NTNU

Kunnskap for en bedre verden

TMA4135 - MATEMATIKK 4D

Execise #1

Author:
Sondre Pedersen

September 27, 2023

Problem 1.

Compute the Laplace transforms.
a) $\mathcal{L}\{(t-2)^4\}$
Here we use that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$$\begin{aligned}
\mathcal{L}\{(t-2)^4\} &= \mathcal{L}\{t^4 - 8t^3 + 24t^2 - 32t + 16\} \\
&= \frac{4!}{s^5} - \frac{8 \times 3!}{s^4} + \frac{2! \times 24}{s^3} - \frac{32}{s^2} + \frac{16}{s} \\
&= \underline{\frac{24}{s^5} - \frac{48}{s^4} + \frac{48}{s^3} - \frac{32}{s^2} + \frac{16}{s}}
\end{aligned}$$

b) $\mathcal{L}\{te^{-t}\}$
If we say $\mathcal{L}\{t\} = \frac{1}{s^2} = F(s)$, we can use the translation property of Laplace transforms. That is, $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$. In our case, $a = -1$, so we get:

$$\mathcal{L}\{te^{-t}\} = F(s+1) = \underline{\frac{1}{(s+1)^2}}$$

c) $\mathcal{L}\{e^{-5t}\sin t\}$
The same strategy can be used here. $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = F(s)$. Translate with $a = -5$, and we get:

$$\mathcal{L}\{e^{-5t}\sin t\} = F(s+5) = \frac{1}{(s+5)^2+1} = \underline{\frac{1}{s^2+10s+26}}$$

d) $\mathcal{L}\{e^{-2t}\cos^2 3t - 3t^2e^{3t}\}$

$$\mathcal{L}\{e^{-2t}\cos^2 3t - 3t^2e^{3t}\} = \mathcal{L}\{e^{-2t}\cos^2 3t\} - 3\mathcal{L}\{t^2e^{3t}\}$$

Because of linearity. We can solve this separately. Since $\mathcal{L}\{t^2\} = \frac{2}{s^3} = F(s)$,

$$-3\mathcal{L}\{t^2e^{3t}\} = -3F(s-3) = -3\frac{2}{(s-3)^2} = -\frac{6}{(s-3)^2}$$

For the other part we need to use $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.

$$\begin{aligned}
\mathcal{L}\{e^{-2t}\cos^2 3t\} &= \mathcal{L}\{e^{-2t}\left(\frac{1}{2} + \frac{1}{2}\cos 6t\right)\} \\
&= \frac{1}{2}\mathcal{L}\{e^{-2t}\} + \frac{1}{2}\mathcal{L}\{e^{-2t}\cos 6t\} \\
&= \frac{1}{2}\frac{1}{s+2} + \frac{s+2}{2((s+2)^2+36)}
\end{aligned}$$

$\mathcal{L}\{e^{-2t}\cos 6t\}$ was solved in the same way as c). Adding the two results we get

$$\mathcal{L}\{e^{-2t}\cos^2 3t - 3t^2e^{3t}\} = \underline{\frac{1}{2s+4} + \frac{s+2}{2((s+2)^2+36)} - \frac{6}{(s-3)^2}}$$

Problem 2.

Find the inverse Laplace transforms.
a) $F(s) = \frac{2s}{s^2-3}$
Here we want to write the expression on the form $\frac{2s}{s^2-3} = \frac{A}{s-\sqrt{3}} + \frac{B}{s+\sqrt{3}}$. Now it is easy to use the known inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$.

$$\begin{aligned}
\frac{2s}{s^2-3} &= \frac{A}{s-\sqrt{3}} + \frac{B}{s+\sqrt{3}} \\
\Rightarrow 2s &= A(s+\sqrt{3}) + B(s-\sqrt{3}) \\
\text{let s } &= -\sqrt{3} : -2\sqrt{3} = B(-\sqrt{3}-\sqrt{3}) \Rightarrow B = 1 \\
\text{let s } &= \sqrt{3} : 2\sqrt{3} = A(\sqrt{3}+\sqrt{3}) \Rightarrow A = 1 \\
\Rightarrow \frac{2s}{s^2-3} &= \frac{1}{s-\sqrt{3}} + \frac{1}{s+\sqrt{3}} \\
\Rightarrow \mathcal{L}^{-1}\left\{\frac{2s}{s^2-3}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-\sqrt{3}}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+\sqrt{3}}\right\} \\
&= \underline{e^{\sqrt{3}t} + e^{-\sqrt{3}t}}
\end{aligned}$$

b) $F(s) = \frac{s^2+s+1}{s^3+s}$
Again it is best to simplify with partial fractions before solving.

