Solar Winds
Midterm project TFY4240 - problem 2

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Abstract. We model the magnetic field of the earth as the field of a magnetic point dipole and the solar winds as protons sent towards it. The qualitative behaviour of the trajectories are compared with what we expect from theory in the first approximation. The requirement of constant energy is used as a test criterion for the validity of the solution.

1 Theory

We model the magnetic field from the earth as a point dipole at its centre, giving the field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{(\mathbf{m}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{m}}{r^5}\right) \tag{1}$$

where $\mathbf{m}$ is the dipole moment. The magnitude of earth's dipole moment is approximately $8 \cdot 10^{22}\mathrm{Am}^2$ [OA06]. The equations of motion for a particle moving in the presence of a magnetic field $\mathbf{B}$ is given by Newton's second law and the Lorentz force (with $\mathbf{E} = 0$):

$$\ddot{\mathbf{r}} = \frac{1}{m} q\dot{\mathbf{r}} \times \mathbf{B}. \tag{2}$$

To solve the equation numerically, we recast it into dimensionless form. Define the following dimensionless quantities

$$\boldsymbol{\xi} := \frac{\mathbf{r}}{a} \quad ; \quad \hat{\mathbf{B}} := \frac{\mathbf{B}}{B_0} \tag{3}$$

where $B_0 = \frac{\mu_0 m_0}{4\pi a^3}$. Here a denotes the average radius of the earth, and m_0 the dipole moment of earth's magnetic field. Inserting these definitions into the equation of motion in 2 yields

$$\ddot{\boldsymbol{\xi}} = \frac{q\mu_0 m_0}{4\pi m a^3}\dot{\boldsymbol{\xi}} \times \hat{\mathbf{B}}. \tag{4}$$

If we now also introduce a dimensionless time $\tau := t/t_0$ we see that we can identify the quantity

$$t_0 = \left(\frac{q\mu_0 m_0}{4\pi m a^3}\right)^{-1} \tag{5}$$

as a natural time scale for our problem. Hence the equation of motion is in dimensionless form written as

$$\frac{\mathrm{d}^2\boldsymbol{\xi}}{\mathrm{d}\tau^2} = \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}\tau} \times \hat{\mathbf{B}}. \tag{6}$$

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Table 1: The constants involved in the dimensionless quantities.

Quantity	Value	Unit
m_0	$8 \cdot 10^{22}$	$\mathrm{A} \cdot \mathrm{m}^2$
q	$1.6 \cdot 10^{-19}$	C
m	$1.67 \cdot 10^{-27}$	kg
a	$6.4 \cdot 10^{6}$	m
μ_0	$4\pi \cdot 10^{-7}$	$\mathrm{N/A}^2$
t_0	$3.42 \cdot 10^{-4}$	s

What we observe here is however that the time scale set by the parameters of the problem is very short. We will adjust the time scale so that velocities $\mathcal{O}(1)$ are typical velocities of the solar winds. These velocities are in the range $250 - 750\,\mathrm{km/s}$ [KOK+18]. This is done by scaling the time by $1 \cdot 10^5$, so that the typical speeds are $\simeq 190\,\mathrm{km/s}$. To keep the simple form of (6) we scale $\hat{B}$ by the same factor. This will ensure that the interesting behaviour of the particles is captured by the simulations.

What kind of motion do we expect?

(This argument is slightly adapted and shortened from [Jac75, sec. 12.4]. For more details, please consult this reference.)

In the case under consideration, a perturbation solution to the motion gives adequate insight into how the particles move. When the distance over which $\mathbf{B}$ changes appreciably is large compared to the gyration radius of the motion[1], the lowest order approximation is a spiralling motion around the field lines, with a frequency given by the local field. The second term in the expansion of the solution will involve a slow change which can be described as drifting of the centre of the orbit.

