NTNU
Kunnskap for en bedre verden

TMA4135 - Matematikk 4D

Execise #8

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Problem 1.

a) Determine order of the following Runge-Kutta method:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{1} & \frac{1}{2} & 0 & 0 \\ \frac{3}{4} & 0 & \frac{3}{4} & 0 \\ \hline & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} \end{array}$$

Can be written

$$\begin{cases} k_1 = f(x_n, t_n) \\ k_2 = f(x_n + h\frac{1}{2}k_1, t_n + h\frac{1}{2}) \\ k_3 = f(x_n + h\frac{3}{4}k_2, t_n + h\frac{3}{4}) \\ x_{n+1} = x_n + h(\frac{2}{9}k_1 + \frac{1}{3}k_2 + \frac{4}{9}k_3) \end{cases}$$

This is Ralston's third-order method. This can be discovered by going to Wikipedia. By looking at the tableu we can see that the method is explicit, because the diagonal (and to the right) contain only 0. We know that it has at least first order convergence, because the bottom row sum to 1.

b) First step of Ralston's method.

let $f(x,t) = x'(t) = t^2e^{-2x}, x(0) = 0, h = 0.48$. The first step is then:

$$\begin{aligned} k_1 &= f(x_0, t_0) = 0^2e^{-2\times 0} = 0 \\ k_2 &= f(0 + 0.48\frac{1}{2} \times 0, 0 + 0.48\frac{1}{2}) = f(0, 0.24) = 0.24^2e^{-2\times 0} = 0.0576 \\ k_3 &= f(0.48\frac{3}{4} \times 0.0576, 0.48\frac{3}{4}) = f(0.021, 0.36) = 0.36^2e^{-2\times 0.021} = 0.124 \\ x_1 &= 0.48(\frac{2}{9} \times 0 + \frac{1}{3}0.0576 + \frac{4}{9}0.124) = \underline{0.0357} \end{aligned}$$

I know this answer is quite far from correct. Will have to figure out what's wrong another time.

Problem 2.

a) Verify that $u(x) = \frac{x}{2}\cos\pi x$ *is the solution to* $u_{xx} + 2u_x + \pi^2 u = \cos\pi x - \pi(x+1)\sin\pi x$.
First find u_{xx} and u_x:

$$\begin{aligned}
u(x) &= \frac{x}{2}\cos\pi x \\
u_x &= \frac{1}{2}\frac{d}{dx}x\cos\pi x + \frac{1}{2}x\frac{d}{dx}\cos\pi x = \frac{1}{2}\cos\pi x - \frac{1}{2}x\pi\sin\pi x \\
u_{xx} &= -\frac{1}{2}\pi\sin\pi x - \frac{1}{2}\pi\sin\pi x - \pi^2\frac{1}{2}x\cos\pi x = -\pi\sin\pi x - \pi^2\frac{1}{2}x\cos\pi x
\end{aligned}$$

Now plug that into the bvp:

$$\begin{aligned}
u_{xx} + 2u_x + \pi^2 u &= -\pi\sin\pi x - \pi^2\frac{1}{2}x\cos\pi x + 2(\frac{1}{2}\cos\pi x - \frac{1}{2}x\pi\sin\pi x) + \pi^2\frac{x}{2}\cos\pi x \\
&= -\pi\sin\pi x - \pi^2\frac{1}{2}x\cos\pi x + \cos\pi x - x\pi\sin\pi x + \pi^2\frac{x}{2}\cos\pi x \\
&= \cos\pi x - \pi\sin\pi x - x\pi\sin\pi x \\
&= \underline{\cos\pi x - \pi(x+1)\sin\pi x}
\end{aligned}$$

Since the left hand side of the eqation equals the right hand side when plugging in the suggested solution, it must be valid.

b) Finite difference scheme using central differences. Let $\Delta x = 2/N$ as the grid size, and let $x_i = i\Delta x, i = 0, 1, 2, \ldots, N$. Central difference approximates the derivative by using the previous and the next point. We find the approximation for the first and second derivative by Taylor expansion of the two points:

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \frac{1}{2}u''(x)\Delta x^2 + \mathcal{O}\left(\Delta x^3\right) \tag{1}$$

$$u(x-\Delta x) = u(x) - u'(x)\Delta x + \frac{1}{2}u''(x)\Delta x^2 + \mathcal{O}\left(\Delta x^3\right) \tag{2}$$

Find approximation of u_x by taking (1) - (2):

$$\begin{aligned}
u(x+\Delta x) - u(x-\Delta x) &= 2u'(x)\Delta x + \mathcal{O}\left(\Delta x^3\right) \\
\Rightarrow u_x &\approx \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}
\end{aligned}$$

Find approximation of u_{xx} by taking (1) + (2):

$$\begin{aligned}
u(x+\Delta x) + u(x-\Delta x) &= 2u(x) + u''(x)\Delta x^2 + \mathcal{O}\left(\Delta x^4\right) \\
\Rightarrow u_{xx} &\approx \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2}
\end{aligned}$$

Now we can plug the approximation into the original equation. Let $f(x) = \cos(\pi x) - \pi(x+1)\sin(\pi x)$ to simplify the notation slightly.

$$\begin{aligned}
f(x) &= u_{xx} + 2u_x + \pi^2 u \\
&= \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2} + 2\frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} + \pi^2 x \\
\Delta x^2 f(x) &= u(x+\Delta x) - 2u(x) + u(x-\Delta x) + \Delta x u(x+\Delta x) - \Delta x u(x-\Delta x) + \Delta x^2 \pi^2 x
\end{aligned}$$

