Exercises

Ex. 1. Let X be a weakly stationary process. Show that $\gamma_X(h) = \gamma_X(-h)$.

Ex. 2. Let X_t be a real valued weakly stationary process, with autocovariance function γ_X. Let

$$Y_t = X_t - X_{t-1}.$$

Show that Y is weakly stationary and compute its covariance function.

Ex. 3. In this exercise we explore a little more the notion of invertibility.

As explained during the lectures, if $X = F_\alpha Z$ is invertible, there exists β such that $F_\beta X = Z$, and both of these process are causal (that is they only depends on the past). In other words: $F_\beta F_\alpha X = X$, which means $F_\beta = F_\alpha^{-1}$.

If we write these results with polynomials of B:

$$P_\alpha^{-1}(B)P_\alpha(B)X = X.$$

1. If $P_\alpha(B) = 1 - \frac{1}{2}B$, show that $P_\alpha^{-1}(B) = \sum_{n=1}^{\infty} \frac{1}{2^n} B^n$.
2. Show that the AR(1) process $X_t = X_{t-1}/2 + w_t$, where w is a white noise with mean 0 and variance σ^2, it can be written as:

$$X_t = \sum_{i=0}^{\infty} \psi_i w_{t-i},$$

where ψ_i is to be determined.

Ex. 4. Consider the MA(2) process defined as:

$$X_t = w_t + 0.7w_{t-1} + 0.2w_{t-2}$$

where w is a white noise with mean 0 and variance σ^2.

1. Show that X is invertible.
2. Find its atocovariance and autocorrelation functions.

Ex. 5 Consider the AR process defined by:

$$X_t = \alpha X_{t-1} - \alpha X_{t-2} + Z_t, \qquad Z_t \sim (0, \sigma^2).$$

Determine for which values of α the process is causal, and for which it is invertible.

Ex. 6 Let (w_t) be an iid family of normally distributed RVs with mean 0 and variance σ^2. Let $a, b, c \in \mathbb{R}$ be constants. Determine if each of the following processes is stationary. If so, compute its mean and autocovariance function.

1. $X_t = a - bw_5$;
2. $X_t = w_0 \sin(ct)$;
3. $X_t = w_t w_{t-1}$;
4. $X_t = w_0 \cos(ct) - w_1 \sin(ct)$.

Ex. 7 Let (x_t) be the time serie of the average yearly temperature in New Haven (USA). We assume these observations comfe from a stationary time series (it was before the largest effects of global warming). The observations are given below. They are also available on `R`, in the `datasets` package (10 first obserbations of `nhtemp`).

Year	1912	1913	1914	1915	1916	1917	1918	1919	1920	1921
Temp (°F)	49,9	52,3	49,4	51,1	49,4	47,9	49,8	50,9	49,3	51,9

Explain mathematically what you do and do the computations and plots with `R`.

1. Estimate $E[X_t]$ and explain why you chose that estimator. Compute the autocorrelation function.
2. Draw a correlogram at the first 3 lags.
3. Under the null hypothesis that (X_t) is a white noise, write your approximate distribution for the estimated autocorrelation coefficients r_1, r_2, r_3.
4. Perform tests to asses whether the process is a white noise using the Binomial distribution and the Ljung-Box test independently.
5. compare those results when using the full set of observations.

Ex. 8 For each of the Figures 1, 2, 3 and 4, observe the correlogram and the PACF. State what model you would choose and why. Give the order of the models and list any approximate parameter that might be estimated from the graphs.

1. AR(1), as the PACF goes to 0 after that. The first autocorrelation parameter can be approximated by $0,8$.
2. White noise: all the autocorrelation coefficient (except at lag 0) are approximately null. The covariance of the model is approximately 1.
3. MA(2) as the correlogram goes to 0 after lag 2.
4. AR(2) as PACF is close to 0 after lag 2. An estimate of the second autocorrelation parameter is $0,5$.

