NTNU

Kunnskap for en bedre verden

TMA4135 - Matematikk 4D

Execise #1

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Problem 1.

$u = t^5 + \sin(xy)$

$$
\begin{aligned}
u_y &= \frac{\partial}{\partial y}\left(t^5 + \sin(xy)\right) = \frac{\partial}{\partial y}\left(t^5\right) + \frac{\partial}{\partial y}(xy)\cos xy = \underline{x\cos xy} \\
u_t &= \frac{\partial}{\partial t}\left(t^5\right) + 0 = \underline{5t^4} \\
u_{xx} &= \frac{\partial}{\partial x}(y\cos xy) = \underline{-y^2\sin xy} \\
u_{xy} &= u_{yx} \text{ (due to function being continuous)} \\
&= \frac{\partial}{\partial x}u_y = \frac{\partial}{\partial x}x\cos xy = \underline{\cos xy - xy\sin xy}
\end{aligned}
$$

$u = \cos txy$

$$
\begin{aligned}
u_y &= \underline{-tx\sin txy} \\
u_t &= \underline{-xy\sin txy} \\
u_{xx} &= \frac{\partial}{\partial x}(-tx\sin txy) = \underline{-t^2y^2\cos txy} \\
u_{xy} &= u_{yx} \\
&= \frac{\partial}{\partial x} - tx\sin txy = \underline{-t\left(txy\cos txy + sintxy\right)}
\end{aligned}
$$

$u = e^{-t}\sin x\ln y$

$$
\begin{aligned}
u_y &= \frac{e^{-t}\sin x}{y} \\
u_t &= \underline{e^{-t}\sin x\ln y} \\
u_{xx} &= \frac{\partial}{\partial x}\left(e^{-t}\cos x\ln y\right) = \underline{-e^{-t}\sin x\ln y} \\
u_{xy} &= u_{yx} \\
&= \frac{\partial}{\partial x}\frac{e^{-t}\sin x}{y} = \frac{e^{-t}\cos x}{y}
\end{aligned}
$$

$u = e^{-x}\sqrt{x^2 + y}$

$$
\begin{aligned}
u_y &= \frac{e^{-x}}{2\sqrt{x^2+y}} \\
u_t &= \underline{0} \\
u_{xx} &= \frac{\partial}{\partial x}\left(-e^{-x}\sqrt{x^2+y} + \frac{e^{-x}x}{\sqrt{x^2+y}}\right) \\
&= \frac{e^{-x}\left(x^4-2x^3+2x^2y-xy+y^2+y\right)}{(x^2+y)\sqrt{x^2+y}} \\
u_{xy} &= \frac{\partial}{\partial x}\left(\frac{e^{-x}}{2\sqrt{x^2+y}}\right) = -\frac{e^{-x}}{2\sqrt{x^2+y}} - \frac{xe^{-x}}{2\left(x^2+x\right)^{3/2}} \\
u_{yx} &= \frac{\partial}{\partial y}\left(-e^{-x}\sqrt{x^2+y} + \frac{e^{-x}x}{\sqrt{x^2+y}}\right) = \frac{-e^{-x}\left(x^2+x+y\right)}{2\left(x^2+y\right)\sqrt{x^2+y}}
\end{aligned}
$$

$u = t^2e^t\cos x$

$$
\begin{aligned}
u_y &= \underline{0} \\
u_t &= \underline{\cos x\left(2te^t+e^tt^2\right)} \\
u_{xx} &= \underline{-e^tt^2\cos x} \\
u_{xy} &= u_{yx} = \underline{0}
\end{aligned}
$$

$u = \sin(t)e^{-y} + \cos te^{-x}$

$$
\begin{aligned}
u_y &= \underline{-e^{-y}t\sin t} \\
u_t &= \underline{e^{-y}\left(t\cos t+\sin t\right) - e^{-x}\sin t} \\
u_{xx} &= \underline{e^{-x}\cos t} \\
u_{xy} &= u_{yx} = \underline{0}
\end{aligned}
$$

Problem 2.

The given PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ can be written $u_{xx} + u_{yy} = 0$. Therefore, an equation u is a solution to the PDE if it can be written on the form $u_{xx} = -u_{yy}$.

$u = -x^2 + y^2$

$$
\begin{aligned}
u_{xx} &= \frac{\partial}{\partial x} - 2x = -2 \\
u_{yy} &= \frac{\partial}{\partial y}2y = 2 = -u_{xx}
\end{aligned}
$$

Since $u_{xx} = -u_{yy}$, we know that $u = -x^2 + y^2$ is a solution to the PDE.

$u = \sin x \cosh y$

$$\begin{aligned}
u_{xx} &= \cosh y \frac{\partial}{\partial x} \cos x = -\sin x \cosh y \\
u_{yy} &= \sin x \frac{\partial}{\partial y} \sinh y = \sin x \cosh y \\
\Rightarrow u_{xx} &= -u_{yy}
\end{aligned}$$

$u = \dfrac{y}{x^2 + y^2}$

$$\begin{aligned}
u_{xx} &= -\frac{\partial}{\partial x} \frac{2xy}{\left(x^2 + y^2\right)^2} = -\frac{2y\left(y^2 - 3x^2\right)}{\left(x^2 + y^2\right)^3} \\
u_{yy} &= \frac{\partial}{\partial y} \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} = -\frac{2y\left(3x^2 - y^2\right)}{\left(y^2 + x^2\right)^3} \\
\Rightarrow u_{xx} &= -u_{yy}
\end{aligned}$$

$u = \arctan \dfrac{y}{x}$

$$\begin{aligned}
u_{xx} &= -\frac{\partial}{\partial x} \frac{y}{y^2 + x^2} = \frac{2xy}{\left(y^2 + x^2\right)^2} \\
u_{yy} &= \frac{\partial}{\partial y} \frac{x}{y^2 + x^2} = -\frac{2xy}{\left(y^2 + x^2\right)^2} \\
\Rightarrow u_{xx} &= -u_{yy}
\end{aligned}$$

This verifies that all the equations solve the PDE.