To simplify the notation further, discretize x by letting $x_i = i\Delta x$, $i = 0, 1, 2, ..., N+1$. Also discretize u(x) and f(x), by letting $u_i = u(x_i), f_i = f(x_i)$. Finally let $\Delta x = \frac{2}{N}$.

$$\begin{aligned}
&\frac{4}{N^2} f_i = u_{i+1} - 2u_i + u_{i-1} + \frac{2}{N}u_{i+1} - \frac{2}{N}u_{i-1} + \frac{4}{N^2}\pi^2 u_i \\
&\underline{(1 - \frac{2}{N})u_{i-1} + (\frac{4}{N^2}\pi^2 - 2)u_i + (1 + \frac{2}{N})u_{i+1} = \frac{4}{N^2} f_i}
\end{aligned}$$

This is the finite difference schema.

c) Approximate by letting N=4 and evaluate for points x_1, x_2, x_3.
It is also given that $u_0 = 0$, and $u_4 = 1$. These are the boundary conditions. Since we now know that N=4, we can include that in the schema. This gives us:

$$\frac{1}{2}u_{i-1} + (\frac{1}{4}\pi^2 - 2)u_i + \frac{3}{2}u_{i+1} = \frac{1}{4}f_i$$

We get a system of 3 equations, one for each i = 1, 2, 3.

$$\begin{aligned}
\frac{1}{2}u_0 + (\frac{1}{4}\pi^2 - 2)u_1 + \frac{3}{2}u_2 = \frac{1}{4}f_1 \\
\frac{1}{2}u_1 + (\frac{1}{4}\pi^2 - 2)u_2 + \frac{3}{2}u_3 = \frac{1}{4}f_2 \\
\frac{1}{2}u_2 + (\frac{1}{4}\pi^2 - 2)u_3 + \frac{3}{2}u_4 = \frac{1}{4}f_3
\end{aligned}$$

This can be written in matrix form. Also note that u_0 and u_4 are known, so subtract those. This gives us:

$$\begin{bmatrix} \frac{\pi^2}{4} - 2 & \frac{3}{2} & 0 \\ \frac{1}{2} & \frac{\pi^2}{4} - 2 & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{\pi^2}{4} - 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 - \frac{1}{2}u_0 \\ f_2 \\ f_3 - \frac{3}{2}u_4 \end{bmatrix}$$

Find the values on the right by using that $f(x) = \cos(\pi x) - \pi(x+1)\sin(\pi x)$. We are finding $x_1 = 0.5, x_2 = 1, x_3 = \frac{3}{2}$.

$$\begin{cases} f_1 - \frac{1}{2}u_0 = f(\frac{1}{2}) - \frac{1}{2} \times 0 = \cos\frac{\pi}{2} - \pi\frac{3}{2}\sin\frac{\pi}{2} = -\frac{3}{2}\pi \\ f_2 = f(1) = \cos\pi - \pi\sin\pi = -1 \\ f_3 - \frac{3}{2}u_4 = f(\frac{3}{2}) - \frac{3}{2} \times 1 = \cos\frac{3\pi}{2} - \pi\frac{5}{2}\sin\frac{3\pi}{2} = \frac{5}{2}\pi \end{cases}$$

Now we can solve the matrix.

$$\begin{bmatrix} \frac{\pi^2}{4} - 2 & \frac{3}{2} & 0 \\ \frac{1}{2} & \frac{\pi^2}{4} - 2 & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{\pi^2}{4} - 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}\pi \\ -1 \\ \frac{5}{2}\pi \end{bmatrix}$$

The solution to this is very wrong. Again, I will have to come back later and have a look.

Problem 3.

a) How many stages does the Ralston's method have? It has 2 stages. We can see this by the number of rows in the matrix part of the Butcher tableu.

b) Determine the polynomial R(h).
Let f(x) = y'(x). The general Ralston's method looks like this:

$$\begin{cases} k_1 = f(t_n, y_n) \\ k_2 = f(t_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_1) \\ y_{n+1} = y_n + h\left(\frac{1}{4}k_1 + \frac{3}{4}k2\right) \end{cases}$$

Express stages k_1, k_2 with y_n:

$$k_1 = -6y_n$$
$$k_2 = -6\left(y_n + \frac{2}{3}hk_1\right) = -6y_n - 4h(-6y_n) = -6y_n + 24hy_n = (24h - 6)y_n$$

Now into y_{n+1}:

$$\frac{1}{4}k_1 + \frac{3}{4}k_2 = \frac{1}{4}(-6y_n) + \frac{3}{4}(24h - 6)y_n = -\frac{3}{2}y_n + (18h - \frac{9}{2})y_n = (18h - 6)y_n$$
$$\Rightarrow y_{n+1} = y_n + h(\frac{1}{4}k_1 + \frac{3}{4}k_2) = y_n + h(18h - 6)y_n = \underline{(18h^2 - 6h + 1)y_n}$$

Here we find $R(h) = 18h^2 - 6h + 1$.

c) What step size is required to keep it stable?
We need the polynomial to be less than 1. This gives us the inequality:

$$18h^2 - 6h + 1 < 1$$
$$18h^2 - 6h < 0$$
$$h(3h - 1) < 0$$

From this we can see that h must be in the interval $\underline{\left(0, \frac{1}{3}\right)}$.