NTNU
Kunnskap for en bedre verden

TMA4135 - Matematikk 4D

Execise #6

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Problem 2.

Compute the Laplace transforms.
a) $y'' + 4y = \delta(t-\pi)$, with $y(0) = 8$ and $y'(0) = 0$.

$$\begin{aligned}
\mathcal{L}\{y'' + 4y\} &= \mathcal{L}\{\delta(t-\pi)\} \\
s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= e^{-\pi s} \\
s^2Y(s) - 8s + 4Y(s) &= e^{-\pi s} \\
Y(s)(s^2+4) &= e^{-\pi s} + 8s \\
Y(s) &= \frac{e^{-\pi s}}{s^2+4} - 8\frac{s}{s^2+4}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} &= \frac{1}{2}u(t-\pi)\sin(2(t-\pi)) \\
\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} &= \cos 2t \\
\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} - 8\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} \\
&= \underline{\frac{1}{2}u(t-\pi)\sin(2(t-\pi)) - 8\cos 2t}
\end{aligned}$$

b) $y'' + 3y' + 2y = 10(\sin(t)\delta(t-1))$, with $y(0) = 1$ and $y'(0) = -1$

$$\begin{aligned}
\mathcal{L}\{y'' + 3y' + 2y\} &= 10\mathcal{L}\{\sin t\} + 10\mathcal{L}\{\delta(t-1)\} \\
s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) &= 10\frac{1}{s^2+1} + 10e^{-t} \\
s^2Y(s) - s + 1 + 3sY(s) - 3 + 2Y(s) &= 10\frac{1}{s^2+1} + 10e^{-t} \\
Y(s)(s^2+3s+2) &= \frac{10}{s^2+1} + 10e^{-s} + s \\
Y(s) = \frac{10}{(s^2+1)(s+2)(s+1)} &+ \frac{10e^{-s}}{(s+2)(s+1)} + \frac{s}{(s+)(s+2)}
\end{aligned}$$

...partial fractions of each one...

$$\begin{aligned}
Y(s) &= -3\frac{s}{s^2+1} + \frac{1}{s^2+1} - 2\frac{1}{s+2} + 5\frac{1}{s+1} + 10e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) + 2\frac{1}{s+2} - \frac{1}{s+1} \\
Y(s) &= -3\frac{s}{s^2+1} + \frac{1}{s^2+1} + 4\frac{1}{s+1} + 10e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) \\
y(t) &= \underline{-3\cos t + \sin t + 4e^{-t} + 10u(t-1)\left(e^{-(t-1)} - e^{-2(t-2)}\right)}
\end{aligned}$$

c) $y'' + 2y' + 5y = 25t - 100\delta(t-\pi)$, with $y(0) = -2$ and $y'(0) = 5$

$$\begin{aligned}
\mathcal{L}\{y'' + 2y' + 5y\} &= \mathcal{L}\{25t - 100\delta(t-\pi)\} \\
s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 5Y(s) &= 25\frac{1}{s^2} - 100e^{-\pi s} \\
s^2Y(s) + 2s - 5 + 2sY(s) + 4 + 5Y(s) &= 25\frac{1}{s^2} - 100e^{-\pi s} \\
Y(s)(s^2 + 2s + 5) &= 25\frac{1}{s^2} - 100e^{-\pi s} - 2s + 5
\end{aligned}$$

$$Y(s) = \frac{25}{(s^2)(s^2+2s+5)} - 100e^{-\pi s}\frac{1}{(s^2+2s+5)} - 2s\frac{1}{(s^2+2s+5)} + 5\frac{1}{(s^2+2s+5)}$$

...lots of calculation (too much for latex)...

$$y(t) = \underline{-2u(t) + 5t - 7500e^{-t+\pi}u(t-\pi)^2 \sin^2 2(t-\pi)\cos 2t \sin 2t}$$

Problem 3.

a) *Convergence of error for three-point finite-difference formula.* First find the Taylor expansions:

$$
\begin{aligned}
u(x+h) &= u(x) + hu'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + \mathcal{O}\left(h^4\right) \\
u(x+2h) &= u(x) + 2hu'(x) + \frac{4h^2}{2!}u''(x) + \frac{8h^3}{3!}u'''(x) + \mathcal{O}\left(h^4\right)
\end{aligned}
$$

Insert the expansions into the given formula.

$$
\begin{aligned}
&\frac{-3u(x) + 4u(x+h) - u(x+2h)}{2h} \\
=&\frac{-3u(x) + 4\left(u(x) + hu'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + \mathcal{O}\left(h^4\right)\right)}{2h} \\
&- \frac{u(x) + 2hu'(x) + \frac{4h^2}{2!}u''(x) + \frac{8h^3}{3!}u'''(x) + \mathcal{O}\left(h^4\right)}{2h} \\
=&\frac{u(x)(-3+4-1) + u'(x)(4h-2h) + u''(x)(\frac{4h^2-4h^2}{2}) + u'''(x)(\frac{4h^3-8h^3}{6}) + \mathcal{O}\left(h^4\right)}{2h} \\
=&\frac{2hu'(x) - \frac{2h^3}{3}u'''(x) + \mathcal{O}\left(h^4\right)}{2h} \\
=&u'(x) - \frac{1}{3}h^2u'''(x) + \mathcal{O}\left(h^3\right)
\end{aligned}
$$

From this we can see that we are left with an error on the order $\mathcal{O}\left(h^2\right)$, that is, $\underline{\text{p} = 2}$.

c) *Determine the order of error when computer roundoff error is similar to error from Taylor polynomial.*
To solve this, we add in the computing error, ϵ, to where we calculate. I am not sure if we should add 2 or 3 errors. I will add 2, one for each of $u(x+h), u(x+2h)$. In the end it should not matter.

$$
\begin{aligned}
u'(x) &= \frac{-3u(x) + 4u(x+h) - u(x+2h) + 2\epsilon}{2h} + \mathcal{O}\left(h^2\right) \\
&= \frac{-3u(x) + 4(x+h) - u(x+2h)}{2h} + \frac{\epsilon}{h} + \mathcal{O}\left(h^2\right) \\
\Rightarrow &|Error| \leq \frac{\epsilon}{h} + \frac{1}{3}h^2 \times m
\end{aligned}
$$

Here, $\frac{\epsilon}{h}$ is the machine error, and $\frac{1}{3}h^2 \times m$ is the Taylor expansion error we got from a). We let m be a constant, $m = \max_{x+h \to x+2h} u'''(x)$. The goal is to find h such that the absolute value of the error is as small as possible. This can be done by finding h such that the partial derivative is 0.

$$
\begin{aligned}
\frac{\partial}{\partial h}\left(\frac{\epsilon}{h}+\frac{1}{3}h^2 m\right) &= -\frac{\epsilon}{h^2}+\frac{2}{3}hm = 0 \\
\Rightarrow h^3 &= \frac{3\epsilon}{2m} \\
\Rightarrow h &= \sqrt[3]{\frac{\epsilon}{2m}}
\end{aligned}
$$

Now we can write this is on the order $h = \mathcal{O}\left(\epsilon^{\frac{1}{3}}\right)$, giving us $\underline{k = \frac{1}{3}}$.