Helicopter lab

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Contents

1 Introduction

In this paper, controlling a helicopter with three degrees of freedom will be discussed. The paper will be divided in four parts, from the linearized mathematical modeling to different realizations of the system. Three different types of control will be tested for the realization. Firstly a mono-variable control, then a multi-variable control of the system and lastly a state estimation will be tested to control the system. The mono-variable control will use standard P and PD controllers for regulating pitch, travel and elevation through one of the system's outputs. The multi-variable controller will control several of the system's outputs simultaneously using LQR. In the state estimation an observer will be developed in order to estimate nonmeasured states. In this way numerical differentiation can be avoided.

The helicopter has a stationary fixed point in the center with a set of propellers on one side and a counterweight on the other side. fig. 1 shows a simplified version of the system. Elevation is accomplished when the propellers rotate at a speed which gives the helicopter an upwards facing force equal to the force contributed by the counterweight and their own gravitational force.

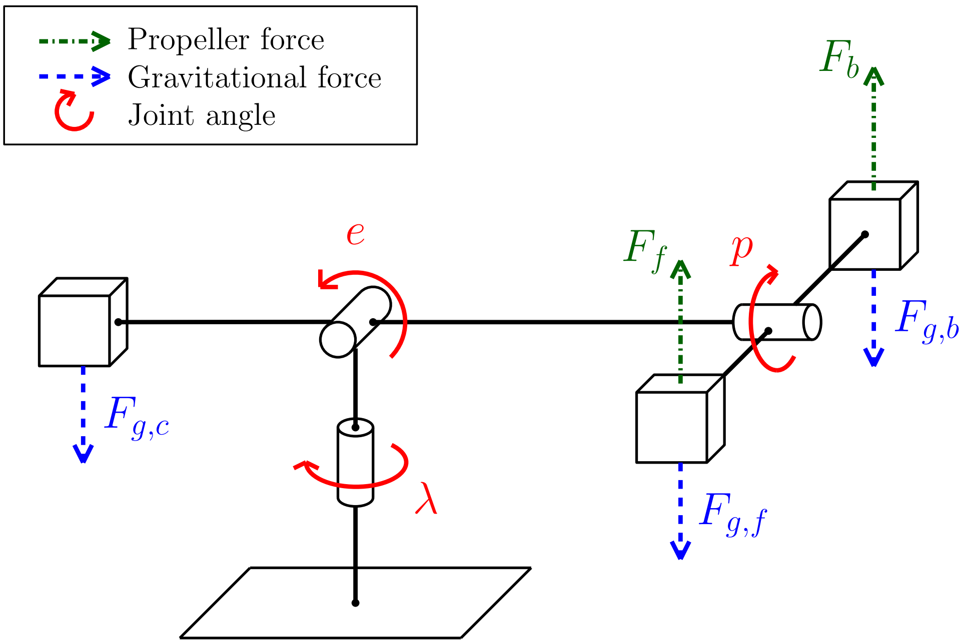


Figure 1: Setup of the system. Image taken from [3]

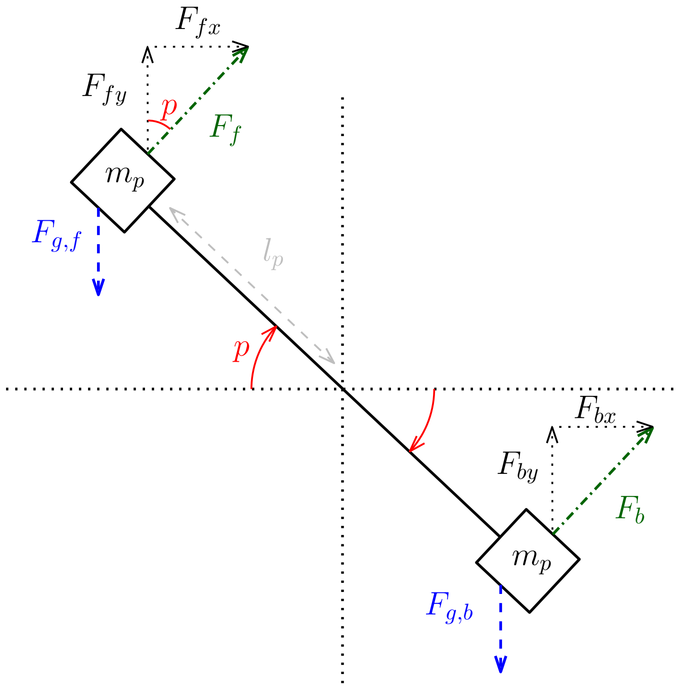


Figure 2: Forces around p joint angle

2 Part I

This part will focus on the mathematical modelling of the helicopter system. Firstly, three motion equations for the pitch angle p, the elevation angle e, and the travel angle λ will be derived. The three differential equations will then be linearized around a suitable working point. It is assumed that the forces produced from the motors are proportional to the input voltages with some constant K_f. The sum of the input voltages will be denoted as V_s, while the difference will be denoted as V_d.

2.1 Problem 1

The differential equation for p is derived from Newton's second law for rotation. From fig. 2 it can be derived that

$$\begin{aligned} J_p\ddot{p} =& \sum \tau_p \\ J_p\ddot{p} =& l_p(F_{g,b} + F_f - F_{g,f} - F_b) \end{aligned}$$

Since both the masses are equal to m_p, the gravitational forces will cancel each other. After some algebraic manipulation, the equation can be written as

$$J_p\ddot{p} = l_pK_fV_d \tag{1}$$

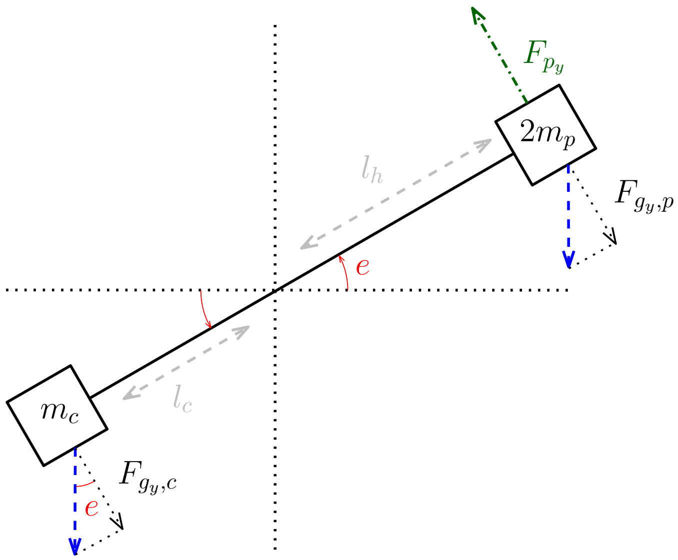


Figure 3: Forces around e joint angle

The differential equation for e is also derived from Newton's second law for rotation. Combining this with the forces illustrated in fig. 3 yields

