NTNU
Kunnskap for en bedre verden

TMA4135 - Matematikk 4D

Execise #3

Author:
Sondre Pedersen

September 6, 2023

Problem 1.

a) Use the trapeziodal rule to approximate the integral $I = \int_{\pi/6}^{\pi/3} \sin x \cos 2x dx$
The trapezoidal rule says that $I_h = \frac{h}{2}(f(a) + f(b))$, where $h = b - a$.

$$\begin{aligned} I_h &= \frac{\pi/3 - \pi/6}{2} \left(\sin\frac{\pi}{6}\cos\frac{2\pi}{6} + \sin\frac{\pi}{3}\cos\frac{2\pi}{3} \right) \\ &= \underline{-0.0479} \end{aligned}$$

b) Compute the error $E = |I - I_h|$.

$$\begin{aligned} I &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x cos 2x dx \\ &\cos 2x = \cos^2 x - \sin^2 x = cos^2 x - (1 - cos^2 x) = 2\cos^2 x - 1 \\ \Rightarrow I &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x \left(2\cos^2 x - 1\right) dx, \qquad \text{let } u = \cos x, du = -\sin x dx \\ \Rightarrow I &= \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} 1 - 2u^2 du = u - 2\frac{u^3}{3}\Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \approx -0.0163 \end{aligned}$$

With the exact and approximate value, we can calculate the error.

$$E = |I - I_h| = |-0.0163 + 0.0479| = \underline{0.0316}$$

c) Check that error is smaller than upper bound
First, find second derivative of f.

$$\begin{aligned} f(x) &= \sin x \cos 2x \\ f'(x) &= \cos x \cos 2x - 2 \sin x \sin 2x \\ f''(x) &= -4 \cos x \sin 2x - 5 \sin x \cos 2x \end{aligned}$$

Using python, we can find that the biggest absolute value the second derivative will take on the interval is 4.25. This happens at x = a. Now we can find the upper bound for the error.

$$\begin{aligned} |I - I_h &\leq \frac{(b-a)^3}{12} max_{\xi \in [a,b]} |f''(x) \\ &= \frac{\left(\frac{\pi}{3} - \frac{\pi}{6}\right)^3}{12} \times 4.25 \\ &\approx \underline{0.0508} \end{aligned}$$

We can see that our calculated error is less than the biggest possible error.

Problem 2.

a) Transfer quadrature
The Gauss-Legendre quadrature is

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

The transfered quadrature is

$$\int_{a}^{b} f(x)dx \approx \frac{b-a}{2} \sum_{i=1}^{n} w_i f\left(\frac{b-a}{2} x_i + \frac{b+a}{2}\right)$$

Plugging in all the values gives us

$$\int_{-3}^{3} e^x \approx \text{...(not writing all that in latex)...} = \underline{29.325}$$

The answer was obtained using python. Steps are also provided in the jupyter notebook.
b) Error in composite Gauss-Legendre quadrature
The formula for error on the interval [a, b] is given as

$$E = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)(2n!)^3} f^{2n}(\xi)$$

When calculating error in a composite quadrature, we simply calculate the error for each interval, and add them up. The only part of E that depends on the interval is $(b-a)^{2n+1}$. We see that a smaller interval will give us a smaller error. The size of this factor decreases much faster than the sum of each new error term. Therefore I expect the error to decrease when splitting into more intervals.

c) Error in simple and composite Gauss-Legendre quadrature

The error of $f(x) = \frac{c^8}{8!}$ on the interval [-3, 3] for the quadrature will begin

$$E_1 = \frac{(3-(-3))^{2n+1}(n!)^4}{(2n+1)(2n!)^3} f^{2n}(\xi) = \frac{(6)^{2n+1}(n!)^4}{(2n+1)(2n!)^3} f^{2n}(\xi)$$

If we split this into two intervals (-3, 0) and (0, 3) and add the error for each one we get

$$\begin{aligned} E_2 &= \frac{(3)^{2n+1}(n!)^4}{(2n+1)(2n!)^3} f^{2n}(\xi) + \frac{(3)^{2n+1}(n!)^4}{(2n+1)(2n!)^3} f^{2n}(\xi) \\ &= \frac{2(3)^{2n+1}(n!)^4}{(2n+1)(2n!)^3} f^{2n}(\xi) \end{aligned}$$

Since all terms in E_1 and E_2 are equal except the parts from $(b-a)^{2n+1}$, these cancel when finding the ratio. We are left with

$$\begin{aligned} \frac{E_2}{E_1} &= \frac{2 \times 3^{2n+1}}{6^{2n+1}} = \frac{2 \times 3^{2n+1}}{(2 \times 3)^{2n+1}} = \frac{2 \times 3^{2n+1}}{2^{2n+1} \times 3^{2n+1}} \\ &= \frac{2}{2^{2n+1}} = \underline{\frac{1}{2^{2n}}} \end{aligned}$$

The ratio is less than 1 for all n bigger than 0. This means that E_2 is the smallest value, and that E_1 has the biggest upper bound for the error.

