Logic: Reasoning with Certainty

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playouts. For example: consider a game with a branching factor of 32, where the average game lasts 100 ply. If we have enough computing power to consider a billion game states before we have to make a move, then minimax can search 6 ply deep, alpha–beta with perfect move ordering can search 12 ply, and Monte Carlo search can do 10 million playouts. Which approach will be better? That depends on the accuracy of the heuristic function versus the selection and playout policies.

The conventional wisdom has been that Monte Carlo search has an advantage over alpha–beta for games like Go where the branching factor is very high (and thus alpha–beta can't search deep enough), or when it is difficult to define a good evaluation function. What alpha–beta does is choose the path to a node that has the highest achievable evaluation function score, given that the opponent will be trying to minimize the score. Thus, if the evaluation function is inaccurate, alpha–beta will be inaccurate. A miscalculation on a single node can lead alpha–beta to erroneously choose (or avoid) a path to that node. But Monte Carlo search relies on the aggregate of many playouts, and thus is not as vulnerable to a single error. It is possible to combine MCTS and evaluation functions by doing a playout for a certain number of moves, but then truncating the playout and applying an evaluation function.

Early playout termination

It is also possible to combine aspects of alpha–beta and Monte Carlo search. For example, in games that can last many moves, we may want to use **early playout termination**, in which we stop a playout that is taking too many moves, and either evaluate it with a heuristic evaluation function or just declare it a draw.

Monte Carlo search can be applied to brand-new games, in which there is no body of experience to draw upon to define an evaluation function. As long as we know the rules of the game, Monte Carlo search does not need any additional information. The selection and playout policies can make good use of hand-crafted expert knowledge when it is available, but good policies can be learned using neural networks trained by self-play alone.

Monte Carlo search has a disadvantage when it is likely that a single move can change the course of the game, because the stochastic nature of Monte Carlo search means it might fail to consider that move. In other words, Type B pruning in Monte Carlo search means that a vital line of play might not be explored at all. Monte Carlo search also has a disadvantage when there are game states that are "obviously" a win for one side or the other (according to human knowledge and to an evaluation function), but where it will still take many moves in a playout to verify the winner. It was long held that alpha–beta search was better suited for games like chess with low branching factor and good evaluation functions, but recently Monte Carlo approaches have demonstrated success in chess and other games.

The general idea of simulating moves into the future, observing the outcome, and using the outcome to determine which moves are good ones is one kind of **reinforcement learning**, which is covered in Chapter 23.

6.5 Stochastic Games

Stochastic game

Stochastic games bring us a little closer to the unpredictability of real life by including a random element, such as the throwing of dice. Backgammon is a typical stochastic game that combines luck and skill. In the backgammon position of Figure 6.12, Black has rolled a 6–5 and has four possible moves (each of which moves one piece forward (clockwise) 5 positions, and one piece forward 6 positions).

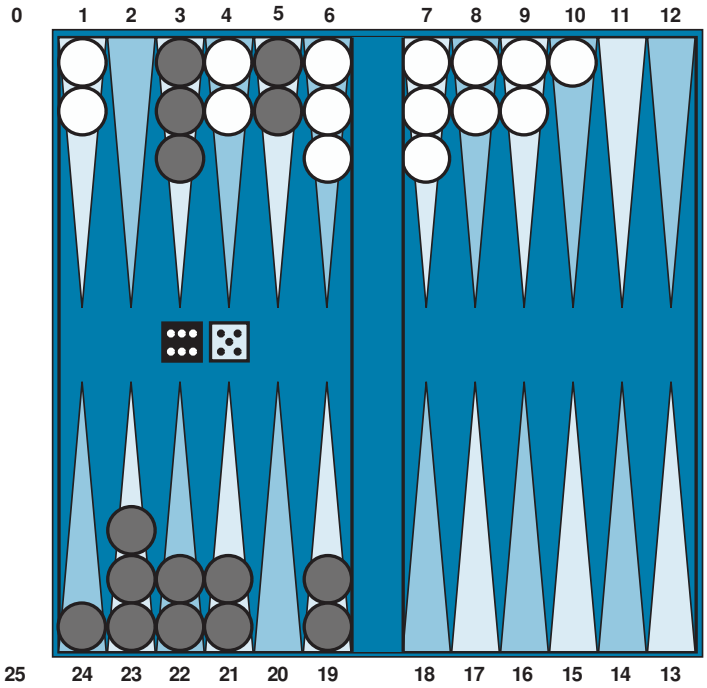


Figure 6.12 A typical backgammon position. The goal of the game is to move all one's pieces off the board. Black moves clockwise toward 25, and White moves counterclockwise toward 0. A piece can move to any position unless multiple opponent pieces are there; if there is one opponent, it is captured and must start over. In the position shown, Black has rolled 6–5 and must choose among four legal moves: (5–11,5–10), (5–11,19–24), (5–10,10–16), and (5–11,11–16), where the notation (5–11,11–16) means move one piece from position 5 to 11, and then move a piece from 11 to 16.

At this point Black knows what moves can be made, but does not know what White is going to roll and thus does not know what White's legal moves will be. That means Black cannot construct a standard game tree of the sort we saw in chess and tic-tac-toe. A game tree in backgammon must include **chance nodes** in addition to MAX and MIN nodes. Chance nodes are shown as circles in Figure 6.13. The branches leading from each chance node denote the possible dice rolls; each branch is labeled with the roll and its probability. There are 36 ways to roll two dice, each equally likely; but because a 6–5 is the same as a 5–6, there are only 21 distinct rolls. The six doubles (1–1 through 6–6) each have a probability of 1/36, so we say $P(1\text{–}1) = 1/36$. The other 15 distinct rolls each have a 1/18 probability.

Chance nodes

The next step is to understand how to make correct decisions. Obviously, we still want to pick the move that leads to the best position. However, positions do not have definite minimax values. Instead, we can only calculate the **expected value** of a position: the average over all possible outcomes of the chance nodes.

Expected value

This leads us to the **expectiminimax value** for games with chance nodes, a generalization of the minimax value for deterministic games. Terminal nodes and MAX and MIN nodes work exactly the same way as before (with the caveat that the legal moves for MAX and MIN will depend on the outcome of the dice roll in the previous chance node). For chance nodes we

Expectiminimax value

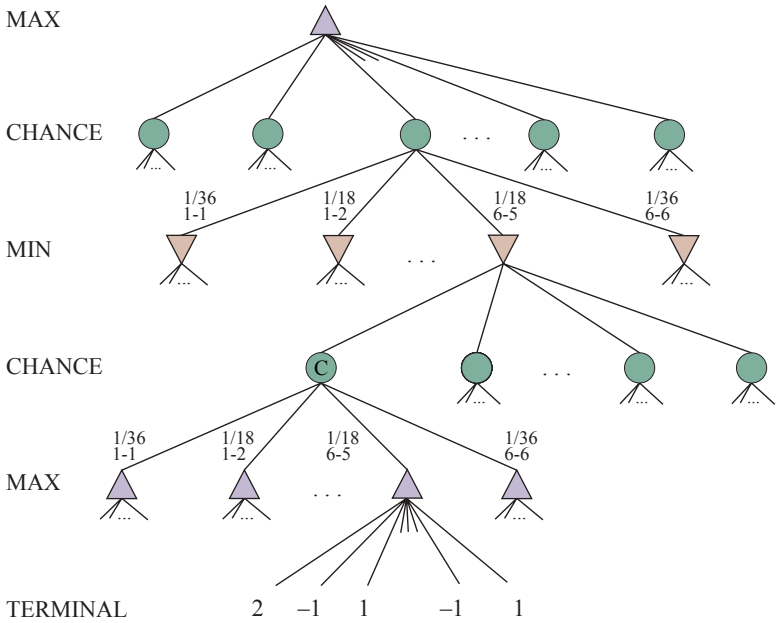


Figure 6.13 Schematic game tree for a backgammon position.

compute the expected value, which is the sum of the value over all outcomes, weighted by the probability of each chance action:

$$\text{EXPECTIMINIMAX}(s) = \begin{cases} \text{UTILITY}(s, \text{MAX}) & \text{if IS-TERMINAL}(s) \\ \max_a \text{EXPECTIMINIMAX}(\text{RESULT}(s,a)) & \text{if TO-MOVE}(s) = \text{MAX} \\ \min_a \text{EXPECTIMINIMAX}(\text{RESULT}(s,a)) & \text{if TO-MOVE}(s) = \text{MIN} \\ \sum_r P(r)\text{EXPECTIMINIMAX}(\text{RESULT}(s,r)) & \text{if TO-MOVE}(s) = \text{CHANCE} \end{cases}$$

where r represents a possible dice roll (or other chance event) and $\text{RESULT}(s, r)$ is the same state as s, with the additional fact that the result of the dice roll is r.

6.5.1 Evaluation functions for games of chance

As with minimax, the obvious approximation to make with expectiminimax is to cut the search off at some point and apply an evaluation function to each leaf. One might think that evaluation functions for games such as backgammon should be just like evaluation functions for chess—they just need to give higher values to better positions. But in fact, the presence of chance nodes means that one has to be more careful about what the values mean.

Figure 6.14 shows what happens: with an evaluation function that assigns the values [1, 2, 3, 4] to the leaves, move a_1 is best; with values [1, 20, 30, 400], move a_2 is best. Hence, the program behaves totally differently if we make a change to some of the evaluation values, even if the preference order remains the same.

It turns out that to avoid this problem, the evaluation function must return values that are a positive linear transformation of the **probability** of winning (or of the expected utility, for games that have outcomes other than win/lose). This relation to probability is an important

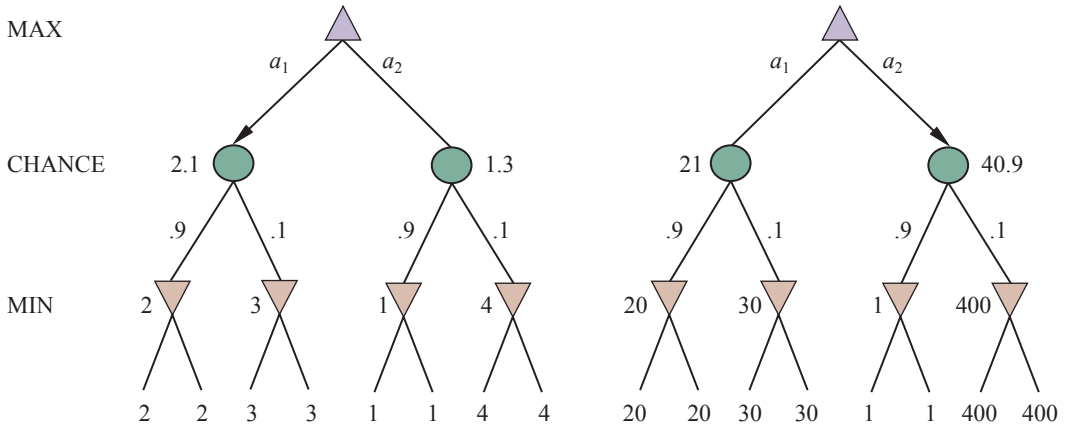


Figure 6.14 An order-preserving transformation on leaf values changes the best move.

and general property of situations in which uncertainty is involved, and we discuss it further in Chapter 15.

If the program knew in advance all the dice rolls that would occur for the rest of the game, solving a game with dice would be just like solving a game without dice, which minimax does in $O(b^m)$ time, where b is the branching factor and m is the maximum depth of the game tree. Because expectiminimax is also considering all the possible dice-roll sequences, it will take $O(b^m n^m)$, where n is the number of distinct rolls.

Even if the search is limited to some small depth d, the extra cost compared with that of minimax makes it unrealistic to consider looking ahead very far in most games of chance. In backgammon n is 21 and b is usually around 20, but in some situations can be as high as 4000 for dice rolls that are doubles. We could probably only manage three ply of search.

Another way to think about the problem is this: the advantage of alpha–beta is that it ignores future developments that just are not going to happen, given best play. Thus, it concentrates on likely occurrences. But in a game where a throw of two dice precedes each move, there are *no* likely sequences of moves; even the most likely move occurs only 2/36 of the time, because for the move to take place, the dice would first have to come out the right way to make it legal. This is a general problem whenever uncertainty enters the picture: the possibilities are multiplied enormously, and forming detailed plans of action becomes pointless because the world probably will not play along.

It may have occurred to you that something like alpha–beta pruning could be applied to game trees with chance nodes. It turns out that it can. The analysis for MIN and MAX nodes is unchanged, but we can also prune chance nodes, using a bit of ingenuity. Consider the chance node C in Figure 6.13 and what happens to its value as we evaluate its children. Is it possible to find an upper bound on the value of C before we have looked at all its children? (Recall that this is what alpha–beta needs in order to prune a node and its subtree.)

At first sight, it might seem impossible because the value of C is the *average* of its children's values, and in order to compute the average of a set of numbers, we must look at all the numbers. But if we put bounds on the possible values of the utility function, then we can

arrive at bounds for the average without looking at every number. For example, say that all utility values are between -2 and $+2$; then the value of leaf nodes is bounded, and in turn we *can* place an upper bound on the value of a chance node without looking at all its children.

In games where the branching factor for chance nodes is high—consider a game like Yahtzee where you roll 5 dice on every turn—you may want to consider forward pruning that samples a smaller number of the possible chance branches. Or you may want to avoid using an evaluation function altogether, and opt for Monte Carlo tree search instead, where each playout includes random dice rolls.

6.6 Partially Observable Games

Bobby Fischer declared that "chess is war," but chess lacks at least one major characteristic of real wars, namely, **partial observability**. In the "fog of war," the whereabouts of enemy units is often unknown until revealed by direct contact. As a result, warfare includes the use of scouts and spies to gather information and the use of concealment and bluff to confuse the enemy.

Partially observable games share these characteristics and are thus qualitatively different from the games in the preceding sections. Video games such as StarCraft are particularly challenging, being partially observable, multi-agent, nondeterministic, dynamic, and unknown.

In *deterministic* partially observable games, uncertainty about the state of the board arises entirely from lack of access to the choices made by the opponent. This class includes children's games such as Battleship (where each player's ships are placed in locations hidden from the opponent) and Stratego (where piece locations are known but piece types are hidden). We will examine the game of **Kriegspiel**, a partially observable variant of chess in which pieces are completely invisible to the opponent. Other games also have partially observable versions: Phantom Go, Phantom tic-tac-toe, and Screen Shogi.

Kriegspiel

6.6.1 Kriegspiel: Partially observable chess

The rules of Kriegspiel are as follows: White and Black each see a board containing only their own pieces. A referee, who can see all the pieces, adjudicates the game and periodically makes announcements that are heard by both players. First, White proposes to the referee a move that would be legal if there were no black pieces. If the black pieces prevent the move, the referee announces "illegal," and White keeps proposing moves until a legal one is found—learning more about the location of Black's pieces in the process.

Once a legal move is proposed, the referee announces one or more of the following: "Capture on square X" if there is a capture, and "Check by D" if the black king is in check, where D is the direction of the check, and can be one of "Knight," "Rank," "File," "Long diagonal," or "Short diagonal." If Black is checkmated or stalemated, the referee says so; otherwise, it is Black's turn to move.

Kriegspiel may seem terrifyingly impossible, but humans manage it quite well and computer programs are beginning to catch up. It helps to recall the notion of a **belief state** as defined in Section 4.4 and illustrated in Figure 4.14—the set of all *logically possible* board states given the complete history of percepts to date. Initially, White's belief state is a singleton because Black's pieces haven't moved yet. After White makes a move and Black responds, White's belief state contains 20 positions, because Black has 20 replies to any

opening move. Keeping track of the belief state as the game progresses is exactly the problem of **state estimation**, for which the update step is given in Equation (4.6) on page 150. We can map Kriegspiel state estimation directly onto the partially observable, nondeterministic framework of Section 4.4 if we consider the opponent as the source of nondeterminism; that is, the RESULTS of White's move are composed from the (predictable) outcome of White's own move and the unpredictable outcome given by Black's reply.[4]

Given a current belief state, White may ask, "Can I win the game?" For a partially observable game, the notion of a **strategy** is altered; instead of specifying a move to make for each possible *move* the opponent might make, we need a move for every possible *percept sequence* that might be received.

