NTNU

Kunnskap for en bedre verden

TMA4135 - MATEMATIKK 4D

Execise #2

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Problem 1.

a) Compute all Taylor polynomials of $f(x) = -2x^4 + 2x^2 - 3x + 2$ *around* $x_0 = -1$.

The general formula for a Taylor polynomial around $x = x_0 = a$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since we are computing all the Taylor polynomials, we can use the fact that the next polynomial builds on the previous like this:

$$T_n(x) = T_{n-1} + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

First we need to evaluate the derivatives at x = -1

$$\begin{aligned}
f'(x) &= -8x^3 + 4x - 3 \\
f''(x) &= -24x^2 + 4 \\
f^{(3)}(x) &= -48x \\
f(-1) &= 5 \\
f'(-1) &= 1 \\
f''(-1) &= -20 \\
f^{(3)}(-1) &= 48
\end{aligned}$$

Now for all the Taylor polynomials at x = -1

$$\begin{aligned}
T_0(x) &= f(-1) = 5 \\
T_1(x) &= f'(-1)(x-(-1)) + T_0(x) = 1 * (x+1) + 5 = x + 6 \\
T_2(x) &= \frac{f''(-1)}{2}(x+1)^2 + T_1(x) = \frac{-20}{2}(x+1)^2 + x + 6 \\
&= -10(x+1)^2 + x + 6 \\
T_3(x) &= \frac{f^{(3)}(-1)}{3!}(x+1)^3 + T_2(x) = \frac{48}{6}(x+1)^3 - 10(x+1)^2 + x + 6 \\
&= 8(x+1)^3 - 10(x+1)^2 + x + 6
\end{aligned}$$

$T_4(x) = f(x)$ because they are of same degree. Perfect approximation. This is why any T_n, $n > 4$ is also equal to f. Mathematically we can see that this is true because the 5th derivative and onwards all equal 0. No new terms added to the previous Taylor polynomial.

b) Compute the Taylor series of $g(x) = e^{1-2x}$ *around* $x_0 = 0$

$$\begin{aligned} g'(x) &= -2e^{1-2x} \\ g''(x) &= 4e^{1-2x} \\ g^{(3)}(x) &= -8e^{1-2x} \\ &\vdots \\ g^{(i)}(x) &= (-1)^i 2^i e^{1-2x} \end{aligned}$$

With this we can find the Taylor series, which is

$$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n e^{1-2x}}{n!} x^n$$

This series diverges, so $f(x) \neq T(x)$

Problem 2.

a) Compute by hand the Lagrangian cardinal functions for the points $x_0 = -1$ $x_1 = 0$ $x_2 = 1$ $x_3 = 2$.
The ith cardinal function for n + 1 datapoints is $L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}$
We have 4 datapoints, so $n = 3$. From this we can calculate the cardinal functions, L_i for $i = 0, 1, 2, 3$.

$$
\begin{aligned}
L_0(x) = \prod_{j=1}^{3} \frac{x-x_j}{x_0-x_j} &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \\
&= \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} \\
&= \frac{x(x^2-3x+2)}{(-6)} \\
&= -\frac{1}{6}(x^3-3x^2+2x) \\
L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{3} \frac{x-x_j}{x_1-x_j} &= \frac{(x+1)(x-1)(x-2)}{(1)(-1)(-2)} \\
&= \frac{(x-2)(x^2-1)}{2} \\
&= \frac{1}{2}(x^3-2x^2-x+2) \\
L_2(x) = \prod_{\substack{j=0 \\ j\neq 2}}^{3} \frac{x-x_j}{x_2-x_j} &= \frac{(x+1)(x)(x-2)}{(1-(-1))(1)(1-2)} \\
&= \frac{x(x^2-x-2)}{-2} \\
&= -\frac{1}{2}(x^3-x^2-2x) \\
L_3(x) = \prod_{j=0}^{2} \frac{x-x_j}{x_1-x_j} &= \frac{(x+1)(x)(x-1)}{(2-(-1))(2)(2-1)} \\
&= \frac{x(x^2-1)}{6} \\
&= \frac{1}{6}(x^3-x)
\end{aligned}
$$

b) Interpolate $f(x) = 2^{x^2-4-x}$ *using the results above.*
First we need to calculate the values $f(x_i)$ for i = 0, 1, 2, 3.

