6.3 Quadratic functions EMA4H

Functions of the form $y = x^2$ EMA4J

Functions of the general form $y = ax^2 + q$ are called parabolic functions. In the equation $y = ax^2 + q$, a and q are constants and have different effects on the parabola.

Worked example 4: Plotting a quadratic function

QUESTION

$y = f(x) = x^2$

Complete the following table for $f(x) = x^2$ and plot the points on a system of axes.

x	-3	-2	-1	0	1	2	3
$f(x)$	9						

1. Join the points with a smooth curve.
2. The domain of f is $x \in \mathbb{R}$. Determine the range.
3. About which line is f symmetrical?
4. Determine the value of x for which $f(x) = 6\frac{1}{4}$. Confirm your answer graphically.
5. Where does the graph cut the axes?

SOLUTION

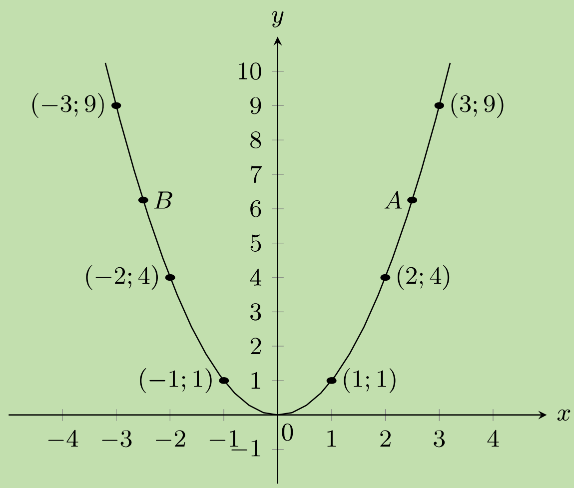
Step 1: Substitute values into the equation

$$
\begin{aligned}
f(x) &= x^2 \\
f(-3) &= (-3)^2 = 9 \\
f(-2) &= (-2)^2 = 4 \\
f(-1) &= (-1)^2 = 1 \\
f(0) &= (0)^2 = 0 \\
f(1) &= (1)^2 = 1 \\
f(2) &= (2)^2 = 4 \\
f(3) &= (3)^2 = 9
\end{aligned}
$$

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	0	4	9

Step 2: Plot the points and join with a smooth curve

From the table, we get the following points: $(-3;9), (-2;4), (-1;1), (0;0), (1;1), (2;4), (3;9)$



Step 3: Determine the domain and range

Domain: $x \in \mathbb{R}$

From the graph we see that for all values of x, $y \geq 0$.

Range: $\{y : y \in \mathbb{R}, y \geq 0\}$

Step 4: Find the axis of symmetry

f is symmetrical about the y-axis. Therefore the axis of symmetry of f is the line $x = 0$.

Step 5: Determine the x-value for which $f(x) = 6\frac{1}{4}$

$$\begin{aligned} f(x) &= \frac{25}{4} \\ \therefore \frac{25}{4} &= x^2 \\ x &= \pm\frac{5}{2} \\ &= \pm 2\frac{1}{2} \end{aligned}$$

See points A and B on the graph.

Step 6: Determine the intercept

The function f intercepts the axes at the origin $(0;0)$.

We notice that as the value of x increases from $-\infty$ to 0, $f(x)$ decreases.

At the turning point $(0;0)$, $f(x) = 0$.

As the value of x increases from 0 to ∞, $f(x)$ increases.

Investigation: The effects of $a$and q on a parabola.

Complete the table and plot the following graphs on the same system of axes:

1. $y_1 = x^2 - 2$
2. $y_2 = x^2 - 1$
3. $y_3 = x^2$
4. $y_4 = x^2 + 1$
5. $y_5 = x^2 + 2$

x	-2	-1	0	1	2
y_1					
y_2					
y_3					
y_4					
y_5					

Use your results to deduce the effect of q.

