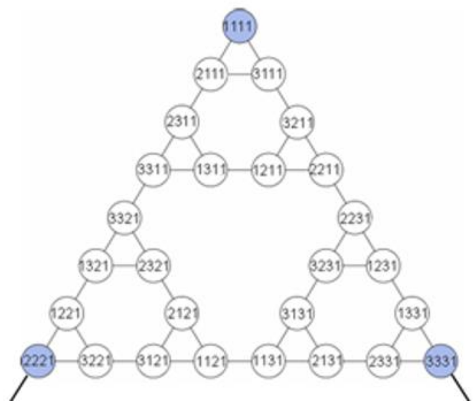

Figure 20: H_{n-1} graph inH_4 (Freiberger, 2012)

The larger piece only comes into play during its move after the D_{n-1} stack is completed, which results in the subsequent bridge into the larger triangle. Then, the D_{n-1} stack is built again above of the D_n disk to solidify the final structure, which causes the two choices branch into a second triangle.

What would the Hanoi graph of an infinite *n* look like? We saw earlier that as n increases, the shape of the Hanoi graph is retained but its complexity and number of vertices increases. Each node in the graph with n-1 disks became a group of three vertices in the graph of n disks, as visualized below:

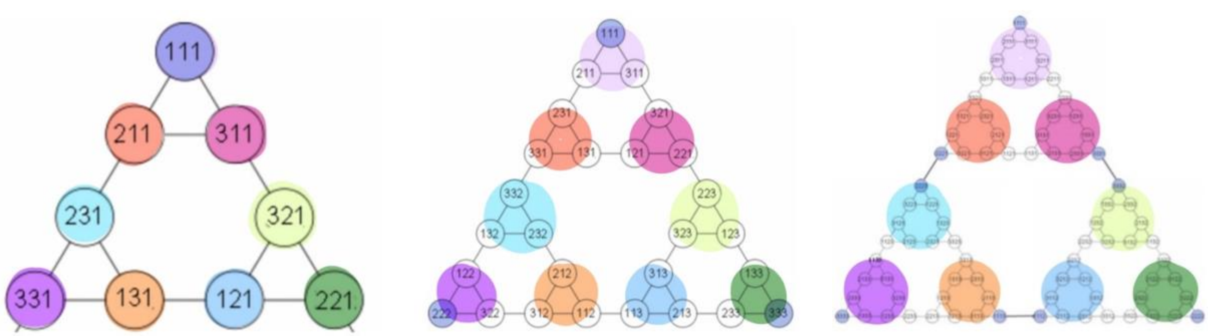

Figure 21: self-similar property of Hanoi Graphs

For subsequent number of disks, we can then conclude that the H_n will consist of three H_{n-1}, each of which consists of three H_{n-2}, and so on, due to its self-similar nature. This leads to the property that the H_n graph consists of three H_{n-1} graphs, and that each H_n graph extends to H_{n+1} from its bottom-left vertex and bottom right vertex, with each vertex joined by one edge to the top vertex of two other H_n graphs. Thus, through constructing the Hanoi graph and identifying some of its properties, the foundation of my investigation is complete, and the main investigation of the problem may begin.

The Need for Restriction

The method, as proposed, is to derive the formula representing the number of all possible ways to solve a game with n disks from starting configuration to destination configuration, through tracing paths in the Hanoi graph and generalizing a pattern. I will begin with a smaller scale with one disk, then expand to cases with greater numbers of disks and compare their results to derive a formula.

First, I must isolate the objective to strictly finding the paths from the source vertex to a *specific* destination rather than either one. In the Tower of Hanoi puzzle, the perfect configurations can be stacked on either peg 2 or peg 3, which are respectively the bottom left-most and bottom right-most vertices in the Hanoi Graph. However, in this investigation, I am only interested in the paths from the source vertex to the bottom right-most vertex. The rationale is that since the Hanoi Graph is symmetrical, the number of paths from source vertex to either of the destination vertices is identical. Subsequently, restricting the to only one destination greatly simplifies the conditions necessary to finding an answer.