$$\begin{aligned}
f(x_0) &= f(-1) = 2^{(-1)^2-4-(-1)} = 2^{1-4+1} = 2^{-2} = \frac{1}{4} \\
f(x_1) &= f(0) = 2^{-4} = \frac{1}{16} \\
f(x_2) &= f(1) = 2^{1-4-1} = \frac{1}{16} \\
f(x_3) &= f(2) = 2^{4-4-2} = \frac{1}{4}
\end{aligned}$$

The formula for Lagrangian interpolation is $P_n(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$.
This gives us:

$$\begin{aligned}
P_3(x) &= \sum_{i=0}^{3} f(x_i)L_i(x) \\
&= f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x) + f(x_3)L_3(x) \\
&= \frac{1}{4}\frac{-1}{6}(x^3 - 3x^2 + 2x) + \frac{1}{16}\frac{1}{2}(x^3 - 2x^2 - x + 2) \\
&\quad + \frac{1}{16}\frac{-1}{2}(x^3 - x^2 - 2x) + \frac{1}{4}\frac{1}{6}(x^3 - x) \\
&= \underline{\frac{3}{32}x^2 - \frac{3}{32}x + \frac{1}{16}}
\end{aligned}$$

Problem 4.

a) Interpolate $f(x) = \cos^2 x$ using the datapoints:

i	0	1	2
x_i	0	$\pi/6$	$\pi/2$
y_i	1	$\frac{3}{4}$	0

It is given that the interpolated function should be written on the form

$$p(x) = a_0 + a_1 \cos(x) + a_2 \cos 2x.$$

This means that we have to find a_0, a_1 and a_2, by solving the following system of equations.

$$\begin{aligned}
p(x_0) &= p(0) = a_0 + a_1 \cos 0 + a_2 \cos(2 * 0) = 1 \\
p(x_1) &= p(\pi/6) = a_0 + a_1 \cos \pi/6 + a_2 \cos \pi/3 = \frac{3}{4} \\
p(x_2) &= p(\pi/2) = a_0 + a_1 \cos \pi/2 + a_2 \cos \pi = 0
\end{aligned}$$

Solve the system on matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{4} \\ 0 \end{bmatrix} \tag{1}$$

$$\begin{aligned}
&\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{3}{4} \\ 1 & 0 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \frac{\sqrt{3}-2}{2} & -\frac{1}{2} & -\frac{1}{4} \\ 0 & -1 & -2 & -1 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \frac{2\sqrt{3}-3}{4} & 0 & 0 \\ 0 & -1 & -2 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & & 1 & \frac{1}{2} \end{array}\right]
\end{aligned}$$

$\Rightarrow \underline{a_0 = \frac{1}{2}, a_1 = 0, a_2 = \frac{1}{2}}$

b) Show that $p(x) = f(x)$ in this case.
Our interpolated function is:

$$p(x) = \frac{1}{2} + \frac{1}{2}\cos 2x$$

Using the fact that $\cos 2x = \cos^2 x - \sin^2 x$ gives us:

$$p(x) = \frac{1}{2} + \frac{1}{2}(\cos^2 x - \sin^2 x) = \frac{1}{2} + \frac{1}{2}\cos^2 x - \frac{1}{2}\sin^2 x$$

Using $\sin^2 x = 1 - \cos^2 x$:

$$= \frac{1}{2} + \frac{1}{2}\cos^2 x - \frac{1}{2}(1 - cos^2 x)$$

which simplifies to $cos^2 x$. <u>This shows that $p(x) = f(x)$ in this particular case.</u>