Complete the table and plot the following graphs on the same system of axes:

1. $y_6 = -2x^2$
2. $y_7 = -x^2$
3. $y_8 = x^2$
4. $y_9 = 2x^2$

x	-2	-1	0	1	2
y_6					
y_7					
y_8					
y_9					

Use your results to deduce the effect of a.

The effect of q

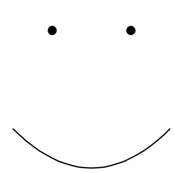
The effect of q is called a vertical shift because all points are moved the same distance in the same direction (it slides the entire graph up or down).

- For $q > 0$, the graph of $f(x)$ is shifted vertically upwards by q units. The turning point of $f(x)$ is above the y-axis.
- For $q < 0$, the graph of $f(x)$ is shifted vertically downwards by q units. The turning point of $f(x)$ is below the y-axis.

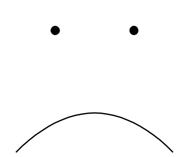
The effect of a

The sign of a determines the shape of the graph.

- For $a > 0$, the graph of $f(x)$ is a "smile" and has a minimum turning point at $(0; q)$. The graph of $f(x)$ is stretched vertically upwards; as a gets larger, the graph gets narrower.
  For $0 < a < 1$, as a gets closer to 0, the graph of $f(x)$ gets wider.
- For $a < 0$, the graph of $f(x)$ is a "frown" and has a maximum turning point at $(0; q)$. The graph of $f(x)$ is stretched vertically downwards; as a gets smaller, the graph gets narrower.
  For $-1 < a < 0$, as a gets closer to 0, the graph of $f(x)$ gets wider.



$a > 0$ (a positive smile)



$a < 0$ (a negative frown)

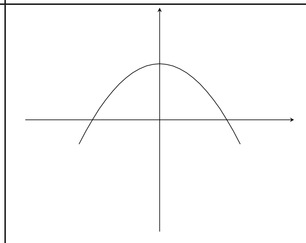
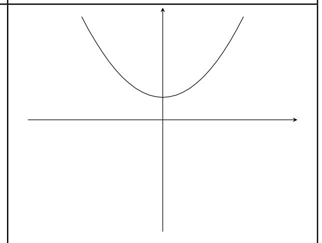
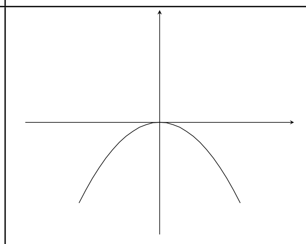
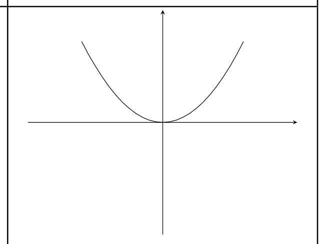
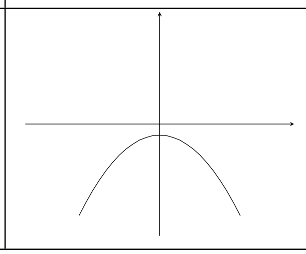
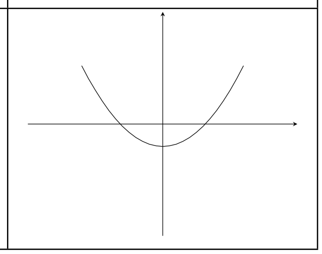
	$a < 0$	$a > 0$
$q > 0$		
$q = 0$		
$q < 0$		

Table 6.2: The effect of a and q on a parabola.

VISIT:
You can use this Phet simulation to help you see the effects of changing a and q for a parabola.

Discovering the characteristics EMA4M

The standard form of the equation of a parabola is $y = ax^2 + q$.

Domain and range

The domain is $\{x : x \in \mathbb{R}\}$ because there is no value for which $f(x)$ is undefined.

