NTNU
Kunnskap for en bedre verden

TMA4135 - Matematikk 4D

Execise #12

Author:
Sondre Pedersen

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Problem 1.

a)
The Riccati equation is non-linear in its original form because of the non-linear operation v^2. Non-linear equations do not admit to superposition.

Problem 2.

a)
Burger's equation is nonlinear and inhomogeneous because of the non-linear operation $u\frac{\partial u}{\partial x}$.
b)
The Laplace equation in polar coordinates is linear and homogeneous because there are no nonlinear operations and no forcing.
c)
The Poisson equation is linear and inhomogeneous because of the $+2$ term.
d)
The convection-reaction is nonlinear because of the u^n term (unless n is 1). Also it is homogeneous.
e)
The one-dimentional heat equation with time-dependent heat source is linear, and inhomogeneous because of the time dependent heat source.
f) Bi-harmonic equation is linear and homogeneous.

Problem 3.

a) Given: $\frac{\partial}{\partial t}u - 2\frac{\partial^2}{\partial x^2}u = 0$, BC $u(0,t) = u(\pi,t) = 0$, and IC $u(x,0) = x(x-\pi)$. Now assume u(x, t) = F(x)G(t). From this we know that

$$u(0,t) = F(0)G(t) = 0 \Rightarrow F(0) = 0$$
$$u(\pi,t) = F(\pi)G(t) = 0 \Rightarrow F(\pi) = 0 \quad \text{, (G(t) = 0 is trivial in both cases.)}$$

$$\begin{aligned}
\frac{\partial u}{\partial t} &= u_t = FG_t \\
\frac{\partial^2 u}{\partial x^2} &= u_{xx} = F_{xx}G \\
\Rightarrow & FG_t = 2F_{xx}G \\
\Rightarrow & \frac{F_{xx}}{F} = \frac{G_t}{2G} = c
\end{aligned}$$

The last line is true, because if a function of space is equal to a funtion of time for all of space and time, they must be equal to a constant.
Now we can solve the ODE $F_{xx} = cF$. There are three cases to consider: c = 0, $c > 0$ and $c < 0$.

CASE I c = 0:

$$\begin{aligned}
F_{xx} &= 0 \\
F &= c_1x + c_2 \\
F(0) &= c_2 = 0 \\
F(\pi) &= c_1\pi = 0 \Rightarrow c_1 = 0 \\
F &\equiv 0 \quad \text{Trivial solution.}
\end{aligned}$$

CASE II $c > 0$:

Let $c = \lambda^2$

$$\begin{aligned}
F_{xx} &= cF \\
F &= c_1e^{\lambda x} + c_2e^{-\lambda x} \\
F(0) &= c_1 = c_2 = 0 \Rightarrow c_2 = -c_1 \\
\Rightarrow F(\pi) &= c_1e^{\lambda\pi} - c_1e^{-\lambda\pi} \Rightarrow c_1(e^{\lambda\pi} - e^{-\lambda\pi}) \\
\Rightarrow & c_1 = 0 \\
F &\equiv 0 \quad \text{Trivial solution.}
\end{aligned}$$

CASE III $c < 0$:

Let $c = -\lambda^2$

$$\begin{aligned}
F_{xx} &= cF \\
F &= c_1 \cos \lambda x + c_2 \sin \lambda x \\
F(0) &= c_1 \cos 0 + c_2 \sin 0 = 0 \\
F(0) &= c_1 = 0 \\
F(\pi) &= c_2 \sin \lambda\pi = 0 \Rightarrow \lambda = n \quad \text{for n = 1, 2, 3, ...} \\
&\Rightarrow \underline{F(x) = \alpha_n \sin nx \quad \text{for n = 1, 2, 3, ...}}
\end{aligned}$$

Now solve G(t).

$$\begin{aligned}
\frac{G_t}{G} &= 2c = -2\lambda^2 = -2n^2 \\
\ln G &= -2n^2 t + \beta \\
&\Rightarrow \underline{G(t) = \beta e^{-2n^2 t}}
\end{aligned}$$

Now finally we can express all solutions satisfying the heat equation with the boundary conditions:

$$u_n(x,t) = F(x)G(t) = B_n e^{-2n^2 t} \sin nx, \quad \text{n = 1, 2, ...}$$

B_n combines the constant terms. Since the heat equation is linear, we can write the superposition to capture all solutions.

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-2n^2 t} \sin nx$$

b)
Let $f(x) = u(x, 0) = x(x - \pi)$. Using this, we can find B_n uniquely, by finding the inner product with sin(kx). All these sine functions will be the basis for u. Note that cos is not needed, because the inner product will be 0 with u anyway, since it contains sin(nx).

$$< u(x,0), sin(kx) > = < f(x), sin(nx) > \tag{1}$$

$$< f(x), sin(kx) > = \int_0^\pi f(x) sin(kx) dx \tag{2}$$

$$< u(x,0), sin(nx) > = \int_0^\pi \sum_{n=0}^{\pi} B_n sin(nx) sin(kx) \tag{3}$$

In equation 3, the inner product will be 0 for all n except when k = n. This allows the expression to be simplified:

$$\begin{aligned}\int_0^\pi \sum_{n=0}^{\pi} B_n sin(nx) sin(kx) &= B_n \int_0^\pi sin(nx) sin(nx) dx \\ &= B_n \frac{\pi}{2}\end{aligned}$$

Using equation 1, we can find B_n:

$$\begin{aligned}B_n &= \frac{2}{\pi} \int_0^\pi f(x) sin(kx) dx \\ &= \underline{\frac{2}{\pi} \int_0^\pi x(x - \pi) sin(kx) dx}\end{aligned}$$