By expanding the magnetic field in the expression for the gyration frequency to first order along the direction perpendicular to the field, $\mathbf{n}$, one obtains[2]

$$\boldsymbol{\omega}_B = \frac{e}{m}\mathbf{B}(\mathbf{x}) \simeq \boldsymbol{\omega}_0 \left[1 + \frac{1}{B_0}\frac{\partial B}{\partial \mathbf{n}}\bigg|_0 \cdot \mathbf{x}\right],$$

where the 0 subscripts denotes the quantity evaluated in the unperturbed case. Since the direction of the perturbed $\mathbf{B}$ is unchanged, the velocity parallel to $\mathbf{B}$ is still uniform. By writing the transverse velocity $\mathbf{v}_\perp = \mathbf{v}_0 + \mathbf{v}_1$, we can substitute the above expression into the equation of motion in 2 to obtain

$$\frac{\mathrm{d}\mathbf{v}_\perp}{\mathrm{d}t} = \mathbf{v}_\perp \times \boldsymbol{\omega}_B(\mathbf{x}),$$

from which it follows that

$$\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} \simeq \left[\mathbf{v}_1 + \mathbf{v_0}\left(\frac{1}{B_0}\frac{\partial B}{\partial \mathbf{n}}\bigg|_0 \cdot \mathbf{x}_0\right)\right] \times \boldsymbol{\omega}_0.$$

[1] This is certainly the case when considering protons in the earth's magnetic field.

[2] Note that [Jac75] treats this case in Gaussian units and uses relativistic equations of motion.

One can show that $\mathbf{v}_1$ has, apart from oscillatory terms, a non-zero time average given by

$$\langle \mathbf{v}_1 \rangle \simeq -\frac{r_l^2}{2}\frac{1}{B_0}\frac{\partial B}{\partial \mathbf{n}} \times \boldsymbol{\omega}_0 = \frac{\omega_B r_l^2}{2B^2}\left(\mathbf{B} \times \boldsymbol{\nabla}_\perp B\right), \tag{7}$$

where the last expression is written in coordinate-independent form. Here r_l denotes the gyration radius (Larmor radius). It is evident from this expression that if the gradient is slowly varying in space, i.e. $r_l|\boldsymbol{\nabla}_\perp B/B| \ll 1$, this time average will be much smaller than the orbital velocity $r_l\omega_B$. Hence, in this case the particles will spiral rapidly while its centre of gyration slowly moves perpendicular to both $\mathbf{B}$ and $\boldsymbol{\nabla}_\perp B$. In the case of a dipole field, this essentially has the direction of $\hat{\theta}$, i.e. the polar direction measured with respect to the symmetry axis of the field.

One can also derive a similar expression for a drift of the centre of gyration caused by the curvature of the magnetic field. This drift velocity is approximately given by

$$\mathbf{v}_c \simeq \frac{v_\parallel^2}{\omega_B R}\frac{\mathbf{R} \times \mathbf{B}_0}{RB_0}, \tag{8}$$

where $\mathbf{R}$ is the radius vector pointing from the effective centre of curvature to the particle. Hence, as for the drift velocity caused by the gradient of B we expect this to cause the particles to drift in the polar direction.

In short, the expected behaviour is helical motion around the field lines with a slow drift around the symmetry axis of the field. As the magnetic moment of the earth points "downwards", we expect the drift to be clockwise seen from above.

2 Numerical implementation

To solve equation 6 we rewrite the second order equation as a coupled system of first order equations as

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}\tau} = \mathbf{v} \tag{9}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau} = \mathbf{v} \times \hat{\mathbf{B}}, \tag{10}$$

where $\mathbf{v}$ is the dimensionless velocity associated to $\boldsymbol{\xi}$ and τ. The system of equations is then solved by stacking $\boldsymbol{\xi}$ and $\mathbf{v}$ into one vector, $\mathbf{X}$, and then applying an ODE-solver to the system.

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau} = \mathbf{f}(\mathbf{X}, \tau), \tag{11}$$

where $f_i = v_i$ and $f_{i+3} = \epsilon_{ijk} v_j \hat{B}_k$ for $i = 1, 2, 3$. We use the built-in ODE-solver in `scipy`, `odeint`, which in turn calls the ODE-solver `lsoda` written in `FORTRAN`, which is an adaptive-step Runge Kutta method for solving ODEs.