If $a > 0$ then we have:

$$\begin{aligned} x^2 &\geq 0 && \text{(Perfect square is always positive)} \\ ax^2 &\geq 0 && \text{(since } a > 0\text{)} \\ ax^2 + q &\geq q && \text{(add } q \text{ to both sides)} \\ \therefore f(x) &\geq q \end{aligned}$$

Therefore if $a > 0$, the range is $[q; \infty)$. Similarly, if $a < 0$ then the range is $(-\infty; q]$.

Worked example 5: Domain and range of a parabola

QUESTION

If $g(x) = x^2 + 2$, determine the domain and range of the function.

SOLUTION

Step 1: Determine the domain

The domain is $\{x : x \in \mathbb{R}\}$ because there is no value for which $g(x)$ is undefined.

Step 2: Determine the range

The range of $g(x)$ can be calculated as follows:

$$\begin{aligned} x^2 &\geq 0 \\ x^2 + 2 &\geq 2 \\ g(x) &\geq 2 \end{aligned}$$

Therefore the range is $\{g\left(x\right) : g\left(x\right) \geq 2\}$.

Intercepts

The y-intercept:

Every point on the y-axis has an x-coordinate of 0, therefore to calculate the y-intercept let $x = 0$.

For example, the y-intercept of $g(x) = x^2 + 2$ is given by setting $x = 0$:

$$\begin{aligned} g(x) &= x^2 + 2 \\ g(0) &= 0^2 + 2 \\ &= 2 \end{aligned}$$

This gives the point $(0; 2)$.

The x-intercepts:

Every point on the x-axis has a y-coordinate of 0, therefore to calculate the x-intercept let $y = 0$.

For example, the x-intercepts of $g(x) = x^2 + 2$ are given by setting $y = 0$:

$$\begin{aligned} g(x) &= x^2 + 2 \\ 0 &= x^2 + 2 \\ -2 &= x^2 \end{aligned}$$

There is no real solution, therefore the graph of $g(x) = x^2 + 2$ does not have x-intercepts.

Turning points

The turning point of the function of the form $f(x) = ax^2 + q$ is determined by examining the range of the function.

- If $a > 0$, the graph of $f(x)$ is a "smile" and has a minimum turning point at $(0; q)$.
- If $a < 0$, the graph of $f(x)$ is a "frown" and has a maximum turning point at $(0; q)$.

Axes of symmetry

The axis of symmetry for functions of the form $f(x) = ax^2 + q$ is the y-axis, which is the line $x = 0$.

Sketching graphs of the form $y = ax^2 + q$ EMA4N

In order to sketch graphs of the form $f(x) = ax^2 + q$, we need to determine the following characteristics:

1. sign of a
2. y-intercept
3. x-intercept
4. turning point

Worked example 6: Sketching a parabola

QUESTION

Sketch the graph of $y = 2x^2 - 4$. Mark the intercepts and turning point.

SOLUTION

Step 1: Examine the standard form of the equation

We notice that $a > 0$. Therefore the graph is a "smile" and has a minimum turning point.

Step 2: Calculate the intercepts

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= 2x^2 - 4 \\ &= 2(0)^2 - 4 \\ &= -4 \end{aligned}$$

This gives the point $(0; -4)$.

For the x-intercepts, let $y = 0$:

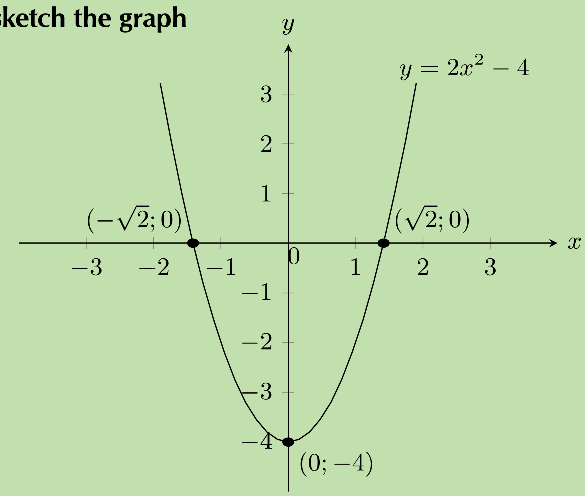
$$\begin{aligned} y &= 2x^2 - 4 \\ 0 &= 2x^2 - 4 \\ x^2 &= 2 \\ \therefore x &= \pm\sqrt{2} \end{aligned}$$