$$\begin{aligned} J_e\ddot{e} =& \sum \tau_e \\ J_e\ddot{e} =& l_h F_{p_y} + l_c F_{g_y,c} - l_h F_{g_y,p} \end{aligned} \tag{2}$$

where the new force F_{p_y} is introduced. This force is the sum of the y-components of the two propeller forces. The new force can be calculated from fig. 2

$$F_{p_y} = cos(p)F_f + cos(p)F_b \tag{3}$$

The two other forces, $F_{g_y,c}$ and $F_{g_y,p}$ are the y-components of a gravitational force with mass m_c and $2m_p$ respectively, as seen in fig. 3. By inserting (3) into (2), combined with algebraic manipulation and further decomposition of forces, (2) can be expressed as

$$J_e\ddot{e} = g(m_c l_c - 2m_p l_h)cos(e) + l_h K_f V_s cos(p) \tag{4}$$

Similar to both p and e, the differential equation for λ is derived from Newton's second law for rotation. The law combined with fig. 4 gives

$$J_\lambda \ddot{\lambda} = \sum \tau_\lambda = -rF_{p_x} \tag{5}$$

The direction of F_{p_x} is dependent on p and a positive p will result in F_{p_x} pointing in the direction illustrated in fig. 4. Therefore, a negative sign

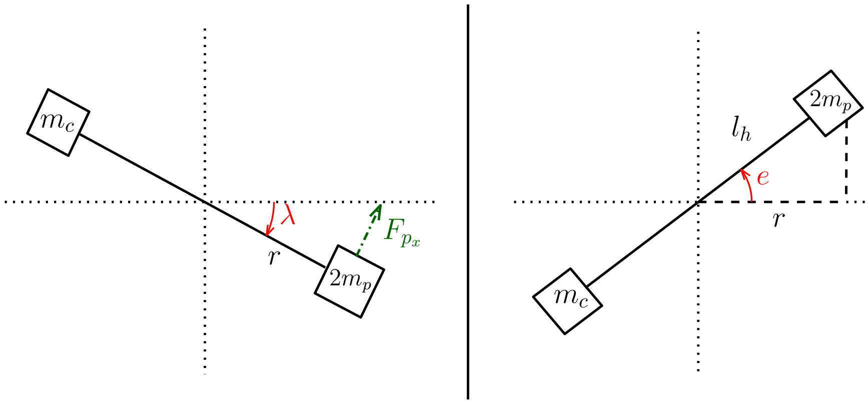


Figure 4: Forces around λ joint angle and how the arm r varies with e

is present in (5). By expressing F_{p_x} and r with the help of trigonometric functions, the equation can be written as

$$J_\lambda \ddot{\lambda} = -l_h K_f V_s cos(e) sin(p) \tag{6}$$

To summarize, a mathematical model for the helicopter system is

$$J_p \ddot{p} = L_1 V_d \tag{7a}$$
$$J_e \ddot{e} = L_2 \cos(e) + L_3 V_s \cos(p) \tag{7b}$$
$$J_\lambda \ddot{\lambda} = L_4 V_s \cos(e) \sin(p) \tag{7c}$$

with

$$L_1 = l_p K_f \tag{8a}$$
$$L_2 = g(m_c - 2m_p l_h) \tag{8b}$$
$$L_3 = l_h K_f \tag{8c}$$
$$L_4 = -L_3 = -l_h K_f \tag{8d}$$

2.2 Problem 2

The equations (7a) - (7c) were used to calculate the values of V_s^* and V_d^* by solving $\dot{p} = \dot{e} = \dot{\lambda} = 0$, which implies $\ddot{p} = \ddot{e} = \ddot{\lambda} = 0$, with respect to V_s and V_d. Thus, by insertion of the working point $p = p^* = e = e^* = 0$, observe that to reach equilibrium around (p^*, e^*, λ^*) the system must have $V_d^* = 0$ by (7a) and $V_s^* = -\frac{L_2}{L_3}$ by (7b).
The linearized system around $[\tilde{p}, \tilde{e}, \tilde{\lambda}]^T = [0, 0, 0]^T$ and $[\tilde{V}_s, \tilde{V}_d]^T = [0, 0]^T$ is by definition is

$$\Delta \begin{bmatrix} \ddot{\tilde{p}} \\ \ddot{\tilde{e}} \\ \ddot{\tilde{\lambda}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \ddot{\tilde{p}}}{\partial \tilde{p}} & \frac{\partial \ddot{\tilde{p}}}{\partial \tilde{e}} & \frac{\partial \ddot{\tilde{p}}}{\partial \tilde{\lambda}} \\ \frac{\partial \ddot{\tilde{e}}}{\partial \tilde{p}} & \frac{\partial \ddot{\tilde{e}}}{\partial \tilde{e}} & \frac{\partial \ddot{\tilde{e}}}{\partial \tilde{\lambda}} \\ \frac{\partial \ddot{\tilde{\lambda}}}{\partial \tilde{p}} & \frac{\partial \ddot{\tilde{\lambda}}}{\partial \tilde{e}} & \frac{\partial \ddot{\tilde{\lambda}}}{\partial \tilde{\lambda}} \end{bmatrix} \Delta \begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} \frac{\partial \ddot{\tilde{p}}}{\partial \tilde{V_s}} & \frac{\partial \ddot{\tilde{p}}}{\partial \tilde{V_d}} \\ \frac{\partial \ddot{\tilde{e}}}{\partial \tilde{V_s}} & \frac{\partial \ddot{\tilde{e}}}{\partial \tilde{V_d}} \\ \frac{\partial \ddot{\tilde{\lambda}}}{\partial \tilde{V_s}} & \frac{\partial \ddot{\tilde{\lambda}}}{\partial \tilde{V_d}} \end{bmatrix} \Delta \begin{bmatrix} \tilde{V_s} \\ \tilde{V_d} \end{bmatrix} \tag{9}$$

Where Δ means a small change. By evaluating (9) at the working point, the expression boils down to. (9), the expression boils down to

$$\Delta \begin{bmatrix} \ddot{\tilde{p}} \\ \ddot{\tilde{e}} \\ \ddot{\tilde{\lambda}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{L_4 L_2}{J_\lambda L_3} & 0 & 0 \end{bmatrix} \Delta \begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} 0 & \frac{L_1}{J_p} \\ \frac{L_3}{J_p} & 0 \\ 0 & 0 \end{bmatrix} \Delta \begin{bmatrix} \tilde{V_s} \\ \tilde{V_d} \end{bmatrix} \tag{10}$$

Thus

$$\ddot{\tilde{p}} = K_1 \tilde{V_d} \tag{11a}$$

$$\ddot{\tilde{e}} = K_2 \tilde{V_s} \tag{11b}$$

$$\ddot{\tilde{\lambda}} = K_3 \tilde{p} \tag{11c}$$

where

$$K_1 = \frac{L_1}{J_p} \tag{12a}$$

$$K_2 = \frac{L_3}{J_e} \tag{12b}$$

$$K_3 = -\frac{L_4 L_2}{J_\lambda L_3} \tag{12c}$$

2.3 Problem 3

The physical interpretation of the system matches well with its actual behavior. As expected by (7a) and (7b), p and e is dependent on V_d and V_s respectively. Around the working point $p^* = e^* = \lambda^* = 0$, both the linearized model and the exact physical representation of the system cohere. On the other hand, when the helicopter reaches points further away from the working point the two models differ. Its angular dependencies, as seen in (4) and (6), are no longer negligible and therefore the linearized model is no longer valid.