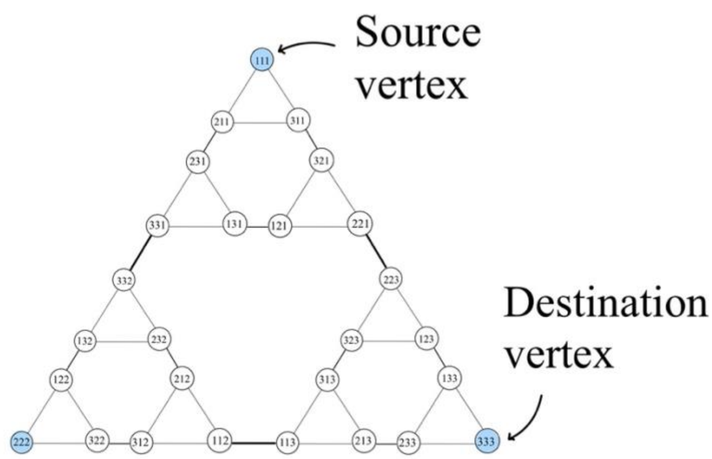

Figure 22: source and destination vertices

Second, a valid path in this investigation must be a simple path. That is, no vertices may be repeated. In terms of the puzzle, this means that no configuration may be visited twice. The most direct reason for this restriction is that if repeated vertices are permitted, instances where an infinite loop between three vertices will be permitted, deviating from the meaningful solutions I aim to investigate.

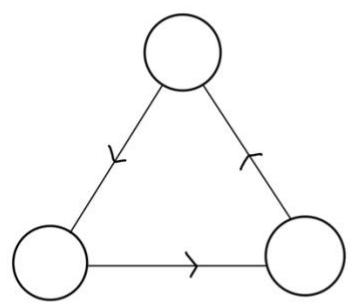

Figure 23: example of non-permissible case

Third, the Hanoi graph must be a directed graph. All vertical edges must be directed down, and all horizontal edges must be directed to the right, towards the destination vertex.

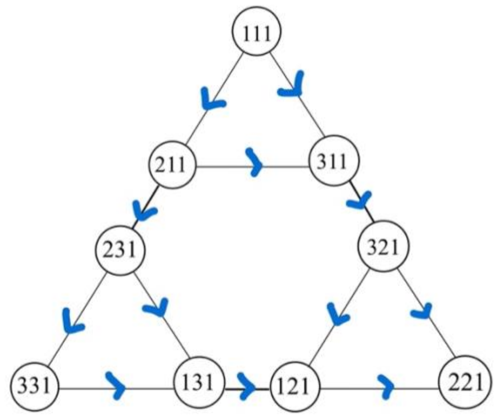

Figure 24: directed Hanoi Graph

In summary, the rationale for the above restrictions is to focus and simplify the investigation so that meaningful observation can be made without the need of computer algorithms. As a result, the conclusions obtained under these restrictions will inevitably not represent all possible paths one may encounter while solving Tower of Hanoi.

Deriving the Formula

I will define a variable P_n for the number of paths from vertices from source vertex to destination vertex for n disks. We will first begin with the H_1 graph in determining the possible paths from (1) to (3).

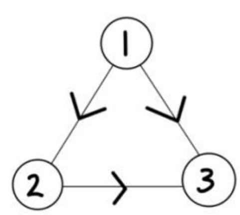

Figure 25: directed H_1 graph

There are two possible paths extending from the source vertex (1):

$$(1) \rightarrow (3)$$

$$(1) \rightarrow (2) \rightarrow (3)$$

If (3) was chosen, the path is complete. If (2) was chosen, another one-way edge must be travelled to arrive at (3). Since it is one-way, there are no additional possibilities. The total paths would be the sum of the two vertex choices, so 2 is the total number of possible paths in H_1.

$$\therefore P_1 = 2$$

For H_2:

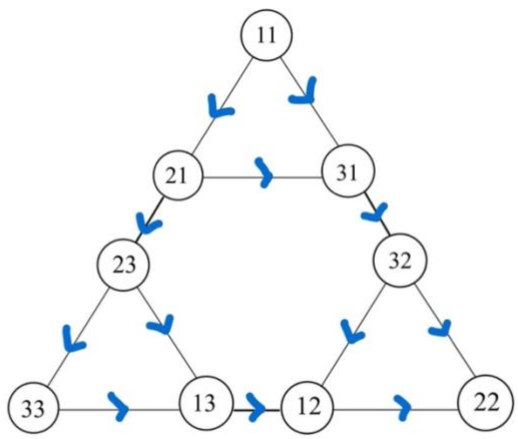

Figure 26: directed H_2 graph

Counting by exhaustion, I found that all paths from the source must either extend from (31) to (32) leading into the bottom right triangle, or (21) to (23), leading to the bottom left. In other words, all paths must either extend from the bottom left-most and right-most vertices of the top H_{n-1} graph to the top vertices of the bottom left and right H_{n-1} graphs. In H_1, there are 4 paths passing through the bottom right triangle, and in the second case, there are only two paths. Adding up the two cases, there are 6 paths in total.