Guaranteed checkmate

For Kriegspiel, a winning strategy, or **guaranteed checkmate**, is one that, for each possible percept sequence, leads to an actual checkmate for every possible board state in the current belief state, regardless of how the opponent moves. With this definition, the opponent's belief state is irrelevant—the strategy has to work even if the opponent can see all the pieces. This greatly simplifies the computation. Figure 6.15 shows part of a guaranteed checkmate for the KRK (king and rook versus king) endgame. In this case, Black has just one piece (the king), so a belief state for White can be shown in a single board by marking each possible position of the Black king.

The general AND-OR search algorithm can be applied to the belief-state space to find guaranteed checkmates, just as in Section 4.4. The incremental belief-state algorithm mentioned in Section 4.4.2 often finds midgame checkmates up to depth 9—well beyond the abilities of most human players.

Probabilistic checkmate

In addition to guaranteed checkmates, Kriegspiel admits an entirely new concept that makes no sense in fully observable games: **probabilistic checkmate**. Such checkmates are still required to work in every board state in the belief state; they are probabilistic with respect to randomization of the winning player's moves. To get the basic idea, consider the problem of finding a lone black king using just the white king. Simply by moving randomly, the white king will *eventually* bump into the black king even if the latter tries to avoid this fate, since Black cannot keep guessing the right evasive moves indefinitely. In the terminology of probability theory, detection occurs *with probability* 1.

The KBNK endgame—king, bishop and knight versus king—is won in this sense; White presents Black with an infinite random sequence of choices, for one of which Black will guess incorrectly and reveal his position, leading to checkmate. On the other hand, the KBBK endgame is won with probability $1 - \epsilon$. White can force a win only by leaving one of his bishops unprotected for one move. If Black happens to be in the right place and captures the bishop (a move that would be illegal if the bishops are protected), the game is drawn. White can choose to make the risky move at some randomly chosen point in the middle of a very long sequence, thus reducing ϵ to an arbitrarily small constant, but cannot reduce ϵ to zero.

Accidental checkmate

Sometimes a checkmate strategy works for *some* of the board states in the current belief state but not others. Trying such a strategy may succeed, leading to an **accidental checkmate**—accidental in the sense that White could not *know* that it would be checkmate—if Black's pieces happen to be in the right places. (Most checkmates in games between humans

[4] Sometimes, the belief state will become too large to represent just as a list of board states, but we will ignore this issue for now; Chapters 7 and 8 suggest methods for compactly representing very large belief states.

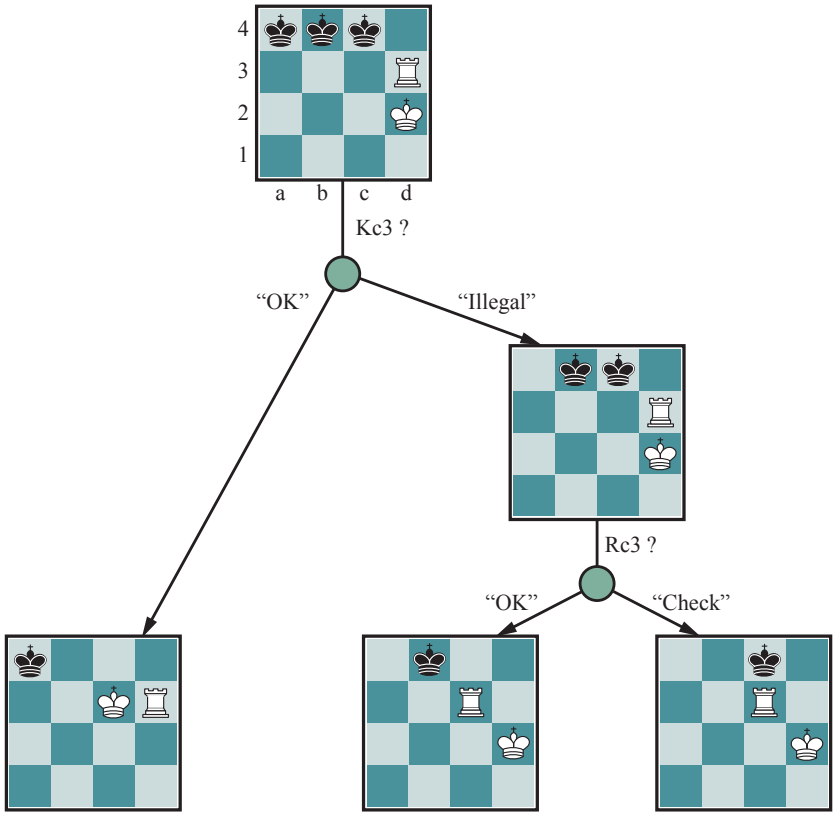


Figure 6.15 Part of a guaranteed checkmate in the KRK endgame, shown on a reduced board. In the initial belief state, Black's king is in one of three possible locations. By a combination of probing moves, the strategy narrows this down to one. Completion of the checkmate is left as an exercise.

are of this accidental nature.) This idea leads naturally to the question of *how likely* it is that a given strategy will win, which leads in turn to the question of *how likely* it is that each board state in the current belief state is the true board state.

One's first inclination might be to propose that all board states in the current belief state are equally likely—but this can't be right. Consider, for example, White's belief state after Black's first move of the game. By definition (assuming that Black plays optimally), Black must have played an optimal move, so all board states resulting from suboptimal moves ought to be assigned zero probability.

This argument is not quite right either, because *each player's goal is not just to move pieces to the right squares but also to minimize the information that the opponent has about their location.* Playing any *predictable* "optimal" strategy provides the opponent with information. Hence, optimal play in partially observable games requires a willingness to play somewhat *randomly*. (This is why restaurant hygiene inspectors do *random* inspection visits.) This means occasionally selecting moves that may seem "intrinsically" weak—but they gain strength from their very unpredictability, because the opponent is unlikely to have prepared any defense against them.

From these considerations, it seems that the probabilities associated with the board states in the current belief state can only be calculated given an optimal randomized strategy; in turn, computing that strategy seems to require knowing the probabilities of the various states the board might be in. This conundrum can be resolved by adopting the game-theoretic notion of an **equilibrium** solution, which we pursue further in Chapter 16. An equilibrium specifies an optimal randomized strategy for each player. Computing equilibria is too expensive for Kriegspiel. At present, the design of effective algorithms for general Kriegspiel play is an open research topic. Most systems perform bounded-depth look-ahead in their own belief-state space, ignoring the opponent's belief state. Evaluation functions resemble those for the observable game but include a component for the size of the belief state—smaller is better! We will return to partially observable games under the topic of Game Theory in Section 17.2.

6.6.2 Card games

Card games such as bridge, whist, hearts, and poker feature *stochastic* partial observability, where the missing information is generated by the random dealing of cards.

At first sight, it might seem that these card games are just like dice games: the cards are dealt randomly and determine the moves available to each player, but all the "dice" are rolled at the beginning! Even though this analogy turns out to be incorrect, it suggests an algorithm: treat the start of the game as a chance node with every possible deal as an outcome, and then use the EXPECTIMINIMAX formula to pick the best move. Note that in this approach the only chance node is the root node; after that the game becomes fully observable. This approach is sometimes called *averaging over clairvoyance* because it assumes that once the actual deal has occurred, the game becomes fully observable to both players. Despite its intuitive appeal, the strategy can lead one astray. Consider the following story:

> Day 1: Road A leads to a pot of gold; Road B leads to a fork. You can see that the left fork leads to two pots of gold, and the right fork leads to you being run over by a bus.
>
> Day 2: Road A leads to a pot of gold; Road B leads to a fork. You can see that the right fork leads to two pots of gold, and the left fork leads to you being run over by a bus.
>
> Day 3: Road A leads to a pot of gold; Road B leads to a fork. You are told that one fork leads to two pots of gold, and one fork leads to you being run over by a bus. Unfortunately you don't know which fork is which.

Averaging over clairvoyance leads to the following reasoning: on Day 1, B is the right choice; on Day 2, B is the right choice; on Day 3, the situation is the same as either Day 1 or Day 2, so B must still be the right choice.

Now we can see how averaging over clairvoyance fails: it does not consider the *belief state* that the agent will be in after acting. A belief state of total ignorance is not desirable, especially when one possibility is certain death. Because it assumes that every future state will automatically be one of perfect knowledge, the clairvoyance approach never selects actions that *gather information* (like the first move in Figure 6.15); nor will it choose actions that hide information from the opponent or provide information to a partner, because it assumes that they already know the information; and it will never **bluff** in poker,[5] because it assumes the opponent can see its cards. In Chapter 16, we show how to construct algorithms that do

Bluff

[5] Bluffing—betting as if one's hand is good, even when it's not—is a core part of poker strategy.

all these things by virtue of solving the true partially observable decision problem, resulting in an optimal equilibrium strategy (see Section 17.2).

Despite the drawbacks, averaging over clairvoyance can be an effective strategy, with some tricks to make it work better. In most card games, the number of possible deals is rather large. For example, in bridge play, each player sees just two of the four hands; there are two unseen hands of 13 cards each, so the number of deals is $\binom{26}{13} = 10{,}400{,}600$. Solving even one deal is quite difficult, so solving ten million is out of the question. One way to deal with this huge number is with **abstraction**: i.e. by treating similar hands as identical. For example, it is very important which aces and kings are in a hand, but whether the hand has a 4 or 5 is not as important, and can be abstracted away.

Another way to deal with the large number is forward pruning: consider only a small random sample of N deals, and again calculate the EXPECTIMINIMAX score. Even for fairly small N—say, 100 to 1,000—this method gives a good approximation. It can also be applied to deterministic games such as Kriegspiel, where we sample over possible states of the game rather than over possible deals, as long as we have some way to estimate how likely each state is. It can also be helpful to do heuristic search with a depth cutoff rather than to search the entire game tree.

So far we have assumed that each deal is equally likely. That makes sense for games like whist and hearts. But for bridge, play is preceded by a bidding phase in which each team indicates how many tricks it expects to win. Since players bid based on the cards they hold, the other players learn something about the probability $P(s)$ of each deal. Taking this into account in deciding how to play the hand is tricky, for the reasons mentioned in our description of Kriegspiel: players may bid in such a way as to minimize the information conveyed to their opponents.

Computers have reached a superhuman level of performance in poker. The poker program Libratus took on four of the top poker players in the world in a 20-day match of no-limit Texas hold 'em and decisively beat them all. Since there are so many possible states in poker, Libratus uses abstraction to reduce the state space: it might consider the two hands AAA72 and AAA64 to be equivalent (they're both "three aces and some low cards"), and it might consider a bet of 200 dollars to be the same as 201 dollars. But Libratus also monitors the other players, and if it detects they are exploiting an abstraction, it will do some additional computation overnight to plug that hole. Overall it used 25 million CPU hours on a supercomputer to pull off the win.

The computational costs incurred by Libratus (and similar costs by ALPHAZERO and other systems) suggests that world champion game play may not be achievable for researchers with limited budgets. To some extent that is true: just as you should not expect to be able to assemble a champion Formula One race car out of spare parts in your garage, there is an advantage to having access to supercomputers or specialty hardware such as Tensor Processing Units. That is particularly true when training a system, but training could also be done via crowdsourcing. For example the open-source LEELAZERO system is a reimplementation of ALPHAZERO that trains through self-play on the computers of volunteer participants. Once trained, the computational requirements for actual tournament play are modest. ALPHASTAR won StarCraft II games running on a commodity desktop with a single GPU, and ALPHAZERO could have been run in that mode.

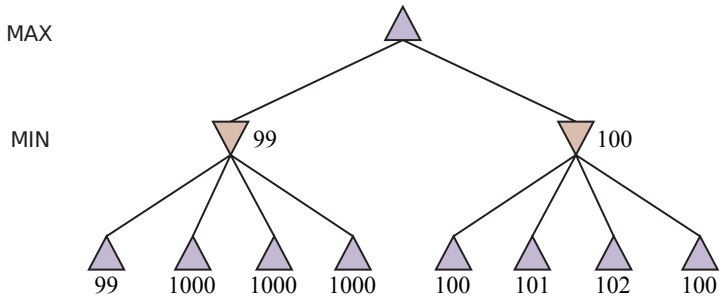


Figure 6.16 A two-ply game tree for which heuristic minimax may make an error.

6.7 Limitations of Game Search Algorithms

Because calculating optimal decisions in complex games is intractable, all algorithms must make some assumptions and approximations. Alpha–beta search uses the heuristic evaluation function as an approximation, and Monte Carlo search computes an approximate average over a random selection of playouts. The choice of which algorithm to use depends in part on the features of each game: when the branching factor is high or it is difficult to define an evaluation function, Monte Carlo search is preferred. But both algorithms suffer from fundamental limitations.

One limitation of alpha–beta search is its vulnerability to errors in the heuristic function. Figure 6.16 shows a two-ply game tree for which minimax suggests taking the right-hand branch because $100 > 99$. That is the correct move if the evaluations are all exactly accurate. But suppose that the evaluation of each node has an error that is independent of other nodes and is randomly distributed with a standard deviation of σ. Then the left-hand branch is actually better 71% of the time when $\sigma = 5$, and 58% of the time when $\sigma = 2$ (because one of the four right-hand leaves is likely to slip below 99 in these cases). If errors in the evaluation function are *not* independent, then the chance of a mistake rises. It is difficult to compensate for this because we don't have a good model of the dependencies between the values of sibling nodes.

A second limitation of both alpha–beta and Monte Carlo is that they are designed to calculate (bounds on) the values of legal moves. But sometimes there is one move that is obviously best (for example when there is only one legal move), and in that case, there is no point wasting computation time to figure out the value of the move—it is better to just make the move. A better search algorithm would use the idea of the *utility of a node expansion*, selecting node expansions of high utility—that is, ones that are likely to lead to the discovery of a significantly better move. If there are no node expansions whose utility is higher than their cost (in terms of time), then the algorithm should stop searching and make a move. This works not only for clear-favorite situations but also for the case of *symmetrical* moves, for which no amount of search will show that one move is better than another.

Metareasoning

This kind of reasoning about what computations to do is called **metareasoning** (reasoning about reasoning). It applies not just to game playing but to any kind of reasoning at all. All computations are done in the service of trying to reach better decisions, all have costs, and all have some likelihood of resulting in a certain improvement in decision quality. Monte

Carlo search does attempt to do metareasoning to allocate resources to the most important parts of the tree, but does not do so in an optimal way.

A third limitation is that both alpha-beta and Monte Carlo do all their reasoning at the level of individual moves. Clearly, humans play games differently: they can reason at a more abstract level, considering a higher-level goal—for example, trapping the opponent's queen—and using the goal to *selectively* generate plausible plans. In Chapter 11 we will study this type of **planning**, and in Section 11.4 we will show how to plan with a hierarchy of abstract to concrete representations.

A fourth issue is the ability to incorporate **machine learning** into the game search process. Early game programs relied on human expertise to hand-craft evaluation functions, opening books, search strategies, and efficiency tricks. We are just beginning to see programs like ALPHAZERO (Silver *et al.*, 2018), which relied on machine learning from self-play rather than game-specific human-generated expertise. We cover machine learning in depth starting with Chapter 19.

