Solar Winds
Semester assignment TFY4240

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We model the magnetic field of the earth as a magnetic point dipole and the solar winds as charged particles that impinge upon it. The qualitative behaviour of the trajectories are compared with what we expect in the first approximation. The requirement of constant energy is used as a test criterion for the validity of the solution.

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1 Theory

We model the magnetic field from the earth as a point dipole at its centre, giving the field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{(\mathbf{m}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{m}}{r^5}\right) \tag{1}$$

where $\mathbf{m}$ is the dipole moment. The magnitude of earth's dipole moment is approximately $8{\cdot}10^{22}\,\mathrm{Am}^2$ [1]. The equations of motion for a particle moving in the presence of a magnetic field $\mathbf{B}$ is given by Newton's second law and the Lorentz force (with $\mathbf{E} = 0$):

$$\ddot{\mathbf{r}} = \frac{1}{m} q\mathbf{v} \times \mathbf{B} \tag{2}$$

To solve the equation numerically, we recast it into dimensionless form. Define the following dimensionless quantities

$$\boldsymbol{\xi} := \frac{\mathbf{r}}{a} \quad ; \quad \hat{\mathbf{B}} := \frac{\mathbf{B}}{B_0} \tag{3}$$

where $B_0 = \frac{\mu_0 m_0}{4\pi a^3}$. Here a denotes the average radius of the earth, and m_0 the dipole moment of earth's magnetic field. Inserting these definitions into the equation of motion in 2 yields

$$\ddot{\boldsymbol{\xi}} = \frac{q\mu_0 m_0}{4\pi m a^3}\dot{\boldsymbol{\xi}} \times \hat{\mathbf{B}}. \tag{4}$$

If we now also introduce a dimensionless time $\tau := t/t_0$ we see that we can identify the quantity

$$t_0 = \left(\frac{q\mu_0 m_0}{4\pi m a^3}\right)^{-1} \tag{5}$$

as a natural time scale for our problem. Hence the equation of motion is in dimensionless form written as

$$\frac{\mathrm{d}^2\boldsymbol{\xi}}{\mathrm{d}\tau^2} = \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}\tau} \times \hat{\mathbf{B}}. \tag{6}$$

Table 1: The constants involved in the dimensionless quantities.

Quantity	Value	Unit
m_0	$8 \cdot 10^{22}$	$\mathrm{A} \cdot \mathrm{m}^2$
q	$1.6 \cdot 10^{-19}$	C
m	$1.67 \cdot 10^{-27}$	kg
a	$6.4 \cdot 10^{6}$	m
μ_0	$4\pi \cdot 10^{-7}$	$\mathrm{N/A}^2$
t_0	$3.42 \cdot 10^{-4}$	s

What we observe here is however that the time scale set by the parameters of the problem is very short. We will adjust the time scale so that velocities $\mathcal{O}(1)$ are typical velocities of the solar winds. These velocities are in the range $250 - 750\,\mathrm{km/s}$ [2]. This is done by scaling the time by $1 \cdot 10^5$, whence the typical speeds are $\simeq 200\,\mathrm{km/s}$ To keep the simple form of (6) we scale $\hat{B}$ by the same factor. This ensures that the interesting behaviour of the particles is captured by the simulations.

What kind of motion do we expect?

This argument is slightly adapted from [3, sec. 12.4]

In the case under consideration, a perturbation solution to the motion gives adequate insight into how the particles move. When the distance over which **B** changes appreciably is large compared to the gyration radius of the motion[1], the lowest order approximation is a spiralling motion around the field lines, with a frequency given by the local field. The second term in the expansion of the solution will involve a slow change which can be described as drifting of the centre of the orbit.

