Exercise 1.1

$\Theta := \text{price} = \mathbb{E}\left[e^{-rT} h(S_T)\right]$

$= \mathbb{E}\left[\hat{h}(Z)\right]$

$B\text{-}S$

$\Theta_N = \text{Estimator (to } \Theta)$

$\text{Error}(\Theta_N, \Theta) =$

$= \text{Mean square error}$

$:= \mathbb{E}\left[|\Theta_N - \Theta|^2\right]$

if Θ_N is unbiased $= \text{Var}(\Theta_N)$

$\approx \frac{1}{N} \sum_{j=1}^{N} \hat{h}(Z_j)$

$=: \Theta_N$

Uniform distro

(i) X, Y are $\mathcal{U}_{[0,1]}$ iid

$\mathbb{E}[\mathbb{1}_{\{X^2+Y^2 \le 1\}}] = \frac{\pi}{4}$

Joint distro

$\mathbb{E}[g(X,Y)] = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x,y) f_{X,Y}(x,y)\, dx\, dy$

$X \sim \mathcal{U}_{[0,1]}$ $\quad F_X(x) = x \in [0,1]$ $\quad$ CDF

$f_X(x) = 1$ $\quad$ pdf

$f_{X,Y}(x,y) = ?$ if dependent

$= f_X(x) \cdot f_Y(y)$ if independent

$= 1 \times 1$ for our case

(ii) $(X_i)_i, (Y_i)_i$ all $\mathcal{U}_{[0,1]}$

indicator function

$$\hat{\pi}_N := \frac{4}{N} \sum_j \mathbb{1}_{\{X_j^2 + Y_j^2 \leq 1\}}$$

$$\approx \pi = 4\, \mathbb{E}[\mathbb{1}_{\{\dots\}}] \quad (i)$$

bias / **Unbiased**: Property of estimators

Θ_N to Θ then $\text{bias}(\Theta_N, \Theta) = \mathbb{E}[\Theta_N] - \Theta$

if bias $= 0 \Rightarrow \Theta_N$ is unbiased

$$\text{bias}(\hat{\pi}_N, \pi) = \mathbb{E}[\hat{\pi}_N] - \pi$$

$$\mathbb{E}\left[\frac{4}{N}\sum_j \mathbb{1}\{x_j^2 + y_j^2 \leq 1\}\right] = \frac{4}{N}\sum_j \mathbb{E}[\mathbb{1}\{x_j^2 + y_j^2 \leq 1\}]$$

$$\stackrel{\text{iid}}{=} \frac{4}{\cancel{N}} \cdot \cancel{N} \cdot \mathbb{E}[\mathbb{1}\{x_1^2 + y_1^2 \leq 1\}]$$

$$\stackrel{(i)}{=} 4\,\frac{\pi}{4} = \pi \quad \checkmark$$

$\hat{\pi}_N$ is unbiased //

$$\tilde{\pi}_N = \frac{4}{N+1}\sum_j \mathbb{1}\{\ldots\} \longleftarrow \text{is biased !}$$

From Video : $MSE(\pi, \hat{\pi}_N) = Var(\hat{\pi}_N)$ || Since unbiased

$$\mathbb{E}\left[\left| \underset{\text{True}}{\pi} - \hat{\pi}_N \right|^2 \right]$$

$$\left\{ \begin{aligned} Var(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ \hline Var(cX) &= c^2 Var(X) \end{aligned} \right.$$

Q : Show $Var(\hat{\pi}_N) = \dfrac{4\pi - \pi^2}{N}$.

$$Var\left(\frac{4}{N} \sum_j \mathbb{1}_{\{\dots\}}\right) = \frac{16}{N^2} Var\left(\sum \mathbb{1}_{\{\dots\}}\right)$$

$$= \frac{16}{N^2} \mathrm{Var}\Big(\sum_i \underset{\uparrow\, x_i^2 + y_i^2}{\mathbb{1}_{\{\ldots\}}} \Big)$$

$$\overset{\text{iid}}{=} \frac{16}{N^2} \sum_{i=1}^{N} \mathrm{Var}\big(\mathbb{1}_{\{x_i^2 + y_i^2 \leq 1\}} \big)$$

$$\overset{\text{iid}}{=} \frac{16}{N} \cdot \mathrm{Var}(\ldots)$$

$$\overset{\text{def}}{=} \frac{16}{N} \Big\{ \mathbb{E}\big[\underbrace{\mathbb{1}^2_{\{\ldots\}}}_{\mathbb{1}_{\{\ldots\}}} \big] - \mathbb{E}\big[\mathbb{1}_{\{\ldots\}} \big]^2 \Big\} = \frac{16}{N} \Big(\overset{\text{From (i)}}{\frac{\pi}{4}} - \Big(\frac{\pi}{4}\Big)^2 \Big)$$

$$= \ldots = \checkmark$$

$$\mathrm{Var}(A+B) = \mathrm{Var}(A) + \mathrm{Var}(B) + 2\,\mathrm{Cov}(A,B)$$