3 Results & Discussion

A plot of the magnetic field from the earth is shown in figure 1.

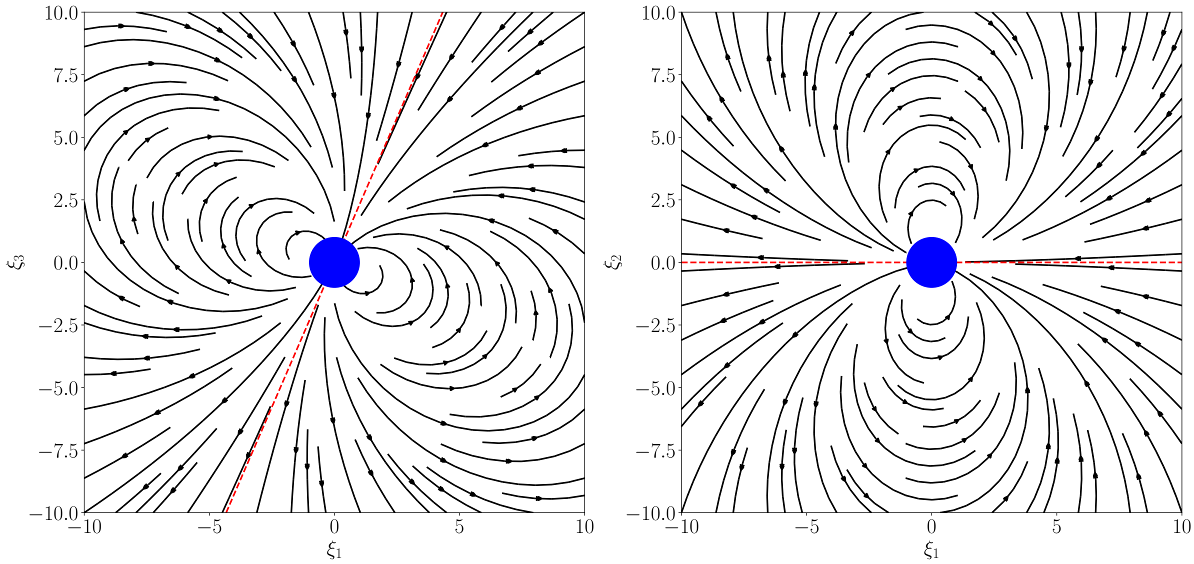

Figure 1: Magnetic field from earth. The tilt of the ecliptic with respect to the equator is 23.5° [CR12].

When sending particles far from the earth towards it they start spiralling around the field lines. This is consistent with the fact that the magnetic force acts perpendicular to the trajectory, and the fact that the motion along $\mathbf{B}$ is uniform in the first approximation, as considered in section 1. This is shown in figure 2 and 3. Moreover, in figure 2 we observe a clockwise drift around the equator, as predicted by equation (7) and (8).

