1. Let X, Y be two random values on $\mathbb{R}$ and $\alpha \in \mathbb{R}$ show that
   (a) $\mathbb{E}\left(\alpha X\right) = \alpha \mathbb{E}\left(X\right)$
   (b) $\mathbb{E}\left(X + Y\right) = \mathbb{E}\left(X\right) + \mathbb{E}\left(Y\right)$
   (c) If in addition X, Y are independent show that

$$V(X+Y) = V(X) + V(Y)$$

Solution:

(a) We have that

$$\mathbb{E}\left(\alpha X\right) = \int_{\mathbb{R}} \alpha x f_X(x) dx = \alpha \int_{\mathbb{R}} x f_X(x) dx = \alpha \mathbb{E}\left(X\right)$$

(b) We have that

$$\begin{aligned}
\mathbb{E}\left(X+Y\right) &= \int_{\mathbb{R}} \int_{\mathbb{R}} (x+y) f_{X,Y}(x,y) dx dy \\
&= \int_{\mathbb{R}} x \left(\int_{\mathbb{R}} f_{X,Y}(x,y) dy \right) dx + \int_{\mathbb{R}} y \left(\int_{\mathbb{R}} f_{X,Y}(x,y) dx \right) dy \\
&= \int_{\mathbb{R}} x f_X(x) dx + \int_{\mathbb{R}} y f_Y(y) dy \\
&= \mathbb{E}\left(X\right) + \mathbb{E}\left(Y\right)
\end{aligned}$$

Both of the previous equalities are true when X, Y are random variables taking values on $\mathbb{R}^n$.

(c) We have that

$$\begin{aligned}
V(X) = \mathbb{E}\left(X - \mathbb{E}\left(X\right)\right)^2 &= \mathbb{E}\left(X^2 - 2X\mathbb{E}\left(X\right) + (\mathbb{E}\left(X\right))^2\right) \\
&= \mathbb{E}\left(X^2\right) - \mathbb{E}\left(2X\mathbb{E}\left(X\right)\right) + (\mathbb{E}\left(X\right))^2 \\
&= \mathbb{E}\left(X^2\right) - 2(\mathbb{E}\left(X\right))^2 + (\mathbb{E}\left(X\right))^2 \\
&= \mathbb{E}\left(X^2\right) - (\mathbb{E}\left(X\right))^2
\end{aligned}$$

we hence have

$$\begin{aligned}
V(X+Y) &= \mathbb{E}\left(X+Y\right)^2 - (\mathbb{E}\left(X+Y\right))^2 \\
&= \mathbb{E}\left(X^2\right) + 2\mathbb{E}\left(XY\right) + \mathbb{E}\left(Y^2\right) - (\mathbb{E}\left(X\right))^2 - 2\mathbb{E}\left(X\right)\mathbb{E}\left(Y\right) + (\mathbb{E}\left(Y\right))^2 \\
&= \mathbb{E}\left(X^2\right) - (\mathbb{E}\left(X\right))^2 + \mathbb{E}\left(Y^2\right) - (\mathbb{E}\left(Y\right))^2 \\
&+ 2\mathbb{E}\left(XY\right) - 2\mathbb{E}\left(X\right)\mathbb{E}\left(Y\right)
\end{aligned}$$

Now since X, Y independent we have that $\mathbb{E}\left(XY\right) = \mathbb{E}\left(X\right)\mathbb{E}\left(Y\right)$ and hence

$$V(X+Y) = \mathbb{E}\left(X^2\right) - (\mathbb{E}\left(X\right))^2 + \mathbb{E}\left(Y^2\right) - (\mathbb{E}\left(Y\right))^2 = V(X) + V(Y)$$

2. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ show directly using the probability density function that

(a) $\mathbb{E}(X) = \mu$,

(b) $V(X) = \sigma^2$.

Solution: We now that the probability density function of the one dimensional distribution is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

(a) Using the definition of $\mathbb{E}(X)$ we see that

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{\infty} \frac{x-\mu}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx + \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \left[-\sigma^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right]' dx + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= 0 + \mu \times 1 = \mu
\end{aligned}$$

where in the last line we have used the fact $\lim_{x\to\pm\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = 0$ as well as since $f_X(x)$ is a probability density function we have that $\int_{-\infty}^{+\infty} f_X(x)dx = 1$

(b) We have that $V(X) = \mathbb{E}(X - \mathbb{E}(X))^2$ and hence we have

$$V(X) = \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

where we have used the fact that $\mathbb{E}(X) = \mu$ from the first part. We now have

$$\begin{aligned}
V(X) &= -\sigma^2 \int_{-\infty}^{+\infty} (x-\mu) \frac{1}{\sqrt{2\pi\sigma^2}} \left[\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right]' dx \\
&= \left[-\sigma^2 (x-\mu) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right]_{-\infty}^{\infty} + \sigma^2 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= 0 + \sigma^2 \times 1 = \sigma^2
\end{aligned}$$