NTNU
Kunnskap for en bedre verden

TTK4135 - Optimization and Control

Exercise #3

Author:
Sondre Pedersen

February 6, 2025

Problem 1: LP and duality

Consider the linear programming problem

$$\begin{aligned} \min_{x\in\mathcal{R}^4} \quad & -2x_1 + 3x_2 \\ \text{s.t.} \quad & -x_1 + x_2 - x_3 = 1 \\ & 3x_1 - x_2 + x_4 \geq 8 \\ & x_1 \geq 0 \\ & x_3 \geq 0 \end{aligned}$$

(a) Transform the LP to standard form, and write down the matrix A and vectors b and c.

(b) Write down the dual form of the standard form LP, and solve it.

(c) Use duality result to calcualte a solution to the original LP.

(a) Standard and matrix form

Here we have to define a positive and negative version of all the variables, so let $x_2 = x_2^+ - x_2^-$, $x_4 = x_4^+ - x_4^-$. Also include the slack variable z as always. Now minimize over $x' = \begin{bmatrix} x_1 & x_2^+ & x_2^- & x_3 & x_4^+ & x_4^- \end{bmatrix}^\top$ and z. Standard form:

$$\begin{aligned} \min_{x',\ z} \quad & -2x_1 + 3x_2^+ - 3x_2^- \\ \text{s.t.} \quad & -x_1 + x_2^+ - x_2^- - x_3 = 1 \\ & 3x_1 - x_2^+ + x_2^- x_4^+ - x_4^- - z = 8 \\ & x',\ z \geq 0 \end{aligned}$$

On matrix form:

$$\min_{x\in\mathcal{R}^7} c^\top x \qquad \text{s.t.} \qquad Ax = b,\ x \geq 0.$$

where

$$c^\top = \begin{bmatrix} -2 & 3 & -3 & 0 & 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & -1 & -1 \end{bmatrix},\ b = \begin{bmatrix} 1 \\ 8 \end{bmatrix}.$$

(b) Dual

Dual form:

$$\max_{y\in\mathcal{R}^2} b^\top y \qquad \text{s.t.} \qquad A^\top y \leq c.$$

where y is just $\begin{bmatrix} y_1 & y_2 \end{bmatrix}^\top$
It is possible to determine y from this from the dual. Writing out $A^\top y \leq 0$ we get

$$\begin{aligned} -y_1 + 3y_2 &\leq -2 \\ y_1 - y_2 &\leq 3 \\ -y_1 + y_2 &\leq -3 \\ -y_1 &\leq 0 \\ y_2 &\leq 0 \\ -y_2 &\leq 0 \\ -y_2 &\leq 0 \end{aligned}$$

From the last rows we can see that $y_2 \leq_0,\ y_2 \geq 0 \implies y_2 = 0$. Now the inequalities can be simplified to

$$\begin{aligned} y_1 &\geq 2 \\ y_1 &\leq 3 \\ y_1 &\geq 3 \\ y_1 &\geq 0 \end{aligned}$$

Clearly $y_1 = 3$. This gives us a solution to the dual (and primal), $b^\top y = \begin{bmatrix}1 & 8\end{bmatrix}\begin{bmatrix}3 & 0\end{bmatrix} = 3$.

(c) Solution to the primal

The result from (b) means that $-2x_1 + 3x_2 = 3$. Putting this together with the other restrictions we get

$$\begin{aligned} -2x_1 + 3x_2 &= 3, \ (i) \\ -x_1 + x_2 - x_3 &= 1, \ (ii) \\ 3x_1 - x_2 + x_4 &\geq 8, \ (iii) \\ x_1 &\geq 0, \ (iv) \\ x_3 &\geq 0, \ (v) \end{aligned}$$

Taking (i) - 2(ii) we get $x_2 + 2x_3 = 1$. Since $x_3 \geq 0$, $x_2 \geq 1$. Using this in (i) we see that $x_1 = 0$, $x_2 = 1$.

With this we can simplify the restrictions and get

$$\begin{aligned} 1 - x_3 &= 1 \\ 3x_1 - 1 + x_4 &\geq 8 \\ x_3 &\geq 0 \end{aligned}$$

$x_3 = 0$, leaving the last restriction $x_4 \geq 9$.
The final solution is then

$$x^* = \begin{bmatrix}0 & 1 & 0 & x_4\end{bmatrix}^\top \qquad x_4 \geq 9.$$

Problem 2: Linear programming problem

Two reactsr, R_1 and R_2, produce two products A and B. To make 1000 kg of A, 2 hours of R_1 and 1 hour of R_2 are required. To make 1000kg of B, 1 hour of R_1 and 3 hours of R_2 are reqauired. The order of R_1 and R_2 does not matter. R_1 and R_2 are available for 8 and 15 hours, respectively. The selling price of A is $\frac{3}{2}$ of the selling price of B (i.e., 50% higher). We want ot maximize the total selling price of the two products.

(a) Formulate this problem as an LP in standard form.

(b) Make a contour plot and sketch the constraints.

(c) Calculate the production of A and B that maximizes the total selling price. Use the simplex method, with startingpoint $x_1 =?x_2 = 0$. Is the solution at a point of intersection between the constraints? Area all contraints active.

(d) Mark all iteration s on the plot made in (b), as well as the iteration number.

(e) Look at the iterations on the plot and the algorithm output. Does it agree with the theory?