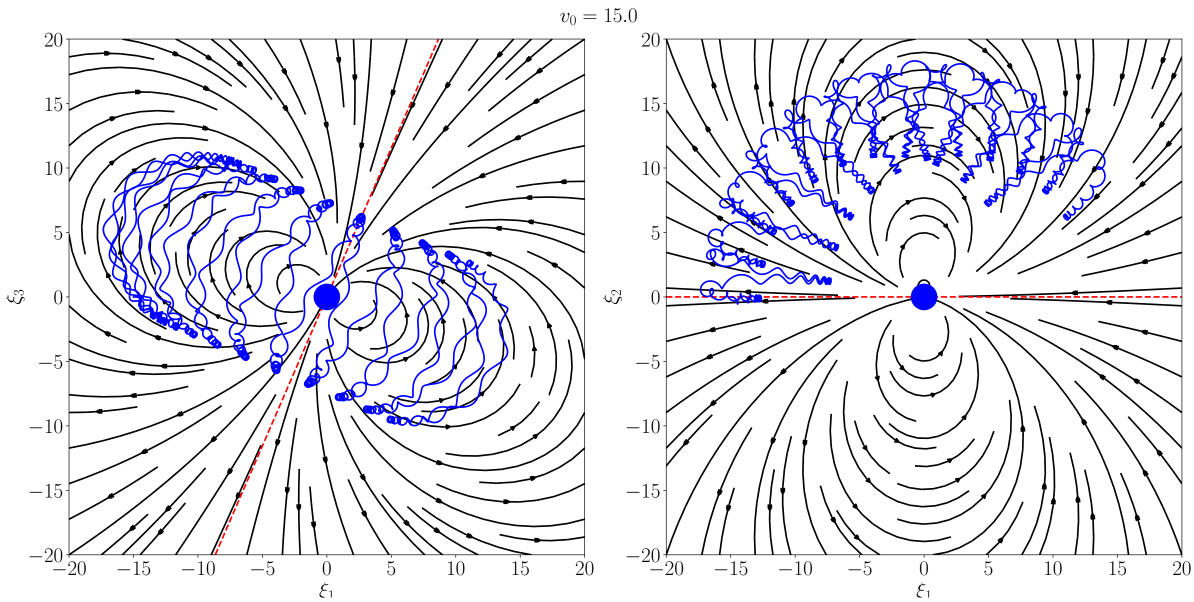

Figure 2: Particle sent towards the earth with high solar wind velocity.

The trajectories of particles sent towards earth at different heights z (ξ_3) is shown in figure 4.

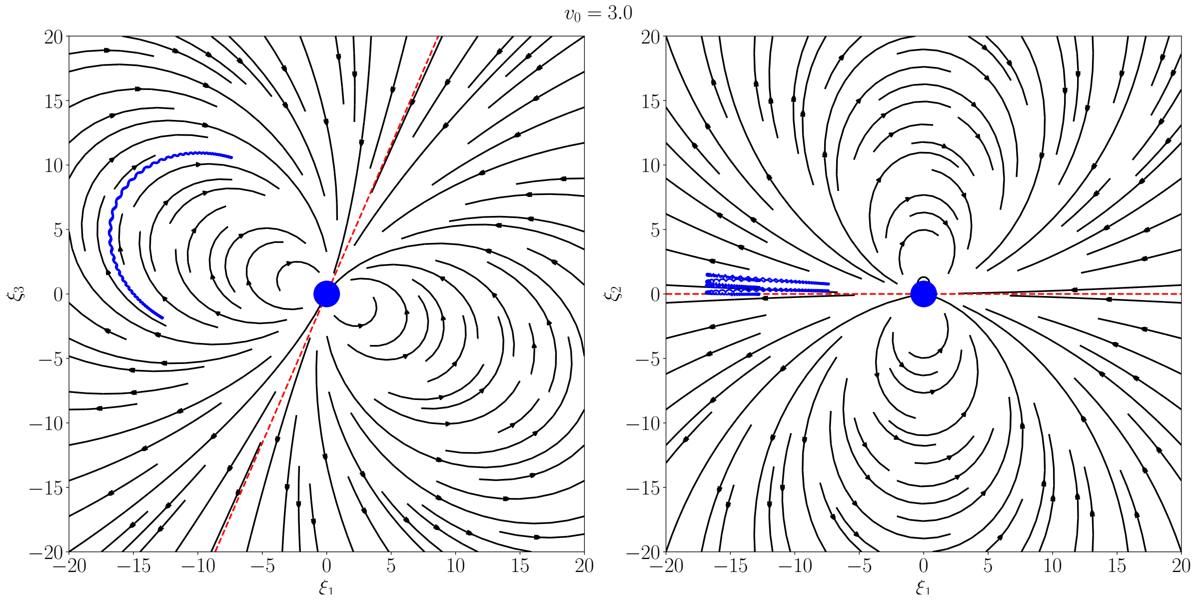

Figure 3: Particle sent towards the earth with typical solar wind velocity.

