Lecture 7: Active Set Method for Quadratic Programming

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1 Background for Active Set Method

Simplified preview:

- Make a guess of which inequality constraints are active
- Solve corresponding EQP
- Check KKT-conditions
  - If KKT OK: done
  - else: update active contraint guess and go to 2.

KKT for QP $\min_{x\in\mathbb{R}^n} \frac{1}{2}x^\top Gx + c^\top x \qquad \text{s.t.} \qquad \begin{cases} a_i^\top x = b_i,\ i \in \mathcal{E} \\ a_i^\top x \geq b_i,\ i \in \mathcal{I} \end{cases}$

Lagrangian:

$$\mathcal{L}(x,\lambda) = \frac{1}{2}x^\top Gx + x^\top x - \sum_{i\in\mathcal{E}\cup\mathcal{I}} \lambda_i(a_i^\top x - b_i).$$

KKT:

$$\begin{aligned} Gx^* + c - \sum_{i\in\mathcal{E}\cup\mathcal{I}} \lambda_i^* a_i &= 0 \\ a_i^\top x^* &= b_i,\ i \in \mathcal{E} \\ a_i^\top x^* &\geq b_i,\ i \in \mathcal{I} \\ \lambda_i^* &\geq,\ i \in \mathcal{I} \\ \lambda_i^*(a_i^\top x - b_i) &= 0, i \in \mathcal{I}\cup\mathcal{E} \end{aligned}$$

To make active set method, we reformulate KKT via the active set, which was defined as:

$$\mathcal{A}(x^*) = \{i \in \mathcal{E}\cup\mathcal{I} | a_i^\top x^* = b_i\}$$

Assume we know $\mathcal{A}(x^*)$, then KKT is

$$\begin{aligned} Gx^* + x - \sum_{\mathcal{A}(x^*)} \lambda_i^* a_i &= 0 \\ a_i^\top x^* &= b_i,\ i \in \mathcal{A}(x^*) \\ a_i^\top x^* &\geq b_i,\ i \in \mathcal{I}\backslash\mathcal{A}(x^*) \\ \lambda_i^* &\geq 0,\ i \in \mathcal{I}\cup\mathcal{A}(x^*) \end{aligned}$$

The lambdas that are not active equals 0.

Theorem 16.4: If x^* satisifies KKT and $G \geq 0$ then x^* is a global solution

Proof: Assume x is feasible, $x \neq x^*$. Note first: $(x-x^*)^\top (Gx^*+c) = (x-x^*)^\top \sum_{i\in\mathcal{A}(x^*)} \lambda_i^* a_i = \sum_{i\in\mathcal{E}} \lambda_i^* a_i^\top (x-x^*) + \sum_{i\in\mathcal{A}(x^*)\cup\mathcal{I}} \lambda_i^* a_i^\top (x-x^*)$

$$\begin{aligned}
q(x) &= \frac{1}{2}(x^* + (x-x^*)^\top) G(x^* + (x-x^*)) + c^\top (x^* + (x-x^*)) \\
&= \frac{1}{2} x^{*\top} G x^* + c^\top x^* + (x-x^*)^\top G(x-x^*) + (x-x^*)^\top Gx + c^\top (x-x^*) \\
&= q(x^*) + \frac{1}{2}(x-x^*)^\top G(x-x^*) + (x-x^*)^\top (Gx^*+c) \\
&\geq q(x^*) \qquad \text{proof complete}
\end{aligned}$$

Note: $Z^\top G Z > 0 \implies q(x) > q(x^*)$, that is: x^* is strict global solution.

If we know $\mathcal{A}(x^*)$, QP can be forumalted as

$$\min_{xin\mathbb{R}^n} \frac{1}{2} x^\top G x + c^\top x \qquad \text{s-t-} \qquad a_i^\top x = b_i,\ i \in \mathcal{A}(x^\top).$$

This can then be solved by solving the KKT-system:

$$\begin{bmatrix} G & -A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix} \begin{cases} A = \begin{bmatrix} \vdots \\ a_i^\top \\ \vdots \end{bmatrix},\ i \in \mathcal{A}(x^*) \\ \\ b = \begin{bmatrix} \vdots \\ b_i \\ \vdots \end{bmatrix},\ i \in \mathcal{A}(x^*) \end{cases}.$$

2 One step of active set method for QP

Iteration k:

- $\mathcal{W}_k$: current 'guess' of $\mathcal{A}(x^*)$.
- x_k: current feasible estimate of x^*.
- g_k: $(Gx_k + c)^\top$

Define $A_x = \begin{bmatrix} \vdots \\ Q_i^\top \\ \vdots \end{bmatrix}, i \in \mathcal{W}_k,\ b_k = \begin{bmatrix} \vdots \\ b_i \\ \vdots \end{bmatrix}, i \in \mathcal{W}_k$

Consider EQP: $x = x_k + p$:

$$\begin{aligned}
&\min_{p\in\mathbb{R}^n} \quad \frac{1}{2}(x_k+p)^\top G(x_k+p) + c^\top (x_k+p) \\
&\quad \text{s.t.} \quad A_k(x_k+p) = b_k \\
\implies &\min_{p\in\mathbb{R}^4} \quad \frac{1}{2} p^\top G p + (Gx_k+c)^\top p \qquad \text{s.t.} \qquad Ap = 0
\end{aligned}$$