This gives the points $(-\sqrt{2}; 0)$ and $(\sqrt{2}; 0)$.

Step 3: Determine the turning point

From the standard form of the equation we see that the turning point is $(0; -4)$.

Step 4: Plot the points and sketch the graph



Domain: $\{x : x \in \mathbb{R}\}$

Range: $\{y : y \geq -4, y \in \mathbb{R}\}$

The axis of symmetry is the line $x = 0$.

Worked example 7: Sketching a parabola

QUESTION

Sketch the graph of $g(x) = -\frac{1}{2}x^2 - 3$. Mark the intercepts and the turning point.

SOLUTION

Step 1: Examine the standard form of the equation

We notice that $a < 0$. Therefore the graph is a "frown" and has a maximum turning point.

Step 2: Calculate the intercepts

For the y-intercept, let $x = 0$:

$$\begin{aligned} g(x) &= -\frac{1}{2}x^2 - 3 \\ g(0) &= -\frac{1}{2}(0)^2 - 3 \\ &= -3 \end{aligned}$$

This gives the point $(0; -3)$.

For the x-intercepts let $y = 0$:

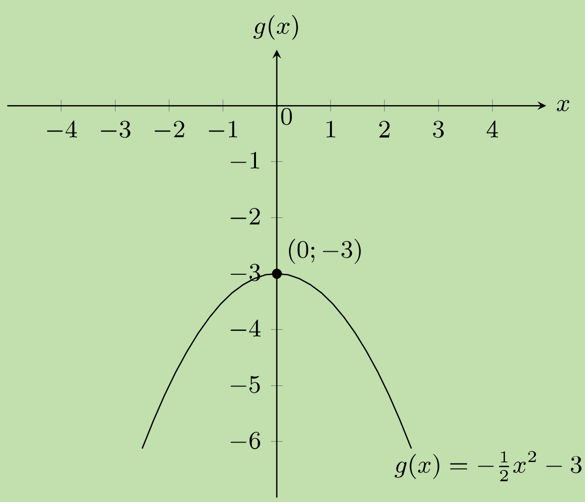
$$\begin{aligned} 0 &= -\frac{1}{2}x^2 - 3 \\ 3 &= -\frac{1}{2}x^2 \\ -2(3) &= x^2 \\ -6 &= x^2 \end{aligned}$$

There is no real solution, therefore there are no x-intercepts.

Step 3: Determine the turning point

From the standard form of the equation we see that the turning point is $(0; -3)$.

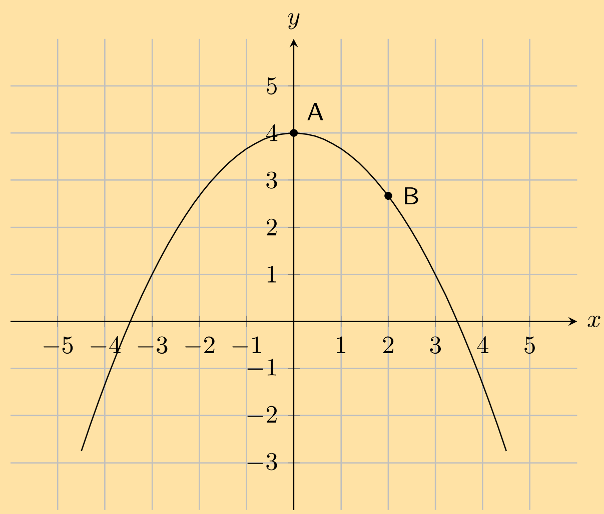
Step 4: Plot the points and sketch the graph