Summary

We have looked at a variety of games to understand what optimal play means, to understand how to play well in practice, and to get a feel for how an agent should act in any type of adversarial environment. The most important ideas are as follows:

- A game can be defined by the **initial state** (how the board is set up), the legal **actions** in each state, the **result** of each action, a **terminal test** (which says when the game is over), and a **utility function** that applies to terminal states to say who won and what the final score is.
- In two-player, discrete, deterministic, turn-taking zero-sum games with **perfect information**, the **minimax** algorithm can select optimal moves by a depth-first enumeration of the game tree.
- The **alpha–beta** search algorithm computes the same optimal move as minimax, but achieves much greater efficiency by eliminating subtrees that are provably irrelevant.
- Usually, it is not feasible to consider the whole game tree (even with alpha–beta), so we need to cut the search off at some point and apply a heuristic **evaluation function** that estimates the utility of a state.
- An alternative called **Monte Carlo tree search** (MCTS) evaluates states not by applying a heuristic function, but by playing out the game all the way to the end and using the rules of the game to see who won. Since the moves chosen during the **playout** may not have been optimal moves, the process is repeated multiple times and the evaluation is an average of the results.
- Many game programs precompute tables of best moves in the opening and endgame so that they can look up a move rather than search.
- Games of chance can be handled by **expectiminimax**, an extension to the minimax algorithm that evaluates a **chance node** by taking the average utility of all its children, weighted by the probability of each child.
- In games of **imperfect information**, such as Kriegspiel and poker, optimal play requires reasoning about the current and future **belief states** of each player. A simple

approximation can be obtained by averaging the value of an action over each possible configuration of missing information.

- Programs have soundly defeated champion human players at chess, checkers, Othello, Go, poker, and many other games. Humans retain the edge in a few games of imperfect information, such as bridge and Kriegspiel. In video games such as StarCraft and Dota 2, programs are competitive with human experts, but part of their success may be due to their ability to perform many actions very quickly.

Bibliographical and Historical Notes

In 1846, Charles Babbage discussed the feasibility of computer chess and checkers (Morrison and Morrison, 1961). He did not understand the exponential complexity of search trees, claiming "the combinations involved in the Analytical Engine enormously surpassed any required, even by the game of chess." Babbage also designed, but did not build, a special-purpose machine for playing tic-tac-toe. The first game-playing machine was built around 1890 by the Spanish engineer Leonardo Torres y Quevedo. It specialized in the "KRK" (king and rook versus king) chess endgame, guaranteeing a win when the side with the rook has the move. The **minimax** algorithm is traced to a 1912 paper by Ernst Zermelo, the developer of modern set theory.

Game playing was one of the first tasks undertaken in AI, with early efforts by such pioneers as Konrad Zuse (1945), Norbert Wiener in his book *Cybernetics* (1948), and Alan Turing (1953). But it was Claude Shannon's article *Programming a Computer for Playing Chess* (1950) that laid out all the major ideas: a representation for board positions, an evaluation function, quiescence search, and some ideas for selective game-tree search. Slater (1950) had the idea of an evaluation function as a linear combination of features, and stressed the mobility feature in chess.

John McCarthy conceived the idea of **alpha–beta** search in 1956, although the idea did not appear in print until later (Hart and Edwards, 1961). Knuth and Moore (1975) proved the correctness of alpha–beta and analysed its time complexity, while Pearl (1982b) showed alpha–beta to be asymptotically optimal among all fixed-depth game-tree search algorithms.

Berliner (1979) introduced B*, a heuristic search algorithm that maintains interval bounds on the possible value of a node in the game tree rather than giving it a single point-valued estimate. David McAllester's (1988) conspiracy number search expands leaf nodes that, by changing their values, could cause the program to prefer a new move at the root of the tree. MGSS* (Russell and Wefald, 1989) uses the decision-theoretic techniques of Chapter 15 to estimate the value of expanding each leaf in terms of the expected improvement in decision quality at the root.

The SSS* algorithm (Stockman, 1979) can be viewed as a two-player A* that never expands more nodes than alpha–beta. The memory requirements make it impractical, but a linear-space version has been developed from the RBFS algorithm (Korf and Chickering, 1996). Baum and Smith (1997) propose a probability-based replacement for minimax, showing that it results in better choices in certain games. The **expectiminimax** algorithm was proposed by Donald Michie (1966). Bruce Ballard (1983) extended alpha–beta pruning to cover trees with chance nodes.

Pearl's book *Heuristics* (1984) thoroughly analyzes many game-playing algorithms.

Monte Carlo simulation was pioneered by Metropolis and Ulam (1949) for calculations related to the development of the atomic bomb. Monte Carlo tree search (MCTS) was introduced by Abramson (1987). Tesauro and Galperin (1997) showed how a Monte Carlo search could be combined with an evaluation function for the game of backgammon. Early playout termination is studied by Lorentz (2015). ALPHAGO terminated playouts and applied an evaluation function (Silver *et al.*, 2016). Kocsis and Szepesvari (2006) refined the approach with the "Upper Confidence Bounds applied to Trees" selection mechanism. Chaslot *et al.* (2008) show how MCTS can be applied to a variety of games and Browne *et al.* (2012) give a survey.

Koller and Pfeffer (1997) describe a system for completely solving **partially observable** games. It handles larger games than previous systems, but not the full version of complex games like poker and bridge. Frank *et al.* (1998) describe several variants of Monte Carlo search for partially observable games, including one where MIN has complete information but MAX does not. Schofield and Thielscher (2015) adapt a general game-playing system for partially observable games.

Ferguson hand-derived randomized strategies for winning Kriegspiel with a bishop and knight (1992) or two bishops (1995) against a king. The first Kriegspiel programs concentrated on finding endgame checkmates and performed AND–OR search in belief-state space (Sakuta and Iida, 2002; Bolognesi and Ciancarini, 2003). Incremental belief-state algorithms enabled much more complex midgame checkmates to be found (Russell and Wolfe, 2005; Wolfe and Russell, 2007), but efficient state estimation remains the primary obstacle to effective general play (Parker *et al.*, 2005). Ciancarini and Favini (2010) apply MCTS to Kriegspiel, and Wang *et al.* (2018b) describe a belief-state version of MCTS for Phantom Go.

Chess milestones have been marked by successive winners of the Fredkin Prize: BELLE (Condon and Thompson, 1982), the first program to achieve master status; DEEP THOUGHT (Hsu *et al.*, 1990), the first to reach international master status; and Deep Blue (Campbell *et al.*, 2002; Hsu, 2004), which defeated world champion Garry Kasparov in a 1997 exhibition match. Deep Blue ran alpha–beta search at over 100 million positions per second, and could generate singular extensions to occasionally reach a depth of 40 ply.

The top chess programs today (e.g., STOCKFISH, KOMODO, HOUDINI) far exceed any human player. These programs have reduced the effective branching factor to less than 3 (compared with the actual branching factor of about 35), searching to about 20 ply at a speed of about a million nodes per second on a standard 1-core computer. They use pruning techniques such as the **null move** heuristic, which generates a good lower bound on the value of a position, using a shallow search in which the opponent gets to move twice at the beginning. Also important is **futility pruning**, which helps decide in advance which moves will cause a beta cutoff in the successor nodes. SUNFISH is a simplified chess program for teaching purposes; the core is less than 200 lines of Python.

Null move

Futility pruning

The idea of retrograde analysis for computing endgame tables is due to Bellman (1965). Using this idea, Ken Thompson (1986, 1996) and Lewis Stiller (1992, 1996) solved all chess endgames with up to five pieces. Stiller discovered one case where a forced mate existed but required 262 moves; this caused some consternation because the rules of chess require a capture or pawn move to occur within 50 moves, or else a draw is declared. In 2012 Vladimir Makhnychev and Victor Zakharov compiled the Lomonosov Endgame Tablebase,

which solved all endgame positions with up to seven pieces—some require over 500 moves without a capture. The 7-piece table consumes 140 terabytes; an 8-piece table would be 100 times larger.

In 2017, ALPHAZERO (Silver *et al.*, 2018) defeated STOCKFISH (the 2017 TCEC computer chess champion) in a 1000-game trial, with 155 wins and 6 losses. Additional matches also resulted in decisive wins for ALPHAZERO, even when it was given only 1/10th the time allotted to STOCKFISH.

Grandmaster Larry Kaufman was surprised at the sucess of this Monte Carlo program and noted, "It may well be that the current dominance of minimax chess engines may be at an end, but it's too soon to say so." Garry Kasparov commented "It's a remarkable achievement, even if we should have expected it after ALPHAGO. It approaches the Type B human-like approach to machine chess dreamt of by Claude Shannon and Alan Turing instead of brute force." He went on to predict "Chess has been shaken to its roots by ALPHAZERO, but this is only a tiny example of what is to come. Hidebound disciplines like education and medicine will also be shaken" (Sadler and Regan, 2019).

Checkers was the first of the classic games played by a computer (Strachey, 1952). Arthur Samuel (1959, 1967) developed a checkers program that learned its own evaluation function through self-play using a form of reinforcement learning. It is quite an achievement that Samuel was able to create a program that played better than he did, on an IBM 704 computer with only 10,000 words of memory and a 0.000001 GHz processor. MENACE—the Machine Educable Noughts And Crosses Engine (Michie, 1963)—also used reinforcement learning to become competent at tic-tac-toe. Its processor was even slower: a collection of 304 matchboxes holding colored beads to represent the best learned move in each position.

In 1992, Jonathan Schaeffer's CHINOOK checkers program challenged the legendary Marion Tinsley, who had been world champion for over 20 years. Tinsley won the match, but lost two games—the fourth and fifth losses in his entire career. After Tinsley retired for health reasons, CHINOOK took the crown. The saga was chronicled by Schaeffer (2008).

In 2007 Schaeffer and his team "solved" checkers (Schaeffer *et al.*, 2007): the game is a draw with perfect play. Richard Bellman (1965) had predicted this: "In checkers, the number of possible moves in any given situation is so small that we can confidently expect a complete digital computer solution to the problem of optimal play in this game." Bellman did not anticipate the scale of the effort: the endgame table for 10 pieces has 39 trillion entries. Given this table, it took 18 CPU-years of alpha–beta search to solve the game.

I. J. Good, who was taught the Game of **Go** by Alan Turing, wrote (1965a) " I think it will be even more difficult to programme a computer to play a reasonable game of Go than of chess." He was right: through 2015, Go programs played only at an amateur level. The early literature is summarized by Bouzy and Cazenave (2001) and Müller (2002).

Visual pattern recognition was proposed as a promising technique for Go by Zobrist (1970), while Schraudolph *et al.* (1994) analyzed the use of reinforcement learning, Lubberts and Miikkulainen (2001) recommended neural networks, and Brügmann (1993) introduced Monte Carlo tree search to Go. ALPHAGO (Silver *et al.*, 2016) put those four ideas together to defeat top-ranked professionals Lee Sedol (by a score of 4–1 in 2015) and Ke Jie (by 3–0 in 2016).

Ke Jie remarked "After humanity spent thousands of years improving our tactics, computers tell us that humans are completely wrong. I would go as far as to say not a single human

has touched the edge of the truth of Go." Lee Sedol retired from Go, lamenting, "Even if I became the number one, there is an entity that cannot be defeated."

In 2018, ALPHAZERO surpassed ALPHAGO at Go, and also defeated top programs in chess and shogi, learning through self-play without any expert human knowledge and without access to any past games. (It does, of course, rely on humans to define the basic architecture as Monte Carlo tree search with deep neural networks and reinforcement learning, and to encode the rules of the game.) The success of ALPHAZERO has led to increased interest in reinforcement learning as a key component of general AI (see Chapter 23). Going one step further, the MUZERO system operates without even being told the rules of the game it is playing—it has to figure out the rules by making plays. MUZERO achieved state-of-the-art results in Pacman, chess, Go, and 75 Atari games (Schrittwieser *et al.*, 2019). It learns to generalize; for example, it learns that in Pacman the "up" action moves the player up a square (unless there is a wall there), even though it has only observed the result of the "up" action in a small percentage of the locations on the board.

Othello, also called Reversi, has a smaller search space than chess, but defining an evaluation function is difficult, because material advantage is not as important as mobility. Programs have been at superhuman level since 1997 (Buro, 2002).

Backgammon, a game of chance, was analyzed mathematically by Gerolamo Cardano (1663), and taken up for computer play with the BKG program (Berliner, 1980b), which used a manually constructed evaluation function and searched only to depth 1. It was the first program to defeat a human world champion at a major game (Berliner, 1980a), although Berliner readily acknowledged that BKG was very lucky with the dice. Gerry Tesauro's (1995) TD-GAMMON learned its evaluation function using neural networks trained by self-play. It consistently played at world champion level and caused human analysts to change their opinion on the best opening move for several dice rolls.

Poker, like Go, has seen surprising advances in recent years. Bowling *et al.* (2015) used game theory (see Section 17.2) to determine the exact optimal strategy for a version of poker with just two players and a fixed number of raises with fixed bet sizes. In 2017, for the first time, champion poker players were beaten at heads-up (two player) no-limit Texas hold 'em in two separate matches against the programs Libratus (Brown and Sandholm, 2017) and DeepStack (Moravčík *et al.*, 2017). In 2019, Pluribus (Brown and Sandholm, 2019) defeated top-ranked professional human players in Texas hold 'em games with six players. Multiplayer games introduce some strategic concerns that we will cover in Chapter 17. Petosa and Balch (2019) implement a multiplayer version of ALPHAZERO.

Bridge: Smith *et al.* (1998) report on how BRIDGE BARON won the 1998 computer bridge championship, using hierarchical plans (see Chapter 11) and high-level actions, such as finessing and squeezing, that are familiar to bridge players. Ginsberg (2001) describes how his GIB program, based on Monte Carlo simulation (first proposed for bridge by Levy (1989)), won the following computer championship and did surprisingly well against expert human players. In the 21st century, the computer bridge championship has been dominated by two commercial programs, JACK and WBRIDGE5. Neither has been described in published articles, but both are believed to use Monte Carlo techniques. In general, bridge programs are at human champion level when actually playing the hands, but lag behind in the bidding phase, because they do not completely understand the conventions used by humans to communicate with their partners. Bridge programmers have concentrated more on producing

useful and educational programs that encourage people to take up the game, rather than on defeating human champions.

Scrabble is a game where amateur human players have difficulty coming up with high-scoring words, but for a computer, it is easy to find the highest possible score for a given hand (Gordon, 1994); the hard part is planning ahead in a partially observable, stochastic game. Nevertheless, in 2006, the QUACKLE program defeated the former world champion, David Boys, 3–2. Boys took it well, stating, "It's still better to be a human than to be a computer." A good description of a top program, MAVEN, is given by Sheppard (2002).

Video games such as **StarCraft II** involve hundreds of partially observable units moving in real time with high-dimensional near-continuous[6] observation and action spaces with complex rules. Oriol Vinyals, who was Spain's StarCraft champion at age 15, described how the game can serve as a testbed and grand challenge for reinforcement learning (Vinyals *et al.*, 2017a). In 2019, Vinyals and the team at DeepMind unveiled the ALPHASTAR program, based on deep learning and reinforcement learning, which defeated expert human players 10 games to 1, and ranks in the top 0.02% of officially ranked human players (Vinyals *et al.*, 2019). ALPHASTAR took steps to limit the number of actions per minute it could perform in critical bursts, in response to critics who felt it had an unfair advantage.

Computers have defeated top humans in other popular video games such as Super Smash Bros. (Firoiu *et al.*, 2017), Quake III (Jaderberg *et al.*, 2019), and Dota 2 (Fernandez and Mahlmann, 2018), all using deep learning techniques.

Physical games such as **robotic soccer** (Visser *et al.*, 2008; Barrett and Stone, 2015), **billiards** (Lam and Greenspan, 2008; Archibald *et al.*, 2009), and **ping-pong** (Silva *et al.*, 2015) have attracted some attention in AI. They combine all the complications of video games with the messiness of the real world.

