Kunnskap for en bedre verden

TTK4135 - Optimization and Control

# Exercise #2

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**Problem Problem 1: LP and KKT conditions**

The following linear program is in standard forM:

$$\min_x c^\top x \qquad s.t. \qquad Ax = b,\ x \geq 0.$$

with $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $bin \in \mathbb{R}^m$.

(a) Derive the KKT conditions

(b) Show that the objective function is convex, by using the definition of a convex function.

## (a) KKT conditions

The Lagrangian:

$$\mathcal{L}(x, \lambda, s) = c^\top x - \lambda^\top (Ax - b) - s^\top x.$$

The KKT conditions:

$$\begin{aligned} \nabla_x \mathcal{L}(x, \lambda, s) = c - A^\top \lambda - s = 0 \\ Ax = b \\ x \geq 0 \\ s \geq 0 \\ x^\top s = 0 \end{aligned}$$

The last one is comes from the assumption that both x and s is greater than 0.

## (b) Convex objective function

The feasible region is convex, because it is constrained by linear functions. Given two feasible points $x^{(1)}$ and $x^{(2)}$, $f(x)$ is convex if

$$f(\alpha x^{(1)} + (1-\alpha)x^{(2)}) \leq \alpha f(x^{(1)}) + (1 - \alpha f(x^{(2)})),\ \alpha \in [0, 1].$$

Check for $f(x) = c^\top x$:

$$f(\alpha x^{(1)} + (1-\alpha)x^{(2)}) = c^\top(\alpha x^{(1)} + (1-\alpha)x^{(2)}) = \alpha c^\top x^{(1)} + (1-\alpha)c^\top x^{(2)} = \alpha f(x^{(1)}) + (1-\alpha) f(x^{(2)}).$$

This shows that f is convex.

**Problem Problem 2: Linear programming problem**

In a plant three products, R, S, and T are made in two process stages A and B. To make a product the following time in each process stage is required:

- 1 tonne of R: 3 hours in stage A plus 2 hours in stage B.
- 1 tonne of S: 2 hours in stage A and 2 hours in stage B.
- 1 tonne of T: 1 hour in stage A and 3 hours in stage B.

During one year, stage A has 7200 hours and stage B has 6000 hours available production time. The rest of the time is needed for maintenance. It is required that the available production time should be fully utilized in both stages.

The profit from the sale of the product is:

- R: 100 NOK per tonne.
- S: 75 NOK per tonne.
- T: 55 NOK per tonne.

We wish to mazimize the yearly profit.

(a) Formulate this as an LP problem on standard form.

(b) Which basic feasible points exist?

(c) Find the solution by checking the KKT conditions at all the feasible points found in (b).

(d) Formulate the dual problem for the LP in (a).

(e) Show that the optimal objective function value for the LP in (a) equals the optimal objective function value for the dual problem in (d) by showing that $c^\top x^* = b^\top \lambda^*$.

(f) If you can make either stage A or stage B more available, which should you choose?

## (a) Standard form

$$\begin{aligned} &\min && -c^\top x \qquad \text{s.t.} \qquad Ax = b,\ x \geq 0 \\ &\text{where} && c = \begin{bmatrix} 100 \\ 75 \\ 65 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix},\ b = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix} \end{aligned}$$

## (b) Feasible points

The feasible points are the intersections between the restrictions. We can find these by picking two indices $i, j \in \{1, 2, 3\}$ and solving $\begin{bmatrix} A_i & A_j \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} = b$

With $i, j = 1, 2$:

$$\left[\begin{array}{cc|c} 3 & 2 & 7200 \\ 2 & 2 & 6000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2/3 & 2400 \\ 0 & 2/3 & 1200 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1200 \\ 0 & 1 & 1800 \end{array}\right]. \qquad x^{(1)} = \begin{bmatrix} 1200 \\ 1800 \\ 0 \end{bmatrix}.$$

With $i, j = 1, 3$:

$$\left[\begin{array}{cc|c} 3 & 1 & 7200 \\ 2 & 3 & 6000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/3 & 2400 \\ 0 & 7/3 & 1200 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 15600/7 \\ 0 & 1 & 3600/7 \end{array}\right]. \qquad x^{(2)} = \begin{bmatrix} 2400 \\ 0 \\ 1200 \end{bmatrix}.$$