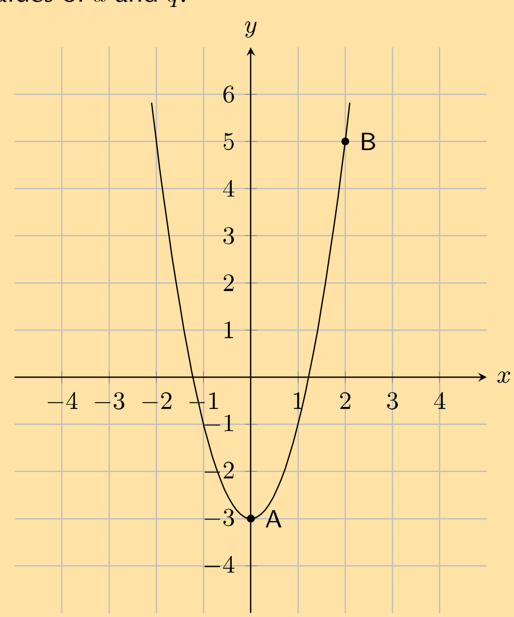
Domain: $x \in \mathbb{R}$. Range: $y \in (-\infty; -3]$. The axis of symmetry is the line $x = 0$.

Exercise 6 – 3:

1. The graph below shows a quadratic function with the following form: $y = ax^2 + q$.
   Two points on the parabola are shown: **Point A**, the turning point of the parabola, at $(0; 4)$, and **Point B** is at $\left(2; \frac{8}{3}\right)$. Calculate the values of a and q.



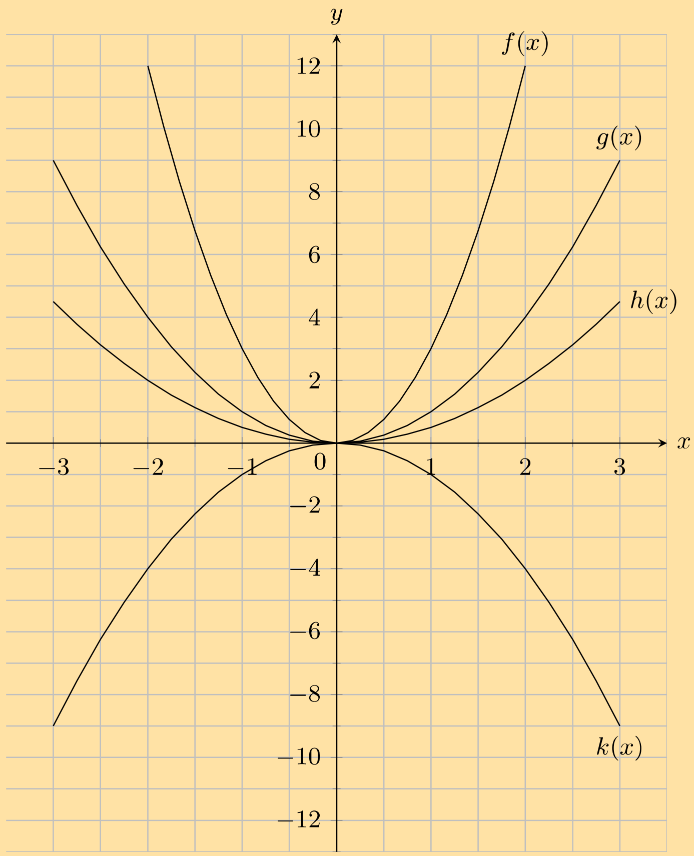
2. The graph below shows a quadratic function with the following form: $y = ax^2 + q$.
   Two points on the parabola are shown: **Point A**, the turning point of the parabola, at $(0; -3)$, and **Point B** is at $(2; 5)$. Calculate the values of a and q.