Forces, such as friction and centripetal forces, are negligible and therefore neglected. The helicopters aerodynamics, and the time delay, as a result of computer sampling, is also left out of the model. This manifests as unexpected behaviour in the helicopters dynamics, such as when close to the ground, the helicopters input to the elevation will double due to the "the ground effect" [4].

2.4 Problem 4

Note that $e = 0$ is defined for when the helicopter is parallel to the ground. To find the motor constant K_f, a value for V_s^* was measured such that the helicopter initializes to reach the desired angle for e. By inserting this value for V_s^*, which was found by measuring to be approximately 6.1V, into (4), and inserting $p^* = e^* = \lambda^* = 0$ into the equations (1), (4) and (6), an expression for K_f can be found.

$$K_f = -\frac{L2}{L3} = -\frac{(m_c l_c - 2m_p l_h)g}{l_h V_s^*} \tag{13}$$

which gave $K_f \approx 0.2432$.

3 Part II

In this part a controller was designed for the pitch angle, $\tilde{p}$, and the travel rate, $\dot{\tilde{\lambda}}$, with a fixed horizontal elevation angle, $\tilde{e_c}$. The joystick's x-axis was used as a reference trajectory.

3.1 Problem 1

A PD controller was implemented to control the helicopter's pitch angle, p, and is given by

$$\tilde{V_d} = K_{pp}(\tilde{p_c} - \tilde{p}) - K_{pd}\dot{\tilde{p}} \tag{14}$$

Reasonable values for K_{pp} and K_{pd} was derived by inserting (14) into (11a) and applying the Laplace transform to obtain the transfer function (assuming all initial values equal to zero).

$$\ddot{\tilde{p}} = K_1(K_{pp}(\tilde{p_c} - \tilde{p}) - K_{kp}\dot{\tilde{p}}) \tag{15a}$$

$$\mathcal{L} \downarrow$$

$$\frac{\tilde{p}(s)}{\tilde{p_c}(s)} = \frac{K_1 K_{pp}}{s^2 + K_1 K_{pd} s + K_1 K_{pp}} \tag{15b}$$

To get reasonable values for K_{pp} and K_{pd}, (15b) was compared to the characteristic solution to the second order equation given by

$$h_0(s) = \frac{\omega_c^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \tag{16}$$

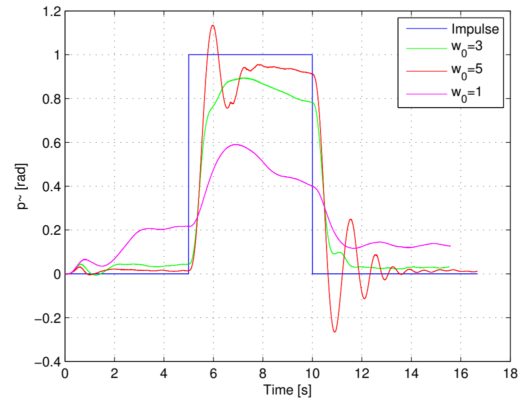
From (16) K_{pp} can be obtained independently from K_{pd}. Choosing a reasonable ω_0, in this case $\omega_0 = 3$, will result in

$$K_{pp} = \frac{\omega_0^2}{K_1} = \frac{9}{K_1} \tag{17}$$

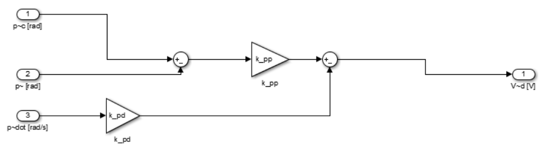
With $\omega_0 = \sqrt{K_1 K_{pp}}$ and critical damping, $\zeta = 1$, the correlation will be

$$K_{pd} = \frac{2\omega_0}{K_1} = \frac{2\sqrt{K_{pp}}}{\sqrt{K_1}} \tag{18}$$

The correlation between K_{pd} and K_{pp} is now given, but it is still possible to tune the poles by adjusting ω_0. It is stated earlier in this section that this value is set to 3. This value was found with the combination of a qualified guess and by repeated tries of tuning with different values close to $\omega_0 = 3$. Poles far in the left plane are necessary for a fast response, but not far enough to challenge the system's stability. Therefore $\omega_0 = 3$ was a reasonable choice. The response for different values of ω_0 is shown in fig. 5a. A snapshot of the PD-controller is shown in fig. 5b.



(a) Impulse response for $\tilde{p}$ with PD-controller



(b) Snapshot of the PD-controller for $\tilde{p}$

Figure 5: PD-controller

3.2 Problem 2

To control the travel rate $\dot{\tilde{\lambda}}$ a P-controller was implemented. Assuming that $\tilde{p} = \tilde{p_c}$, a P-controller ($\tilde{p_c} = K_{rp}(\dot{\tilde{\lambda}}_c - \dot{\tilde{\lambda}})$) can be inserted into (11c) resulting in

$$\ddot{\tilde{\lambda}} = K_3 K_{rp} \dot{\tilde{\lambda}}_c - K_3 K_{rp} \dot{\tilde{\lambda}} \tag{19a}$$

$$\mathcal{L} \downarrow$$

$$\frac{\dot{\tilde{\lambda}}(s)}{\dot{\tilde{\lambda}}_c(s)} = \frac{K_3 K_{rp}}{s + K_3 K_{rp}} = \frac{\rho}{s + \rho} \tag{19b}$$

After testing and tuning, it was decided that $K_{rp} = -1.5$ gave a good response. The response for different values of K_{rp} is shown in fig. 6.

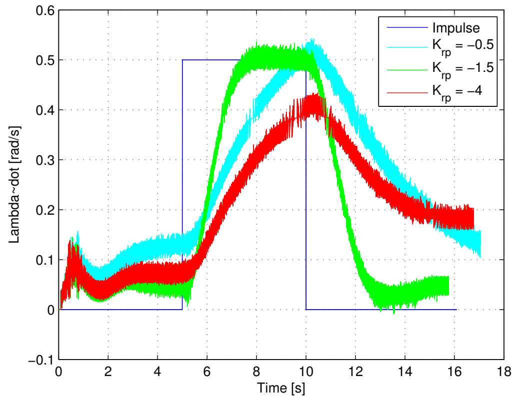


Figure 6: Impulse response for $\dot{\tilde{\lambda}}$

4 Part III

This part will have the designing of a multivariable controller as its main focus. The pitch angle $\tilde{p}$ and the elevation rate $\dot{\tilde{e}}$ will be controlled, where the reference for the pitch angle and the elevation rate is provided by the output of the joystick.