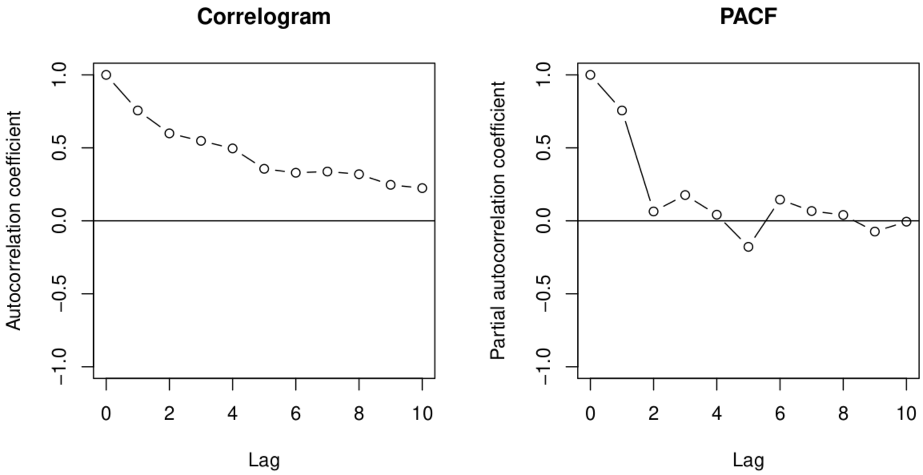

Figure 1: (a)

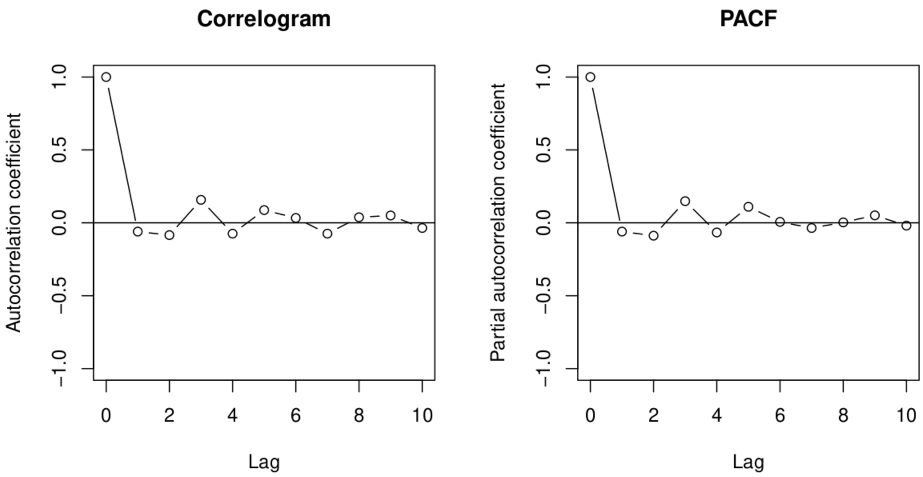

Figure 2: (b)

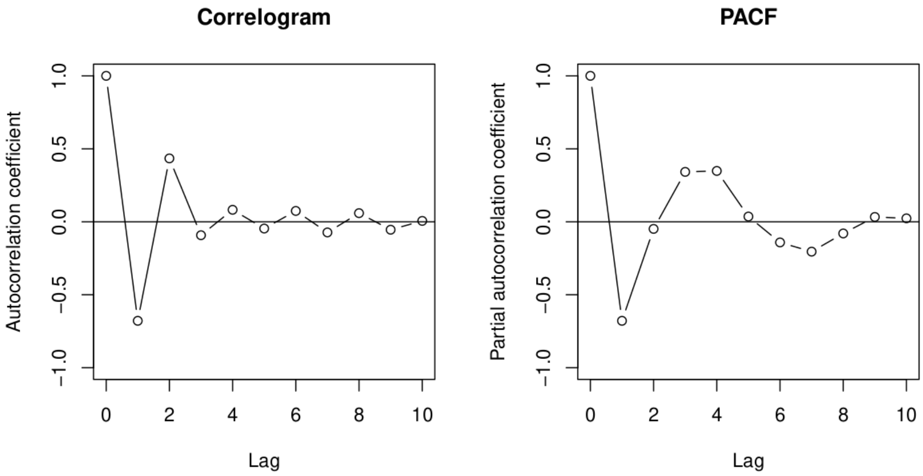

Figure 3: (c)