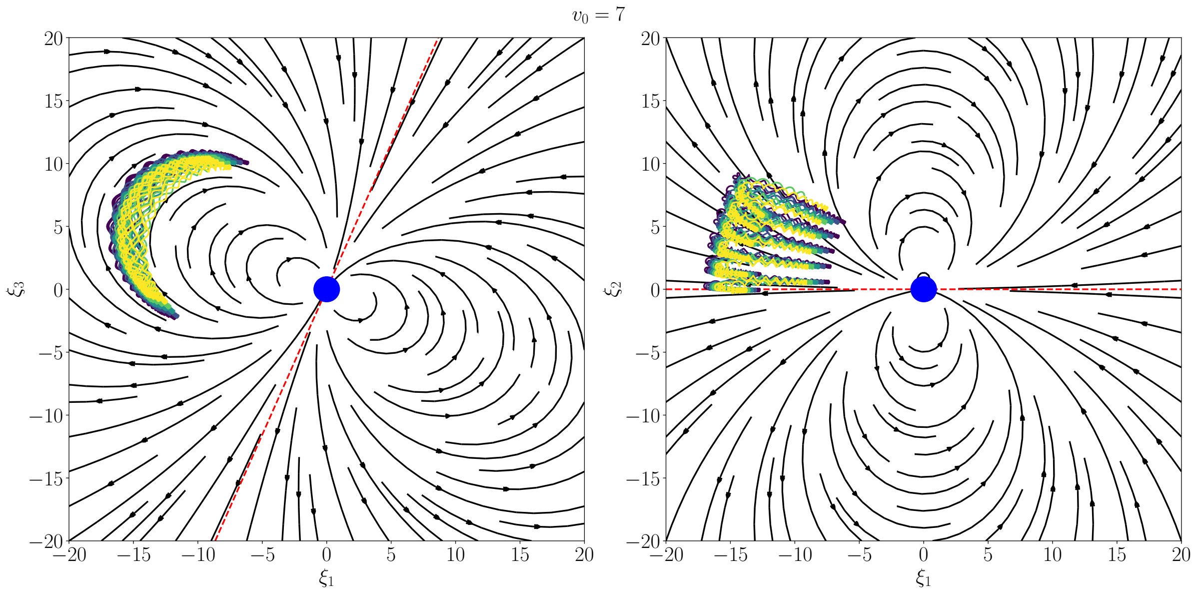

Figure 4: Particle sent towards the earth with typical solar wind velocity, for different heights z_0.

When doing this exact same simulation but with a weaker field ($\hat{B} \to 1/100\hat{B}$) we observe that the same qualitative behaviour is captured. Moreover, we see here more clearly that the particles tend to spiral in towards the poles. This is exactly what causes the Aurora to be visible here and not near the equator. Note however that the strength of this field has nothing to do with that of earth's, so this demonstration is therefore not to be taken too seriously.