4.1 Problem 1

The goal is to put the system of equations given by the relations for pitch and elevation in (11a) - (11b) in a state-space formulation of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{20}$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices. It is further given that the state vector and input should be

$$\mathbf{x} = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{p}} \\ \dot{\tilde{e}} \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} \tag{21}$$

The state vector and input combined with (11a) - (11b) yields

$$\begin{bmatrix} \dot{\tilde{p}} \\ \ddot{\tilde{p}} \\ \ddot{\tilde{e}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \dot{\tilde{p}} \\ \dot{\tilde{e}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ K_2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} \tag{22}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ K_2 & 0 \end{bmatrix} \tag{23}$$

4.2 Problem 2

The controllability matrix for a system is generally given by [2, p. 185]

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{24}$$

Observe that the pair ($\mathbf{A}$, $\mathbf{B}$) is 3-dimensional. Thus the controllability matrix for the given system is

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} \tag{25}$$

Inserted values yields

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & K_1 & 0 & 0 \\ 0 & K_1 & 0 & 0 & 0 & 0 \\ K_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

The matrix has full row rank, $rank(\mathcal{C}) = n = 3$, so the system is <u>controllable</u> [2, p. 184].
The goal is to track a reference $\mathbf{r} = \begin{bmatrix} \tilde{p}_c & \dot{\tilde{e}}_c \end{bmatrix}^T$ for the pitch angle $\tilde{p}$ and elevation rate $\dot{\tilde{e}}$, which is fed in with the output of the joystick. The controller is based on state feedback and reference feed forward. Therefore, the input has the form

$$\mathbf{u} = \mathbf{Pr} - \mathbf{Kx} \tag{27}$$

where the matrix $\mathbf{K}$ corresponds to the linear quadratic regulator (LQR) for which the control input $\mathbf{u} = -\mathbf{Kx}$ optimizes the cost function

$$J = \int_0^\infty (\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t))dt \tag{28}$$

For simplicity, the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ are diagonal. By inserting the chosen controller, the system can be expressed as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{B}(\mathbf{Pr} - \mathbf{Kx}) \\ \dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BPr} \end{aligned} \tag{29}$$

Since the system is controllable, the eigenvalues of the matrix $\mathbf{A} - \mathbf{BK}$ can be chosen arbitrarily by choosing an appropriate $\mathbf{K}$ matrix. The poles of the closed-loop system are equivalent to the eigenvalues of $\mathbf{A} - \mathbf{BK}$, so the system's dynamics can therefore be chosen as desired.

The matrix $\mathbf{K}$ is obtained by using the MATLAB command $K = lqr(A, B, Q, R)$. The matrix $\mathbf{P}$, however, should be chosen such that (in theory) $\lim_{t\to\infty} \tilde{p} = \tilde{p}_c$ and $\lim_{t\to\infty} \dot{\tilde{e}} = \dot{\tilde{e}}_c$ for fixed values of $\tilde{p}_c$ and $\dot{\tilde{e}}_c$. This implies that $\dot{\mathbf{x}} \to \mathbf{0}$ as $t \to \infty$. Thus

$$\mathbf{x}_\infty = -(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{BPr} \tag{30}$$

assuming the matrix $\mathbf{A} - \mathbf{BK}$ is invertible. The goal is to control $\tilde{p}$ and $\dot{\tilde{e}}$, so the measurements are defined as

$$\mathbf{y} = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{e}} \end{bmatrix} \tag{31}$$

As time goes, it is desirable to obtain zero deviation between the reference and the output. Thus

$$\mathbf{y}_\infty = \begin{bmatrix} \tilde{p_c} \\ \dot{\tilde{e_c}} \end{bmatrix} = \mathbf{r} \tag{32}$$

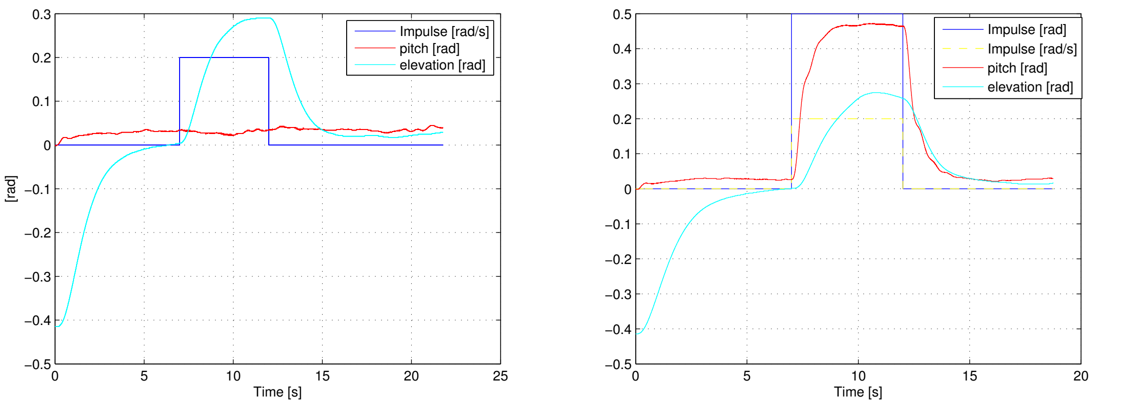
By using the relationship $\mathbf{y}_\infty = \mathbf{C}\mathbf{x}_\infty$, (30) yields

$$\mathbf{r} = -\mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{BPr}$$
$$\Downarrow$$
$$\mathbf{P} = [-\mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}]^{-1} \tag{33}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{34}$$

In the beginning, the **Q** matrix and the **R** matrix in the LQR regulator were inserted with 1's on the diagonal. From there, the elevation was tuned until a desired response was achieved. The response is shown in fig. 7a.



(a) Step response for $\tilde{e}$ with P-controller.

(b) Step response for $\tilde{p}$ and $\tilde{e}$ with P-controller

Figure 7: Step response with P-controller

Then, when the elevation was tuned, the pitch and pitch rate were tuned. Which resulted in the **Q** matrix and the **R** matrix equal to

$$\mathbf{Q} = \begin{bmatrix} 140 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 90 \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{35}$$

The tuning of the regulator is relatively aggressive with high values on the diagonal of $\mathbf{Q}$ and low values on the diagonal of $\mathbf{R}$. The cost for deviation between the output and the reference is therefore high, combined with a low limitation for input. This was necessary to obtain a fast and accurate response. A relatively high value on the element representing the pitch rate $\dot{\tilde{p}}$ was needed to smooth out the pitch response, although it made the response slightly slower. By lowering this value, the motors would start to vibrate which was undesired.

With the chosen tuning, the system behaves desirable. The response is fast and accurate with minimal oscillations, as seen in fig. 7b. However, a problem occurs with the use of a P-regulator. A deviation between the reference and the output can be observed. This is due to the lack of an integral effect, and is further described in classical control [1, p. 386]. Especially, the elevation struggles to stabilize on a non-zero value after the reference for $\dot{\tilde{e}}$ is set to zero.