By expanding the magnetic field in the expression for the gyration frequency to first order along the direction perpendicular to the field **n**, one obtains

$$\boldsymbol{\omega}_B = \frac{e}{\gamma mc}\mathbf{B}(\mathbf{x}) \simeq \boldsymbol{\omega}_0 \left[1 + \frac{1}{B_0}\frac{\partial B}{\partial \mathbf{n}}\bigg|_0 \cdot \mathbf{x}\right],$$

where the 0 subscripts denotes the quantity evaluated in the unperturbed case. Writing the transverse velocity $\mathbf{v}_\perp = \mathbf{v}_0 + \mathbf{v}_1$, we can substitute the above expression into the equation of motion in 2 to obtain

$$\frac{\mathrm{d}\mathbf{v}_\perp}{\mathrm{d}t} = \mathbf{v}_\perp \times \boldsymbol{\omega}_B(\mathbf{x}),$$

whence

$$\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} \simeq \left[\mathbf{v}_1 + \mathbf{v_0}\left(\frac{1}{B_0}\frac{\partial B}{\partial \mathbf{n}}\bigg|_0 \cdot \mathbf{x}_0\right)\right] \times \boldsymbol{\omega}_0.$$

One can show that $\mathbf{v}_1$ has, apart from oscillatory terms, a non-zero average value given by

$$\langle \mathbf{v}_1 \rangle \approx -\frac{r_l^2}{2}\frac{1}{B_0}\frac{\partial B}{\partial \mathbf{n}} \times \boldsymbol{\omega}_0 = \frac{\omega_B r_l^2}{2B^2}\left(\mathbf{B} \times \boldsymbol{\nabla}_\perp B\right), \tag{7}$$

where the last expression is the same only written in coordinate-independent form. r_l here denotes the gyration radius (Larmor radius). It is evident from this expression that if the gradient is slowly varying in space, i.e. $r_l|\boldsymbol{\nabla}_\perp B/B| \ll 1$, this time average will be much smaller than the orbital velocity $r_l\omega_B$. Hence, the particles will spiral rapidly while its centre of gyration slowly moves perpendicular to both **B** and $\boldsymbol{\nabla}_\perp B$.

One can also derive a similar expression for a drift of the centre of gyration caused by the curvature of the magnetic field. This drift velocity is approximately given by

$$\mathbf{v}_c = \frac{v_\parallel^2}{\omega_B R}\frac{\mathbf{R} \times \mathbf{B}}{RB_0}, \tag{8}$$

where **R** is the radius of curvature of the field, pointing from the centre of curvature to the particle. Hence, in the presence of spherical field lines the particle trajectories will tend to drift in the direction of $\mathbf{R} \times \mathbf{B}$, i.e. around the equator.

[1] This is certainly the case when considering protons in the earth's magnetic field.

2 Numerical implementation

To solve equation 6 we rewrite the second order equation as a coupled system of first order equations as

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}\tau} = \boldsymbol{\chi} \tag{9}$$

$$\frac{\mathrm{d}\boldsymbol{\chi}}{\mathrm{d}\tau} = \boldsymbol{\chi} \times \hat{\mathbf{B}}, \tag{10}$$

where $\boldsymbol{\chi}$ is the dimensionless velocity associated to $\boldsymbol{\xi}$ and τ. The system of equations is then solved by stacking $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$ into one vector, $\mathbf{X}$, and then applying an ODE-solver to the system.

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau} = \mathbf{f}(\mathbf{X}, \tau), \tag{11}$$

where $f_i = \chi_i$ and $f_{i+3} = \epsilon_{ijk}\chi_j\hat{B}_k$ for $i = 1, 2, 3$. We use the built-in ODE-solver in `scipy`, `odeint`, which in turn calls the ODE-solver `lsoda` written in `FORTRAN`, which is an adaptive-step Runge Kutta method for solving ODEs.

3 Results

A plot of the magnetic field from the earth is shown in figure 1 with the dimensionless length scales on the axes.