Computer game competitions occur annually, including the Computer Olympiads since 1989. The General Game Competition (Love *et al.*, 2006) tests programs that must learn to play an unknown game given only a logical description of the rules of the game. The International Computer Games Association (ICGA) publishes the *ICGA Journal* and runs two alternating biennial conferences, The International Conference on Computers and Games (ICCG or CG) and the International Conference on Advances in Computer Games (ACG). The IEEE publishes *IEEE Transactions on Games* and runs an annual Conference on Computational Intelligence and Games.

[6] To a human player, it appears that objects move continuously, but they are actually discrete at the level of a pixel on the screen.

CHAPTER 7

LOGICAL AGENTS

In which we design agents that can form representations of a complex world, use a process of inference to derive new representations about the world, and use these new representations to deduce what to do.

Knowledge-based agents
Reasoning
Representation

Humans, it seems, know things; and what they know helps them do things. In AI, **knowledge-based agents** use a process of **reasoning** over an internal **representation** of knowledge to decide what actions to take.

The problem-solving agents of Chapters 3 and 4 know things, but only in a very limited, inflexible sense. They know what actions are available and what the result of performing a specific action from a specific state will be, but they don't know general facts. A route-finding agent doesn't know that it is impossible for a road to be a negative number of kilometers long. An 8-puzzle agent doesn't know that two tiles cannot occupy the same space. The knowledge they have is very useful for finding a path from the start to a goal, but not for anything else.

The atomic representations used by problem-solving agents are also very limiting. In a partially observable environment, for example, a problem-solving agent's only choice for representing what it knows about the current state is to list all possible concrete states. I could give a human the goal of driving to a U.S. town with population less than 10,000, but to say that to a problem-solving agent, I could formally describe the goal only as an explicit set of the 16,000 or so towns that satisfy the description.

Chapter 5 introduced our first factored representation, whereby states are represented as assignments of values to variables; this is a step in the right direction, enabling some parts of the agent to work in a domain-independent way and allowing for more efficient algorithms. In this chapter, we take this step to its logical conclusion, so to speak—we develop **logic** as a general class of representations to support knowledge-based agents. These agents can combine and recombine information to suit myriad purposes. This can be far removed from the needs of the moment—as when a mathematician proves a theorem or an astronomer calculates the Earth's life expectancy. Knowledge-based agents can accept new tasks in the form of explicitly described goals; they can achieve competence quickly by being told or learning new knowledge about the environment; and they can adapt to changes in the environment by updating the relevant knowledge.

We begin in Section 7.1 with the overall agent design. Section 7.2 introduces a simple new environment, the wumpus world, and illustrates the operation of a knowledge-based agent without going into any technical detail. Then we explain the general principles of **logic** in Section 7.3 and the specifics of **propositional logic** in Section 7.4. Propositional logic is a factored representation; while less expressive than **first-order logic** (Chapter 8), which is the canonical structured representation, propositional logic illustrates all the basic concepts

```
function KB-AGENT(percept) returns an action
  persistent: KB, a knowledge base
              t, a counter, initially 0, indicating time

  TELL(KB, MAKE-PERCEPT-SENTENCE(percept, t))
  action ← ASK(KB, MAKE-ACTION-QUERY(t))
  TELL(KB, MAKE-ACTION-SENTENCE(action, t))
  t ← t + 1
  return action
```

Figure 7.1 A generic knowledge-based agent. Given a percept, the agent adds the percept to its knowledge base, asks the knowledge base for the best action, and tells the knowledge base that it has in fact taken that action.

of logic. It also comes with well-developed inference technologies, which we describe in sections 7.5 and 7.6. Finally, Section 7.7 combines the concept of knowledge-based agents with the technology of propositional logic to build some simple agents for the wumpus world.

7.1 Knowledge-Based Agents

The central component of a knowledge-based agent is its **knowledge base**, or KB. A knowledge base is a set of **sentences**. (Here "sentence" is used as a technical term. It is related but not identical to the sentences of English and other natural languages.) Each sentence is expressed in a language called a **knowledge representation language** and represents some assertion about the world. When the sentence is taken as being given without being derived from other sentences, we call it an **axiom**.

Knowledge base

Sentence

Knowledge representation language

Axiom

There must be a way to add new sentences to the knowledge base and a way to query what is known. The standard names for these operations are TELL and ASK, respectively. Both operations may involve **inference**—that is, deriving new sentences from old. Inference must obey the requirement that when one ASKs a question of the knowledge base, the answer should follow from what has been told (or TELLed) to the knowledge base previously. Later in this chapter, we will be more precise about the crucial word "follow." For now, take it to mean that the inference process should not make things up as it goes along.

Inference

Figure 7.1 shows the outline of a knowledge-based agent program. Like all our agents, it takes a percept as input and returns an action. The agent maintains a knowledge base, *KB*, which may initially contain some **background knowledge**.

Background knowledge

Each time the agent program is called, it does three things. First, it TELLs the knowledge base what it perceives. Second, it ASKs the knowledge base what action it should perform. In the process of answering this query, extensive reasoning may be done about the current state of the world, about the outcomes of possible action sequences, and so on. Third, the agent program TELLs the knowledge base which action was chosen, and returns the action so that it can be executed.

The details of the representation language are hidden inside three functions that implement the interface between the sensors and actuators on one side and the core representation and reasoning system on the other. MAKE-PERCEPT-SENTENCE constructs a sentence as-

serting that the agent perceived the given percept at the given time. MAKE-ACTION-QUERY constructs a sentence that asks what action should be done at the current time. Finally, MAKE-ACTION-SENTENCE constructs a sentence asserting that the chosen action was executed. The details of the inference mechanisms are hidden inside TELL and ASK. Later sections will reveal these details.

The agent in Figure 7.1 appears quite similar to the agents with internal state described in Chapter 2. Because of the definitions of TELL and ASK, however, the knowledge-based agent is not an arbitrary program for calculating actions. It is amenable to a description at the

Knowledge level

knowledge level, where we need specify only what the agent knows and what its goals are, in order to determine its behavior.

For example, an automated taxi might have the goal of taking a passenger from San Francisco to Marin County and might know that the Golden Gate Bridge is the only link between the two locations. Then we can expect it to cross the Golden Gate Bridge *because it knows that that will achieve its goal*. Notice that this analysis is independent of how the taxi

Implementation level

works at the **implementation level**. It doesn't matter whether its geographical knowledge is implemented as linked lists or pixel maps, or whether it reasons by manipulating strings of symbols stored in registers or by propagating noisy signals in a network of neurons.

A knowledge-based agent can be built simply by TELLing it what it needs to know. Starting with an empty knowledge base, the agent designer can TELL sentences one by one until

Declarative

the agent knows how to operate in its environment. This is called the **declarative** approach

Procedural

to system building. In contrast, the **procedural** approach encodes desired behaviors directly as program code. In the 1970s and 1980s, advocates of the two approaches engaged in heated debates. We now understand that a successful agent often combines both declarative and procedural elements in its design, and that declarative knowledge can often be compiled into more efficient procedural code.

We can also provide a knowledge-based agent with mechanisms that allow it to learn for itself. These mechanisms, which are discussed in Chapter 19, create general knowledge about the environment from a series of percepts. A learning agent can be fully autonomous.

7.2 The Wumpus World

In this section we describe an environment in which knowledge-based agents can show their

Wumpus world

worth. The **wumpus world** is a cave consisting of rooms connected by passageways. Lurking somewhere in the cave is the terrible wumpus, a beast that eats anyone who enters its room. The wumpus can be shot by an agent, but the agent has only one arrow. Some rooms contain bottomless pits that will trap anyone who wanders into these rooms (except for the wumpus, which is too big to fall in). The only redeeming feature of this bleak environment is the possibility of finding a heap of gold. Although the wumpus world is rather tame by modern computer game standards, it illustrates some important points about intelligence.

A sample wumpus world is shown in Figure 7.2. The precise definition of the task environment is given, as suggested in Section 2.3, by the PEAS description:

- **Performance measure**: +1000 for climbing out of the cave with the gold, –1000 for falling into a pit or being eaten by the wumpus, –1 for each action taken, and –10 for using up the arrow. The game ends either when the agent dies or when the agent climbs out of the cave.

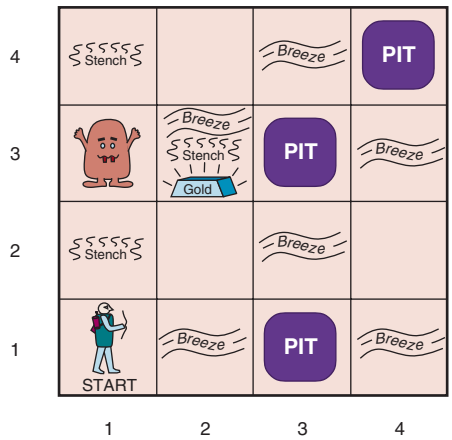


Figure 7.2 A typical wumpus world. The agent is in the bottom left corner, facing east (rightward).

- **Environment**: A 4×4 grid of rooms, with walls surrounding the grid. The agent always starts in the square labeled [1,1], facing to the east. The locations of the gold and the wumpus are chosen randomly, with a uniform distribution, from the squares other than the start square. In addition, each square other than the start can be a pit, with probability 0.2.
- **Actuators**: The agent can move *Forward*, *TurnLeft* by 90°, or *TurnRight* by 90°. The agent dies a miserable death if it enters a square containing a pit or a live wumpus. (It is safe, albeit smelly, to enter a square with a dead wumpus.) If an agent tries to move forward and bumps into a wall, then the agent does not move. The action *Grab* can be used to pick up the gold if it is in the same square as the agent. The action *Shoot* can be used to fire an arrow in a straight line in the direction the agent is facing. The arrow continues until it either hits (and hence kills) the wumpus or hits a wall. The agent has only one arrow, so only the first *Shoot* action has any effect. Finally, the action *Climb* can be used to climb out of the cave, but only from square [1,1].
- **Sensors**: The agent has five sensors, each of which gives a single bit of information:
  - In the squares directly (not diagonally) adjacent to the wumpus, the agent will perceive a *Stench*.[1]
  - In the squares directly adjacent to a pit, the agent will perceive a *Breeze*.
  - In the square where the gold is, the agent will perceive a *Glitter*.
  - When an agent walks into a wall, it will perceive a *Bump*.
  - When the wumpus is killed, it emits a woeful *Scream* that can be perceived anywhere in the cave.

  The percepts will be given to the agent program in the form of a list of five symbols; for example, if there is a stench and a breeze, but no glitter, bump, or scream, the agent program will get $[Stench, Breeze, None, None, None]$.

[1] Presumably the square containing the wumpus also has a stench, but any agent entering that square is eaten before being able to perceive anything.

Figure 7.3 The first step taken by the agent in the wumpus world. (a) The initial situation, after percept $[None, None, None, None, None]$. (b) After moving to [2,1] and perceiving $[None, Breeze, None, None, None]$.

We can characterize the wumpus environment along the various dimensions given in Chapter 2. Clearly, it is deterministic, discrete, static, and single-agent. (The wumpus doesn't move, fortunately.) It is sequential, because rewards may come only after many actions are taken. It is partially observable, because some aspects of the state are not directly perceivable: the agent's location, the wumpus's state of health, and the availability of an arrow. As for the locations of the pits and the wumpus: we could treat them as unobserved parts of the state—in which case, the transition model for the environment is completely known, and finding the locations of pits completes the agent's knowledge of the state. Alternatively, we could say that the transition model itself is unknown because the agent doesn't know which *Forward* actions are fatal—in which case, discovering the locations of pits and wumpus completes the agent's knowledge of the transition model.

For an agent in the environment, the main challenge is its initial ignorance of the configuration of the environment; overcoming this ignorance seems to require logical reasoning. In most instances of the wumpus world, it is possible for the agent to retrieve the gold safely. Occasionally, the agent must choose between going home empty-handed and risking death to find the gold. About 21% of the environments are utterly unfair, because the gold is in a pit or surrounded by pits.

Let us watch a knowledge-based wumpus agent exploring the environment shown in Figure 7.2. We use an informal knowledge representation language consisting of writing down symbols in a grid (as in Figures 7.3 and 7.4).

The agent's initial knowledge base contains the rules of the environment, as described previously; in particular, it knows that it is in [1,1] and that [1,1] is a safe square; we denote that with an "A" and "OK," respectively, in square [1,1].

The first percept is $[None, None, None, None, None]$, from which the agent can conclude that its neighboring squares, [1,2] and [2,1], are free of dangers—they are OK. Figure 7.3(a) shows the agent's state of knowledge at this point.

(a) (b)

Figure 7.4 Two later stages in the progress of the agent. (a) After moving to [1,1] and then [1,2], and perceiving $[Stench, None, None, None, None]$. (b) After moving to [2,2] and then [2,3], and perceiving $[Stench, Breeze, Glitter, None, None]$.

A cautious agent will move only into a square that it knows to be OK. Let us suppose the agent decides to move forward to [2,1]. The agent perceives a breeze (denoted by "B") in [2,1], so there must be a pit in a neighboring square. The pit cannot be in [1,1], by the rules of the game, so there must be a pit in [2,2] or [3,1] or both. The notation "P?" in Figure 7.3(b) indicates a possible pit in those squares. At this point, there is only one known square that is OK and that has not yet been visited. So the prudent agent will turn around, go back to [1,1], and then proceed to [1,2].

The agent perceives a stench in [1,2], resulting in the state of knowledge shown in Figure 7.4(a). The stench in [1,2] means that there must be a wumpus nearby. But the wumpus cannot be in [1,1], by the rules of the game, and it cannot be in [2,2] (or the agent would have detected a stench when it was in [2,1]). Therefore, the agent can infer that the wumpus is in [1,3]. The notation W! indicates this inference. Moreover, the lack of a breeze in [1,2] implies that there is no pit in [2,2]. Yet the agent has already inferred that there must be a pit in either [2,2] or [3,1], so this means it must be in [3,1]. This is a fairly difficult inference, because it combines knowledge gained at different times in different places and relies on the lack of a percept to make one crucial step.

The agent has now proved to itself that there is neither a pit nor a wumpus in [2,2], so it is OK to move there. We do not show the agent's state of knowledge at [2,2]; we just assume that the agent turns and moves to [2,3], giving us Figure 7.4(b). In [2,3], the agent detects a glitter, so it should grab the gold and then return home.

Note that in each case for which the agent draws a conclusion from the available information, that conclusion is *guaranteed* to be correct if the available information is correct. This is a fundamental property of logical reasoning. In the rest of this chapter, we describe how to build logical agents that can represent information and draw conclusions such as those described in the preceding paragraphs.

7.3 Logic

This section summarizes the fundamental concepts of logical representation and reasoning. These beautiful ideas are independent of any of logic's particular forms. We therefore postpone the technical details of those forms until the next section, using instead the familiar example of ordinary arithmetic.

Syntax

In Section 7.1, we said that knowledge bases consist of sentences. These sentences are expressed according to the **syntax** of the representation language, which specifies all the sentences that are well formed. The notion of syntax is clear enough in ordinary arithmetic: "$x+y=4$" is a well-formed sentence, whereas "$x4y+=$" is not.

Semantics
Truth
Possible world

A logic must also define the **semantics**, or meaning, of sentences. The semantics defines the **truth** of each sentence with respect to each **possible world**. For example, the semantics for arithmetic specifies that the sentence "$x+y=4$" is true in a world where x is 2 and y is 2, but false in a world where x is 1 and y is 1. In standard logics, every sentence must be either true or false in each possible world—there is no "in between."[2]

Model

When we need to be precise, we use the term **model** in place of "possible world." Whereas possible worlds might be thought of as (potentially) real environments that the agent might or might not be in, models are mathematical abstractions, each of which has a fixed truth value (true or false) for every relevant sentence. Informally, we may think of a possible world as, for example, having x men and y women sitting at a table playing bridge, and the sentence $x+y=4$ is true when there are four people in total. Formally, the possible models are just all possible assignments of nonnegative integers to the variables x and y. Each such assignment determines the truth of any sentence of arithmetic whose variables are x and y. If a sentence α is true in model m, we say that m **satisfies** α or sometimes m **is a model of** α. We use the notation $M(\alpha)$ to mean the set of all models of α.