4.3 Problem 3

To include an integral effect for the pitch angel and elevation rate, the controller presented in 4.2 is modified. This results in two new states, γ and ζ, and their differential equations are given by

$$\dot{\gamma} = \tilde{p} - \tilde{p}_c \tag{36}$$

$$\dot{\zeta} = \dot{\tilde{e}} - \dot{\tilde{e}}_c \tag{37}$$

The state vector and input is now given by

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x_a} \end{bmatrix} = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{p}} \\ \dot{\tilde{e}} \\ \gamma \\ \zeta \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} \tag{38}$$

where $\mathbf{x_a}$ is the augmented state space vector given by the two new states γ and ζ. The augmented state vector can further be expressed as

$$\mathbf{x_a} = \int_0^t \mathbf{y}(\tau) - \mathbf{r}(\tau) d\tau \tag{39}$$

which implies that

$$\dot{\mathbf{x}}_\mathbf{a} = \mathbf{C}\mathbf{x}(t) - \mathbf{r}(t) \tag{40}$$

By introducing the two additional states, the state space model must also be extended. The state space model can now be written in the form

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_\mathbf{a} \end{bmatrix} = \bar{\mathbf{A}} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_a} \end{bmatrix} + \bar{\mathbf{B}}\mathbf{u} + \mathbf{D}\mathbf{r} \tag{41}$$

where

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} \tag{42}$$

As a consequence of changing the state space model, the matrix $\mathbf{Q}$ must be extended with two additional values on the diagonal. Furthermore, the new state space model also implies a modified input

$$\mathbf{u} = -\bar{\mathbf{K}} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_a} \end{bmatrix} + \bar{\mathbf{P}}\mathbf{r} \tag{43}$$

where $\bar{\mathbf{K}}$ can be divided according to the new state vector $\bar{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & \mathbf{K_a} \end{bmatrix}$. The matrix $\bar{\mathbf{K}}$ can be found using the MATLAB function from 4.2, $\bar{K} = lqr(\bar{A}, \bar{B}, \bar{Q}, \bar{R})$. The matrix $\bar{\mathbf{P}}$ can be found by inserting (43) into (41)

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_\mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_a} \end{bmatrix} - \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} \mathbf{K} & \mathbf{K_a} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_a} \end{bmatrix} + \bar{\mathbf{P}}\mathbf{r} \right) + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} \mathbf{r} \tag{44}$$

After simplifying, the equation is reduced to

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_\mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & -\mathbf{BK_a} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_a} \end{bmatrix} + \begin{bmatrix} \mathbf{B}\bar{\mathbf{P}} \\ -1 \end{bmatrix} \mathbf{r} \tag{45}$$

Similar to section 4.2, the goal is to attain zero deviation between the output and the reference. This implies that $\begin{bmatrix} \dot{\mathbf{x}} & \dot{\mathbf{x}}_\mathbf{a} \end{bmatrix}^T \to \mathbf{0}$ as $t \to \infty$ and $\mathbf{y}_\infty = \mathbf{C}\mathbf{x}_\infty = \mathbf{r}$. Combining these conditions with (45) gives

$$\begin{bmatrix} \mathbf{A} - \mathbf{BK} & -\mathbf{BK_a} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_\infty \\ \mathbf{x}_{\mathbf{a},\infty} \end{bmatrix} + \begin{bmatrix} \mathbf{B}\bar{\mathbf{P}} \\ -1 \end{bmatrix} \mathbf{r} = \mathbf{0} \tag{46}$$

which can be rewritten as

$$\begin{bmatrix} (\mathbf{A} - \mathbf{BK})\mathbf{x}_\infty - \mathbf{BK_a}\mathbf{x}_{\mathbf{a},\infty} + \mathbf{B}\bar{\mathbf{P}}\mathbf{r} \\ \mathbf{C}\mathbf{x}_\infty - \mathbf{r} \end{bmatrix} = \mathbf{0} \tag{47}$$

Observe that the bottom equation in (47) is fulfilled because $\mathbf{C}\mathbf{x}_\infty = \mathbf{r}$, which is a direct consequence of the applied conditions. This is the case no matter how $\bar{\mathbf{P}}$ is chosen. To terminate the transients of the system $\bar{\mathbf{P}}$ is chosen such that

$$(\mathbf{A} - \mathbf{BK})\mathbf{x}_\infty - \mathbf{BK_a}\mathbf{x}_{\mathbf{a},\infty} + \mathbf{B}\bar{\mathbf{P}}\mathbf{r} = \mathbf{0} \tag{48}$$

The augmented state vector $\mathbf{x_a}$ is equal to zero when the reference is met (i.e $\mathbf{y} = \mathbf{r}$). Thus

$$(\mathbf{A} - \mathbf{BK})\mathbf{x}_\infty + \mathbf{B}\bar{\mathbf{P}}\mathbf{r} = \mathbf{0} \tag{49}$$

which gives

$$\mathbf{x}_\infty = -(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}\bar{\mathbf{P}}\mathbf{r} \tag{50}$$

By using the same procedure presented in 4.2, (49) is fulfilled if

$$\bar{\mathbf{P}} = [-\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}]^{-1} \tag{51}$$

After tuning as described in 4.2, with the previous $\mathbf{Q}$ matrix and $\mathbf{R}$ matrix as an initial guess, the following $\bar{\mathbf{Q}}$ and $\bar{\mathbf{R}}$ were chosen

$$\bar{\mathbf{Q}} = \begin{bmatrix} 180 & 0 & 0 & 0 & 0 \\ 0 & 70 & 0 & 0 & 0 \\ 0 & 0 & 180 & 0 & 0 \\ 0 & 0 & 0 & 1.8 & 0 \\ 0 & 0 & 0 & 0 & 30 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{R}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{52}$$

The first element in $\bar{\mathbf{R}}$ was doubled in order to limit the input which regulates the elevation, V_s. Input needed to regulate the pitch would then be prioritized. The system's overall response is shown in fig. 8.