Problem 3.

a) Derive composite formula for Simpson's rule
We have an interval [a, b]. Now split the interval into m subintervals, each of length $h = \frac{b-a}{m}$ and points $x_i = ih, i = 0, ..., m$. To improve the approximation over this interval, we instead apply the Simpson's rule over pairs of subintervals, and sum them together. Simpson's rule over a single interval $[x_0, x_m]$ is

$$\int_{x_0}^{x_m} f(x)dx \approx \frac{x_m - x_0}{6}\left(f(x_0) + 4f\left(\frac{x_0 + x_m}{2}\right) + f(x_m)\right)$$

Splitting this over pairs of intervals before applying Simpson's rule gives us

$$\begin{aligned}
\int_{x_0}^{x_m} f(x)dx &= \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{m-2}}^{x_m} f(x)dx \\
&= \frac{x_2 - x_0}{6}\left(f(x_0) + 4f\left(\frac{x_2 - x_0}{2}\right) + f(x_2)\right) \\
&+ \frac{x_4 - x_2}{6}\left(f(x_2) + 4f\left(\frac{x_4 - x_2}{2}\right) + f(x_4)\right) \\
&+ \quad \vdots \\
&+ \frac{x_m - x_{m-2}}{6}\left(f(x_{m-2}) + 4f\left(\frac{x_m - x_{m-2}}{2}\right) + f(x_m)\right)
\end{aligned}$$

This can be simplified a bit, by noticing that $\frac{x_i - x_{i-2}}{2} = x_{i-1}$. Also, the factors in front can be written $\frac{x_i - x_{i-2}}{6} = \frac{2h}{6} = \frac{h}{3}$. Now we have

$$\begin{aligned}
&= \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \frac{h}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right) \\
&+ \cdots + \frac{h}{3}\left(f(x_{m-2}) + 4f(x_{m-1}) + f(x_m)\right) \\
&= \frac{h}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{m-2} + 4f(x_{m-1} + f(x_m))
\end{aligned}$$

Which is what we wanted to find.

c) Find m for low enough error

Given $f(x) = e^{-x}$, $a = 0, b = 1$ and $E = \frac{h^4}{180}(b-a)\max_{\xi\in|a,b|}\left|f^{(4)}(\xi)\right|$. Notice that E can be simplified. $b - a = 1$ and $f^{(4)}(x) = e^{-x}$. With some knowledge of how e^{-x} behave, we can conclude that its value is decreasing. That means it will never get bigger than at x = 0. Since $f(0) = f^{(4)}(0) = 1$, we can simplify $\max_{\xi\in|a,b|}\left|f^{(4)}(\xi)\right| = 1$. This gives us

$$E = \frac{h^4}{180}$$

Now solve for m (h = 1/m), when $E \leq 10^{-3}$.

$$\begin{aligned}
E =& \frac{h^4}{180} \leq 10^{-3} \\
\Rightarrow& h^4 \leq 0.18 \\
\Rightarrow& h \leq 0.6514 \quad \text{(we only care about the positive solution)} \Rightarrow \quad h > 1.54 \\
\Rightarrow& \underline{h \geq 2}
\end{aligned}$$

This shows that 2 or more subintervals are enough to make the error less than 10^{-3}. It holds true for the calculations done in the notebook.

Problem 4.

The numerical method that is implemented is the composite trapezoid rule. The error is that s also have to be multiplied with h / 2. Here is a corrected version:

```
import numpy as numpy

def f(x):
    return x**2

def Method(f, a, b, m):
    h = (b - a) / m
    xs = np.linspace(a, b, m+1)
    ys = [f(x) for x in xs]
    s = (h / 2) * (ys[0] + ys[-1] + 2*sum(ys[1:-1]))
    return s
```