

Solar Winds

Semester assignment TFY4240

Sondre Duna Lundemo[†]

Department of Physics, Norwegian University of Science and Technology, Trondheim
Norway

(Last updated on March 30, 2021)

We model the magnetic field of the earth as a magnetic point dipole and simulate the motion of charge particles in it. The requirement of constant energy is used as a test criterion for the validity of the solution.

1 Theory

We model the magnetic field from the earth by a point dipole at its centre, giving the field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left(\frac{(\mathbf{m} \cdot \mathbf{r})\mathbf{r} - r^2\mathbf{m}}{r^5} \right) \quad (1)$$

where \mathbf{m} is the dipole moment. The magnitude of earth's dipole moment is approximately $8 \cdot 10^{22} \text{Am}^2$ [1]. The equations of motion for a particle moving in the presence is given by Newton's second law and the Lorentz force (with $\mathbf{E} = 0$):

$$\ddot{\mathbf{r}} = \frac{1}{m} q \mathbf{v} \times \mathbf{B} \quad (2)$$

To solve the equation numerically, we recast it into dimensionless form. Define the following dimensionless quantities

$$\boldsymbol{\xi} := \frac{\mathbf{r}}{a} \quad ; \quad \hat{\mathbf{B}} := \frac{\mathbf{B}}{B_0} \quad (3)$$

where $B_0 = \frac{\mu_0 m_0}{4\pi a^3}$. Here a denotes the average radius of the earth, and m_0 the dipole moment of earth's magnetic field. Inserting these definitions into the equation of motion in 2 yields

$$\ddot{\boldsymbol{\xi}} = \frac{q\mu_0 m_0}{4\pi m a^3} \dot{\boldsymbol{\xi}} \times \hat{\mathbf{B}}. \quad (4)$$

If we now also introduce a dimensionless time $\tau := t/t_0$ we see that we can identify that the quantity

$$t_0 = \left(\frac{q\mu_0 m_0}{4\pi m a^3} \right)^{-1} \quad (5)$$

sets a natural time scale for our problem. Hence the equation in question is in dimensionless form written as

$$\frac{d^2 \boldsymbol{\xi}}{d\tau^2} = \frac{d\boldsymbol{\xi}}{d\tau} \times \hat{\mathbf{B}}. \quad (6)$$

Table 1: The constants involved in the dimensionless quantities.

Quantity	Value	Unit
m_0	$8 \cdot 10^{22}$	$\text{A} \cdot \text{m}^2$
q	$1.6 \cdot 10^{-19}$	C
m	$1.67 \cdot 10^{-27}$	kg
a	$6.4 \cdot 10^6$	m
μ_0	$4\pi \cdot 10^{-7}$	N/A^2
t_0	$3.42 \cdot 10^{-4}$	s

What we observe here is however that the time scale set by the parameters of the problem is very short. We will adjust the time scale so that velocities $\mathcal{O}(1)$ are typical velocities of the solar winds. These velocities are in the range $250 - 750 \text{ km/s}$ [2]. This is done by scaling the time by $1 \cdot 10^5$, whence the typical speeds are $\simeq 200 \text{ km/s}$. To keep the simple form of (6) we scale $\hat{\mathbf{B}}$ by the same factor. This ensures that the interesting behaviour of the particles is captured by the simulations.

What kind of motion do we expect?

This argument is slightly adapted from [3, sec. 12.4]

In the case under consideration, a perturbation solution to the motion gives adequate insight into how the particles move. When the distance over which \mathbf{B} changes appreciably is large compared to the gyration radius of the motion¹, the lowest order approximation is a spiralling motion around the field lines, with a frequency given by the local field. The second term in the expansion of the solution will involve a slow change which can be described as drifting of the centre of the orbit.

2 Numerical implementation

To solve equation 6 we rewrite the second order equation as a coupled system of first order equations as

$$\frac{d\boldsymbol{\xi}}{d\tau} = \boldsymbol{\chi} \quad (7)$$

$$\frac{d\boldsymbol{\chi}}{d\tau} = \boldsymbol{\chi} \times \hat{\mathbf{B}}, \quad (8)$$

where $\boldsymbol{\chi}$ is the dimensionless velocity associated to $\boldsymbol{\xi}$ and τ . The system of equations is then solved by stacking $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$ into one vector, \mathbf{X} , and then applying an ODE-solver to the system.

$$\frac{d\mathbf{X}}{d\tau} = \mathbf{f}(\mathbf{X}, \tau), \quad (9)$$

where $f_i = X_{i+3}$ for $i = 1, 2, 3$, and $f_i = \epsilon_{ijk} X_j \hat{B}_k$ for $i = 4, 5, 6$. We use the built-in ODE-solver in `scipy`, `odeint`, which in turn calls the ODE-solver `lsoda` written in FORTRAN, which is an adaptive-step Runge Kutta method for solving ODEs.