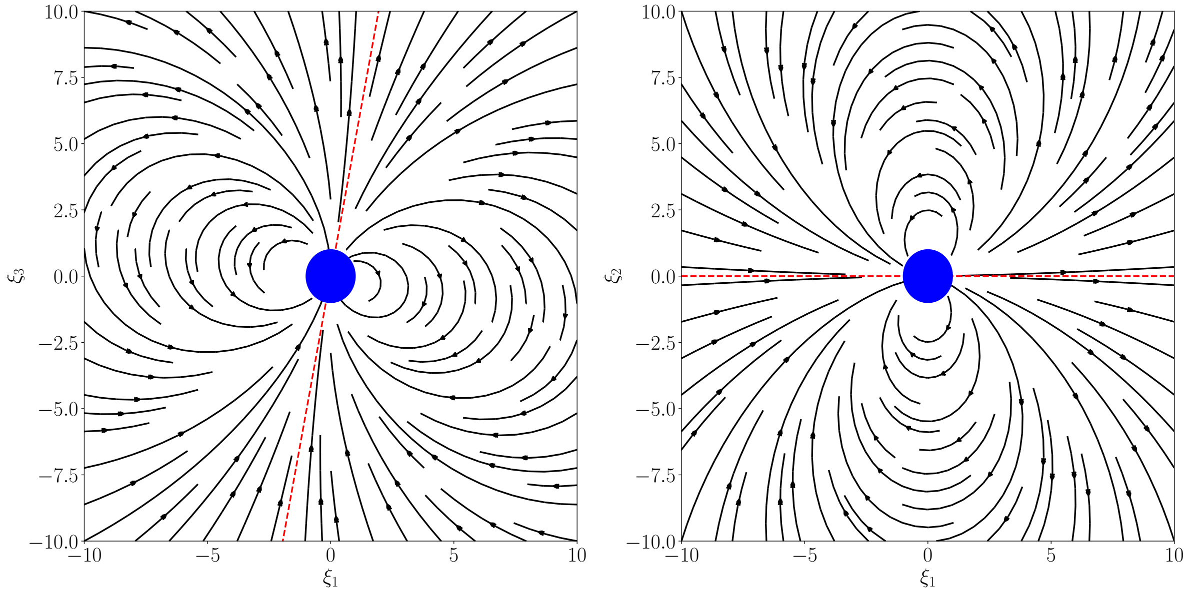

Figure 1: Magnetic field from earth.

When sending particles far from the earth towards it they start oscillating around the field lines as they approach the earth. This is consistent with the more familiar and simple case of Larmor precession in a uniform field. This is shown in figure 2 and 3. Moreover, in figure 2 we observe both drifts along $\mathbf{B}$ and $\boldsymbol{\nabla}_{\perp} B$, as predicted by equation (7), and a drift around the equator, as predicted by equation (8).

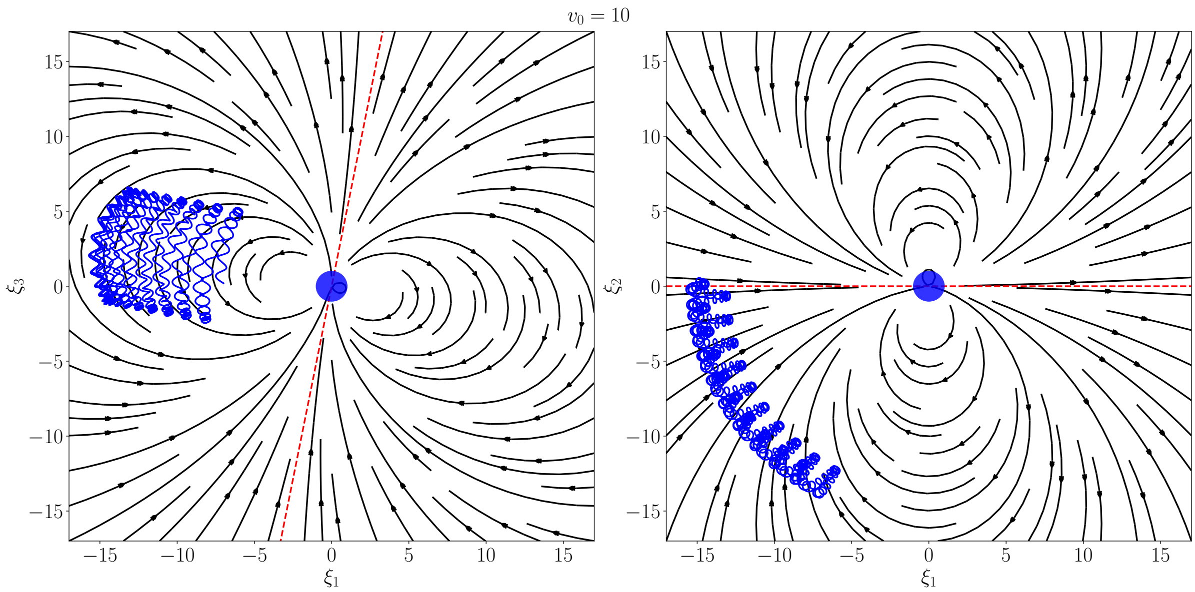

Figure 2: Particle sent towards the earth with typical solar wind velocity.