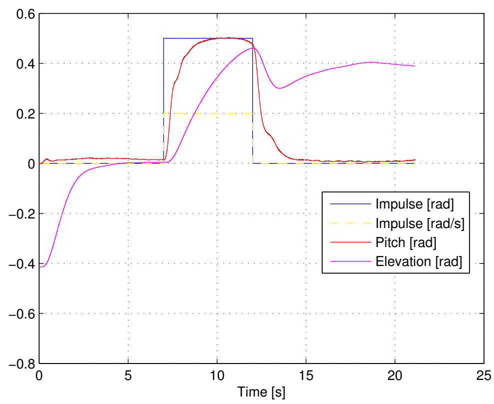
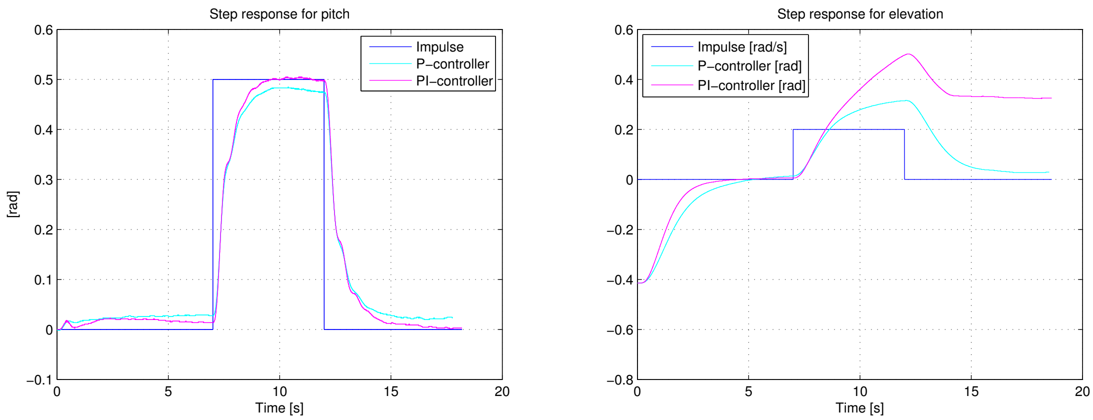


Figure 8: Simultaneous step response for $\tilde{p}$ and $\tilde{e}$ with PI-controller

With an integral effect, both the elevation and the pitch reach the reference. This stands in contrast with the P-controller, where there was a deviation between the reference and the output. The difference in response between the controllers are shown in fig. 9a and fig. 9b. A reason for this difference is the inclusion of the two new states γ and ζ. These two states represent the deviation between the output and the reference. Minimizing the transient of these states with the use of the LQR controller will be the equivalent of making the output equal to the reference.



(a) Comparison between step response for $\tilde{p}$ (b) Comparison between step response for $\tilde{e}$

Figure 9: P and PI controller comparison

5 Part IV

In this section state estimation will be presented and discussed. In the previous problems, the angular velocities corresponding to the pitch, elevation and travel have been computed using numerical differentiation. In this part of the report, an observer will be developed in order to estimate these non-measured states.

5.1 Problem 1

The goal is to put the system of equations given by (11a) - (11c) in a state-space formulation on the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{53}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{54}$$

Where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices describing the system. The state vector, the input vector and the output vector are given as

$$\mathbf{x} = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{p}} \\ \tilde{e} \\ \dot{\tilde{e}} \\ \tilde{\lambda} \\ \dot{\tilde{\lambda}} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} \tag{55}$$

The vectors combined with (11a) - (11c) yields

$$\begin{bmatrix} \dot{\tilde{p}} \\ \ddot{\tilde{p}} \\ \dot{\tilde{e}} \\ \ddot{\tilde{e}} \\ \dot{\tilde{\lambda}} \\ \ddot{\tilde{\lambda}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ K_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \dot{\tilde{p}} \\ \tilde{e} \\ \dot{\tilde{e}} \\ \tilde{\lambda} \\ \dot{\tilde{\lambda}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ 0 & 0 \\ K_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{V_s} \\ \tilde{V_d} \end{bmatrix} \tag{56}$$

and

$$\begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{p}} \\ \ddot{\tilde{p}} \\ \dot{\tilde{e}} \\ \ddot{\tilde{e}} \\ \dot{\tilde{\lambda}} \\ \ddot{\tilde{\lambda}} \end{bmatrix} \tag{57}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ K_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{58a}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ 0 & 0 \\ K_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{58b}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{58c}$$

5.2 Problem 2

The observability matrix is generally given by [2, p. 197]

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ . \\ . \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{59}$$

Observe that the pair $(\mathbf{A}, \mathbf{C})$ has six columns. Thus the observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ . \\ . \\ \mathbf{CA}^5 \end{bmatrix} \tag{60}$$

Row reduction in MATLAB gives

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ & & . & . & & \\ & & . & . & & \end{bmatrix} \tag{61}$$

The observability matrix has full column rank, $rank(\mathcal{O}) = n = 6$. Therefore, the system is observable [2, p. 197]. A linear observer for the system is given on the form

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{62}$$

where $\mathbf{L}$ is the observer gain matrix. An implementation of the state estimator is shown in fig. 14. The error between the states and the observer is defined as

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}} \tag{63}$$

which means that

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} \tag{64}$$

Given that the measurement has noise, $\mathbf{y} = \mathbf{y}_m + \mathbf{n}$, the following equation can be derived from (64)

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC})\mathbf{e} - \mathbf{Ln} \tag{65}$$

The eigenvalues of the $\mathbf{A} - \mathbf{LC}$ matrix represents the dynamics of $\mathbf{e}$. In other words, the eigenvalues of $\mathbf{A} - \mathbf{LC}$ are equivalent to the poles of the closed-loop system for $\mathbf{e}$. Since the system is observable, the eigenvalues of $\mathbf{A} - \mathbf{LC}$ can be chosen arbitrarily by choosing an appropriate $\mathbf{L}$. Thus, the dynamics of $\mathbf{e}$ can be chosen as desired.

To ensure a correct value for the estimated states, the error dynamics must be faster than the dynamics of the plant. The plant dynamics are given by the eigenvalues of $\mathbf{A} - \mathbf{BK}$, so the eigenvalues for $\mathbf{A} - \mathbf{LC}$ must lie further into the left plane than the ones given by $\mathbf{A} - \mathbf{BK}$. Equivalently, the poles of the closed-loop system for $\mathbf{e}$ must lie further into the left plane than the poles of the plant's closed-loop system.

The placement of the poles were done by using the *place* function in MATLAB. The *place* function finds the $\mathbf{L}$ matrix which gives $\mathbf{A} - \mathbf{LC}$ the desired eigenvalues. As an initial guess, the poles were placed in a fan with radius ten times further into the left plane than the most negative pole given by the plant's dynamics [2, p. 302]. After some tuning, the poles were placed as shown in fig. 10, where the MATLAB code used for placing the poles are shown in section 6.