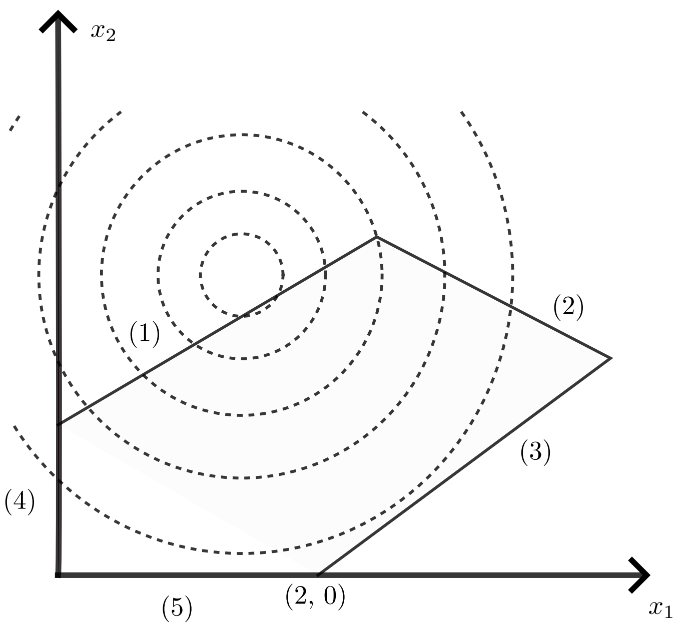
- If $p_k = 0$: solve $\sum_{i\in\mathcal{W}_k} a_i \hat{\lambda}_i = g_k$ for $\hat{\lambda}$ $(A_k \hat{\lambda} = g_k)$
  - If $\hat{\lambda} \geq 0$:
    * All KKT-conditions fulfulled
    * x_k is our solution

- If $\hat{\lambda} \ngeq 0$:
  * Pick index of one negative $\hat{\lambda}_i$
  * Remove this index from $\mathcal{W}_k$
  * Start over.

- If $p_k \neq 0$:
  - If $x_{k+1} = x_k + p_k$ is feasible: Set $\mathcal{W}_{k+1} = \mathcal{W}_k$, start over
  - If $x_{k+1} = x_k + p_k$ is not feasible: find blocking constraint
    * For $i \in \{i | a_i^\top p_k < 0\}$
    * want $a_i^\top (x_k + \alpha_i p_k) \geq b_i$
      $\implies \lambda_i = \frac{b_i - a_i^\top x_k}{\alpha_i^\top p_k}$
    * Set j = "i with smallest α_i"
    * $x_{k+1} = x_n + \alpha_j p_k$, $\mathcal{W}_{k+1} = \mathcal{W}_k + \{j\}$
    * start over...

This is one of the key algorithms in the course.

Example 16.4 -



$$
\begin{aligned}
\min_x q(x) &= (x_1 - 1)^2 + (x_2 - 2.5)^2 \\
\text{s.t.} \quad x_1 - 2x_2 + 2 &\geq 0 \qquad (1) \\
-x_1 - 2x_2 + 6 &\geq 0 \qquad (2) \\
-x_1 + 2x_2 + 2 &\geq 0 \qquad (3) \\
x_1 &\geq 0 \qquad (4) \\
x_2 &\geq 0 \qquad (5)
\end{aligned}
$$

$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$
\begin{aligned}
a_1 &= \begin{bmatrix} 1 & -2 \end{bmatrix}^\top & b_1 &= -2 \\
a_2 &= \begin{bmatrix} -1 & -2 \end{bmatrix}^\top & b_2 &= -6 \\
a_3 &= \begin{bmatrix} -1 & 2 \end{bmatrix}^\top & b_3 &= -2 \\
a_4 &= \begin{bmatrix} 1 & 0 \end{bmatrix}^\top & b_4 &= 0 \\
a_5 &= \begin{bmatrix} 0 & 1 \end{bmatrix}^\top & b_5 &= 0
\end{aligned}
$$

We have two different indices, one for iteration and one for variable. Denote iteration k like x^k.

$x^0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathcal{W} = \{3, 5\}$, $g_k = Gx^0 + c = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

EQP:

$$\begin{aligned}&\min_p \frac{1}{2} p^\top G p + g_0^\top p\\&\text{s.t. } A_0 p = 0\end{aligned}$$

$$\begin{aligned}&\min_p p_1^2 + p_2^2 + 2p_1 - 5p_2\\&\text{s.t. } -p_1 + 2p_2 = 0\\&p_2 = 0\end{aligned}$$

$$\sum_{i \in \mathcal{W}_0} a_i \hat{\lambda}_i = g_0 : \begin{bmatrix}-1\\-2\end{bmatrix} \hat{\lambda}_3 + \begin{bmatrix}0\\1\end{bmatrix} \hat{\lambda}_5 = \begin{bmatrix}2\\-5\end{bmatrix}$$

Remove $\{3\}$ from $\mathcal{W}_0$. Continue with next iteration. We have $x^1 = \begin{bmatrix}2\\0\end{bmatrix},\ \mathcal{W}_1 = \{5\},\ g_1 = \begin{bmatrix}2\\-5\end{bmatrix}$

EQP gives $p^1 = \begin{bmatrix}-1\\0\end{bmatrix}$.

$x^2 = x^1 + p^1 = \begin{bmatrix}1\\0\end{bmatrix}$, which is feasible. $\mathcal{W}_2 = \mathcal{W}^1$