Lecture 6: Quadratic Programming & Equality-constrained QPs

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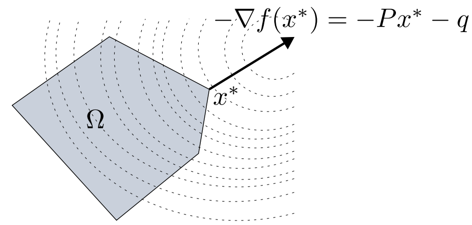
- Quadratic programming
  - Convex problem if $G \geq 0$
  - Feasible set polyhedron

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \frac{1}{2} x^\top G x + q^\top x \\ \text{s.t.} \quad & c_i(x) = a_i^\top x - b_i = 0, \ i \in \mathcal{E} \\ & c_i(x) = a_i^\top x - b_i \geq 0, \ i \in \mathcal{I} \end{aligned}$$

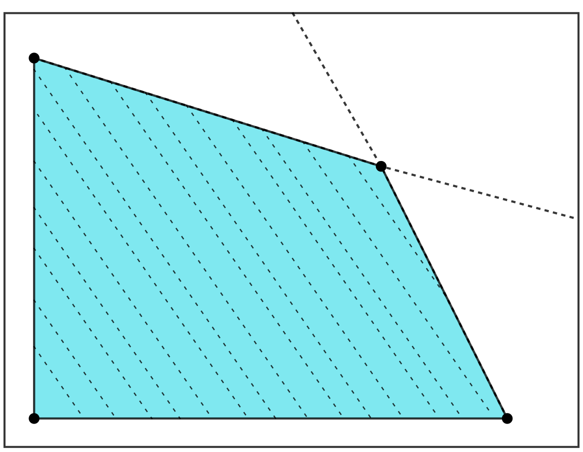


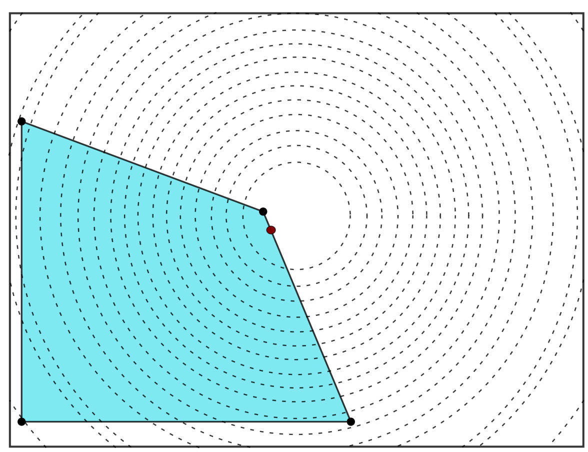
The easy case is when $G \geq 0$, because the QP is convex. When $G \ngeq 0$, then the problem is non-convex, and much harder.

Quadratic programming is common in control. Quadratic programming also shows up in production optimization when the price varies with ammount produced. This is a realistic example. Here is an example comparing linear profit with profit that depends on how much you produce. :

$$\begin{aligned} \max_{x_1, x_2} \quad & 7000x_1 + 6000x_2 \\ \text{s.t.} \quad & 4000x_1 + 3000x_2 \geq 100000 \\ & 60x_1 + 80x_2 \leq 2000 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max_{x_1, x_2} \quad & (7000 - 200x_1)x_1 + (6000 - 140x_2)x_2 \\ \text{s.t.} \quad & 4000x_1 + 3000x_2 \leq 100000 \\ & 60x_1 + 80x_2 \leq 2000 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$





The level curves are no longer linear, but real curves. Also, the optimal point may no longer lie in the corner, which is the property used in LP solvers.

Rewriting the objective function for the quadratic version, we get

$$\min_{x_1, x_2} 200x_1^2 + 140x_2^2 - 7000x_1 - 6000x_2$$

$$\min_{x \in \mathbb{R}^2} \frac{1}{2} x^\top \begin{bmatrix} 400 & 0 \\ 0 & 280 \end{bmatrix} x + \begin{bmatrix} -7000 & -6000 \end{bmatrix} x$$

The matrix here is G, and it is always symmetric.