¹This is certainly the case when considering protons in the earth's magnetic field.

3 Results

A plot of the magnetic field from the earth is shown in figure 1 with the dimensionless length scales on the axes.

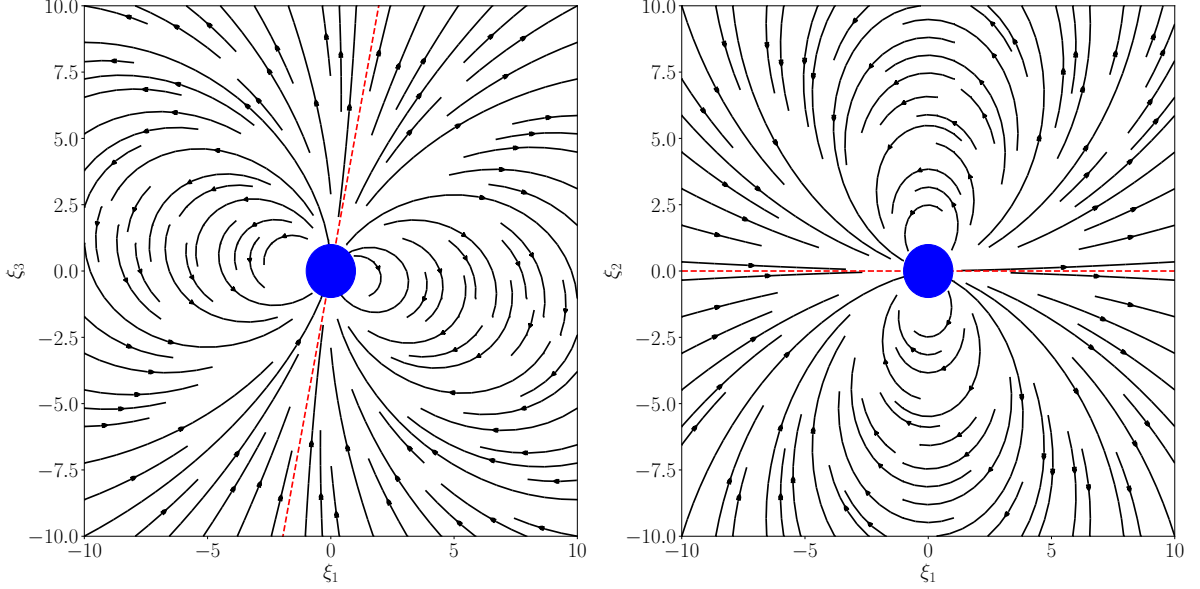


Figure 1: Magnetic field from earth.

When sending particles far from the earth towards it they start oscillating around the field lines as they approach the earth. This is consistent with the more familiar and simple case of Larmor precession in a uniform field. This is shown in figure 2 and 3. Also apparent from these plots is that the radius of precession is larger when the particles are sent towards the earth with larger speed.