With $i, j = 2, 3$:

$$\left[\begin{array}{cc|c} 2 & 1 & 7200 \\ 2 & 3 & 6000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/2 & 3600 \\ 0 & 2 & -1200 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3900 \\ 0 & 1 & -600 \end{array}\right]. \qquad x^{(3)} = \begin{bmatrix} 0 \\ 3900 \\ -600 \end{bmatrix}.$$

## (c) Check KKT

Look at $x^{(1)}$:

Find s such that $s^\top x^{(1)} = 0$:

$$\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} 1200 \\ 1800 \\ 0 \end{bmatrix} = 0 \implies s = \begin{bmatrix} 0 \\ 0 \\ s_3 \end{bmatrix}.$$

Now check KKT-1: $A^\top \lambda^* + s^* = c$:

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s_3 \end{bmatrix} = - \begin{bmatrix} 100 \\ 75 \\ 55 \end{bmatrix} \implies \left[\begin{array}{ccc|c} 3 & 2 & 0 & -100 \\ 2 & 2 & 0 & -75 \\ 1 & 3 & 1 & -55 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & 0 & \frac{-100}{3} \\ 0 & \frac{2}{3} & 0 & \frac{-25}{3} \\ 0 & \frac{7}{3} & 1 & \frac{-65}{3} \end{array}\right] \to \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} -25 \\ -25/2 \\ 15/2 \end{bmatrix}.$$

This is a valid solution. Since satisfying the KKT conditions is both necessary and sufficient for LP problems, we do not need to check the other points. Anyway,

Look at $x^{(2)}$:

Find s such that $s^\top x^{(2)}$:

$$\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} 2400 \\ 0 \\ 1200 \end{bmatrix} = 0 \implies s = \begin{bmatrix} 0 \\ s_2 \\ 0 \end{bmatrix}.$$

Now check KKT-1: $A^\top \lambda^* + s^* = c$:

$$\begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ s_2 \\ 0 \end{bmatrix} = - \begin{bmatrix} 100 \\ 75 \\ 55 \end{bmatrix} \implies \left[\begin{array}{ccc|c} 3 & 2 & 0 & -100 \\ 2 & 2 & 1 & -75 \\ 1 & 3 & 0 & -55 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & 0 & -100/3 \\ 0 & \frac{2}{3} & 1 & -25/3 \\ 0 & \frac{7}{3} & 0 & -65/3 \end{array}\right] \to m \left[\begin{array}{ccc|c} 1 & 0 & -1 & -25 \\ 0 & 1 & \frac{3}{2} & -25/2 \\ 0 & 0 & \frac{-7}{2} & 15/2 \end{array}\right].$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -190/7 \\ 0 & 1 & 0 & -65/7 \\ 0 & 0 & 1 & -15/7 \end{array}\right] \implies \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} -190/7 \\ -65/7 \\ -15/7 \end{bmatrix}.$$

We see that $x^{(2)}$ does not satisfy the KKT conditions because $s_2 < 0$.

No need to look at $x^{(3)}$, because it is not greater than 0, breaking a constraint.
The optimal solution is now $f(x^{(1)}) = - \begin{bmatrix} 100 & 75 & 55 \end{bmatrix}^\top \begin{bmatrix} 1200 & 1800 & 0 \end{bmatrix} = -255000$

## (d) Dual

The dual is $\max b^\top \lambda \qquad \text{s.t.} \qquad A^\top \lambda \leq c$

## (e) Primal and dual are equal

Using:

$$\begin{aligned} A^\top \lambda^* + s^* &= c \\ A x^* &= b \\ s^{*T} x^* &= 0 \end{aligned}$$

Gives:

$$\begin{aligned} c^\top x^* &= (A^\top \lambda^* + s^*)^\top x^* \\ &= \lambda^{*\top} A x^* + s^{*\top} x^* \\ &= \lambda^{*\top} b + s^{*\top} x^* \\ &= \lambda^{*\top} b + 0 \\ &= b^\top \lambda^* \end{aligned}$$

## (d) Increase availability

To decide which stage to increase availability, we only need to look at the shadow price ($\lambda^*$). The value of these show how much we gain by increasing the availability. Since $25 = |\lambda_1^*| > |\lambda_2^*| = 25/2$, we choose to relax constraint 1 (increase availability of A). One unit relaxation (one hour added) improves the objective by $\lambda_1 = 25$. So the new optimal value is -255025.