The trajectories of particles sent towards earth at different heights z (ξ_3) is shown in figure 4.

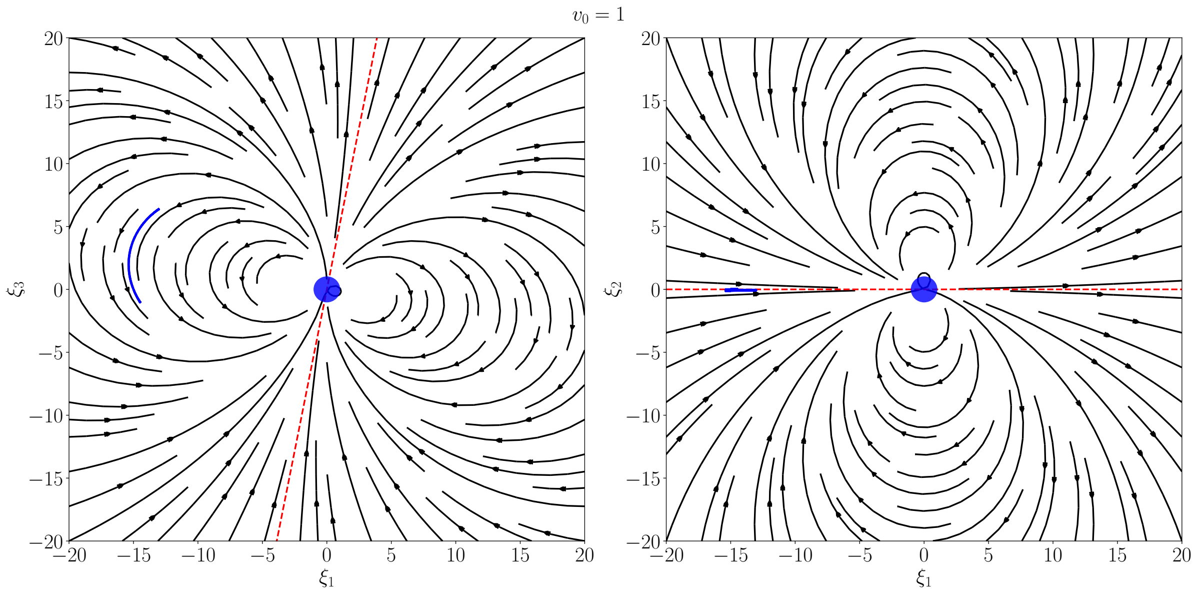

Figure 3: Particle sent towards the earth with lower typical solar wind velocity.