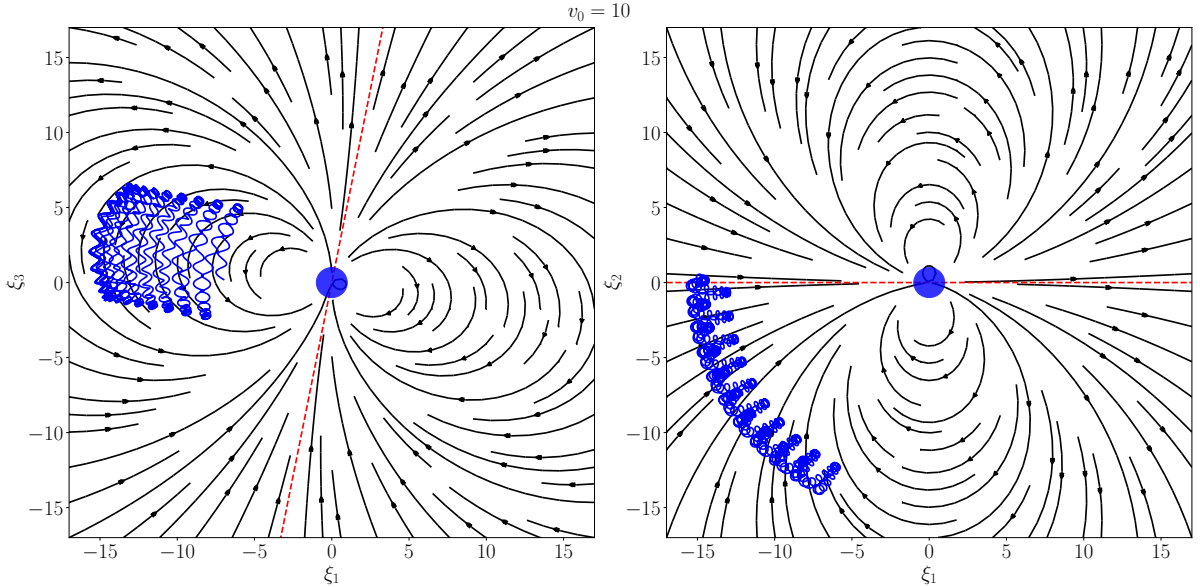


Figure 2: Particle sent towards the earth with typical solar wind velocity.

The trajectories of particles sent towards earth at different heights z (ξ_3) is shown in figure 4.

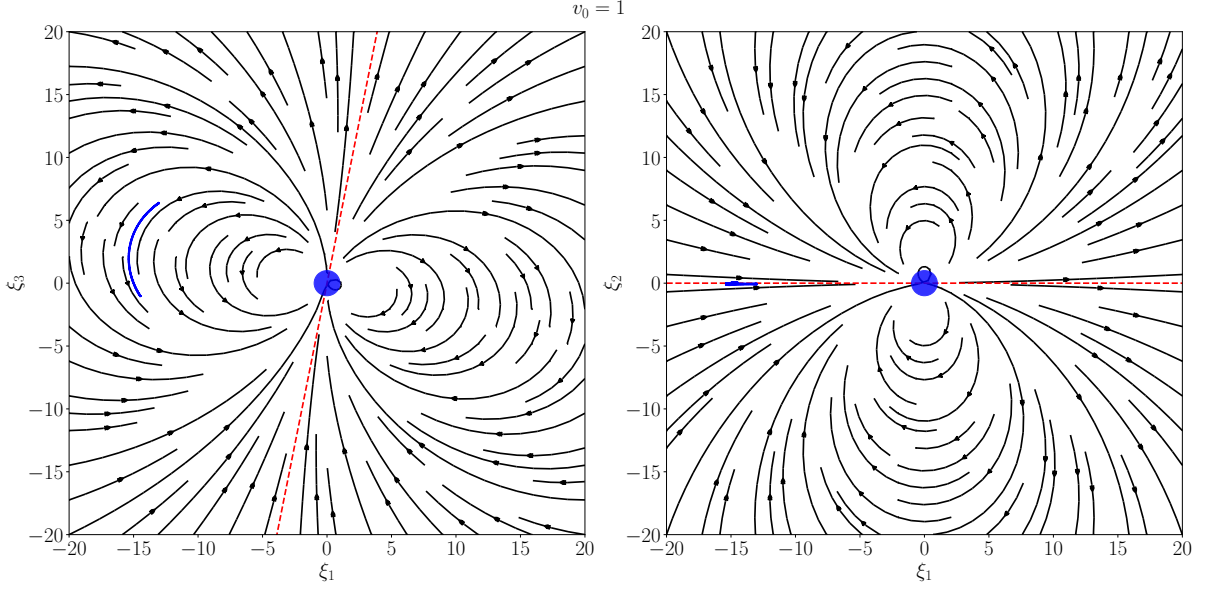


Figure 3: Particle sent towards the earth with lower typical solar wind velocity.