1 Equality-constrained QPs

In general

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \frac{1}{2}x^\top Gx + c^\top x \\ \text{s.t.} \quad & a_i^\top x - b_i = 0,\ i \in \mathcal{E} = \{1,,\dots,m\},\ \mathcal{I} = \emptyset \end{aligned}$$

Define $A = \begin{bmatrix} a_1^\top \\ a_2^\top \\ \vdots \\ a_m^\top \end{bmatrix} \in \mathbb{R}^{m\times n},\ m < n.\ b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ and we assume $rank(A) = m$ (full row rank). Now we can write it (EQP) as

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \frac{1}{2}x^\top Gx + c^\top x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

KKT for equality-constrained QP
Since we have equality constraints, only the two KKT conditions need to be considered. The lagrangian:

$$\mathcal{L}(x,\lambda) = \frac{1}{2}x^\top Gx + c^\top x - \lambda^\top (Ax - b).$$

KKT:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*,\lambda^*) = Gx^* + c - A^\top\lambda^* = 0 \\ Ax^* = b \end{aligned}$$

The KKT conditions are linear here, so we can write them in matrix form.

$$\begin{bmatrix} G & -A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}.$$

Solving this system will solve the KKT condition. An alternative form that can be more practical ($x^* = x + p$):

$$\begin{bmatrix} G & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} -p \\ \lambda^* \end{bmatrix} = \begin{bmatrix} c - Gx^* \\ Ax - b \end{bmatrix}.$$

These are called KKT-systems, and they show up a lot.

Nullspace is needed to know when solution to KKT is also the solution to EQP.

Given feasible x (Ax = b). Rewrite the EQP as

$$\begin{aligned} \min_{p} \quad & \frac{1}{2}(x+p)^\top G(x+p) + x^\top (x+p) \\ \text{s.t.} \quad & A(x+p) = b \qquad \text{(Ap = 0 since Ax = b)} \end{aligned}$$

We are searching for solutions p in $Null(A) = \{w|Aw = 0\}$.

Let columns of $Z \in \mathbb{R}^{n\times(n-m)}$ span $Null(A)$. Use QR to find this.

Example: $A_p = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0 \qquad \begin{pmatrix} m = 1 \\ n = 3 \end{pmatrix}$

$A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0,\ A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \implies Z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \implies AZ = 0$

When can EQP be solved - Assume A full row rank. Also, $Z^\top GZ > 0$, then $\begin{bmatrix} G & A^\top \\ A & S \end{bmatrix}$ is non-singluar. This implies $\begin{cases} x^* = x + p & \text{is unique solution} \\ \lambda^* & \text{to KKT system} \end{cases}$

Theorem 1.

$$\left.\begin{aligned} &A \textit{ full row rank} \\ &Z^\top GZ > 0 \end{aligned}\right\} \implies x^* \textit{ unique solution to EQP.}$$

Proof Theorem 1 (16.2)

A x is feasible, x^* is a solution to KKT-system, $x + p = x^*$.

$$Ap = A(x^* - x) = Ax^* - Ax = b - b = 0 \implies p \in Null(A)$$

We want to show $q(x) > q(x^*),\ x \neq x^*$.

$$\begin{aligned} q(x) &= \frac{1}{2}x^\top Gx + x^\top x = \frac{1}{2}(A^* - p) + c^\top(x^* - p) \\ &= \frac{1}{2}x^{*\top}GA^* + \frac{1}{2}p^\top Gp + c^\top x^* - c^\top p \\ &= q(x^*) - p^\top(GA^* + c) + \frac{1}{2}p^\top Gp \\ &= q(x^*) + \frac{1}{2}u^\top Z^\top GZu \\ &> q(x^*) \end{aligned}$$

Nullspace method/Elimination of varialbes - ?