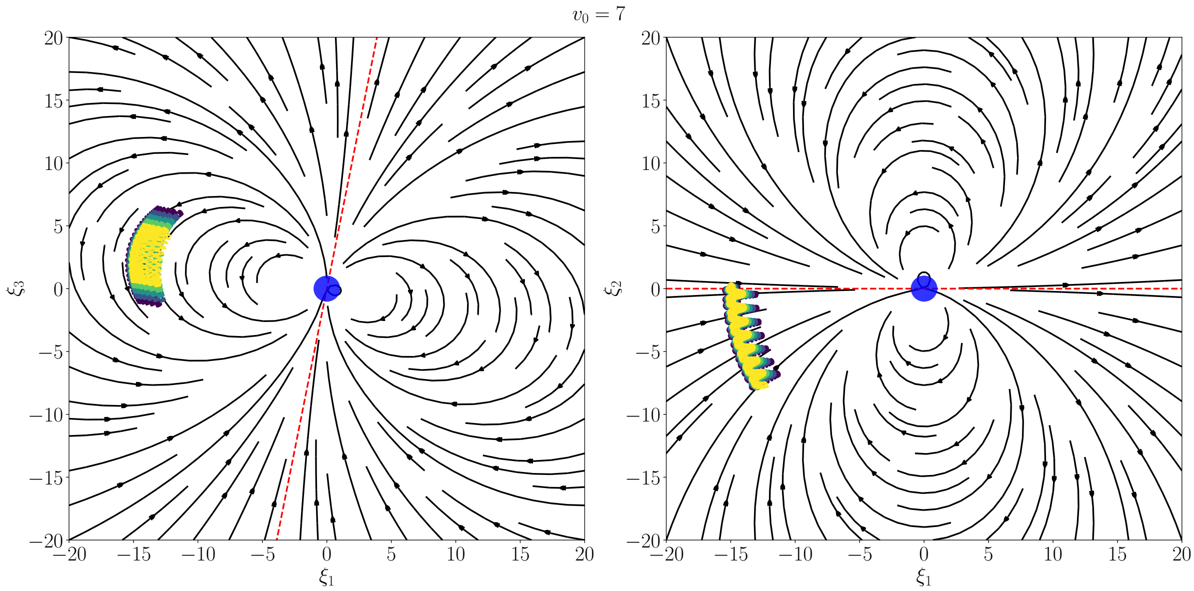

Figure 4: Particle sent towards the earth with typical solar wind velocity, for different heights $z0$

When doing this exact same simulation but with a weaker field ($\hat{B} \to 1/100\hat{B}$) we observe that the same qualitative behaviour is captured. Moreover, we see here that the particles spiral in towards the poles. This is exactly what causes the Aurora. As seen from equation (2), scaling the field is equivalent to scaling the velocity by the reciprocal factor. Hence, we could suspect that the behaviour shown in 5 is due to *fast* charged particles impinging on earth's magnetic field. Note however that scaling the velocity by a factor of 100 means we are effectively using velocities near c. The fact demonstrated here is therefore not to be taken too seriously.

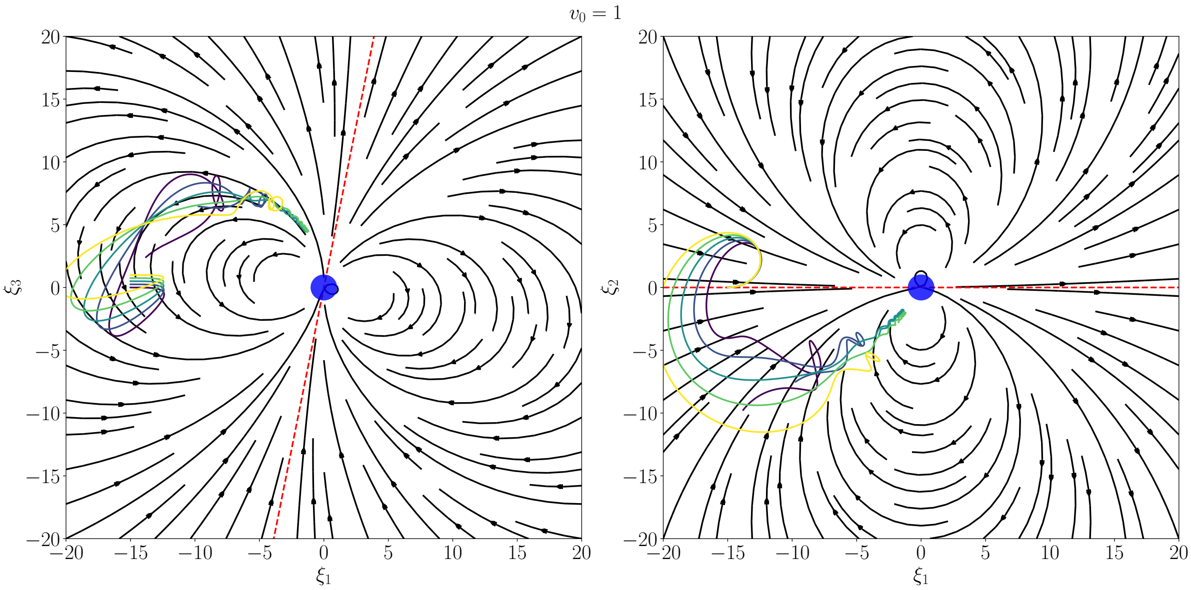

Figure 5: Particle sent towards the earth with lower typical solar wind velocity, for different heights $z0$, with a weaker field.

