2025/26 Semester 2

Numerical Probability and Monte Carlo Methods

Problem Sheet 1 - Friday 16 & 23 January 2025[1]

> ▷**Note:** Exercise 1.1 (iii) and (iv) are exactly the same question but (iii) is set for *Matlab* while (iv) is set for *Python*. Solve the one you prefer.

Some useful identities

Let X, Y be (square) integrable random variables and recall that

$$\begin{aligned} \mathrm{Var}[X] &:= \mathbb{E}\Big[(X-\mathbb{E}[X])^2\Big] = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2, \\ \mathrm{Cov}[X,Y] &:= \mathbb{E}\Big[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\Big] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]. \end{aligned}$$

If λ, μ are constants, then

$$\begin{aligned} \mathbb{E}[\mu + X] &= \mathbb{E}[X] + \mu, \\ \mathrm{Var}[\mu + X] &= \mathrm{Var}[X], \\ \mathbb{E}[\lambda X] &= \lambda\,\mathbb{E}[X], \\ \mathrm{Var}[\lambda X] &= \lambda^2\,\mathrm{Var}[X], \\ \mathbb{E}[X+Y] &= \mathbb{E}[X] + \mathbb{E}[Y], \\ \mathrm{Var}[X+Y] &= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y]. \end{aligned}$$

Exercise 1.1 (Monte Carlo Approximation of π). [5 marks]

(i) Let X, Y be independent and uniformly distributed on $[0,1]$. Show that

$$\mathbb{E}[\mathbf{1}_{\{X^2+Y^2\le 1\}}] = \tfrac{\pi}{4}.$$

(ii) Let $(X_i)_{i=1}^N$ and $(Y_i)_{i=1}^N$ be independent random variables distributed uniformly on $[0,1]$ and let

$$\hat{\pi}_N = \frac{4}{N}\sum_{i=1}^N \mathbf{1}_{\{X_i^2+Y_i^2\le 1\}}.$$

Show that $\mathbb{E}[\hat{\pi}_N] = \pi$ (i.e. $\hat{\pi}_N$ is an unbiased estimator for π) and that the variance of $\hat{\pi}_N$ is given by

$$\mathrm{Var}(\hat{\pi}_N) = \tfrac{4\pi-\pi^2}{N}$$

(iii) Use **Matlab** to approximate $\hat{\pi}_N$. Plot the error (i.e. $|\pi-\hat{\pi}_N|$) for $N = 10^2, 10^3, \ldots, 10^7$. Please print the code you wrote and the figure you get .

Hints:

- Use `rand` to generate samples from uniform distribution.
- Use `doc rand` to learn more about the `rand` function.
- If `R` is a vector in Matlab then $R < 1$ will give me a vector of ones and zeros depending on whether the condition was satisfied or not.
- `sum(R<1)` would then count the number of entries in vector `R` that are smaller than one.

[1] Last updated 14th January 2026

- For plotting errors from numerical simulations it is best to use logarithm of the error (for us $\ln|\pi - \hat{\pi}_N|$) versus the logarithm of the simulation parameter (for us $\ln N$).

(iv) Use **Python** to approximate $\hat{\pi}_N$. Plot the error (i.e. $|\pi - \hat{\pi}_N|$) for $N = 10^2, 10^3, \ldots, 10^7$. Please print the code you wrote and the figure you get .

Hints:

- Use `np.random.random` to generate samples from the unifrom distribution.
- Note that Python modules must be imported to function. Start your code with `import numpy as np`.
- Use `help(np.random)` to learn more about `np.random` or search the numpy docs: `https://docs.scipy.org/doc/numpy/`.
- Python module `numpy` is very friendly for Matlab users. For a comparison see: `https://numpy.org/devdocs/user/numpy-for-matlab-users.html`.
- `np.sum(R < 1)` functions just as in Matlab, the only difference is here `R<1` produces a vector of `True`'s and `False`'s.
- Use `matplotlib.pyplot` for a Matlab like plotting framework.

(v) Comment on the plot you obtain in relation to the Central limit theorem.

Exercise 1.2 (Characteristic functions). [2 marks]

i) Let X be a real-valued random variable that is equal to a constant m (i.e. $X(\omega) = m$ for all $\omega \in \Omega$) and let $\varphi(t) = \mathbb{E}[e^{itX}]$. Show that

$$\varphi(t) = e^{itm}.$$

ii) Let X be now $X \sim N(0, 1)$ (i.e. X is normally distributed with mean 0, variance 1) and let $\varphi(t) = \mathbb{E}[e^{itX}]$. Show that

$$\varphi(t) = e^{-\frac{t^2}{2}}.$$

Exercise 1.3 (Acceptance-Rejection Part 1). [5 marks]
Let Z be an $\mathbb{R}^d$-valued random variable and let D be a measurable subset of $\mathbb{R}^d$ such that $\alpha := \mathbb{P}(Z \in D) > 0$.
Our goal is to find a method that allows to sample from the distribution ρ defined as

$$\rho(A) := \mathbb{P}(Z \in A | Z \in D), \quad \text{for all measurable } A \subseteq \mathbb{R}^d.$$