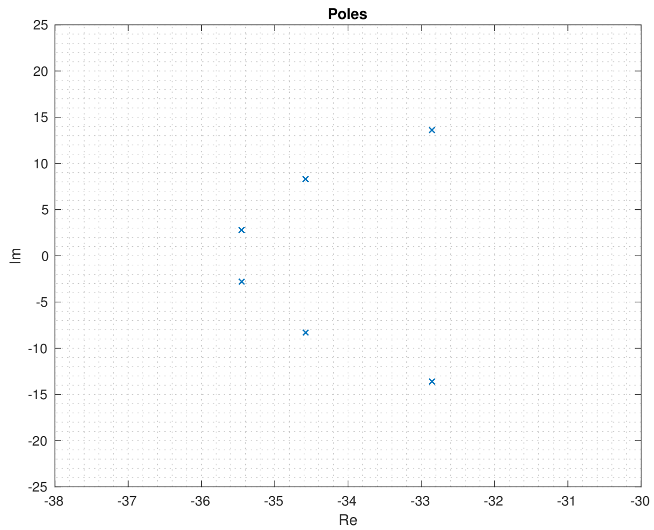


Figure 10: The system's poles

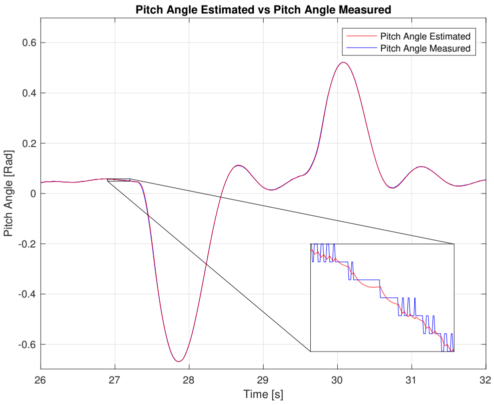
Choosing different $\mathbf{L}$ matrices will result in different error dynamics. As seen in (65), as the larger $\mathbf{L}$ is chosen the more the noise will be amplified. This is an undesired behaviour, and will affect the state estimation. However, choosing a smaller $\mathbf{L}$ will result in slower error dynamics. It is also showed in (65) that a smaller $\mathbf{L}$ will result in the model of the state estimator having a larger impact. This will leave the state estimator more vulnerable to modelling inaccuracies.

Plots for the estimated states are based on the PI-controller from section 4.3 with

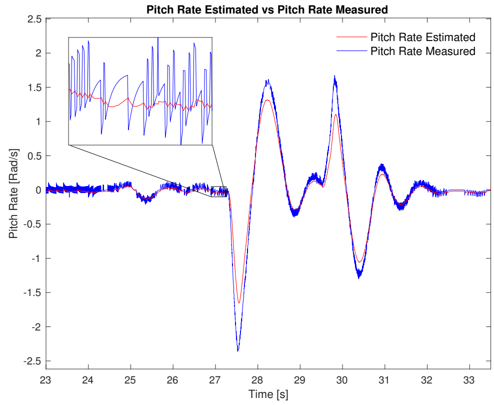
$$\bar{\mathbf{Q}} = \begin{bmatrix} 75 & 0 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{R}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{66}$$

Differences between the measured states and the estimated states are shown in fig. 11a, fig. 11b, fig. 11c

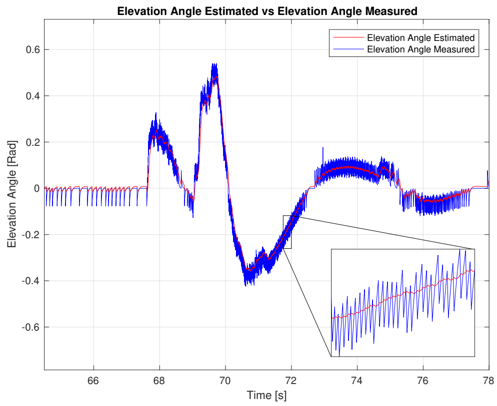
When choosing values for the estimator, a choice was made of whether focus



(a) Pitch Angle Estimated/Measured



(b) Pitch Rate Estimated/Measured



(c) Travel Rate Estimated/Measured

Figure 11: Estimator plotted with measured data

on the responsiveness or focus on the robustness. If the estimator was to be very fast and/or accurate, it would include the noise, which course is not desired. Therefore a trade off between robustness and responsiveness was made and shown in fig. 11a, fig. 11b and fig. 11c. The system subdues the noise, but is still affected by it in some degree. It is still fast, but can struggle in some instances, for instance in fig. 11c, where the estimator is lagging when the pitch rate is high. The estimator could almost be compared to a low-pass filter of the measured data in a sense.

5.3 Problem 3

If only the linearized elevation angle and travel, $\tilde{e}$ and $\tilde{\lambda}$, is measured, the output and the C-matrix is given by

$$\mathbf{y} = \begin{bmatrix} \tilde{e} \\ \tilde{\lambda} \end{bmatrix} \tag{67a}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{67b}$$

The pair $(\mathbf{A}, \mathbf{C})$ is still 6-dimensional, so the observability matrix is still given as before (60). MATLAB was used to calculate the observability matrix[1] and is given by

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ K3 & 0 & 0 & 0 & 0 & 0 \\ 0 & K3 & 0 & 0 & 0 & 0 \\ & & \cdot & \cdot & & \\ & & \cdot & \cdot & & \end{bmatrix} \tag{68}$$

The matrix has full rank and the system therefore <u>observable</u>.

However, if only the linearized pitch and elevation is measured, the observability matrix will look quite different. The output of the system will be given by

$$\mathbf{y} = \begin{bmatrix} \tilde{p} \\ \tilde{e} \end{bmatrix} \tag{69a}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{69b}$$

Again by using (60) and MATLAB, the observability matrix[2] is given by

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ & & \cdot & \cdot & & \\ & & \cdot & \cdot & & \end{bmatrix} \tag{70}$$

This matrix has a rank of four, which is not full rank, and the system is therefore <u>not observable</u>. This is because the relationship given by the differential

[1] Last four rows are zero
[2] Last eight rows are zero

equations describing the the system (11). The linearized travel acceleration is proportional with the linearized pitch angle and is given by (11c). Finding $\tilde{\lambda}$ requires double integration of $\tilde{p}$ (and multiplying the result with the constant $K3$). This leads to unknown integral constants that affects both the angle position and the angle velocity of the pitch.

When realizing the system with the new output from (67), only changing the **C**-matrix of the total system was necessary. The rest of the system is equal the PI-control system given in (58a) and (58b). Tuning this system proved to be more difficult than it had been before, with the reason that double differentiation was involved. This made the system very sensitive to small changes in the system, ie. noise. Therefore a slower and a more "accepting" system was required if the helicopter was to hover at all. This was done by tuning the **R**-matrix to be very large for its correspond value for pitch rate. The new **R**-matrix was given by

$$\mathbf{R} = \begin{bmatrix} 2 & 0 \\ 0 & 10000000 \end{bmatrix} \tag{71a}$$

The Q-matrix was valuing pitch rate and pitch and was given by[3]

$$\mathbf{Q} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{71b}$$

The new **R**-matrix changed the system's **K**-matrix and moved the poles more to the right (made the system slower). It also changed the **L**-matrix to have lower values. However, due to the double derivation, the system was still sensitive to fast and/or small changes. In fig. 12[4] the helicopter takes off at $t = 0$ and gets a push at the travel angle at $t \approx 13$. The helicopter is slowly decreasing the oscillating behaviour after the takeoff, but will never be asymptotically stable at the pitch angle because the inertia of the system and noise affecting it. When the helicopter is pushed the pitch angle and pitch rate work against the travel rate. This is to oppose the change of travel rate and make the helicopter still. When the helicopter is still, the oscillating behaviour will start again.