Satisfaction

Entailment

Now that we have a notion of truth, we are ready to talk about logical reasoning. This involves the relation of logical **entailment** between sentences—the idea that a sentence *follows logically* from another sentence. In mathematical notation, we write

$$\alpha \models \beta$$

to mean that the sentence α entails the sentence β. The formal definition of entailment is this: $\alpha \models \beta$ if and only if, in every model in which α is true, β is also true. Using the notation just introduced, we can write

$$\alpha \models \beta \text{ if and only if } M(\alpha) \subseteq M(\beta)\,.$$

(Note the direction of the $\subseteq$ here: if $\alpha \models \beta$, then α is a *stronger* assertion than β: it rules out *more* possible worlds.) The relation of entailment is familiar from arithmetic; we are happy with the idea that the sentence $x=0$ entails the sentence $xy=0$. Obviously, in any model where x is zero, it is the case that xy is zero (regardless of the value of y).

We can apply the same kind of analysis to the wumpus-world reasoning example given in the preceding section. Consider the situation in Figure 7.3(b): the agent has detected nothing in [1,1] and a breeze in [2,1]. These percepts, combined with the agent's knowledge of the rules of the wumpus world, constitute the KB. The agent is interested in whether the adjacent squares [1,2], [2,2], and [3,1] contain pits. Each of the three squares might or might

[2] **Fuzzy logic**, discussed in Chapter 13, allows for degrees of truth.

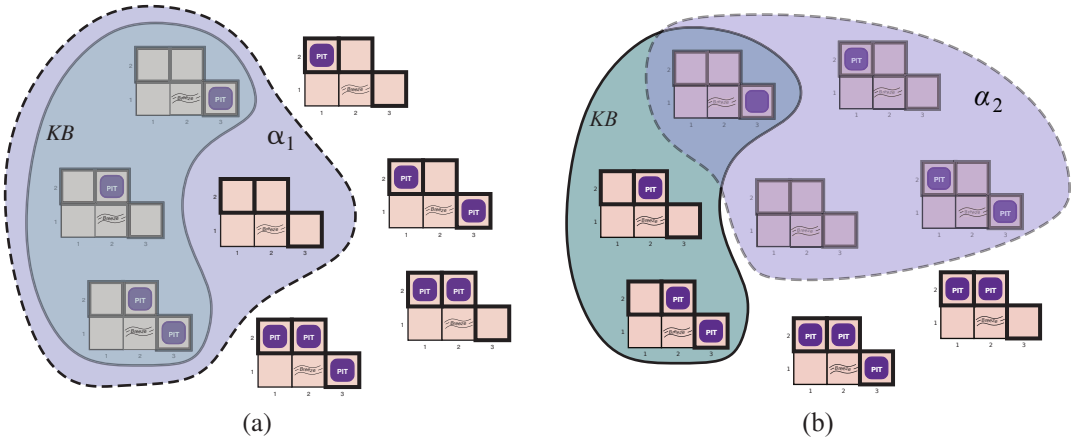


Figure 7.5 Possible models for the presence of pits in squares [1,2], [2,2], and [3,1]. The KB corresponding to the observations of nothing in [1,1] and a breeze in [2,1] is shown by the solid line. (a) Dotted line shows models of α_1 (no pit in [1,2]). (b) Dotted line shows models of α_2 (no pit in [2,2]).

not contain a pit, so (ignoring other aspects of the world for now) there are $2^3 = 8$ possible models. These eight models are shown in Figure 7.5.[3]

The KB can be thought of as a set of sentences or as a single sentence that asserts all the individual sentences. The KB is false in models that contradict what the agent knows—for example, the KB is false in any model in which [1,2] contains a pit, because there is no breeze in [1,1]. There are in fact just three models in which the KB is true, and these are shown surrounded by a solid line in Figure 7.5. Now let us consider two possible conclusions:

α_1 = "There is no pit in [1,2]." α_2 = "There is no pit in [2,2]."

We have surrounded the models of α_1 and α_2 with dotted lines in Figures 7.5(a) and 7.5(b), respectively. By inspection, we see the following:

in every model in which *KB* is true, α_1 is also true.

Hence, $KB \models \alpha_1$: there is no pit in [1,2]. We can also see that

in some models in which *KB* is true, α_2 is false.

Hence, *KB* does not entail α_2: the agent *cannot* conclude that there is no pit in [2,2]. (Nor can it conclude that there *is* a pit in [2,2].)[4]

The preceding example not only illustrates entailment but also shows how the definition of entailment can be applied to derive conclusions—that is, to carry out **logical inference**. The inference algorithm illustrated in Figure 7.5 is called **model checking**, because it enumerates all possible models to check that α is true in all models in which *KB* is true, that is, that $M(KB) \subseteq M(\alpha)$.

Logical inference

Model checking

[3] Although the figure shows the models as partial wumpus worlds, they are really nothing more than assignments of *true* and *false* to the sentences "there is a pit in [1,2]" etc. Models, in the mathematical sense, do not need to have 'orrible 'airy wumpuses in them.

[4] The agent can calculate the *probability* that there is a pit in [2,2]; Chapter 12 shows how.

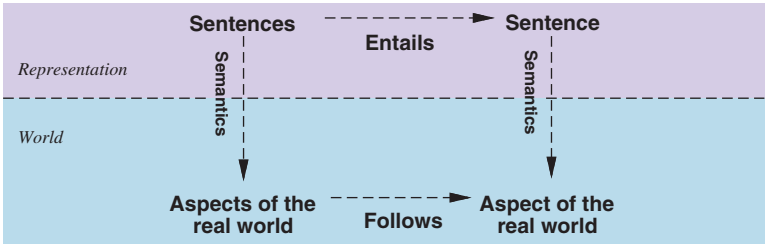


Figure 7.6 Sentences are physical configurations of the agent, and reasoning is a process of constructing new physical configurations from old ones. Logical reasoning should ensure that the new configurations represent aspects of the world that actually follow from the aspects that the old configurations represent.

In understanding entailment and inference, it might help to think of the set of all consequences of *KB* as a haystack and of α as a needle. Entailment is like the needle being in the haystack; inference is like finding it. This distinction is embodied in some formal notation: if an inference algorithm i can derive α from *KB*, we write

$$KB \vdash_i \alpha ,$$

which is pronounced "α is derived from *KB* by i" or "i derives α from *KB*."

Sound
Truth-preserving

An inference algorithm that derives only entailed sentences is called **sound** or **truth-preserving**. Soundness is a highly desirable property. An unsound inference procedure essentially makes things up as it goes along—it announces the discovery of nonexistent needles. It is easy to see that model checking, when it is applicable,[5] is a sound procedure.

Completeness

The property of **completeness** is also desirable: an inference algorithm is complete if it can derive any sentence that is entailed. For real haystacks, which are finite in extent, it seems obvious that a systematic examination can always decide whether the needle is in the haystack. For many knowledge bases, however, the haystack of consequences is infinite, and completeness becomes an important issue.[6] Fortunately, there are complete inference procedures for logics that are sufficiently expressive to handle many knowledge bases.

We have described a reasoning process whose conclusions are guaranteed to be true in any world in which the premises are true; in particular, *if KB is true in the* real *world, then any sentence* α *derived from KB by a sound inference procedure is also true in the real world.* So, while an inference process operates on "syntax"—internal physical configurations such as bits in registers or patterns of electrical blips in brains—the process *corresponds* to the real-world relationship whereby some aspect of the real world is the case by virtue of other aspects of the real world being the case.[7] This correspondence between world and representation is illustrated in Figure 7.6.

Grounding

The final issue to consider is **grounding**—the connection between logical reasoning processes and the real environment in which the agent exists. In particular, *how do we know that*

[5] Model checking works if the space of models is finite—for example, in wumpus worlds of fixed size. For arithmetic, on the other hand, the space of models is infinite: even if we restrict ourselves to the integers, there are infinitely many pairs of values for x and y in the sentence $x + y = 4$.

[6] Compare with the case of infinite search spaces in Chapter 3, where depth-first search is not complete.

[7] As Wittgenstein (1922) put it in his famous *Tractatus*: "The world is everything that is the case."

KB is true in the real world? (After all, *KB* is just "syntax" inside the agent's head.) This is a philosophical question about which many, many books have been written. (See Chapter 28.) A simple answer is that the agent's sensors create the connection. For example, our wumpus-world agent has a smell sensor. The agent program creates a suitable sentence whenever there is a smell. Then, whenever that sentence is in the knowledge base, it is true in the real world. Thus, the meaning and truth of percept sentences are defined by the processes of sensing and sentence construction that produce them. What about the rest of the agent's knowledge, such as its belief that wumpuses cause smells in adjacent squares? This is not a direct representation of a single percept, but a general rule—derived, perhaps, from perceptual experience but not identical to a statement of that experience. General rules like this are produced by a sentence construction process called **learning**, which is the subject of Part V. Learning is fallible. It could be the case that wumpuses cause smells *except on February 29 in leap years*, which is when they take their baths. Thus, *KB* may not be true in the real world, but with good learning procedures, there is reason for optimism.

7.4 Propositional Logic: A Very Simple Logic

We now present **propositional logic**. We describe its syntax (the structure of sentences) and its semantics (the way in which the truth of sentences is determined). From these, we derive a simple, syntactic algorithm for logical inference that implements the semantic notion of entailment. Everything takes place, of course, in the wumpus world.

Propositional logic

7.4.1 Syntax

The **syntax** of propositional logic defines the allowable sentences. The **atomic sentences** consist of a single **proposition symbol**. Each such symbol stands for a proposition that can be true or false. We use symbols that start with an uppercase letter and may contain other letters or subscripts, for example: P, Q, R, $W_{1,3}$ and *FacingEast*. The names are arbitrary but are often chosen to have some mnemonic value—we use $W_{1,3}$ to stand for the proposition that the wumpus is in [1,3]. (Remember that symbols such as $W_{1,3}$ are *atomic*, i.e., W, 1, and 3 are not meaningful parts of the symbol.) There are two proposition symbols with fixed meanings: *True* is the always-true proposition and *False* is the always-false proposition. **Complex sentences** are constructed from simpler sentences, using parentheses and operators called **logical connectives**. There are five connectives in common use:

Atomic sentences
Proposition symbol
Complex sentences
Logical connectives

- $\neg$ (not). A sentence such as $\neg W_{1,3}$ is called the **negation** of $W_{1,3}$. A **literal** is either an atomic sentence (a **positive literal**) or a negated atomic sentence (a **negative literal**). — Negation, Literal
- $\wedge$ (and). A sentence whose main connective is $\wedge$, such as $W_{1,3} \wedge P_{3,1}$, is called a **conjunction**; its parts are the **conjuncts**. (The $\wedge$ looks like an "A" for "And.") — Conjunction
- $\vee$ (or). A sentence whose main connective is $\vee$, such as $(W_{1,3} \wedge P_{3,1}) \vee W_{2,2}$, is a **disjunction**; its parts are **disjuncts**—in this example, $(W_{1,3} \wedge P_{3,1})$ and $W_{2,2}$. — Disjunction
- $\Rightarrow$ (implies). A sentence such as $(W_{1,3} \wedge P_{3,1}) \Rightarrow \neg W_{2,2}$ is called an **implication** (or conditional). Its **premise** or **antecedent** is $(W_{1,3} \wedge P_{3,1})$, and its **conclusion** or **consequent** is $\neg W_{2,2}$. Implications are also known as **rules** or **if–then** statements. The implication symbol is sometimes written in other books as $\supset$ or $\rightarrow$. — Implication, Premise, Conclusion, Rules
- $\Leftrightarrow$ (if and only if). The sentence $W_{1,3} \Leftrightarrow \neg W_{2,2}$ is a **biconditional**. — Biconditional

$$
\begin{aligned}
\mathit{Sentence} &\rightarrow \mathit{AtomicSentence} \mid \mathit{ComplexSentence} \\
\mathit{AtomicSentence} &\rightarrow \mathit{True} \mid \mathit{False} \mid P \mid Q \mid R \mid \ldots \\
\mathit{ComplexSentence} &\rightarrow (\ \mathit{Sentence}\ ) \\
&\mid \neg\ \mathit{Sentence} \\
&\mid \mathit{Sentence} \wedge \mathit{Sentence} \\
&\mid \mathit{Sentence} \vee \mathit{Sentence} \\
&\mid \mathit{Sentence} \Rightarrow \mathit{Sentence} \\
&\mid \mathit{Sentence} \Leftrightarrow \mathit{Sentence} \\
\text{OPERATOR PRECEDENCE} &: \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow
\end{aligned}
$$

Figure 7.7 A BNF (Backus–Naur Form) grammar of sentences in propositional logic, along with operator precedences, from highest to lowest.

Figure 7.7 gives a formal grammar of propositional logic. (BNF notation is explained on page 1081.) The BNF grammar is augmented with an operator precedence list to remove ambiguity when multiple operators are used. The "not" operator ($\neg$) has the highest precedence, which means that in the sentence $\neg A \wedge B$ the $\neg$ binds most tightly, giving us the equivalent of $(\neg A) \wedge B$ rather than $\neg(A \wedge B)$. (The notation for ordinary arithmetic is the same: $-2+4$ is 2, not –6.) When appropriate, we also use parentheses and square brackets to clarify the intended sentence structure and improve readability.

7.4.2 Semantics

Having specified the syntax of propositional logic, we now specify its semantics. The semantics defines the rules for determining the truth of a sentence with respect to a particular model. In propositional logic, a model simply sets the **truth value**—*true* or *false*—for every proposition symbol. For example, if the sentences in the knowledge base make use of the proposition symbols $P_{1,2}$, $P_{2,2}$, and $P_{3,1}$, then one possible model is

Truth value

$$m_1 = \{P_{1,2} = \mathit{false}, P_{2,2} = \mathit{false}, P_{3,1} = \mathit{true}\}.$$

With three proposition symbols, there are $2^3 = 8$ possible models—exactly those depicted in Figure 7.5. Notice, however, that the models are purely mathematical objects with no necessary connection to wumpus worlds. $P_{1,2}$ is just a symbol; it might mean "there is a pit in [1,2]" or "I'm in Paris today and tomorrow."

The semantics for propositional logic must specify how to compute the truth value of *any* sentence, given a model. This is done recursively. All sentences are constructed from atomic sentences and the five connectives; therefore, we need to specify how to compute the truth of atomic sentences and how to compute the truth of sentences formed with each of the five connectives. Atomic sentences are easy:

- *True* is true in every model and *False* is false in every model.
- The truth value of every other proposition symbol must be specified directly in the model. For example, in the model m_1 given earlier, $P_{1,2}$ is false.

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	*false*	*true*	*false*	*false*	*true*	*true*
false	*true*	*true*	*false*	*true*	*true*	*false*
true	*false*	*false*	*false*	*true*	*false*	*false*
true	*true*	*false*	*true*	*true*	*true*	*true*

Figure 7.8 Truth tables for the five logical connectives. To use the table to compute, for example, the value of $P \vee Q$ when P is true and Q is false, first look on the left for the row where P is *true* and Q is *false* (the third row). Then look in that row under the $P \vee Q$ column to see the result: *true*.

For complex sentences, we have five rules, which hold for any subsentences P and Q (atomic or complex) in any model m (here "iff" means "if and only if"):

- $\neg P$ is true iff P is false in m.
- $P \wedge Q$ is true iff both P and Q are true in m.
- $P \vee Q$ is true iff either P or Q is true in m.
- $P \Rightarrow Q$ is true unless P is true and Q is false in m.
- $P \Leftrightarrow Q$ is true iff P and Q are both true or both false in m.