The following algorithm is proposed: Take a sequence $(Z_i)_{i \geq 1}$ of i.i.d random variables sampled from the distribution of the r.v. Z. Then set

$$\begin{aligned} \nu_1 :=& \inf\{k \geq 1 : Z_k \in D\} \\ Y_1 :=& Z_{\nu_1}, n \geq 1. \end{aligned}$$

i) Show that $\mathbb{P}(\nu_1 = k, Z_k \in A \cap D) = (1 - \alpha)^{k-1} \mathbb{P}(Z_k \in A \cap D)$.

ii) Show that $\mathbb{P}(Z_{\nu_1} \in A) = \sum_{k \geq 1} \mathbb{P}(\nu_1 = k, Z_k \in A \cap D)$.

iii) Hence show that

$$\mathbb{P}(Z_{\nu_1} \in A) = \frac{\mathbb{P}(Z \in A \cap D)}{\mathbb{P}(Z \in D)}.$$

Exercise 1.4 (Acceptance Rejection Part 2). The problem with the previous method was that we got only one sample from Y out of a whole sequence $(Z_i)_{i \in \mathbb{N}}$ of i.i.d. r.v.s. The following modification is proposed.

Let Z be an $\mathbb{R}^d$-valued random variable and let D be a measurable subset of $\mathbb{R}^d$ such that $\alpha := \mathbb{P}(Z \in D) > 0$.
Our goal is to find a method that allows to to sample from the distribution ρ defined as

$$\rho(A) := \mathbb{P}(Z \in A | Z \in D), \quad \text{for all measurable } A \subseteq \mathbb{R}^d.$$

The following algorithm is proposed: Take a sequence $(Z_i)_{i\geq 1}$ of i.i.d random variables sampled from the distribution of the r.v. Z. Then set

$$\begin{aligned} \nu_1 &:= \inf\{k \geq 1 : Z_k \in D\} \\ \nu_{n+1} &:= \inf\{k > \nu_n : Z_k \in D\} \\ Y_n &:= Z_{\nu_n}, n \geq 1. \end{aligned}$$

The goal of is to show that $(Y_n)_{n\geq 0}$ are i.i.d with probability distribution ρ.

i) Given an integer $n \geq 2$ and n positive integers $k_1, \ldots, k_n$, show that

$$\begin{aligned} &\mathbb{P}(Z_{\nu_1} \in A_1, \ldots, Z_{\nu_n} \in A_n, \nu_1 = k_1, \nu_2 - \nu_1 = k_2, \ldots, \nu_n - \nu_{n-1} = k_n) \\ &= \left(\prod_{j=1}^{n} \mathbb{P}(Z \in A_j \cap D)\right)\left(\prod_{j=1}^{n}(1-\alpha)^{k_j - 1}\right). \end{aligned}$$

ii) Given an $n \in \mathbb{N}$, show that

$$\mathbb{P}(Z_{\nu_i} \in A_i, i = 1, \ldots, n) = \prod_{i=1}^{n} \frac{\mathbb{P}(Z \in A_i \cap D)}{\mathbb{P}(Z \in D)}$$

iii) Hence show that

$$\mathbb{P}(Y_i \in A_i, i = 1, \ldots, n) = \prod_{i=1}^{n} \mathbb{P}(Y_i \in A_i)$$

Hint: Observe that we have

$$\begin{aligned} &\{\nu_1 = k_1, \nu_2 - \nu_1 = k_2, \ldots, \nu_n - \nu_{n-1} = k_n\} \\ &= \{Z_{k_1} \in D, Z_{k_1+k_2} \in D, \ldots, Z_{\sum_{i=1}^{n} k_i} \in D\} \cap \bigcap_{j=0}^{n-1}\{Z_p \notin D, \sum_{l=1}^{j} k_l < p < \sum_{l=0}^{j+1} k_l\}, \end{aligned}$$

or alternatively

$$\begin{aligned} &\{\nu_1 = k_1, \nu_2 - \nu_1 = k_2, \ldots, \nu_n - \nu_{n-1} = k_n\} \\ &= \{Z_{\nu_1} \in D, \ldots, Z_{\nu_n} \in D\} \cap \bigcap_{j=0}^{n-1}\{Z_p \notin D, \sum_{l=1}^{j} k_l < p < \sum_{l=0}^{j+1} k_l\}. \end{aligned}$$

Here the first bracket on the RHS of both identities is the event, where the sequence Z_i is anchored to the domain D at certain time-points (sequence of stopping times). However, this event is much wider as the one on the LHS as it does not prescribe how the sequence behave at other time-points. Hence we have to take into account that the sequence Z_i is outside D in between of ν_i and ν_{i+1}. The intersection captures that the sequence is outside D **for all** of the possible intermediate timesteps.