NTNU
Kunnskap for en bedre verden

TMA4135 - Matematikk 4D

Execise #9

Author:
Sondre Pedersen

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Problem 1.

a) Smallest fundamental period.
Let f be periodic, such that $f(x+p) = f(x)$ for all x when $p > 0$. Claim: *Every periodic function has a fundamental period.*. In other words, there exist some smallest $p > 0$ satisfying f(x + p) = f(x) for all x.
This claim is false, shown with a counter example. Consider the function $f(x) = 0$. This is a periodic function, because there exist at least one $p > 0$ such that $f(x+p) = f(x)$. In fact every real number bigger than 0 is satisfactory: $f(x+p) = f(x) = 0$.
For any p satisfying the equation, $p' = \frac{p}{2}$, will also satisfy the equation. There is no smallest real number bigger than 0.

b) Find fundamental periods
$\sin ax + b$ and $\cos ax + b$ are 2π periodic. That means the fundamental period is given by $T = \frac{2\pi}{a}$. One way to find this formula is to realise that the fundamental period must be equal to the distance between peaks of the functions. With sine for example, $T = x_1 - x_0$, where $x_1 = \frac{5\pi}{2}$ and $x_0 = \frac{\pi}{2}$.
To find this difference expressed with a and b:

$$\begin{aligned}
&\frac{d}{dx}\sin ax + b = a\cos ax + b = 0 \\
\Rightarrow &ax_1 + b = 2\pi,\ ax_0 + b = 0 \\
\Rightarrow &x_1 = \frac{2\pi - b}{a},\ x_0 = -\frac{b}{a} \\
\Rightarrow &T = x_1 - x_0 = \frac{2\pi}{a}
\end{aligned}$$

Similarly for cosine. From this we can find the fundamental periods of each function.

- $f(x) = \sin(3x + 2)$

  From formula above: $T = \frac{2\pi}{3}$
- f(x) = $\pi \cos 2\pi x$

  $T = \frac{2\pi}{2\pi} = 1$
- $f(x) = \cos\frac{2\pi}{m+1}x + \sin\frac{2\pi}{n-1}x$

  $T_1 = \frac{2\pi}{\frac{2\pi}{m+1}} = m + 1$, $T_2 = \frac{2\pi}{\frac{2\pi}{n-1}} = n - 1$

  The period for f is the least common multiple of m+1 and n-1.

Problem 2.

Fourier series of functions. We have that L = π.

a)

Let f(x) = $\begin{cases} 0 & -\pi < x < 0, \frac{\pi}{2} < x < \pi \\ x & 0 \le x \le \frac{\pi}{2} \end{cases}$. To find the fourier series, we first need to find a_0, a_k and b_k:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2\pi}\int_0^{\pi/2} xdx = \frac{2}{2\pi}\frac{1}{2}x^2\Big|_0^{\pi/2} = \frac{\pi}{16}$$

$$\begin{aligned} a_k &= \frac{1}{L}\int_{-L}^{L} f(x)cos(kx)dx = \frac{1}{\pi}\int_0^{\pi/2} xcos(kx)dx \\ &= \frac{1}{k\pi}sin(kx)\Big|_0^{\pi/2} - \frac{1}{k\pi}\int_0^{\pi/2} sin(kx)dx = \frac{sin\left(\frac{k\pi}{2}\right)}{2k} + \frac{1}{k^2\pi}cos(kx)\Big|_0^{\pi/2} \\ &= \frac{1}{2k}sin\left(\frac{k\pi}{2}\right) + \frac{1}{k^2\pi}\left(\cos\left(\frac{k\pi}{2}\right) - 1\right) \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{L}\int_{-L}^{L} f(x)sin(kx)dx = \frac{1}{\pi}\int_0^{\pi/2} xsin(kx)dx \\ &= -\frac{1}{k\pi}cos(kx)\Big|_0^{\pi/2} + \frac{1}{k\pi}\int_0^{\pi/2} cos(kx)dx = -\frac{cos\left(\frac{k\pi}{2}\right)}{2k} + \frac{1}{k^2\pi}sin(kx)\Big|_0^{\pi/2} \\ &= \frac{1}{k^2\pi}\sin\left(\frac{k\pi}{2}\right) - \frac{1}{2k}\cos\left(\frac{k\pi}{2}\right) \end{aligned}$$

We can see that in both a_k and b_k the sine and cosine functions are evaluated at every $\frac{\pi}{2}$. This is really annoying, because there is no clean way to simplify the expressions (as I am aware of at least). The only solution I could come up with was to look at what happens for k=1, 5, ..., k=2, 6, ... and so on.