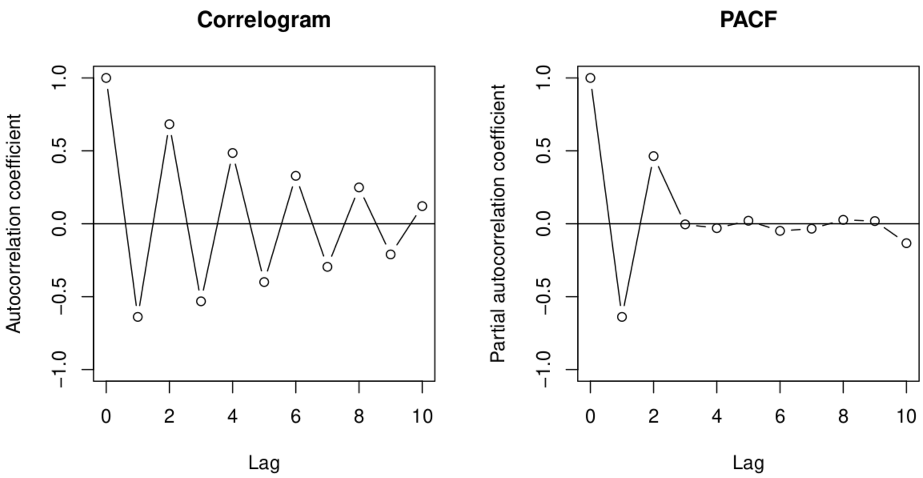

Figure 4: (d)

Ex. 9 We define the seasonal difference $\Delta_d^k = (1 - B^d)^k$, and for short $\Delta_: := \Delta_1^1$; $\Delta_d := \Delta_d^1$; $\Delta^k = \Delta_1^k$. Show that:

1. if (Y_t) is a weakly stationary process, then for arbitrary $k \in \mathbb{N}^\star$ the process $(\Delta^k Y_t)$ is again weakly stationary.
2. If (Y_t) is a weakly stationary process then for arbitrary $d \in \mathbb{N}^\star$ the process $(\Delta_d Y_t)$ is again weakly stationary.

Ex. 10 Let $x_1, \ldots, x_n$ be n observations of an AR(p) process associated with white noise of parameters $(0, \sigma^2)$. The coefficients $\hat{\alpha_1}, \ldots, \hat{\alpha_n}$ have been estimated by minimizing

$$S = \frac{1}{n} \sum_{t=p+1}^{n} \left(x_t - \sum_{i=1}^{p} \alpha_i x_{t-i} \right)^2 .$$

1. Assume that the observations come from an AR(1), show that the least square estimators can be obtained by minimising:

$$S = \frac{1}{n} \sum_{t=2}^{n} (x_t - \alpha x_{t-1})^2 .$$

2. Solve the previous minimising problem.
3. Compare your estimators to the estimation of the autocorrelation coefficient given in the lecture notes.
4. Assume you observe the following time series:

$$1,658; -3,194; -1,402; -1,426; 0,609; 0,575; -0,241; 0,030; 0.171; -0,260.$$

   Fit the AR(1) model with zero mean given by: $X_t = \alpha_1 X_{t-1} + Z_t$, $(Z_t) \sim (0, \sigma^2)$.

Ex. 11 Consider the ARMA$(2,2)$ model defined by:

$$X_t - \frac{13}{2O} X_{t-1} - \frac{3}{25} X_{t-2} = w_t - \frac{2}{10} w_{t-1} - \frac{1}{10} w_{t-2}$$

with $(w_t) \sim (0, \sigma^2)$. Determine if the process is regular.

Ex. 12 Consider the ARMA$(2,2)$ model

$$X_t + 0.3 X_{t-1} - 0.1 X_t = w_t + 0.1 w_{t-1} - 0.2 w_{t-2}$$

with $(w_t) \sim (0, \sigma^2)$.

1. Write the model under th form $\Phi(B) X_t = \Theta(B) w_t$, and show that the model is not regular.

2. Show that the process defined by:

$$Y_t - 0.2Y_{t-1} = w_t - 0.5w_{t-1}$$

with $(w_t) \sim (0, \sigma^2)$, is such that $Y_t + 0,5Y_{t-1}$ is equivalent to the process defined at the last question.