Truth table

The rules can also be expressed with **truth tables** that specify the truth value of a complex sentence for each possible assignment of truth values to its components. Truth tables for the five connectives are given in Figure 7.8. From these tables, the truth value of any sentence s can be computed with respect to any model m by a simple recursive evaluation. For example, the sentence $\neg P_{1,2} \wedge (P_{2,2} \vee P_{3,1})$, evaluated in m_1, gives $true \wedge (false \vee true) = true \wedge true = true$. Exercise 7.TRUV asks you to write the algorithm PL-TRUE?(s, m), which computes the truth value of a propositional logic sentence s in a model m.

The truth tables for "and," "or," and "not" are in close accord with our intuitions about the English words. The main point of possible confusion is that $P \vee Q$ is true when P is true or Q is true *or both*. A different connective, called "exclusive or" ("xor" for short), yields false when both disjuncts are true.[8] There is no consensus on the symbol for exclusive or; some choices are $\veebar$ or $\neq$ or $\oplus$.

The truth table for $\Rightarrow$ may not quite fit one's intuitive understanding of "P implies Q" or "if P then Q." For one thing, propositional logic does not require any relation of *causation* or *relevance* between P and Q. The sentence "5 is odd implies Tokyo is the capital of Japan" is a true sentence of propositional logic (under the normal interpretation), even though it is a decidedly odd sentence of English. Another point of confusion is that any implication is true whenever its antecedent is false. For example, "5 is even implies Sam is smart" is true, regardless of whether Sam is smart. This seems bizarre, but it makes sense if you think of "$P \Rightarrow Q$" as saying, "If P is true, then I am claiming that Q is true; otherwise I am making no claim." The only way for this sentence to be *false* is if P is true but Q is false.

The biconditional, $P \Leftrightarrow Q$, is true whenever both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true. In English, this is often written as "P if and only if Q." Many of the rules of the wumpus world are best

[8] Latin uses two separate words: "vel" is inclusive or and "aut" is exclusive or.

written using $\Leftrightarrow$. For example, a square is breezy *if* a neighboring square has a pit, and a square is breezy *only if* a neighboring square has a pit. So we need a biconditional,

$$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1}),$$

where $B_{1,1}$ means that there is a breeze in [1,1].

7.4.3 A simple knowledge base

Now that we have defined the semantics for propositional logic, we can construct a knowledge base for the wumpus world. We focus first on the *immutable* aspects of the wumpus world, leaving the mutable aspects for a later section. For now, we need the following symbols for each $[x,y]$ location:

$P_{x,y}$ is true if there is a pit in $[x,y]$.
$W_{x,y}$ is true if there is a wumpus in $[x,y]$, dead or alive.
$B_{x,y}$ is true if there is a breeze in $[x,y]$.
$S_{x,y}$ is true if there is a stench in $[x,y]$.
$L_{x,y}$ is true if the agent is in location $[x,y]$.

The sentences we write will suffice to derive $\neg P_{1,2}$ (there is no pit in [1,2]), as was done informally in Section 7.3. We label each sentence R_i so that we can refer to them:

- There is no pit in [1,1]:

$$R_1 : \quad \neg P_{1,1}.$$

- A square is breezy if and only if there is a pit in a neighboring square. This has to be stated for each square; for now, we include just the relevant squares:

$$R_2 : \quad B_{1,1} \quad \Leftrightarrow \quad (P_{1,2} \vee P_{2,1}).$$
$$R_3 : \quad B_{2,1} \quad \Leftrightarrow \quad (P_{1,1} \vee P_{2,2} \vee P_{3,1}).$$

- The preceding sentences are true in all wumpus worlds. Now we include the breeze percepts for the first two squares visited in the specific world the agent is in, leading up to the situation in Figure 7.3(b).

$$R_4 : \quad \neg B_{1,1}.$$
$$R_5 : \quad B_{2,1}.$$

7.4.4 A simple inference procedure

Our goal now is to decide whether $KB \models \alpha$ for some sentence α. For example, is $\neg P_{1,2}$ entailed by our *KB*? Our first algorithm for inference is a model-checking approach that is a direct implementation of the definition of entailment: enumerate the models, and check that α is true in every model in which *KB* is true. Models are assignments of *true* or *false* to every proposition symbol. Returning to our wumpus-world example, the relevant proposition symbols are $B_{1,1}$, $B_{2,1}$, $P_{1,1}$, $P_{1,2}$, $P_{2,1}$, $P_{2,2}$, and $P_{3,1}$. With seven symbols, there are $2^7 = 128$ possible models; in three of these, *KB* is true (Figure 7.9). In those three models, $\neg P_{1,2}$ is true, hence there is no pit in [1,2]. On the other hand, $P_{2,2}$ is true in two of the three models and false in one, so we cannot yet tell whether there is a pit in [2,2].

Figure 7.9 reproduces in a more precise form the reasoning illustrated in Figure 7.5. A general algorithm for deciding entailment in propositional logic is shown in Figure 7.10. Like the BACKTRACKING-SEARCH algorithm on page 176, TT-ENTAILS? performs a recursive

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	R_1	R_2	R_3	R_4	R_5	KB
false	*false*	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*false*	*false*
false	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*false*	*true*	*false*	*false*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
false	*true*	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*false*	*true*	*true*	*false*
false	*true*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*false*	*false*	*true*	*false*	*true*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*false*	*true*	*false*	*false*	*true*	*false*	*false*	*true*	*true*	*false*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
true	*true*	*true*	*true*	*true*	*true*	*true*	*false*	*true*	*true*	*false*	*true*	*false*

Figure 7.9 A truth table constructed for the knowledge base given in the text. *KB* is true if R_1 through R_5 are true, which occurs in just 3 of the 128 rows (the ones underlined in the right-hand column). In all 3 rows, $P_{1,2}$ is false, so there is no pit in [1,2]. On the other hand, there might (or might not) be a pit in [2,2].

```
function TT-ENTAILS?(KB, α) returns true or false
  inputs: KB, the knowledge base, a sentence in propositional logic
          α, the query, a sentence in propositional logic

  symbols ← a list of the proposition symbols in KB and α
  return TT-CHECK-ALL(KB, α, symbols, { })

function TT-CHECK-ALL(KB, α, symbols, model) returns true or false
  if EMPTY?(symbols) then
      if PL-TRUE?(KB, model) then return PL-TRUE?(α, model)
      else return true      // when KB is false, always return true
  else
      P ← FIRST(symbols)
      rest ← REST(symbols)
      return (TT-CHECK-ALL(KB, α, rest, model ∪ {P = true})
              and
              TT-CHECK-ALL(KB, α, rest, model ∪ {P = false })
```

Figure 7.10 A truth-table enumeration algorithm for deciding propositional entailment. (TT stands for truth table.) PL-TRUE? returns *true* if a sentence holds within a model. The variable *model* represents a partial model—an assignment to some of the symbols. The keyword **and** here is an infix function symbol in the pseudocode programming language, not an operator in propositional logic; it takes two arguments and returns *true* or *false*.

enumeration of a finite space of assignments to symbols. The algorithm is **sound** because it implements directly the definition of entailment, and **complete** because it works for any *KB* and α and always terminates—there are only finitely many models to examine.

Of course, "finitely many" is not always the same as "few." If *KB* and α contain n symbols in all, then there are 2^n models. Thus, the time complexity of the algorithm is $O(2^n)$. (The space complexity is only $O(n)$ because the enumeration is depth-first.) Later in this chapter we show algorithms that are much more efficient in many cases. Unfortunately, propositional entailment is co-NP-complete (i.e., probably no easier than NP-complete—see Appendix A), so *every known inference algorithm for propositional logic has a worst-case complexity that is exponential in the size of the input.*

7.5 Propositional Theorem Proving

So far, we have shown how to determine entailment by *model checking*: enumerating models and showing that the sentence must hold in all models. In this section, we show how entailment can be done by **theorem proving**—applying rules of inference directly to the sentences in our knowledge base to construct a proof of the desired sentence without consulting models. If the number of models is large but the length of the proof is short, then theorem proving can be more efficient than model checking.

Theorem proving

Before we plunge into the details of theorem-proving algorithms, we will need some additional concepts related to entailment. The first concept is **logical equivalence**: two sentences α and β are logically equivalent if they are true in the same set of models. We write this as $\alpha \equiv \beta$. (Note that $\equiv$ is used to make claims about sentences, while $\Leftrightarrow$ is used as part of a sentence.) For example, we can easily show (using truth tables) that $P \wedge Q$ and $Q \wedge P$ are logically equivalent; other equivalences are shown in Figure 7.11. These equivalences play much the same role in logic as arithmetic identities do in ordinary mathematics. An alternative definition of equivalence is as follows: any two sentences α and β are equivalent if and only if each of them entails the other:

Logical equivalence

$$\alpha \equiv \beta \quad \text{if and only if} \quad \alpha \models \beta \text{ and } \beta \models \alpha\,.$$

The second concept we will need is **validity**. A sentence is valid if it is true in *all* models. For example, the sentence $P \vee \neg P$ is valid. Valid sentences are also known as **tautologies**—they are *necessarily* true. Because the sentence *True* is true in all models, every valid sentence is logically equivalent to *True*. What good are valid sentences? From our definition of entailment, we can derive the **deduction theorem**, which was known to the ancient Greeks:

Validity

Tautology

Deduction theorem

For any sentences α and β, $\alpha \models \beta$ if and only if the sentence $(\alpha \Rightarrow \beta)$ is valid.

(Exercise 7.DEDU asks for a proof.) Hence, we can decide if $\alpha \models \beta$ by checking that $(\alpha \Rightarrow \beta)$ is true in every model—which is essentially what the inference algorithm in Figure 7.10 does—or by proving that $(\alpha \Rightarrow \beta)$ is equivalent to *True*. Conversely, the deduction theorem states that every valid implication sentence describes a legitimate inference.

The final concept we will need is **satisfiability**. A sentence is satisfiable if it is true in, or satisfied by, *some* model. For example, the knowledge base given earlier, $(R_1 \wedge R_2 \wedge R_3 \wedge R_4 \wedge R_5)$, is satisfiable because there are three models in which it is true, as shown in Figure 7.9. Satisfiability can be checked by enumerating the possible models until one is found that satisfies the sentence. The problem of determining the satisfiability of sentences

Satisfiability

$$
\begin{aligned}
(\alpha \wedge \beta) &\equiv (\beta \wedge \alpha) && \text{commutativity of } \wedge \\
(\alpha \vee \beta) &\equiv (\beta \vee \alpha) && \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) &\equiv (\alpha \wedge (\beta \wedge \gamma)) && \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) &\equiv (\alpha \vee (\beta \vee \gamma)) && \text{associativity of } \vee \\
\neg(\neg\alpha) &\equiv \alpha && \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) && \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \vee \beta) && \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) && \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) &\equiv (\neg\alpha \vee \neg\beta) && \text{De Morgan} \\
\neg(\alpha \vee \beta) &\equiv (\neg\alpha \wedge \neg\beta) && \text{De Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) &\equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) && \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) &\equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) && \text{distributivity of } \vee \text{ over } \wedge
\end{aligned}
$$

Figure 7.11 Standard logical equivalences. The symbols α, β, and γ stand for arbitrary sentences of propositional logic.

in propositional logic—the **SAT** problem—was the first problem proved to be NP-complete. SAT
Many problems in computer science are really satisfiability problems. For example, all the constraint satisfaction problems in Chapter 5 ask whether the constraints are satisfiable by some assignment.

Validity and satisfiability are of course connected: α is valid iff $\neg\alpha$ is unsatisfiable; contrapositively, α is satisfiable iff $\neg\alpha$ is not valid. We also have the following useful result:

$\alpha \models \beta$ *if and only if the sentence* $(\alpha \wedge \neg\beta)$ *is unsatisfiable.*

Proving β from α by checking the unsatisfiability of $(\alpha \wedge \neg\beta)$ corresponds exactly to the standard mathematical proof technique of *reductio ad absurdum* (literally, "reduction to an absurd thing"). It is also called proof by **refutation** or proof by **contradiction**. One assumes a sentence β to be false and shows that this leads to a contradiction with known axioms α. This contradiction is exactly what is meant by saying that the sentence $(\alpha \wedge \neg\beta)$ is unsatisfiable.

Reductio ad absurdum
Refutation
Contradiction

7.5.1 Inference and proofs

This section covers **inference rules** that can be applied to derive a **proof**—a chain of conclusions that leads to the desired goal. The best-known rule is called **Modus Ponens** (Latin for *mode that affirms*) and is written

Inference rules
Proof
Modus Ponens

$$\frac{\alpha \Rightarrow \beta, \quad \alpha}{\beta}$$

The notation means that, whenever any sentences of the form $\alpha \Rightarrow \beta$ and α are given, then the sentence β can be inferred. For example, if $(\mathit{WumpusAhead} \wedge \mathit{WumpusAlive}) \Rightarrow \mathit{Shoot}$ and $(\mathit{WumpusAhead} \wedge \mathit{WumpusAlive})$ are given, then *Shoot* can be inferred.

Another useful inference rule is **And-Elimination**, which says that, from a conjunction, any of the conjuncts can be inferred:

And-Elimination

$$\frac{\alpha \wedge \beta}{\alpha}.$$

For example, from $(\mathit{WumpusAhead} \wedge \mathit{WumpusAlive})$, *WumpusAlive* can be inferred.

By considering the possible truth values of α and β, one can easily show once and for all that Modus Ponens and And-Elimination are sound. These rules can then be used in any particular instances where they apply, generating sound inferences without the need for enumerating models.

All of the logical equivalences in Figure 7.11 can be used as inference rules. For example, the equivalence for biconditional elimination yields the two inference rules

$$\frac{\alpha \Leftrightarrow \beta}{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)} \quad \text{and} \quad \frac{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)}{\alpha \Leftrightarrow \beta}.$$

Not all inference rules work in both directions like this. For example, we cannot run Modus Ponens in the opposite direction to obtain $\alpha \Rightarrow \beta$ and α from β.

Let us see how these inference rules and equivalences can be used in the wumpus world. We start with the knowledge base containing R_1 through R_5 and show how to prove $\neg P_{1,2}$, that is, there is no pit in [1,2]:

1. Apply biconditional elimination to R_2 to obtain
$$R_6 : \quad (B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1}).$$
2. Apply And-Elimination to R_6 to obtain
$$R_7 : \quad ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1}).$$
3. Logical equivalence for contrapositives gives
$$R_8 : \quad (\neg B_{1,1} \Rightarrow \neg(P_{1,2} \vee P_{2,1})).$$
4. Apply Modus Ponens with R_8 and the percept R_4 (i.e., $\neg B_{1,1}$), to obtain
$$R_9 : \quad \neg(P_{1,2} \vee P_{2,1}).$$
5. Apply De Morgan's rule, giving the conclusion
$$R_{10} : \quad \neg P_{1,2} \wedge \neg P_{2,1}.$$
That is, neither [1,2] nor [2,1] contains a pit.

Any of the search algorithms in Chapter 3 can be used to find a sequence of steps that constitutes a proof like this. We just need to define a proof problem as follows:

- INITIAL STATE: the initial knowledge base.
- ACTIONS: the set of actions consists of all the inference rules applied to all the sentences that match the top half of the inference rule.
- RESULT: the result of an action is to add the sentence in the bottom half of the inference rule.
- GOAL: the goal is a state that contains the sentence we are trying to prove.

Thus, searching for proofs is an alternative to enumerating models. In many practical cases *finding a proof can be more efficient because the proof can ignore irrelevant propositions, no matter how many of them there are.* For example, the proof just given leading to $\neg P_{1,2} \wedge \neg P_{2,1}$ does not mention the propositions $B_{2,1}$, $P_{1,1}$, $P_{2,2}$, or $P_{3,1}$. They can be ignored because the goal proposition, $P_{1,2}$, appears only in sentence R_2; the other propositions in R_2 appear only in R_4 and R_2; so R_1, R_3, and R_5 have no bearing on the proof. The same would hold even if we added a million more sentences to the knowledge base; the simple truth-table algorithm, on the other hand, would be overwhelmed by the exponential explosion of models.