It is important to mention that the estimated states are no way as close to the real values in this section as they have been earlier. As shown in figures fig. 13a and fig. 13b, the estimated states are not as close to the real system as they have been. This is caused by the fast changes that were applied to the

[3] This might seem arbitrary because of the high value in the R-matrix. Both could perhaps have been lowered, but this combination worked well

[4] Estimated states plotted, not "real" behaviour

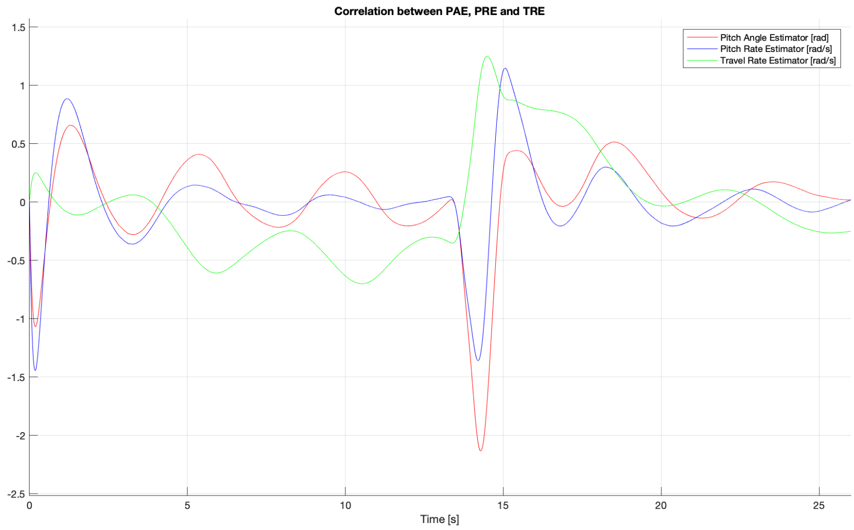
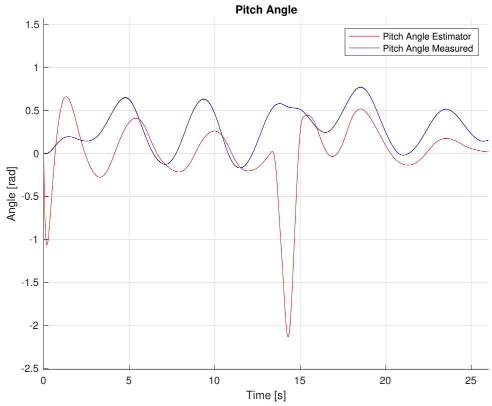
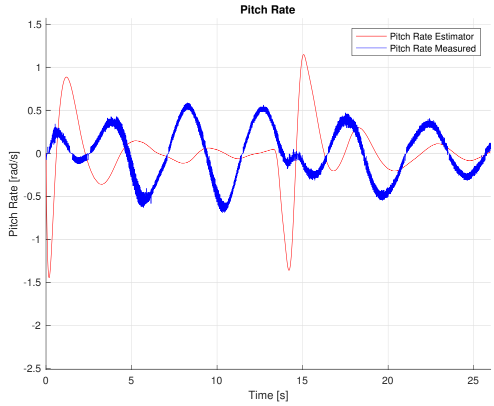


Figure 12: Pitch angle, pitch rate and travel rate estimators plotted in correlation

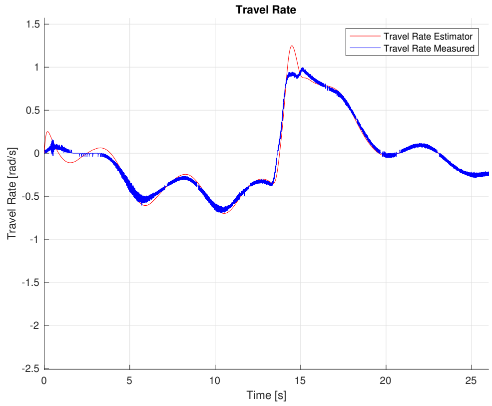
system. The takeoff and the push is especially off. In fig. 13a the estimator suggest a fast change in pitch to counteract change in travel rate caused by the push. In fig. 13b, this is also the cause, but it also has second spike to straighten the helicopter's pitch. The reason the travel rate estimator in fig. 13c is working well is because it is estimated based on the measurements. The pitch estimator is based on the travel rate and is therefore not as good as travel rate estimator.



(a) Pitch Angle Estimated/Measured



(b) Pitch Rate Estimated/Measured



(c) Travel Rate Estimated/Measured

Figure 13: Estimator plotted with measured data

6 Appendix

MATLAB code used for pole placement

```
K = lqr(A, B, Q, R);
P = -(C*((A-B*K)^(-1))*B)^(-1);

%Placing poles for estimator (Placing in an arc)
systemPoles = eig(A-B*K);

%Longest distance from origo for systempoles ...
    (Pole farthest away)
r0 = max(abs(systemPoles));

%How many times the arc is going to be bigger ...
    than r0
fr = 11.5;
phi = pi/8; % angle between each pole-par
r = r0*fr; % Radius of the arc
spread = -phi:(phi/(2.5)):phi;
poles = -r*exp(1i*spread);

L = transpose(place(transpose(A),transpose(C), ...
    poles));
```

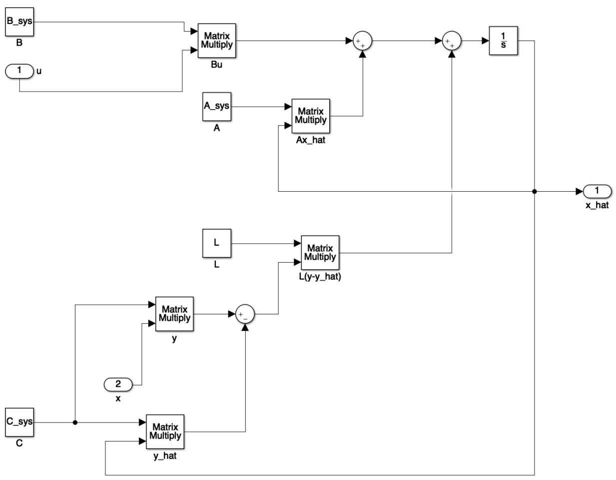


Figure 14: Snapshot of the implementation of the estimator in Simulink

References

[1] J. G. Balchen, T. Andresen, and A. F. Foss. *Reguleringsteknikk*. Institutt for teknisk kybernetikk, NTNU, 2016.

[2] Chi-Tsong Chen. *Linear System Theory and Design*. Oxford University Press, Incorporated, 2014.

[3] *Helicopter lab assignment*. https://ntnu.blackboard.com/bbcswebdav/pid-420173-dt-content-rid-17161604_1/courses/194_TTK4115_1_2018_H_1/194_TTK4115_1_2018_H_1_ImportedContent_20180815022347/Helicopter_lab_assignment.pdf. Accessed: 2018-10-20.

[4] *The Ground Effect*. https://en.wikipedia.org/wiki/Ground_effect_(aerodynamics). Accessed: 2018-10-20.