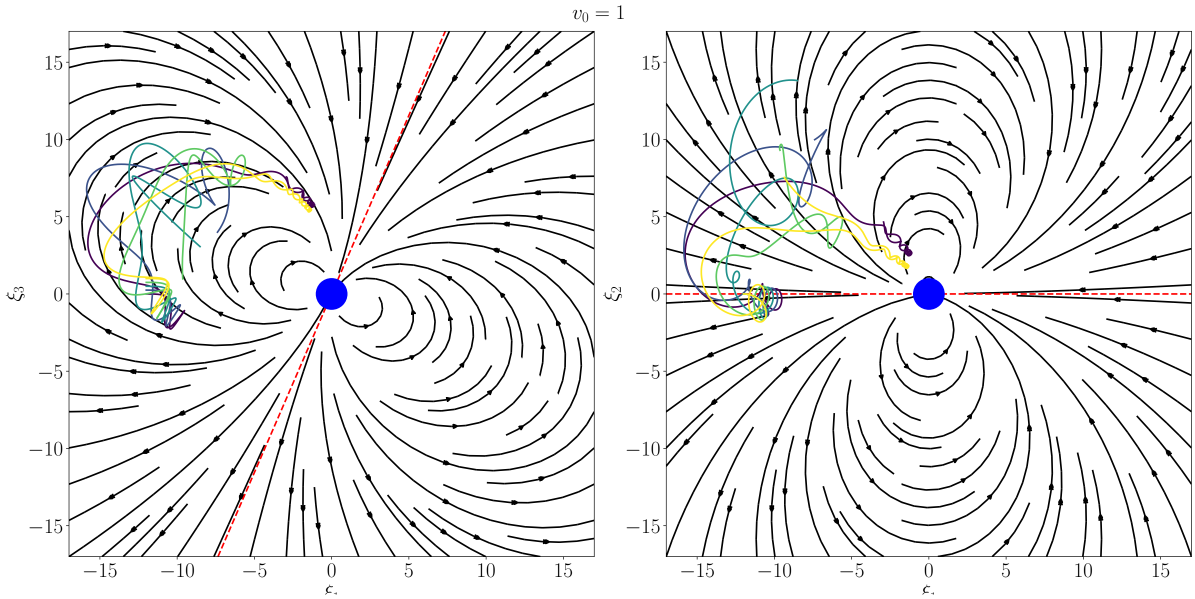

Figure 5: Particle sent towards the earth with lower typical solar wind velocity, for different heights z_0, with a weaker field.

3.1 Validity of solution

To have an idea of how good the numerical solution is we investigate how well energy is conserved for the particles. Since magnetic forces do no work, the energy should be invariant. The (normalised) energy difference as a function of time is shown in figure 6 for some of the paths plotted above.

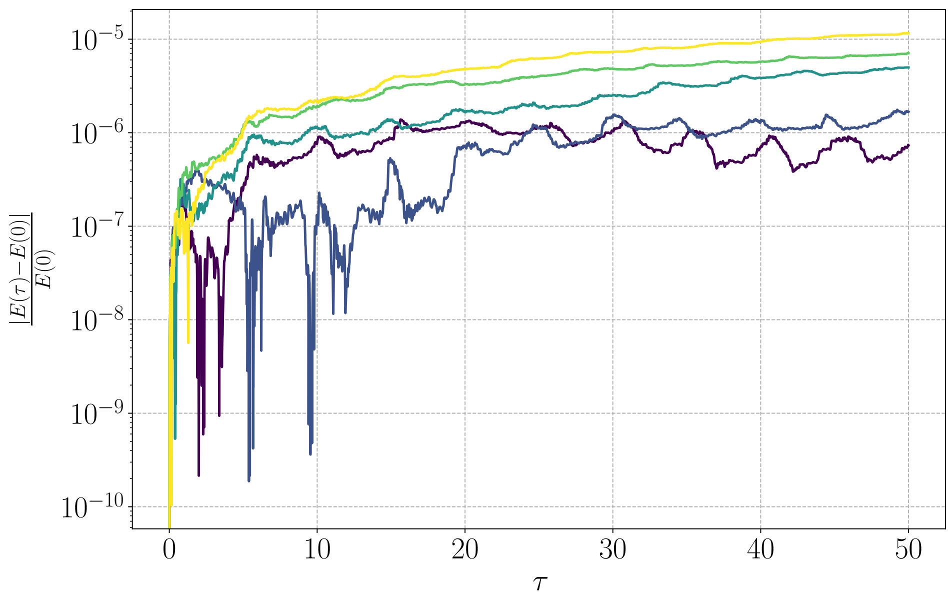

Figure 6: Energy difference as a function of time for the 5 trajectories shown in figure 4.

As the variations in the energy shown in figure 6 are $\ll 1$ we conclude that the numerical validity of the solutions are good. There is however a slight drift in the energy, but that is inevitable when we take into consideration numerical round-off errors.

4 Conclusion

Despite the fact that the model used is extremely simple, it captures the essence of the physics involved. The simulations show that the particles are trapped in helical trajectories following the field lines, exactly as one would predict using electromagnetic theory. Moreover, a drift of the trajectories around the equator is observed for sufficiently large initial velocities, as is also predictable from theory.

References

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[Jac75] John David Jackson. *Classical electrodynamics; 2nd ed.* Wiley, New York, NY, 1975.

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