k	a_k	b_k
k=1,5,...	$\frac{\pi}{2k} - \frac{1}{k^2}$	$\frac{1}{k^2}$
k=2,6,...	$-\frac{2}{k^2}$	$\frac{\pi}{2k}$
k=3,7,...	$-\frac{\pi}{2k} - \frac{1}{k^2}$	$-\frac{1}{k^2}$
k=4,8,...	0	$-\frac{\pi}{2k}$

So the final Fourier series is:

$$f(x) = \frac{\pi}{16} + \sum_{k=1,5,...}^{\infty} (\frac{\pi}{2k} - \frac{1}{k^2})cos(kx) + (\frac{1}{k^2})sin(kx)$$
$$+ \sum_{k=2,6,...}^{\infty} (-\frac{2}{k^2})cos(kx) + (\frac{\pi}{2k})sin(kx)$$
$$+ \sum_{k=3,7,...}^{\infty} (-\frac{\pi}{2k} - \frac{1}{k^2})cos(kx) + (-\frac{1}{k^2})sin(kx)$$
$$+ \sum_{k=4,8,...}^{\infty} (-\frac{\pi}{2k})sin(kx)$$

b)

Let f(x) = $\begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x \leq \pi \end{cases}$. Find a_0, a_k and b_k:

$a_0 = \frac{\pi}{8}$, because it must be twice as large as in a). Last time f was half a triangle, and now it is double in size.

$$a_k = \frac{1}{\pi}\int_0^{\pi} f(x)cos(kx)dx = \frac{1}{\pi}\int_0^{\pi/2} xcos(kx)dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} (\pi - x)cos(kx)dx$$
$$= \frac{sin\left(\frac{k\pi}{2}\right)}{2k} + \frac{cos\left(\frac{k\pi}{2}\right)}{k^{\pi}} - \frac{1}{k^2\pi} - \frac{cos(k\pi)}{k^2\pi} - \frac{sin\left(\frac{k\pi}{2}\right)}{2k} - \frac{cos\left(\frac{k\pi}{2}\right)}{k^{\pi}}$$
$$= -\frac{1}{k^2\pi}(cos(k\pi) + 1) \qquad \text{cos}(k\pi) \text{ is -1 for all k.}$$
$$= 0$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} f(x)sin(kx)dx = \frac{1}{\pi}\int_0^{\pi/2} xsin(kx)dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} (\pi - x)sin(kx)dx$$
$$= \frac{sin(k\pi/2)}{k^2\pi} - \frac{cos(k\pi/2)}{2k} + \frac{sin(k\pi/2)}{k^2\pi} + \frac{cos(k\pi/2)}{2k} - \frac{sin(k\pi)}{k^2\pi} \qquad \text{sin(k}\pi\text{) is 0 for all k}$$
$$= \frac{2sin(k\pi/2)}{k^2\pi} \qquad sin(k\pi/2) \text{ evaluates to 1 for k=1,5,... and -1 for k=3,7,...}$$

$$f(x) = \frac{\pi}{8} + \frac{2}{\pi}\sum_{k=1,5,...}^{\infty} \frac{1}{k^2}sin(kx) - \frac{2}{\pi}\sum_{k=3,7,...}^{\infty} \frac{1}{k^2}sin(kx)$$

See sketches at the bottom of the page.

Problem 3.

Find Fourier coefficients of the following functions.
a) $f(x) = 1/2 + cos(2x) - 4sin(4x)$
The fundamental period is π because period of cos(2x) is $\frac{2\pi}{2} = \pi$ and sin(4x) is $\frac{2\pi}{4} = \frac{\pi}{2}$. $LCM(\pi, \frac{\pi}{2}) = \pi$.

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} \frac{1}{2} + cos(2x) - 4sin(4x)dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2} + cos(2x) - 4sin(4x)dx$$

Here sine and cosine integrate to 0, because they are integrated over full periods.

$$\Rightarrow a_0 = \frac{2}{2\pi}\int_{0}^{\pi/2} dx = \frac{1}{\pi}x\Big|_0^{\pi/2} = \frac{1}{\pi}\frac{\pi}{2} = \underline{\frac{1}{2}}$$

$$a_k = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} + cos(2x) - 4sin(4x)\right) cos(kx)dx$$

$\frac{cos(kx)}{2}$ will integrate to 0 for the same reason as above. cos(2x)cos(kx) will integrate to 0 unless k = 2. This is because they form orthogonal basis in the vector space. But not when dotted with itself. sin(4x)cos(kx) will integrate to 0 because they are orthogonal for all k. This gives us

$$a_k = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} cos(2x)cos(2x)dx = \frac{2}{\pi}\frac{\pi}{2} = \underline{1}$$

b_k will also be 1 for the same reason.