3.1 Validity of solution

To have an idea of how good the numerical solution is we investigate how well energy is conserved for the particles. Since magnetic forces do no work, the energy should be invariant. The (normalised) energy as a function of time is shown in figure 6 for some of the paths plotted above.

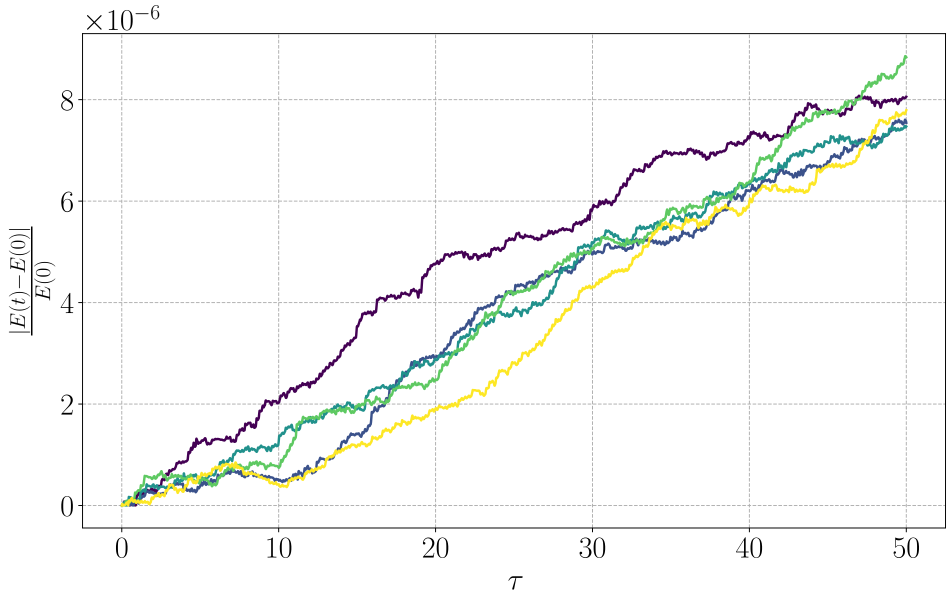

Figure 6: Energy as a function of time for the 5 trajectories shown in figure 4.

As the variations in the energy shown in figure 6 are $\ll 1$ we conclude that the numerical validity of the solutions are good. There is however a slight drift in the energy, but that is inevitable when we take into consideration numerical round off errors.

4 Conclusion

Despite the fact that the model used is extremely simple, it captures the essence of the physics involved. The plots show that the particles are trapped in helical trajectories following the field lines, exactly as one would predict using electromagnetic theory.

Moreover, through modifying the parameters of the model slightly, the particles are shown to be spiralling in towards the poles. This is in accordance with the fact that the Aurora is observed here. However, this fact is based on a rather flimsy demonstration, as special relativity starts playing an important role when the energy of the particles increases this much.

A more careful analysis is needed to include the relativistic corrections to the trajectories. This is not embarked on here.

References

[1] Peter Olson and Hagay Amit. Changes in earth's dipole., aug 2006.

[2] O. V. Khabarova, V. N. Obridko, R. A. Kislov, H. V. Malova, A. Bemporad, L. M. Zelenyi, V. D. Kuznetsov, and A. F. Kharshiladze. Evolution of the solar wind speed with heliocentric distance and solar cycle. surprises from ulysses and unexpectedness from observations of the solar corona. *Plasma Physics Reports*, 44(9):840–853, Sep 2018.

[3] John David Jackson. *Classical electrodynamics; 2nd ed.* Wiley, New York, NY, 1975.