(a) Standard form

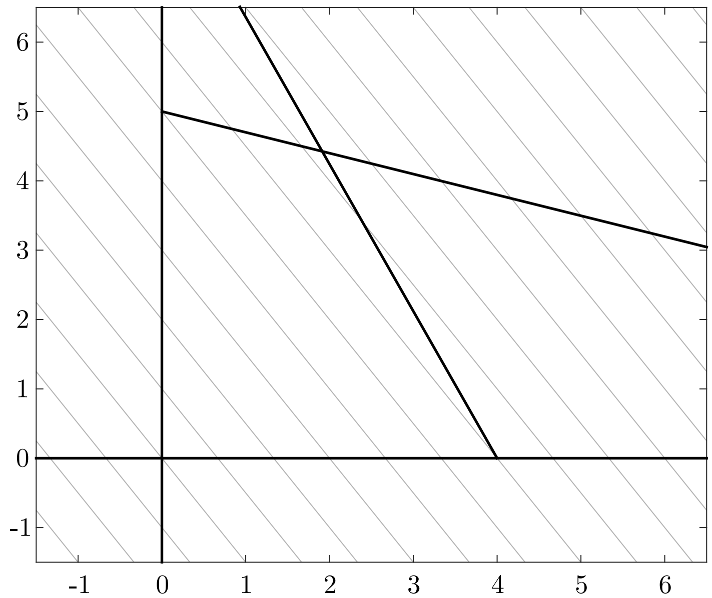
Standard form:

$$\min c^\top x \qquad \text{s.t.} \qquad Ax = b, \ x \geq 0.$$

Where

$$c = \begin{bmatrix}-3 & -2 & 0 & 0\end{bmatrix}^\top, \ A = \begin{bmatrix}2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1\end{bmatrix}^\top, \ b = \begin{bmatrix}8 & 15\end{bmatrix}^\top.$$

(b) Contour plot



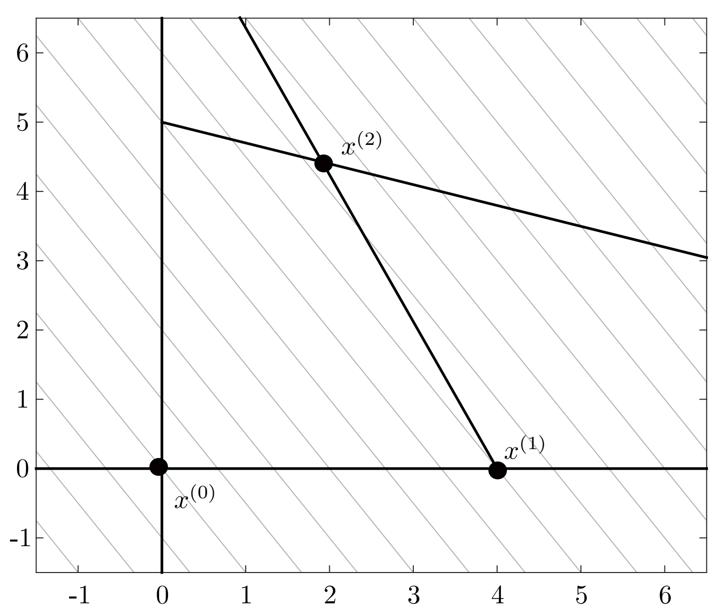
(c) Solution

The simplex algorithm gives these ouputs:

- $x^{(0)} = \begin{bmatrix} 0 & 0 & 8 & 15 \end{bmatrix}^\top$
- $x^{(1)} = \begin{bmatrix} 4 & 0 & 0 & 11 \end{bmatrix}^\top$
- $x^{(2)} = \begin{bmatrix} 1.8 & 4.4 & 0 & 0 \end{bmatrix}^\top$

The equality constraints are active. The non-negativity constraints are not active.

(d) Iteration points



(e) Theory

I am not able to detect anything that does not follow the theory.

Problem 3: QP and KKT

A quadratic program (QP) can be formulated as

$$\begin{aligned} \min_{x} \quad & q(x) = \frac{1}{2}x^\top G x + x^\top c \\ \text{s.t.} \quad & a_i^\top x = b_i, \ i \in \mathcal{E} \\ & a_i^\top x \geq b_i, \ i \in \mathcal{I} \end{aligned}$$

where G is a symmetric $n \times n$ matrix, $\mathcal{E}$ and $\mathcal{I}$ are finite sets of indices, and c, x, and $\{a_i\}, \ i \in \mathcal{E} \cup \mathcal{I}$ are vectors in $\mathcal{R}^n$

(a) Define the active set $\mathcal{A}(x^*)$ for problem (2).

(b) Derive the KKT conditions for problem (2), using the active set in the formulation.

(a) Active set

The active set includes the indices of active constraints. It looks like

$$\mathcal{A}(x^*) = \{i \in \mathcal{E} \cup \mathcal{I} | a_i^\top x^* = b_i\}.$$

(b) KKT

The Lagrangian

$$\mathcal{L}(x, \lambda) = q(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i (a_i^\top x - b_i).$$

λ_i for negativity constraints are 0 (these constraints are not active). We get the derivate of the Lagrangian:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= \frac{1}{2}(Gx^* + G^\top x^*) + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i, \ G^\top = G \\ &= Gx^* + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i \end{aligned}$$

The remaining KKT conditions say that constraints must be respected, and that λ for active constraints must be non-negative. Summarized we get:

$$\begin{aligned} Gx^* + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i = 0 & \\ a_i^\top x^* = b_i & \qquad \text{(active constraints)} \\ a_i^\top x^* \geq b_i & \qquad \text{(non-active constraints)} \\ \lambda_i^* \geq 0 & \qquad \text{(active constraints)} \end{aligned}$$