Alternatively, a simpler method would be to utilize our findings from H_1 through a recursive solution. We previously discovered that there are 2 possible paths in H_1. Since, the H_2 consists of three H_1 graphs, which was a property we found earlier in the Hanoi Graph. The three edges joining the H_1 graphs: (21, 23), (31, 32) and (13, 12) will be inevitably passed when the path travels from one H_1 to another and will therefore not contribute to variations.

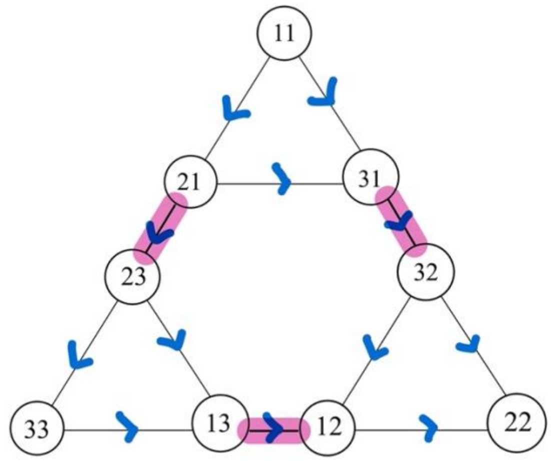

Figure 27: edges joining perfect configurations of H_1 graphs in H_2 graph

In consideration to the shape and directional constraints, we can then categorize the paths in one of two cases:

1. Passes through bottom right H_1 only
2. Passes through both bottom right and bottom H_1.

In analyzing the first case, I find that there are two paths in the top H_1 graph that arrive at (31), then for each of the two paths, there are two paths in the bottom right H_1 graph to reach the desired vertex. Since P_1 is 2, the total number of possible paths is:

$$P_1 \times P_1$$
$$= 2 \times 2$$
$$= 4$$

In analyzing the second case, I find that pertaining to the directed graph, there is only one vertex (21) that the paths can pass through in the top H_1. In the bottom left H_1, there are two paths which we know from P_1. Then in the bottom right H_1, there is only one path leading up to the bottom right-most vertex (22).

$$1 \times P_1 \times 1$$
$$= 1 \times 2 \times 1$$
$$= 2$$

Adding the number of paths for the two cases, we obtain:

$$P_1 \times P_1 + P_1$$
$$= 2 \times 2 + 2$$
$$= 6$$
$$\therefore\ P_2 = 6$$

By deriving the answer for P_2 in terms of P_1, we can also extrapolate that

$$P_2 = {P_1}^2 + P_1$$

Let us now examine H_3:

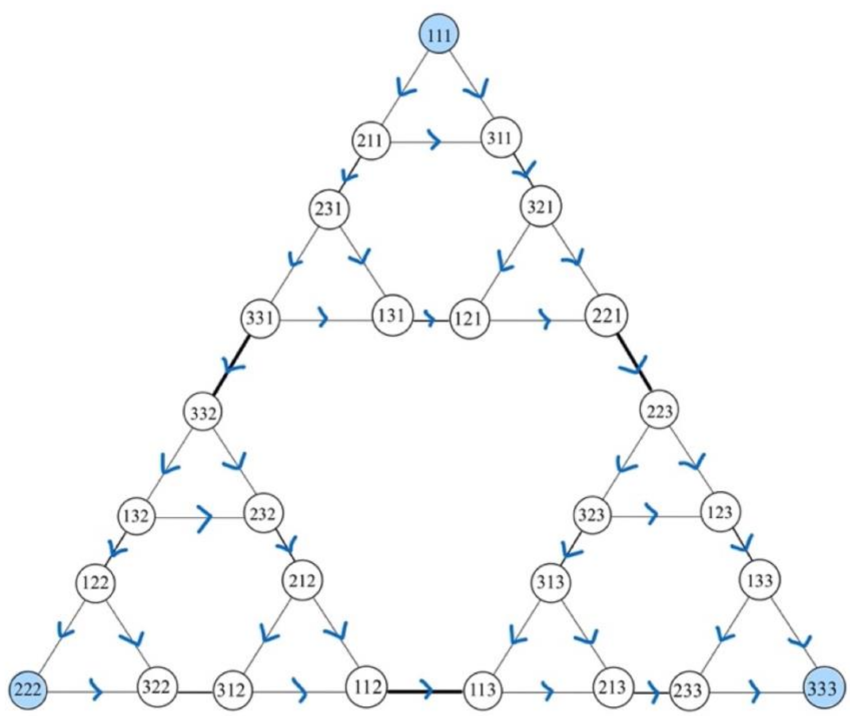

Figure 28: directed H_3 graph

A likewise approach will be used in the investigation for P_3. The H_3 graph consists of three H_2 graphs, for which we know that there are 6 possible paths. Again, the possible paths could be separated into two cases:

1. Passes through the bottom right H_2 only
2. Passes through both the bottom left and right H_2

Upon examining the first case, for which we use a similar approach as the previous graph, we must consider the top H_2 as well as the bottom right H_2. We know from the previous graph that there are P_2 paths arriving at vertex (221) in the top H_2, and for each of the P_2 paths there are another P_2 paths to arrive at the bottom right-most vertex (333).

$$P_2 \times P_2$$
$$= {P_2}^2$$
$$= 6^2$$
$$= 36$$

Likewise, the second case can be analyzed in a similar manner as we did in the previous graph. Pertaining to the direction of the directional graph, there is only one possible path arriving at the vertex (331):

$$(111) \rightarrow (211) \rightarrow (231) \rightarrow (331)$$

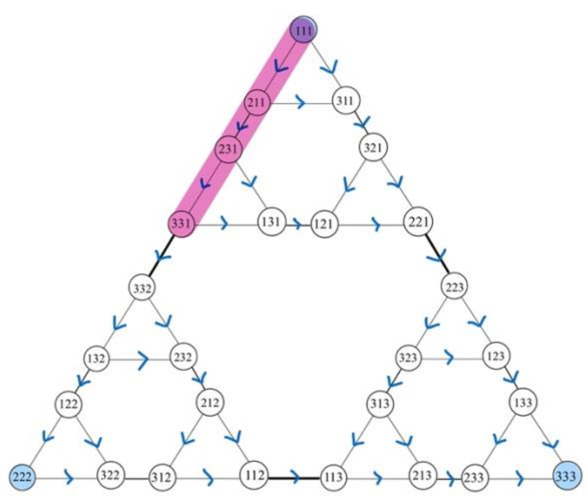

Figure 29: only path from (111) to (331) pertaining to direction

Subsequently, there are P_2 distinct paths in the bottom left H_2 to arrive at (112), and only one path in the bottom right H_2 to arrive at vertex (333). The resulting number of paths for the second case is therefore,

$$1 \times P_2 \times 1$$
$$= 6$$

Summing the number of possible paths, we obtain P_3,

$$P_3 = case\ one + case\ two$$
$$= {P_2}^2 + P_2$$
$$= 6^2 + 6$$
$$= 42$$
$$\therefore P_3 = 42$$

Recursive Formula

By comparing my solutions for P_1, P_2, and P_3, I can extrapolate a property that P_n can be defined in relation to P_{n-1}. The following hypothesis can be written for P_n,

$$P_1 = 2$$
$$P_n = {P_{n-1}}^2 + P_{n-1}$$
$$P_n = P_{n-1}(P_{n-1} + 1)$$

The above relation is a nonlinear recurrence relation, specifically a quadratic one. (Weisstein, 2021) Quadratic recurrence relation is a second-degree polynomial in the form

$$x_n = a\, {x_{n-1}}^2 + b\, x_{n-1} + c$$

Proof

We can prove that the recurrence relation

$$P_1 = 2$$

$$P_n = {P_{n-1}}^2 + P_{n-1}$$

is true for all $n \in \mathbb{Z}^+$ through mathematical induction.

Base Case

$$n = 2$$

$$P_2 = {P_1}^2 + P_1$$

$$P_2 = 2^2 + 2$$

Since we derived earlier that $P_2 = 6$,

$$6 = 6$$

Hence, the base case $n = 2$ is proven true.

Inductive Step

Assume for H_k, the number of paths is:

$$P_k = {P_{k-1}}^2 + P_{k-1}$$

The graph H_{k+1} is formed from H_k extending its bottom left-most and right-most vertices into the top vertices of two other identical H_k graphs without loss of generality. The two H_k graphs are also directly joined to each other through one edge between the bottom right-most vertex of the bottom left H_k and the bottom left-most vertex of the bottom right H_k. Pertaining to the directed graph, each extension creates P_k paths from the top vertex to bottom right vertex of each H_k.

