NTNU

Kunnskap for en bedre verden

TMA4135 - Matematikk 4D

Execise #4

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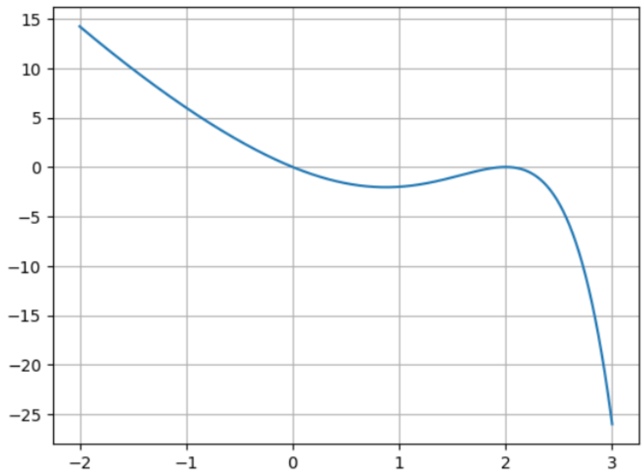
Problem 1.

a) Plot the function $f(x) = (1 - 3^x)x^2 + 4(x-1)3^x + 4(1-x)$ *on the interval [-2, 3]*



b) Do 4 iterations of the bisection method by hand.
We start with $a_0 = -2$, $b_0 = 3$. $f(a_0) \times f(b_0) < 0$, which suggests that there is at least one root between the points.

$$
\begin{aligned}
c_0 &= \frac{a_0 + b_0}{2} = \frac{-2+3}{2} = 0.5 \\
f(0.5) &< 0 \\
&\Rightarrow I_1 = [a_1, b_1] = [-2, 0.5]
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \frac{a_1 + b_1}{2} = \frac{-2+0.5}{2} = -0.75 \\
f(-0.75) &> 0 \\
&\Rightarrow I_2 = [a_2, b_2] = [-0.75, 0.5]
\end{aligned}
$$

$$
\begin{aligned}
c_2 &= \frac{a_2 + b_2}{2} = \frac{-0.75+0.5}{2} = -0.125 \\
f(-0.125) &> 0 \\
&\Rightarrow I_3 = [a_3, b_3] = [-0.125, 0.5]
\end{aligned}
$$

$$
\begin{aligned}
c_3 &= \frac{a_3 + b_3}{2} = \frac{-0.125+0.5}{2} = 0.375 \\
f(0.375) &< 0 \\
&\Rightarrow I_4 = [a_4, b_4] = [-0.125, 0.375]
\end{aligned}
$$

Now we can calculate the solution:

$$\widetilde{x} = f(c_4) = f((a_4, b_4)/2) = f((-0.125 + 0.375)/2) \approx \underline{-0.52}$$

The maximal error is:

$$E = |c_k - r| \leq \frac{1}{2}(b_4 - a_4) \leq \frac{1}{2^{k+1}}(b - a) = \frac{1}{2^5}(3 - (-2)) = \underline{0.16}$$

c) Iterations to guarantee small error
If we want to make sure the error is less than $tol = 10^{-3}$, we need at least k iterations.

$$k \geq \frac{\log \frac{b-a}{2tol}}{\log 2} = \frac{\log \frac{3-(-2)}{2 \times 10^{-3}}}{\log 2} \approx 13.28$$

This means that we need at most <u>14 iterations</u> to guarantee an error smaller than 10^{-3}.

Problem 2.

a) Does g converge?
Given the function $g(x) = \frac{\cos^2 e^{-x}}{4}$, we know that it will converge if g' is less than 1 in the relevant interval. We are only interested in $\hat{x} \geq 0$

$$\begin{aligned} g'(x) &= \frac{2}{4}\cos e^{-x} \times \left(\cos e^{-x}\right)' = \frac{1}{2}\cos e^{-x}(-\sin e^{-x})\left(e^{-x}\right)' \\ &= \frac{1}{2}\cos e^{-x}\sin(e^{-x})e^{-x} \end{aligned}$$

We know that e^{-x} is always greater than 1 for $x > 0$. We also know that sin(x) and cosin(x) cannot be greater than 1 for any value. This means that g' cannot be greater than 1 for any x on the interval. Thus, g converges.

Problem 3.

a) Two steps of Newton's method by hand for $f(x) = \cos x - \sqrt{x}$.
For Newton's method we need to find the derivative of f.

$$f'(x) = -\sin x - \frac{1}{2\sqrt{x}}$$

Compute two steps starting at $x_0 = 0$. This is not possible, because f'(0) is not defined. I will instead use $x_0 = 0.1$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.1 - \frac{\cos 0.1 - \sqrt{0.1}}{-\sin 0.1 - \frac{1}{2\sqrt{0.1}}} = 0.5038$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5038 - \frac{\cos 0.5038 - \sqrt{0.5038}}{-\sin 0.5038 - \frac{1}{2\sqrt{0.5038}}} = \underline{0.6436}$$

Problem 4.

a) What method is supposed to be implemented?
This looks like a version of Newton's method. The function is hard coded unlike the above implementation.
b) What is the function? The function must be $\cos x + \log x$.
c) Where is the error in the code? The change in x is not updated properly. Instead of x += - dx it should be x += dx.