$$\begin{aligned}
\frac{s^2+s+1}{s^3+s} &= \frac{A}{s} + \frac{Bs+x}{s^2+1} \\
\Rightarrow s^2+s+1 &= As^2 + A + Bs^2 + Cs \\
\Rightarrow 1 &= A \\
\Rightarrow 1 &= C \\
\Rightarrow 1 &= A + B = 1 + B \Rightarrow B = 0 \\
\Rightarrow \frac{s^2+s+1}{s^3+s} &= \frac{1}{s} + \frac{1}{s^2+1} \\
\Rightarrow \mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^3+s}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= \underline{H(s) + \sin t}
\end{aligned}$$

c) $F(s) = \frac{1}{(s-2)^2(s+1)}$
Partial fractions again?

$$\begin{aligned}
\frac{1}{(s-1)^2(s+1)} &= \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s+1} \\
&\Rightarrow 1 = A(s+1) + B(s-1)(s+1) + C(s-1)^2 \\
&\quad \text{let s = -1: } 1 = C(-2)^2 \Rightarrow C = \frac{1}{4} \\
&\quad \text{let s = 1: } 1 = A(2) \Rightarrow A = \frac{1}{2} \\
&\quad \text{let s = 0: } 1 = A - B + C \Rightarrow 1 = \frac{1}{2} - B + \frac{1}{4} \Rightarrow B = -\frac{1}{4} \\
\Rightarrow \frac{1}{(s-1)^2(s+1)} &= \frac{1}{2}\frac{1}{(s-1)^2} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{4}\frac{1}{s+1} \\
\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2(s+1)}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\
&= \underline{\frac{1}{2}e^t t^2 - \frac{1}{4}e^t + \frac{1}{4}e^{-t}}
\end{aligned}$$

Problem 3.

Evaluate the statements.
a) $\mathcal{L}\{f-g\} = \mathcal{L}\{f\} - \mathcal{L}\{g\}$? *True.*

$$\begin{aligned}
\mathcal{L}\{f-g\} &= \int_0^\infty e^{-st}(f-g)dt \\
&= \int_0^\infty e^{-st}f - e^{-st}gdt \\
&= \int_0^\infty e^{-st}fdt - \int_0^\infty e^{-st}gdt \\
&= \mathcal{L}\{f\} - \mathcal{L}\{g\}
\end{aligned}$$

b) $\mathcal{L}\{f \times g\} = \mathcal{L}\{f\} \times \mathcal{L}\{g\}$? *False.*

We know $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$. Let $f(t) = e^{a_1 t}$ and $g(t) = e^{a_2 t}$. $\mathcal{L}\{f(t) \times g(t)\} =$

$\mathcal{L}\{e^{a_1 t}e^{a_2 t}\} = \mathcal{L}\left\{e^{(a_1+a_2)t}\right\} = \frac{1}{s-(a_1+a_2)}$
$\mathcal{L}\{f\} \times \mathcal{L}\{g\} = \mathcal{L}\{e^{a_1 t}\} \times \mathcal{L}\{a_2 t\} = \frac{1}{s-a_1}\frac{1}{s-a_2}$
Since these two results are not equal, we can conclude that the statement is false.

c) $\mathcal{L}\{f\}(s) \geq 0$ for all s if $f(t) \geq 0$ for all $t \geq 0$? *False.*

Consider $f(t) = e^t$. f is positive for all t bigger than 0. The Laplace transform is $\mathcal{L}\{e^t\} = \int_0^\infty e^{-st}e^t = \int_0^\infty e^{-(s-1)t} = \frac{1}{s-1}$. We see that $\mathcal{L}\{f\}(s)$ can be negative for small s.

d) $\mathcal{L}\{f\}s$ exist when f is continuous and satisfies $0 \leq f(t) \leq 1$ for all $t \geq 0$? *True.*
We can see this from the definition of the Laplace transform. $\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st}f(t)dt$ converges, since f(t) does not grow faster than exponential.