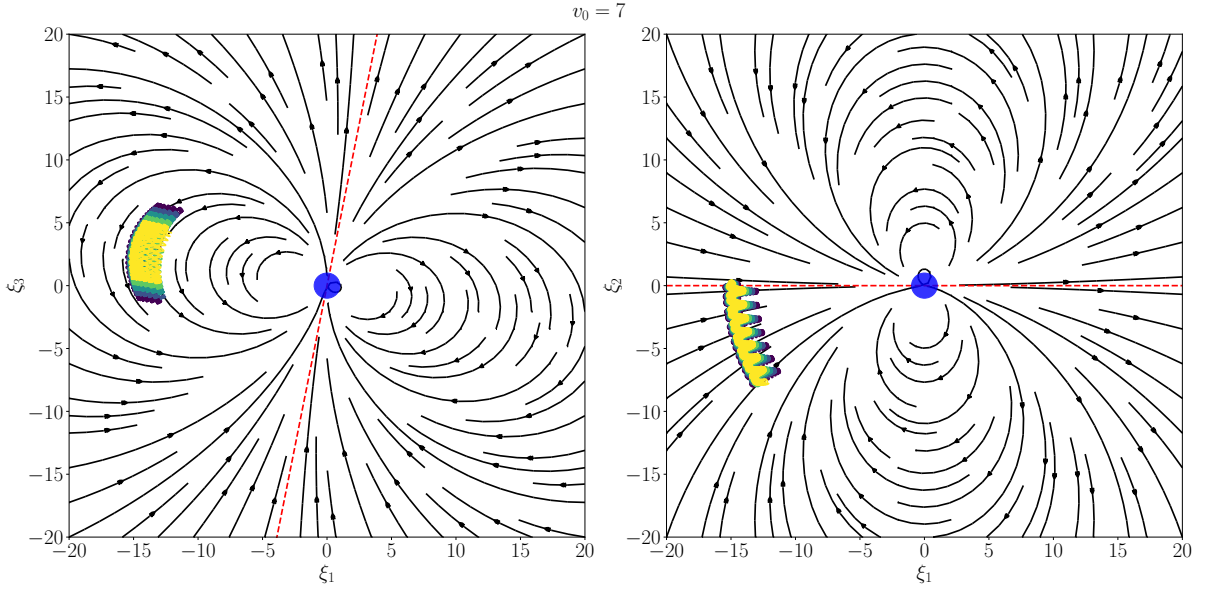


Figure 4: Particle sent towards the earth with typical solar wind velocity, for different heights z_0

When doing this exact same simulation but with a weaker field ($\hat{B} \rightarrow 1/100\hat{B}$) we observe that the same qualitative behaviour is captured. Moreover, we see here that the particles spiral in towards the poles. This is exactly what causes the Aurora. As seen from equation (2), scaling the field is equivalent to scaling the velocity by the reciprocal factor. Hence, the behaviour shown in 5 should be due to the *fast* charged particles impinging on earth's magnetic field.