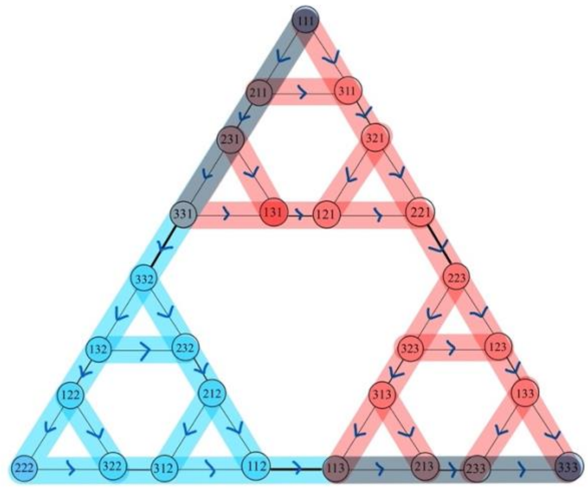

Figure 30: H_3 graph showing paths that pass through the extension from bottom right(red) and bottom left (blue) vertices of the top H_2 graph

Due to the direction, there are P_k paths from the top vertex of a H_k to its bottom right vertex, which we have assumed true, and only one path to the bottom left. Therefore, assuming that

$$P_k = {P_{k-1}}^2 + P_{k-1}$$

holds, we know that there are $P_k \times P_k$ paths passing through the extension from the bottom right vertex of the top H_k, and P_k paths passing through its extension from the bottom left vertex. The total number of paths in H_{k+1} is therefore the sum of the two cases, which results in the conclusion which proves our induction hypothesis:

$$P_{k+1} = {P_k}^2 + P_k$$

To conclude, the recurrence relation,

$$P_1 = 2$$

$$P_n = {P_{n-1}}^2 + P_{n-1}$$

is proven true for $n \in \mathbb{Z}^+$.

Conclusion

In conclusion, I have successfully derived and proved a recursive formula that answers my research question, "how can the Hanoi Graph be used to determine the number of ways that the Tower of Hanoi with 3 pegs can be solved?". Under the restrictions I had stated above, the total number of possible ways to solve a Tower of Hanoi puzzle with n disks, P_n, can be represented in the following recurrence relation,

$$P_1 = 2$$
$$P_n = {P_{n-1}}^2 + P_{n-1}$$

However, due to the following restrictions in my research methodology,

1. Only consider paths from specified source vertex to specified destination vertex
2. Only simple paths permitted
3. Only paths pertaining to the directed Hanoi Graph permitted

my solution was not a holistic representation of all the possible solutions, but rather an idealistic approximation. Realistically, there are far more possible cases one may encounter while attempting to solve the Tower of Hanoi puzzle with factors such as human error. The most notable restriction was the third one, for which the investigation was focused towards the directed graph. By human means, it is extremely difficult to generalize patterns in the undirected graph, which contains more variability and far more solutions. However, a possible future approach could be to represent the undirected graph using code and generalize the numerical patterns produced in the sequence.

References

The Trustees of Princeton University. (n.d.). *Digraphs.* Princeton University. Retrieved November 23, 2021, from https://algs4.cs.princeton.edu/42digraph/.

Freiberger, M. (2017, October 13). *The Tower of Hanoi: Where maths meets psychology*. plus.maths.org. Retrieved August 10, 2021, from https://plus.maths.org/content/tower-hanoi-where-maths-meets-psychology.

De Brabandere, S. (2017, October 26). *The Tower of Hanoi*. Scientific American. Retrieved December August 25, 2021, from https://www.scientificamerican.com/article/the-tower-of-hanoi/.

Carlson, S. C. (2020, November 24). *Graph theory*. Encyclopædia Britannica. Retrieved November 19, 2021, from https://www.britannica.com/topic/graph-theory.

Nykamp , D. Q. (n.d.). *Math insight*. Undirected graph definition - Math Insight. Retrieved November 19, 2021, from https://mathinsight.org/definition/undirected_graph.

Lewis, H. (2021, November 15). *Tower of Hanoi*. Maths Careers. Retrieved December 2, 2021, from https://www.mathscareers.org.uk/tower-of-hanoi/.

plusadmin. (2010, July 7). *Towers of Hanoi Solution*. plus.maths.org. Retrieved August 10, 2021, from https://plus.maths.org/content/puzzle-page-68.

Calcworkshop. (2021, February 15). *Recursive Formula Explained w/ 25 Step-by-Step Examples!* Calcworkshop. Retrieved October 21, 2021, from https://calcworkshop.com/combinatorics/recursive-formula/.

Freiberger, M. (2012). *The Hanoi graph H_3.* [digital image]. Plus Magazine. https://plus.maths.org/content/tower-hanoi-where-maths-meets-psychology

Time Base Technology Limited. (2011-2021). *GoodNotes* (5.7.50)[mobile app]. App Store. https://www.goodnotes.com/zh-cn/