b) $f(x) = |cos(3\pi x)|$
Period is 2/3 for normal $cos(3\pi x)$, but for absolute value, the negative values gets mirrored over the x-axis. This halves the period, giving us 1/3.
If we only look at one period of the function, we can work with it as if the absolute value was not there. This makes life easier. Period from x = -1/6 to x = 1/6.

$$a_0 = 3\int_{-1/6}^{1/6} cos(3\pi)dx = 6\int_0^{1/6} cos(3\pi x)dx = \frac{2}{\pi} sin(3\pi x)\Big|_0^{1/6} = \frac{2}{\pi}$$

$$a_k = \frac{1}{6}\int_{-1/6}^{1/6} cos(3x)cos(kx)dx = \frac{1}{3}\int_0^{1/6} cos(3x)cos(3x)dx \qquad \text{similar to a)}$$
$$= \underline{\frac{\pi}{6}}$$

b_k is 0, because all sin(kx) is orthogonal to cos(3x).

Problem 4.

a) Show that if f(x) is odd, then h(x) := $f^2(x) is even$*.*
$h(-x) = f^2(-x)$. Square this equation and use f(-x) = -f(x)
$[f(-x)^2] = (f(x))^2 = f^2(x)$.
Shows that $h(-x) = f^2(x)$.
Since h(-x) = h(x) = $f^2(x)$, h must be even.

b) Show that if f(x) is odd, g(x) is even, then h(x):= f(x)g(x) is odd.
h(-x) = f(-x)g(-x)=(-f(x))(g(x)) = -[f(x)g(x)] = -h(x).
h satifies the definition of an odd function.

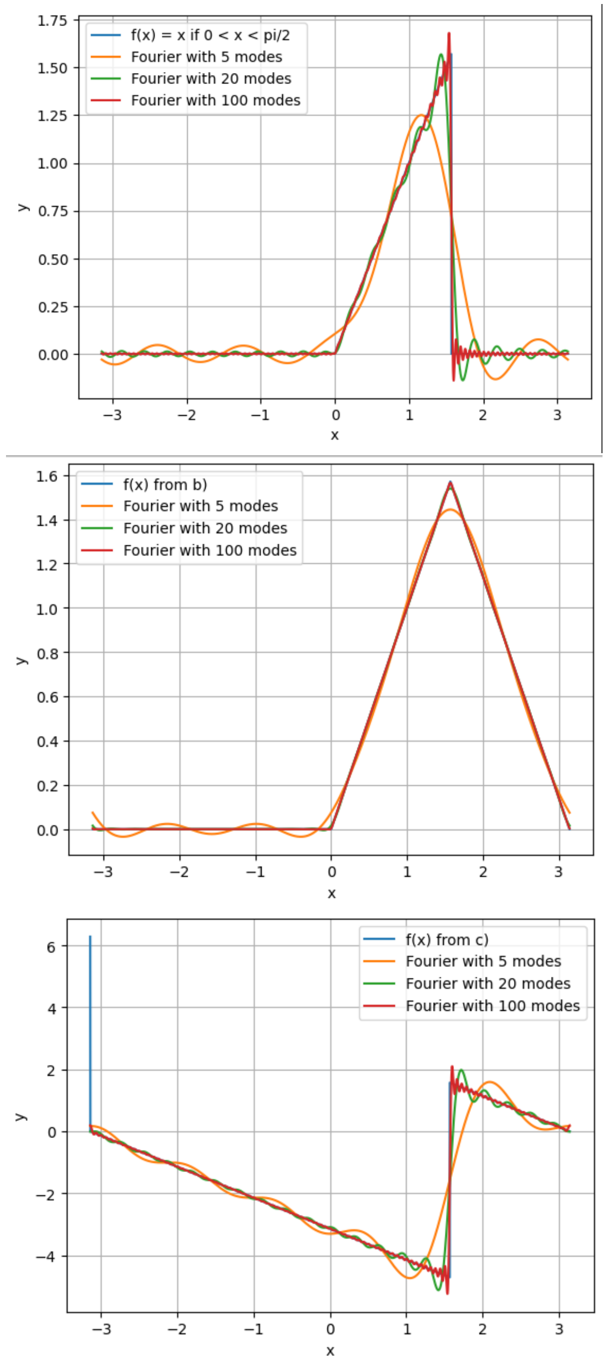

Figure 1: Sketching of the fourier series in exercise 2.