3. Show that the ARMA model of the previous question can be obtained from the equation of the first question.

Ex. 13 Consider the ARMA$(1,2)$ process defined by:

$$X_t = \alpha X_{t-1} + w_t + \beta w_{t-1} + \beta^2 w_{t-2}$$

where $(w_t) \sim (0, \sigma^2)$ and $|\alpha|, |\beta| < 1$.

1. Show that this process is causal and invertible.
2. Show that the process can be written under the form:

$$X_t = \sum_{i=0}^{\infty} \psi_i w_{t-i}$$

where ψ is to be defined.

3. Find an expression for $Var(X_t)$.

Ex. 14 Consider the process

$$X_t = X_{t-1} + 0,2X_{t-1} - 0,2X_{t-2} + w_t - 0,5w_{t-1}$$

with $(w_t) \sim (0, \sigma^2)$.

1. Classify the model as an ARIMA(p, d, q) that is determine the orders.
2. Show that the model can be written $\Phi(B)X_t = \Theta(B)w_t$.
3. Find the roots of Φ and Θ.
4. Show that w_t can be written:

$$w_t = \sum_{i=0}^{\infty} \psi_i X_{t-i}$$

and evaluate ψ_0, ψ_1, ψ_2 and ψ_3.

Ex. 15 Consider the moving average model

$$X_t = w_t + \theta w_{t-1},$$

for $|\theta| < 1$, where $\{w_t\} \sim (0, \sigma^2)$.

1. Using the Box-Jenkins forecasting method, show that $X_{n+1}^n = \theta z_n$, where X_{n+1}^n is the forecast of time $n+1$ from time n and z_n is the observed error at time n.
2. Show that $X_{n+h}^n = 0$ for $h = 2, 3 \ldots$.
3. Show that the variance of the h-step ahead forecast error ($e_n(h)$) is σ^2 for $h = 1$ and $(1+\theta^2)\sigma^2$ for $h \geq 2$, if θ is known.

Ex. 16 (May 2012 Q1 on exam paper).
Consider the time series model

$$X_t = X_{t-1} + w_t - \theta w_{t-1},$$

where $0 < \theta < 1$ and $\{w_t\} \sim (0, \sigma^2)$.

1. Classify this process as an ARIMA(p, d, q) model, where you should specify p, d and q.
2. Write the model in the form
$$\Phi(B) X_t = \theta(B) w_t,$$
where $\Phi(B)$ and $\theta(B)$ are polynomials in B. Find the roots of these polynomials.
3. Show that we may write
$$w_t = \pi(B) X_t,$$
where $\pi(B) = 1 + \pi_1 B + \pi_2 B^2 + \ldots$, and give an expression for π_i.
4. Let X_{n+h}^n denote the Box-Jenkins forecast at time n for X_{n+h}. Write down an expression for X_{n+1}^n in terms of θ and $X_n, X_{n-1}, \ldots$.
5. What is the name of this forecasting method?
6. In practice, X_t may only be available for $t = n, n-1, \ldots 1$. Modify the formula of (d) so that the sum of the coefficients of $X_n, X_{n-1}, \ldots X_n$ is equal to one.

Ex. 17 Let X_t denote the daily income (\$000's) of a store in Michigan, USA. The owner is interested in forecasting the daily income for tomorrow. The income of the store is thought to follow the process

$$\Delta^2 X_t = w_t + 0.748 w_{t-1},$$

where $\{w_t\} \sim (0, \sigma^2)$.

1. Describe the model using the ARIMA(p,d,q) notation.
2. Find a formula for X_{n+1}^n, using the Box-Jenkins method, in terms of X_n, X_{n-1} and X_n^{n-1}.

Ex. 18 Let the time series $\{X_t\}$ denote the average height (in feet) of Lake Huron at time t. Observations of $\{X_t\}$ for each of the years from 1963 to 1972 are given in the following table

Year	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972
Height	576.89	575.96	576.80	577.68	578.38	578.52	579.74	579.31	579.89	579.96

1. Take the forecast for 1963 (made in 1962) to be 579 (so that $X_{1963}^{1962} = 579$). Use exponential smoothing (the recursive formula) to obtain a forecast for the level of Lake Huron in the year 1973, setting $\theta = 0.5$.