One final property of logical systems is **monotonicity**, which says that the set of entailed sentences can only *increase* as information is added to the knowledge base.[9] For any sentences α and β, Monotonicity

if $KB \models \alpha$ then $KB \wedge \beta \models \alpha$.

For example, suppose the knowledge base contains the additional assertion β stating that there are exactly eight pits in the world. This knowledge might help the agent draw *additional* conclusions, but it cannot invalidate any conclusion α already inferred—such as the conclusion that there is no pit in [1,2]. Monotonicity means that inference rules can be applied whenever suitable premises are found in the knowledge base—the conclusion of the rule must follow *regardless of what else is in the knowledge base*.

7.5.2 Proof by resolution

We have argued that the inference rules covered so far are *sound*, but we have not discussed the question of *completeness* for the inference algorithms that use them. Search algorithms such as iterative deepening search (page 99) are complete in the sense that they will find any reachable goal, but if the available inference rules are inadequate, then the goal is not reachable—no proof exists that uses only those inference rules. For example, if we removed the biconditional elimination rule, the proof in the preceding section would not go through. The current section introduces a single inference rule, **resolution**, that yields a complete inference algorithm when coupled with any complete search algorithm.

We begin by using a simple version of the resolution rule in the wumpus world. Let us consider the steps leading up to Figure 7.4(a): the agent returns from [2,1] to [1,1] and then goes to [1,2], where it perceives a stench, but no breeze. We add the following facts to the knowledge base:

R_{11} : $\neg B_{1,2}$.
R_{12} : $B_{1,2} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{1,3})$.

By the same process that led to R_{10} earlier, we can now derive the absence of pits in [2,2] and [1,3] (remember that [1,1] is already known to be pitless):

R_{13} : $\neg P_{2,2}$.
R_{14} : $\neg P_{1,3}$.

We can also apply biconditional elimination to R_3, followed by Modus Ponens with R_5, to obtain the fact that there is a pit in [1,1], [2,2], or [3,1]:

R_{15} : $P_{1,1} \vee P_{2,2} \vee P_{3,1}$.

Now comes the first application of the resolution rule: the literal $\neg P_{2,2}$ in R_{13} *resolves with* the literal $P_{2,2}$ in R_{15} to give the **resolvent** Resolvent

R_{16} : $P_{1,1} \vee P_{3,1}$.

In English; if there's a pit in one of [1,1], [2,2], and [3,1] and it's not in [2,2], then it's in [1,1] or [3,1]. Similarly, the literal $\neg P_{1,1}$ in R_1 resolves with the literal $P_{1,1}$ in R_{16} to give

R_{17} : $P_{3,1}$.

In English: if there's a pit in [1,1] or [3,1] and it's not in [1,1], then it's in [3,1]. These last two inference steps are examples of the **unit resolution** inference rule Unit resolution

[9] **Nonmonotonic** logics, which violate the monotonicity property, capture a common property of human reasoning: changing one's mind. They are discussed in Section 10.6.

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k}$$

Complementary literals
Clause

where each ℓ is a literal and ℓ_i and m are **complementary literals** (i.e., one is the negation of the other). Thus, the unit resolution rule takes a **clause**—a disjunction of literals—and a literal and produces a new clause. Note that a single literal can be viewed as a disjunction of one literal, also known as a **unit clause**.

Unit clause
Resolution

The unit resolution rule can be generalized to the full **resolution** rule

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

where ℓ_i and m_j are complementary literals. This says that resolution takes two clauses and produces a new clause containing all the literals of the two original clauses *except* the two complementary literals. For example, we have

$$\frac{P_{1,1} \vee P_{3,1}, \qquad \neg P_{1,1} \vee \neg P_{2,2}}{P_{3,1} \vee \neg P_{2,2}}.$$

You can resolve only one pair of complementary literals at a time. For example, we can resolve P and $\neg P$ to deduce

$$\frac{P \vee \neg Q \vee R, \qquad \neg P \vee Q}{\neg Q \vee Q \vee R},$$

but you can't resolve on both P and Q at once to infer R. There is one more technical aspect of the resolution rule: the resulting clause should contain only one copy of each literal.[10] The removal of multiple copies of literals is called **factoring**. For example, if we resolve $(A \vee B)$ with $(A \vee \neg B)$, we obtain $(A \vee A)$, which is reduced to just A by factoring.

Factoring

The *soundness* of the resolution rule can be seen easily by considering the literal ℓ_i that is complementary to literal m_j in the other clause. If ℓ_i is true, then m_j is false, and hence $m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n$ must be true, because $m_1 \vee \cdots \vee m_n$ is given. If ℓ_i is false, then $\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k$ must be true because $\ell_1 \vee \cdots \vee \ell_k$ is given. Now ℓ_i is either true or false, so one or other of these conclusions holds—exactly as the resolution rule states.

What is more surprising about the resolution rule is that it forms the basis for a family of *complete* inference procedures. *A resolution-based theorem prover can, for any sentences α and β in propositional logic, decide whether $\alpha \models \beta$.* The next two subsections explain how resolution accomplishes this.

Conjunctive normal form

The resolution rule applies only to clauses (that is, disjunctions of literals), so it would seem to be relevant only to knowledge bases and queries consisting of clauses. How, then, can it lead to a complete inference procedure for all of propositional logic? The answer is that *every sentence of propositional logic is logically equivalent to a conjunction of clauses.*

A sentence expressed as a conjunction of clauses is said to be in **conjunctive normal form** or **CNF** (see Figure 7.12). We now describe a procedure for converting to CNF. We illustrate the procedure by converting the sentence $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$ into CNF. The steps are as follows:

Conjunctive normal form
CNF

[10] If a clause is viewed as a *set* of literals, then this restriction is automatically respected. Using set notation for clauses makes the resolution rule much cleaner, at the cost of introducing additional notation.

$$
\begin{aligned}
\mathit{CNFSentence} &\rightarrow \mathit{Clause}_1 \wedge \cdots \wedge \mathit{Clause}_n \\
\mathit{Clause} &\rightarrow \mathit{Literal}_1 \vee \cdots \vee \mathit{Literal}_m \\
\mathit{Fact} &\rightarrow \mathit{Symbol} \\
\mathit{Literal} &\rightarrow \mathit{Symbol} \mid \neg \mathit{Symbol} \\
\mathit{Symbol} &\rightarrow P \mid Q \mid R \mid \ldots \\
\mathit{HornClauseForm} &\rightarrow \mathit{DefiniteClauseForm} \mid \mathit{GoalClauseForm} \\
\mathit{DefiniteClauseForm} &\rightarrow \mathit{Fact} \mid (\mathit{Symbol}_1 \wedge \cdots \wedge \mathit{Symbol}_l) \Rightarrow \mathit{Symbol} \\
\mathit{GoalClauseForm} &\rightarrow (\mathit{Symbol}_1 \wedge \cdots \wedge \mathit{Symbol}_l) \Rightarrow \mathit{False}
\end{aligned}
$$

Figure 7.12 A grammar for conjunctive normal form, Horn clauses, and definite clauses. A CNF clause such as $\neg A \vee \neg B \vee C$ can be written in definite clause form as $A \wedge B \Rightarrow C$.

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.

   $$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1}) .$$

2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$:

   $$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1}) .$$

3. CNF requires $\neg$ to appear only in literals, so we "move $\neg$ inwards" by repeated application of the following equivalences from Figure 7.11:

   $$\neg(\neg\alpha) \equiv \alpha \quad \text{(double-negation elimination)}$$
   $$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta) \quad \text{(De Morgan)}$$
   $$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta) \quad \text{(De Morgan)}$$

   In the example, we require just one application of the last rule:

   $$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1}) .$$

4. Now we have a sentence containing nested $\wedge$ and $\vee$ operators applied to literals. We apply the distributivity law from Figure 7.11, distributing $\vee$ over $\wedge$ wherever possible.

   $$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1}) .$$

The original sentence is now in CNF, as a conjunction of three clauses. It is much harder to read, but it can be used as input to a resolution procedure.

A resolution algorithm

Inference procedures based on resolution work by using the principle of proof by contradiction introduced on page 241. That is, to show that $KB \models \alpha$, we show that $(KB \wedge \neg\alpha)$ is unsatisfiable. We do this by proving a contradiction.

A resolution algorithm is shown in Figure 7.13. First, $(KB \wedge \neg\alpha)$ is converted into CNF. Then, the resolution rule is applied to the resulting clauses. Each pair that contains complementary literals is resolved to produce a new clause, which is added to the set if it is not already present. The process continues until one of two things happens:

- there are no new clauses that can be added, in which case KB does not entail α; or,
- two clauses resolve to yield the *empty* clause, in which case KB entails α.

```
function PL-RESOLUTION(KB, α) returns true or false
  inputs: KB, the knowledge base, a sentence in propositional logic
          α, the query, a sentence in propositional logic

  clauses ← the set of clauses in the CNF representation of KB ∧ ¬α
  new ← { }
  while true do
    for each pair of clauses C_i, C_j in clauses do
      resolvents ← PL-RESOLVE(C_i, C_j)
      if resolvents contains the empty clause then return true
      new ← new ∪ resolvents
    if new ⊆ clauses then return false
    clauses ← clauses ∪ new
```

Figure 7.13 A simple resolution algorithm for propositional logic. PL-RESOLVE returns the set of all possible clauses obtained by resolving its two inputs.

The empty clause—a disjunction of no disjuncts—is equivalent to *False* because a disjunction is true only if at least one of its disjuncts is true. Moreover, the empty clause arises only from resolving two contradictory unit clauses such as P and $\neg P$.

We can apply the resolution procedure to a very simple inference in the wumpus world. When the agent is in [1,1], there is no breeze, so there can be no pits in neighboring squares. The relevant knowledge base is

$$KB = R_2 \wedge R_4 = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$$

and we wish to prove α, which is, say, $\neg P_{1,2}$. When we convert $(KB \wedge \neg\alpha)$ into CNF, we obtain the clauses shown at the top of Figure 7.14. The second row of the figure shows clauses obtained by resolving pairs in the first row. Then, when $P_{1,2}$ is resolved with $\neg P_{1,2}$, we obtain the empty clause, shown as a small square. Inspection of Figure 7.14 reveals that many resolution steps are pointless. For example, the clause $B_{1,1} \vee \neg B_{1,1} \vee P_{1,2}$ is equivalent to $True \vee P_{1,2}$ which is equivalent to *True*. Deducing that *True* is true is not very helpful. Therefore, any clause in which two complementary literals appear can be discarded.

Completeness of resolution

Resolution closure

To conclude our discussion of resolution, we now show why PL-RESOLUTION is complete. To do this, we introduce the **resolution closure** $RC(S)$ of a set of clauses S, which is the set of all clauses derivable by repeated application of the resolution rule to clauses in S or their derivatives. The resolution closure is what PL-RESOLUTION computes as the final value of the variable *clauses*. It is easy to see that $RC(S)$ must be finite: thanks to the factoring step, there are only finitely many distinct clauses that can be constructed out of the symbols $P_1, \ldots, P_k$ that appear in S. Hence, PL-RESOLUTION always terminates.

Ground resolution theorem

The completeness theorem for resolution in propositional logic is called the **ground resolution theorem**:

> If a set of clauses is unsatisfiable, then the resolution closure of those clauses contains the empty clause.

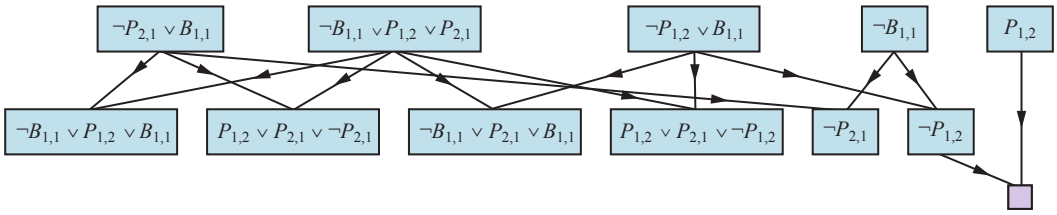


Figure 7.14 Partial application of PL-RESOLUTION to a simple inference in the wumpus world to prove the query $\neg P_{1,2}$. Each of the leftmost four clauses in the top row is paired with each of the other three, and the resolution rule is applied to yield the clauses on the bottom row. We see that the third and fourth clauses on the top row combine to yield the clause $\neg P_{1,2}$, which is then resolved with $P_{1,2}$ to yield the empty clause, meaning that the query is proven.

This theorem is proved by demonstrating its contrapositive: if the closure $RC(S)$ does *not* contain the empty clause, then S is satisfiable. In fact, we can construct a model for S with suitable truth values for $P_1, \ldots, P_k$. The construction procedure is as follows:

For i from 1 to k,

- If a clause in $RC(S)$ contains the literal $\neg P_i$ and all its other literals are false under the assignment chosen for $P_1, \ldots, P_{i-1}$, then assign *false* to P_i.
- Otherwise, assign *true* to P_i.

This assignment to $P_1, \ldots, P_k$ is a model of S. To see this, assume the opposite—that, at some stage i in the sequence, assigning symbol P_i causes some clause C to become false. For this to happen, it must be the case that all the *other* literals in C must already have been falsified by assignments to $P_1, \ldots, P_{i-1}$. Thus, C must now look like either $(false \vee false \vee \cdots false \vee P_i)$ or like $(false \vee false \vee \cdots false \vee \neg P_i)$. If just one of these two is in $RC(S)$, then the algorithm will assign the appropriate truth value to P_i to make C true, so C can only be falsified if *both* of these clauses are in $RC(S)$.

Now, since $RC(S)$ is closed under resolution, it will contain the resolvent of these two clauses, and that resolvent will have all of its literals already falsified by the assignments to $P_1, \ldots, P_{i-1}$. This contradicts our assumption that the first falsified clause appears at stage i. Hence, we have proved that the construction never falsifies a clause in $RC(S)$; that is, it produces a model of $RC(S)$. Finally, because S is contained in $RC(S)$, any model of $RC(S)$ is a model of S itself.

7.5.3 Horn clauses and definite clauses

The completeness of resolution makes it a very important inference method. In many practical situations, however, the full power of resolution is not needed. Some real-world knowledge bases satisfy certain restrictions on the form of sentences they contain, which enables them to use a more restricted and efficient inference algorithm.

One such restricted form is the **definite clause**, which is a disjunction of literals of which *exactly one is positive*. For example, the clause $(\neg L_{1,1} \vee \neg Breeze \vee B_{1,1})$ is a definite clause, whereas $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1})$ is not, because it has two positive clauses. Definite clause

Slightly more general is the **Horn clause**, which is a disjunction of literals of which *at* Horn clause

most one is positive. So all definite clauses are Horn clauses, as are clauses with no positive literals; these are called **goal clauses**. Horn clauses are closed under resolution: if you resolve two Horn clauses, you get back a Horn clause. One more class is the k-CNF sentence, which is a CNF sentence where each clause has at most k literals.

Goal clauses

Knowledge bases containing only definite clauses are interesting for three reasons:

1. Every definite clause can be written as an implication whose premise is a conjunction of positive literals and whose conclusion is a single positive literal. (See Exercise 7.DISJ.) For example, the definite clause $(\neg L_{1,1} \vee \neg Breeze \vee B_{1,1})$ can be written as the implication $(L_{1,1} \wedge Breeze) \Rightarrow B_{1,1}$. In the implication form, the sentence is easier to understand: it says that if the agent is in [1,1] and there is a breeze percept, then [1,1] is breezy. In Horn form, the premise is called the **body** and the conclusion is called the **head**. A sentence consisting of a single positive literal, such as $L_{1,1}$, is called a **fact**. It too can be written in implication form as $True \Rightarrow L_{1,1}$, but it is simpler to write just $L_{1,1}$.