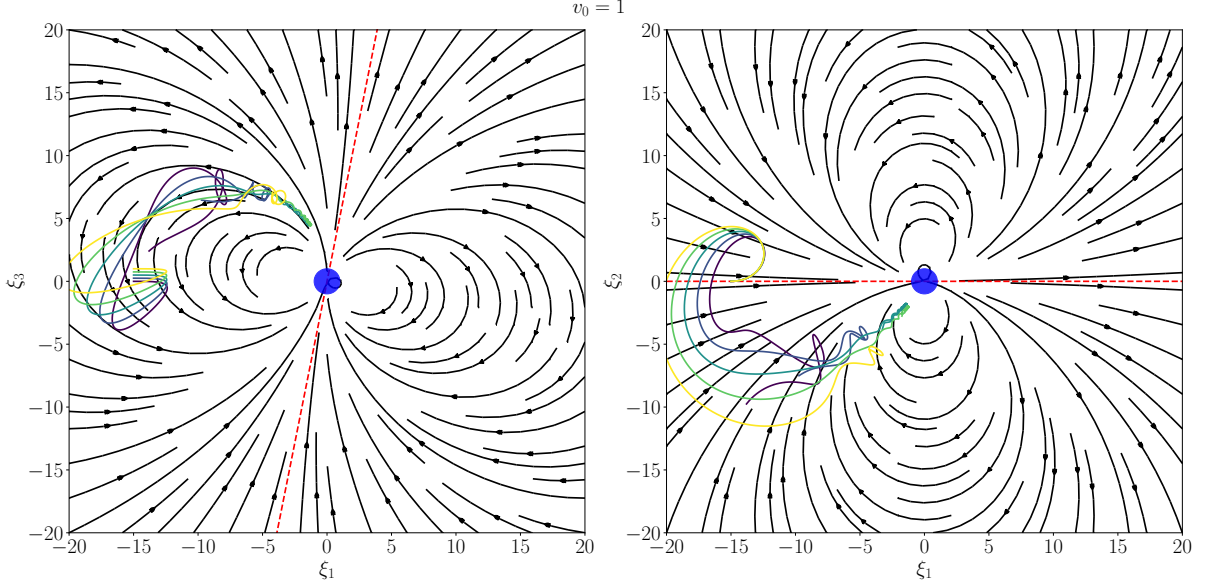


Figure 5: Particle sent towards the earth with lower typical solar wind velocity, for different heights z_0 , with a weaker field.

3.1 Validity of solution

To have an idea of how good the numerical solution is we investigate how well energy is conserved for the particles. Since magnetic forces do no work, the energy should be invariant. The (normalised) energy as a function of time is shown in figure 6 for some of the paths plotted above.

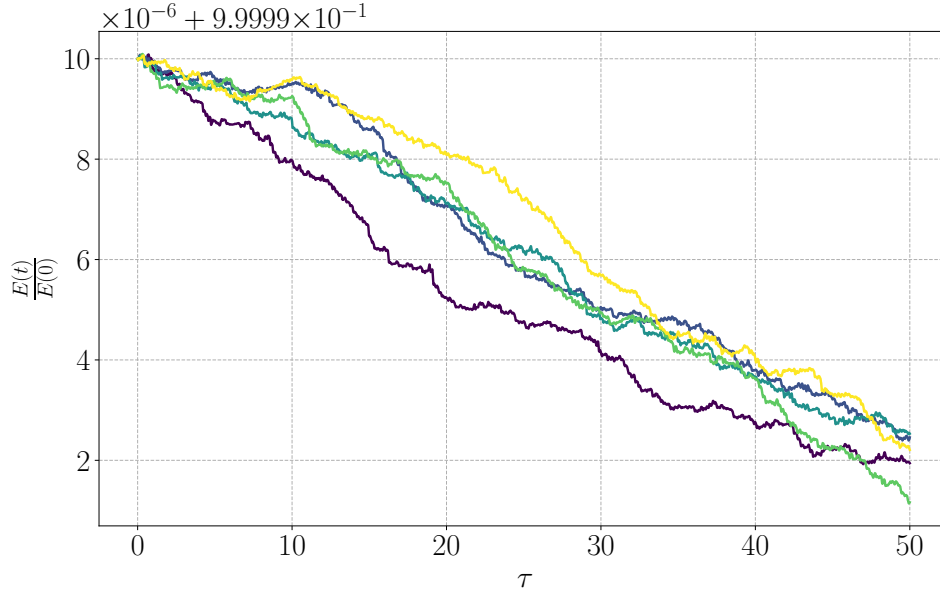


Figure 6: Energy as a function of time for the 5 trajectories shown in figure 4.

As the variations in the energy shown in figure 6 are $\ll 1$ we conclude that the numerical validity of the solutions are good. There is however a slight drift in the energy, but that is inevitable when we take into consideration numerical round off errors.

4 Conclusion

Despite the fact that the model used is extremely simple, it captures the essence of the physics involved. The plots show that the particles are trapped in helical trajectories following the field lines, exactly as one would predict using electromagnetic theory.

Moreover, through modifying the parameters of the model slightly, the particles are shown to be spiralling in towards the poles. This is in accordance with the fact that the Aurora is observed here. However, this rather flimsily demonstrated fact is not based on a careful enough analysis, as special relativity starts playing an important role when the energy of the particles increases.

References

- [1] Peter Olson and Hagay Amit. Changes in earth's dipole., aug 2006.
- [2] O. V. Khabarova, V. N. Obridko, R. A. Kislov, H. V. Malova, A. Bemporad, L. M. Zelenyi, V. D. Kuznetsov, and A. F. Kharshiladze. Evolution of the solar wind speed with heliocentric distance and solar cycle. surprises from ulysses and unexpectedness from observations of the solar corona. *Plasma Physics Reports*, 44(9):840–853, Sep 2018.
- [3] John David Jackson. *Classical electrodynamics; 2nd ed.* Wiley, New York, NY, 1975.