3. Given the following equation:
   $y = 5x^2 - 2$
   a) Calculate the y-coordinate of the y-intercept.
   b) Now calculate the x-intercepts. Your answer must be correct to 2 decimal places.

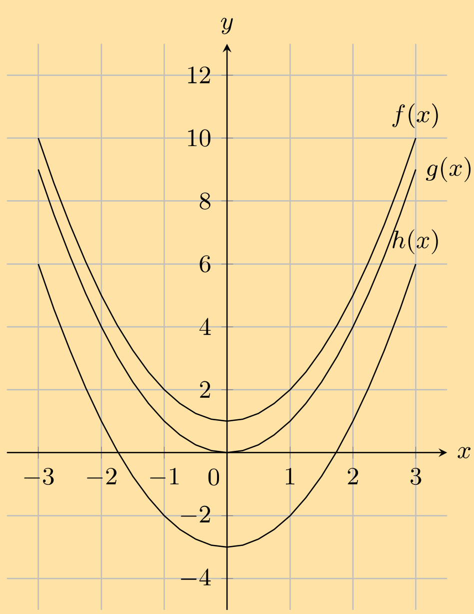
4. Given the following equation:
   $y = -2x^2 + 1$
   a) Calculate the y-coordinate of the y-intercept.
   b) Now calculate the x-intercepts. Your answer must be correct to 2 decimal places.

5. Given the following graph, identify a function that matches each of the following equations:



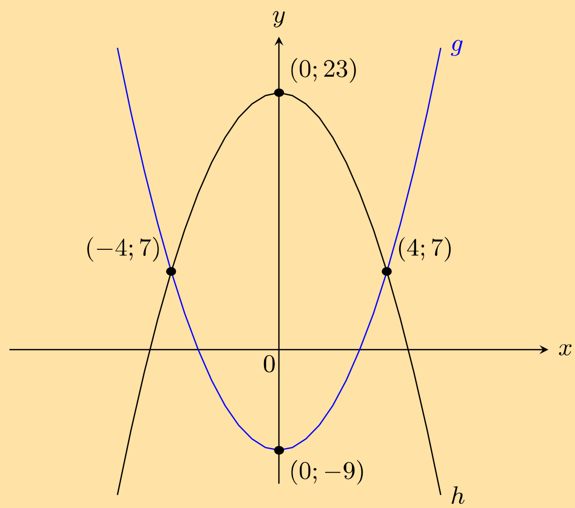
a) $y = 0{,}5x^2$ b) $y = x^2$ c) $y = 3x^2$ d) $y = -x^2$

6. Given the following graph, identify a function that matches each of the following equations:



a) $y = x^2 - 3$ b) $y = x^2 + 1$ c) $y = x^2$

7. Two parabolas are drawn: $g : y = ax^2 + p$ and $h : y = bx^2 + q$.



a) Find the values of a and p.
b) Find the values of b and q.
c) Find the values of x for which $g(x) \geq h(x)$.
d) For what values of x is g increasing?

8. Show that if $a < 0$ the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \leq q\}$.
9. Draw the graph of the function $y = -x^2 + 4$ showing all intercepts with the axes.

For more exercises, visit www.everythingmaths.co.za and click on 'Practise Maths'.

1. 2FY6 2. 2FY7 3. 2FY8 4. 2FY9 5. 2FYB 6. 2FYC 7. 2FYD 8. 2FYF 9. 2FYG

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6.4 Hyperbolic functions EMA4P

Functions of the form $y = \frac{1}{x}$ EMA4Q

Functions of the general form $y = \frac{a}{x} + q$ are called hyperbolic functions.

Worked example 8: Plotting a hyperbolic function

QUESTION

$y = h(x) = \frac{1}{x}$

Complete the following table for $h(x) = \frac{1}{x}$ and plot the points on a system of axes.