Problem 4.

Solve the IVPs. (I will not show partial fraction solving here, since I have done this earlier)

a) $-y'' + 2y' - 3y = 0, y(0) = 1, y(0) = 2$

$$\begin{aligned}
-y'' + 2y' - 3y &= 0 \\
\mathcal{L}\left\{-y'' + 2y' - 3y\right\} &= \mathcal{L}\left\{0\right\} \\
-\mathcal{L}\left\{y''\right\} + 2\mathcal{L}\left\{y'\right\} - 3\mathcal{L}\left\{y\right\} &= 0 \\
-s^2Y(s) + sy(0) + y'(0) + 2sY(s) - 2y(0) - 3Y(s) &= 0 \\
-s^2Y(s) + s + 2 + 2sY(s) - 2 - 3Y(s) &= 0 \\
Y(s)(-s^2 + 2s - 3) + s &= 0
\end{aligned}$$

$$\begin{aligned}
Y(s) &= \frac{s}{s^2 - 2s + 3} \\
&= \frac{s-1+1}{(s-1)^2+2} = \frac{s-1}{(s-1)^2+2} + \frac{1}{(s-1)^2+2} \\
&= \frac{s-1}{(s-1)^2+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s-1)^2+2} \\
\mathcal{L}^{-1}\left\{Y(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s-1)^2+2}\right\} \\
y(t) &= \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2}\right\} + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s-1)^2+2}\right\} \\
y(t) &= \underline{e^t \cos\sqrt{2}t + \frac{1}{\sqrt{2}}e^t \sin\sqrt{2}t}
\end{aligned}$$

b) $y'' - 3y' + 2y = e^{3t}$, $y(0) = 1, y'(0) = 0$

$$\begin{aligned}
y'' - 3y' + 2y &= e^{3t} \\
\mathcal{L}\left\{y'' - 3y' + 2y\right\} &= \mathcal{L}\left\{e^{3t}\right\} \\
s^2Y(s) - sy(s) - y'(s) - 3sY(s) + 3y(0) + 2Y(s) &= \frac{1}{s-3} \\
Y(s)(s^2 - 3s + 2) - s + 3 &= \frac{1}{s-3} \\
Y(s)(s^2 - 3s + 2) &= \frac{1}{s-3} + s - 3 = \frac{s^2 - 6s + 10}{s-3} \\
&= \frac{s^2 - 6s + 10}{(s-3)(s-2)(s-1)}
\end{aligned}$$

... partial fractions here...

$$\begin{aligned}
Y(s) &= \frac{1}{2}\frac{1}{s-3} - 2\frac{1}{s-2} + \frac{5}{2}\frac{1}{s-1} \\
y(t) &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\
y(t) &= \underline{\frac{1}{2}e^{3t} - 2e^{2t} + \frac{5}{2}e^{t}}
\end{aligned}$$

c) $y'' - 10y' + 9y = 5t,\ y(0) = -1, y'(0) = 2$

$$\begin{aligned}
y'' - 10y' + 9y &= 5t \\
s^2Y(s) - sy(0) - y'(0) - 10sY(s) + 10y(0) + 9Y(s) &= \mathcal{L}\left\{5t\right\} \\
Y(s)(s^2 - 10s + 9) + s - 2 - 10 &= \frac{5}{s^2} \\
Y(s) &= \frac{-s^3 + 12s^2 + 5}{s^2(s-9)(s-1)}
\end{aligned}$$

...partial fractions...

$$\begin{aligned}
Y(s) &= \frac{50}{81}\frac{1}{s} + \frac{5}{9}\frac{1}{s^2} + \frac{31}{81}\frac{1}{s-9} - 2\frac{1}{s-1} \\
y(t) &= \frac{50}{81}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{31}{81}\mathcal{L}^{-1}\left\{\frac{1}{s-9}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\
y(t) &= \underline{\frac{50}{81}t + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^t}
\end{aligned}$$