Body
Head
Fact

2. Inference with Horn clauses can be done through the **forward-chaining** and **backward-chaining** algorithms, which we explain next. Both of these algorithms are natural, in that the inference steps are obvious and easy for humans to follow. This type of inference is the basis for **logic programming**, which is discussed in Chapter 9.

Forward-chaining
Backward-chaining

3. Deciding entailment with Horn clauses can be done in time that is *linear* in the size of the knowledge base—a pleasant surprise.

7.5.4 Forward and backward chaining

The forward-chaining algorithm PL-FC-ENTAILS?(KB, q) determines if a single proposition symbol q—the query—is entailed by a knowledge base of definite clauses. It begins from known facts (positive literals) in the knowledge base. If all the premises of an implication are known, then its conclusion is added to the set of known facts. For example, if $L_{1,1}$ and $Breeze$ are known and $(L_{1,1} \wedge Breeze) \Rightarrow B_{1,1}$ is in the knowledge base, then $B_{1,1}$ can be added. This process continues until the query q is added or until no further inferences can be made. The algorithm is shown in Figure 7.15; the main point to remember is that it runs in linear time.

The best way to understand the algorithm is through an example and a picture. Figure 7.16(a) shows a simple knowledge base of Horn clauses with A and B as known facts. Figure 7.16(b) shows the same knowledge base drawn as an **AND–OR graph** (see Chapter 4). In AND–OR graphs, multiple edges joined by an arc indicate a conjunction—every edge must be proved—while multiple edges without an arc indicate a disjunction—any edge can be proved. It is easy to see how forward chaining works in the graph. The known leaves (here, A and B) are set, and inference propagates up the graph as far as possible. Wherever a conjunction appears, the propagation waits until all the conjuncts are known before proceeding. The reader is encouraged to work through the example in detail.

It is easy to see that forward chaining is **sound**: every inference is essentially an application of Modus Ponens. Forward chaining is also **complete**: every entailed atomic sentence will be derived. The easiest way to see this is to consider the final state of the *inferred* table (after the algorithm reaches a fixed point where no new inferences are possible). The table contains *true* for each symbol inferred during the process, and *false* for all other symbols. We can view the table as a logical model; moreover, *every definite clause in the original KB is true in this model.*

```
function PL-FC-ENTAILS?(KB, q) returns true or false
  inputs: KB, the knowledge base, a set of propositional definite clauses
          q, the query, a proposition symbol
  count ← a table, where count[c] is initially the number of symbols in clause c's premise
  inferred ← a table, where inferred[s] is initially false for all symbols
  queue ← a queue of symbols, initially symbols known to be true in KB

  while queue is not empty do
    p ← POP(queue)
    if p = q then return true
    if inferred[p] = false then
      inferred[p] ← true
      for each clause c in KB where p is in c.PREMISE do
        decrement count[c]
        if count[c] = 0 then add c.CONCLUSION to queue
  return false
```

Figure 7.15 The forward-chaining algorithm for propositional logic. The *queue* keeps track of symbols known to be true but not yet "processed." The *count* table keeps track of how many premises of each implication are not yet proven. Whenever a new symbol p from the agenda is processed, the count is reduced by one for each implication in whose premise p appears (easily identified in constant time with appropriate indexing.) If a count reaches zero, all the premises of the implication are known, so its conclusion can be added to the agenda. Finally, we need to keep track of which symbols have been processed; a symbol that is already in the set of inferred symbols need not be added to the agenda again. This avoids redundant work and prevents loops caused by implications such as $P \Rightarrow Q$ and $Q \Rightarrow P$.

To see this, assume the opposite, namely that some clause $a_1 \wedge \ldots \wedge a_k \Rightarrow b$ is false in the model. Then $a_1 \wedge \ldots \wedge a_k$ must be true in the model and b must be false in the model. But this contradicts our assumption that the algorithm has reached a fixed point, because we would now be licensed to add b to the KB. We can conclude, therefore, that the set of atomic sentences inferred at the fixed point defines a model of the original KB. Furthermore, any atomic sentence q that is entailed by the KB must be true in all its models and in this model in particular. Hence, every entailed atomic sentence q must be inferred by the algorithm.

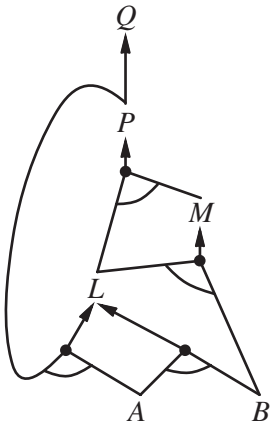
Forward chaining is an example of the general concept of **data-driven** reasoning—that is, reasoning in which the focus of attention starts with the known data. It can be used within an agent to derive conclusions from incoming percepts, often without a specific query in mind. For example, the wumpus agent might TELL its percepts to the knowledge base using an incremental forward-chaining algorithm in which new facts can be added to the agenda to initiate new inferences. In humans, a certain amount of data-driven reasoning occurs as new information arrives. For example, if I am indoors and hear rain starting to fall, it might occur to me that the picnic will be canceled. Yet it will probably not occur to me that the seventeenth petal on the largest rose in my neighbor's garden will get wet; humans keep forward chaining under careful control, lest they be swamped with irrelevant consequences.

Data-driven

The backward-chaining algorithm, as its name suggests, works backward from the query. If the query q is known to be true, then no work is needed. Otherwise, the algorithm finds

$P \Rightarrow Q$
$L \wedge M \Rightarrow P$
$B \wedge L \Rightarrow M$
$A \wedge P \Rightarrow L$
$A \wedge B \Rightarrow L$
A
B

(a)



(b)

Figure 7.16 (a) A set of Horn clauses. (b) The corresponding AND–OR graph.

those implications in the knowledge base whose conclusion is q. If all the premises of one of those implications can be proved true (by backward chaining), then q is true. When applied to the query Q in Figure 7.16, it works back down the graph until it reaches a set of known facts, A and B, that forms the basis for a proof. The algorithm is essentially identical to the AND-OR-GRAPH-SEARCH algorithm in Figure 4.11. As with forward chaining, an efficient implementation runs in linear time.

Goal-directed reasoning

Backward chaining is a form of **goal-directed reasoning**. It is useful for answering specific questions such as "What shall I do now?" and "Where are my keys?" Often, the cost of backward chaining is *much less* than linear in the size of the knowledge base, because the process touches only relevant facts.

7.6 Effective Propositional Model Checking

In this section, we describe two families of efficient algorithms for general propositional inference based on model checking: one approach based on backtracking search, and one on local hill-climbing search. These algorithms are part of the "technology" of propositional logic. This section can be skimmed on a first reading of the chapter.

The algorithms we describe are for checking satisfiability: the SAT problem. (As noted in Section 7.5, testing entailment, $\alpha \models \beta$, can be done by testing *un*satisfiability of $\alpha \wedge \neg\beta$.) We mentioned on page 241 the connection between finding a satisfying model for a logical sentence and finding a solution for a constraint satisfaction problem, so it is perhaps not surprising that the two families of propositional satisfiability algorithms closely resemble the backtracking algorithms of Section 5.3 and the local search algorithms of Section 5.4. They are, however, extremely important in their own right because so many combinatorial problems in computer science can be reduced to checking the satisfiability of a propositional sentence. Any improvement in satisfiability algorithms has huge consequences for our ability to handle complexity in general.

7.6.1 A complete backtracking algorithm

Davis–Putnam algorithm

The first algorithm we consider is often called the **Davis–Putnam algorithm**, after the seminal paper by Martin Davis and Hilary Putnam (1960). The algorithm is in fact the version described by Davis, Logemann, and Loveland (1962), so we will call it DPLL after the initials of all four authors. DPLL takes as input a sentence in conjunctive normal form—a set of clauses. Like BACKTRACKING-SEARCH and TT-ENTAILS?, it is essentially a recursive, depth-first enumeration of possible models. It embodies three improvements over the simple scheme of TT-ENTAILS?:

- *Early termination*: The algorithm detects whether the sentence must be true or false, even with a partially completed model. A clause is true if *any* literal is true, even if the other literals do not yet have truth values; hence, the sentence as a whole could be judged true even before the model is complete. For example, the sentence $(A \vee B) \wedge (A \vee C)$ is true if A is true, regardless of the values of B and C. Similarly, a sentence is false if *any* clause is false, which occurs when each of its literals is false. Again, this can occur long before the model is complete. Early termination avoids examination of entire subtrees in the search space.
- *Pure symbol heuristic*: A **pure symbol** is a symbol that always appears with the same "sign" in all clauses. For example, in the three clauses $(A \vee \neg B)$, $(\neg B \vee \neg C)$, and $(C \vee A)$, the symbol A is pure because only the positive literal appears, B is pure because only the negative literal appears, and C is impure. It is easy to see that if a sentence has a model, then it has a model with the pure symbols assigned so as to make their literals *true*, because doing so can never make a clause false. Note that, in determining the purity of a symbol, the algorithm can ignore clauses that are already known to be true in the model constructed so far. For example, if the model contains $B = \mathit{false}$, then the clause $(\neg B \vee \neg C)$ is already true, and in the remaining clauses C appears only as a positive literal; therefore C becomes pure.

  Pure symbol
- *Unit clause heuristic*: A **unit clause** was defined earlier as a clause with just one literal. In the context of DPLL, it also means clauses in which all literals but one are already assigned *false* by the model. For example, if the model contains $B = \mathit{true}$, then $(\neg B \vee \neg C)$ simplifies to $\neg C$, which is a unit clause. Obviously, for this clause to be true, C must be set to *false*. The unit clause heuristic assigns all such symbols before branching on the remainder. One important consequence of the heuristic is that any attempt to prove (by refutation) a literal that is already in the knowledge base will succeed immediately (Exercise 7.KNOW). Notice also that assigning one unit clause can create another unit clause—for example, when C is set to *false*, $(C \vee A)$ becomes a unit clause, causing *true* to be assigned to A. This "cascade" of forced assignments is called **unit propagation**. It resembles the process of forward chaining with definite clauses, and indeed, if the CNF expression contains only definite clauses then DPLL essentially replicates forward chaining. (See Exercise 7.DPLL.)

  Unit propagation

The DPLL algorithm is shown in Figure 7.17, which gives the essential skeleton of the search process without the implementation details.

What Figure 7.17 does not show are the tricks that enable SAT solvers to scale up to large problems. It is interesting that most of these tricks are in fact rather general, and we have seen them before in other guises:

```
function DPLL-SATISFIABLE?(s) returns true or false
  inputs: s, a sentence in propositional logic

  clauses ← the set of clauses in the CNF representation of s
  symbols ← a list of the proposition symbols in s
  return DPLL(clauses, symbols, { })

function DPLL(clauses, symbols, model) returns true or false

  if every clause in clauses is true in model then return true
  if some clause in clauses is false in model then return false
  P, value ← FIND-PURE-SYMBOL(symbols, clauses, model)
  if P is non-null then return DPLL(clauses, symbols – P, model ∪ {P=value})
  P, value ← FIND-UNIT-CLAUSE(clauses, model)
  if P is non-null then return DPLL(clauses, symbols – P, model ∪ {P=value})
  P ← FIRST(symbols); rest ← REST(symbols)
  return DPLL(clauses, rest, model ∪ {P=true}) or
         DPLL(clauses, rest, model ∪ {P=false})
```

Figure 7.17 The DPLL algorithm for checking satisfiability of a sentence in propositional logic. The ideas behind FIND-PURE-SYMBOL and FIND-UNIT-CLAUSE are described in the text; each returns a symbol (or null) and the truth value to assign to that symbol. Like TT-ENTAILS?, DPLL operates over partial models.

1. **Component analysis** (as seen with Tasmania in CSPs): As DPLL assigns truth values to variables, the set of clauses may become separated into disjoint subsets, called **components**, that share no unassigned variables. Given an efficient way to detect when this occurs, a solver can gain considerable speed by working on each component separately.
2. **Variable and value ordering** (as seen in Section 5.3.1 for CSPs): Our simple implementation of DPLL uses an arbitrary variable ordering and always tries the value *true* before *false*. The **degree heuristic** (see page 177) suggests choosing the variable that appears most frequently over all remaining clauses.
3. **Intelligent backtracking** (as seen in Section 5.3.3 for CSPs): Many problems that cannot be solved in hours of run time with chronological backtracking can be solved in seconds with intelligent backtracking that backs up all the way to the relevant point of conflict. All SAT solvers that do intelligent backtracking use some form of **conflict clause learning** to record conflicts so that they won't be repeated later in the search. Usually a limited-size set of conflicts is kept, and rarely used ones are dropped.
4. **Random restarts** (as seen on page 131 for hill climbing): Sometimes a run appears not to be making progress. In this case, we can start over from the top of the search tree, rather than trying to continue. After restarting, different random choices (in variable and value selection) are made. Clauses that are learned in the first run are retained after the restart and can help prune the search space. Restarting does not guarantee that a solution will be found faster, but it does reduce the variance on the time to solution.
5. **Clever indexing** (as seen in many algorithms): The speedup methods used in DPLL itself, as well as the tricks used in modern solvers, require fast indexing of such things

```
function WALKSAT(clauses, p, max_flips) returns a satisfying model or failure
  inputs: clauses, a set of clauses in propositional logic
          p, the probability of choosing to do a "random walk" move, typically around 0.5
          max_flips, number of value flips allowed before giving up

  model ← a random assignment of true/false to the symbols in clauses
  for each i = 1 to max_flips do
      if model satisfies clauses then return model
      clause ← a randomly selected clause from clauses that is false in model
      if RANDOM(0, 1) ≤ p then
          flip the value in model of a randomly selected symbol from clause
      else flip whichever symbol in clause maximizes the number of satisfied clauses
  return failure
```

Figure 7.18 The WALKSAT algorithm for checking satisfiability by randomly flipping the values of variables. Many versions of the algorithm exist.

as "the set of clauses in which variable X_i appears as a positive literal." This task is complicated by the fact that the algorithms are interested only in the clauses that have not yet been satisfied by previous assignments to variables, so the indexing structures must be updated dynamically as the computation proceeds.

With these enhancements, modern solvers can handle problems with tens of millions of variables. They have revolutionized areas such as hardware verification and security protocol verification, which previously required laborious, hand-guided proofs.

7.6.2 Local search algorithms

We have seen several local search algorithms so far in this book, including HILL-CLIMBING (page 129) and SIMULATED-ANNEALING (page 133). These algorithms can be applied directly to satisfiability problems, provided that we choose the right evaluation function. Because the goal is to find an assignment that satisfies every clause, an evaluation function that counts the number of unsatisfied clauses will do the job. In fact, this is exactly the measure used by the MIN-CONFLICTS algorithm for CSPs (page 182). All these algorithms take steps in the space of complete assignments, flipping the truth value of one symbol at a time. The space usually contains many local minima, to escape from which various forms of randomness are required. In recent years, there has been a great deal of experimentation to find a good balance between greediness and randomness.

One of the simplest and most effective algorithms to emerge from all this work is called WALKSAT (Figure 7.18). On every iteration, the algorithm picks an unsatisfied clause and picks a symbol in the clause to flip. It chooses randomly between two ways to pick which symbol to flip: (1) a "min-conflicts" step that minimizes the number of unsatisfied clauses in the new state and (2) a "random walk" step that picks the symbol randomly.

When WALKSAT returns a model, the input sentence is indeed satisfiable, but when it returns *failure*, there are two possible causes: either the sentence is unsatisfiable or we need to give the algorithm more time. If we set $max\_flips = \infty$ and $p > 0$, WALKSAT will eventually return a model (if one exists), because the random-walk steps will eventually hit upon the