2. Now use Holt's method to estimate the level of Lake Huron in the year 1973, taking the forecast for 1964 made in 1963 to be 579.5 (so that $X_{1964}^{1963} = 579.5$). Assume that $\alpha = 0.3$ and $\gamma = 0.5$.

Ex. 19 (May 2012 Q5 on exam paper)

1. Let $\{x_t\}$ denote a set of observations from a quarterly time series which is thought to be subject to seasonal variations. Describe briefly how to estimate the quarterly seasonal variations by first calculating a suitable adjusted average, y_t.

   The following table refers to a quarterly index of vegetable prices in the UK from the first quarter of 1955 to the first quarter of 1959.

Date	x_t	y_t	$x_t - y_t$
Q1 1955	333.7		
Q2 1955	323.9		
Q3 1955	312.8	318.8375	-6.0375
Q4 1955	310.2	319.9000	-9.7000
Q1 1956	323.2	320.7125	2.4875
Q2 1956	342.9	319.1000	23.8000
Q3 1956	300.3	316.6875	-16.3875
Q4 1956	309.8	307.2000	2.6000
Q1 1957	304.3	299.0750	5.2250
Q2 1957	285.9	296.6875	-10.7875
Q3 1957	292.3	296.3250	-4.0250
Q4 1957	298.7	303.6250	-4.9250
Q1 1958	312.5	310.3000	2.2000
Q2 1958	336.1	313.1625	22.9375
Q3 1958	295.5	314.0750	-18.5750
Q4 1958	318.4		
Q1 1959	300.1		

2. Estimate the seasonal variations in quarters 1 to 4 respectively.

3. Use the estimated variations to find the quarter in which vegetable prices tended to be the highest.

4. Find the seasonally-adjusted series.

Ex. 20 (May 2011 exam question) Woolhouse's adjusted-average formula is central, with coefficients

$$\frac{1}{125}\left(-3,-2,0,3,7,21,24,25,24,21,7,3,0,-2,-3\right).$$

1. State the length of this filter.
2. Show that this filter is exact on cubics.

Ex. 21 (From Chatfield (2004)) Suppose we have a seasonal series of monthly observations $\{X_t\}$, for which the seasonal factor at time t is denoted by $\{S_t\}$. Further suppose that the seasonal pattern is constant through time so that $S_t = S_{t-12}$ for all t. Denote a weakly stationary series of random observations by $\{\epsilon_t\}$.

1. Consider the model

   $$X_t = a + bt + S_t + \epsilon_t,$$

   having a linear trend and additive seasonality. Show that the difference operator $(1 - B^{12})$ acts on $\{X_t\}$ to produce a weakly stationary series.

2. Now consider the model

   $$X_t = (a + bt)S_t + \epsilon_t.$$

   This model is said to have **multiplicative** seasonality (as well as a linear trend). Does the difference operator $(1 - B^{12})$ transform $\{X_t\}$ to stationarity? If not, find a difference operator that does.

3. Consider the linear filter

   $$\sum_{j=\alpha}^{\beta} K_j X_{t+j},$$

   applied to the time series $\{X_t\}$. Show that if

   (a) $\sum_{j=\alpha}^{\beta} K_j = 1$, and

   (b) $\sum_{j=\alpha}^{\beta} j^i K_j = 0$, for all $i \leq k$,

   then the filter does not distort trends which are polynomials of degree k (so is exact on polynomials of degree k).

4. Consider the symmetric linear filter

   $$\sum_{j=-3}^{3} K_j X_{t+j},$$

   applied to the time series $\{X_t\}$.

   (a) Find expressions for K_1, K_2 and K_3 in terms of K_0 such that the filter is exact on polynomials of degree five (you will need to use the theorem you proved in question 3).

(b) Hence, find a symmetric linear filter that is exact on polynomials of degree five and has coefficients which minimise the expression

$$R_0^2 = \sum_{j=-3}^{3} (K_j^2).$$