Itô's Formula An informal idea

$$\frac{d}{dt} f(t, g(t)) = f_t(t, g(t)) + f_x(t, g(t)) \cdot \frac{d}{dt} g(t)$$

$$f(X(t)) = \frac{1}{0!} f(X(s)) \cdot (X(t) - X(s))^0 + \frac{1}{1!} f'(X(s)) (X(t) - X(s))^1$$
$$+ \frac{1}{2!} f''(X(s)) (X(t) - X(s))^2 + \frac{1}{3!} \dots$$

$$\Leftrightarrow \frac{f(X(t)) - f(X(s))}{t-s} = \frac{1}{1!} f' \frac{(X(t) - X(s))^1}{t-s} + \frac{1}{2!} f'' \frac{(X(t) - X(s))^2}{t-s}$$

$$d f(X) = f_t \, dt + f_x \, dX + \frac{1}{2} f_{xx} \, dX \, dX$$

$$f(t, x) = x^2$$

$$df(t, X(t)) = 0 \, dt + 2X(t) \, dX(t) + \frac{1}{2} 2 \, dX(t) \, dX(t)$$

$$dX = \mu X \, dt + \sigma X \, dW$$

$$\hookrightarrow df(t, X(t)) = 2X(t) \mu X(t) \, dt + 2X(t) \sigma X(t) \, dW(t)$$
$$+ X^2 \sigma^2 dt$$
$$= (2\mu + \sigma^2) X^2(t) \, dt + 2X(t) \sigma \, dW(t)$$

Basic Ideas of Pricing in detail later

Arbitrage

No Arbitrage = "There is no such thing as a free lunch."

Arbitrage oportunity = There is the possibility to make money without the risk of loosing any.

$B \quad 1 - 1$

$S \quad 1 \begin{cases} 2 & -\omega_2 \\ 1 & -\omega_1 \end{cases}$

	$t=0$	$t=1$, ω_1	ω_2
B	-1	-1	-1
S	1	1	2
	0	0	$+1$

Replication

A contingent claim C is replicable if there exists a self financing trading strategy h such that

$$C(T) = C(0) + \sum_{i=0}^{d} \int_0^T h_i(t) dS_i(t).$$

I.e. we can achive the value/payoff of C by investing the initial price and then simply buying and selling assets.

$$\begin{array}{ccc} C(0) & \rightarrow & C(T) \\ \neq & & = \\ R(0) & \rightarrow & R(T) \end{array}$$

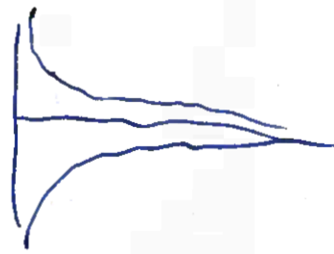
$$\underbrace{\lambda \cdot (C(0) - R(0))}_{\neq 0} \rightarrow \lambda (C(T) - R(T)) = 0$$

Martingale

Corollary 1.1. *Let*

- *M be adapted to $(\mathcal{F}_t)_{t\in[0,T]}$,*
- *$\mathbb{E}[|M_t|] < \infty$ for all $0 \le t \le T$ and*
- *for all $0 \le s \le t \le T$ we have that*

$$\mathbb{E}[M(t)|\mathcal{F}_s] \overset{\le}{\underset{\ge}{=}} M(s) \text{ then we call } M \text{ a } \overset{\text{super-}}{\underset{\text{sub-}}{\text{martingale}}}.$$

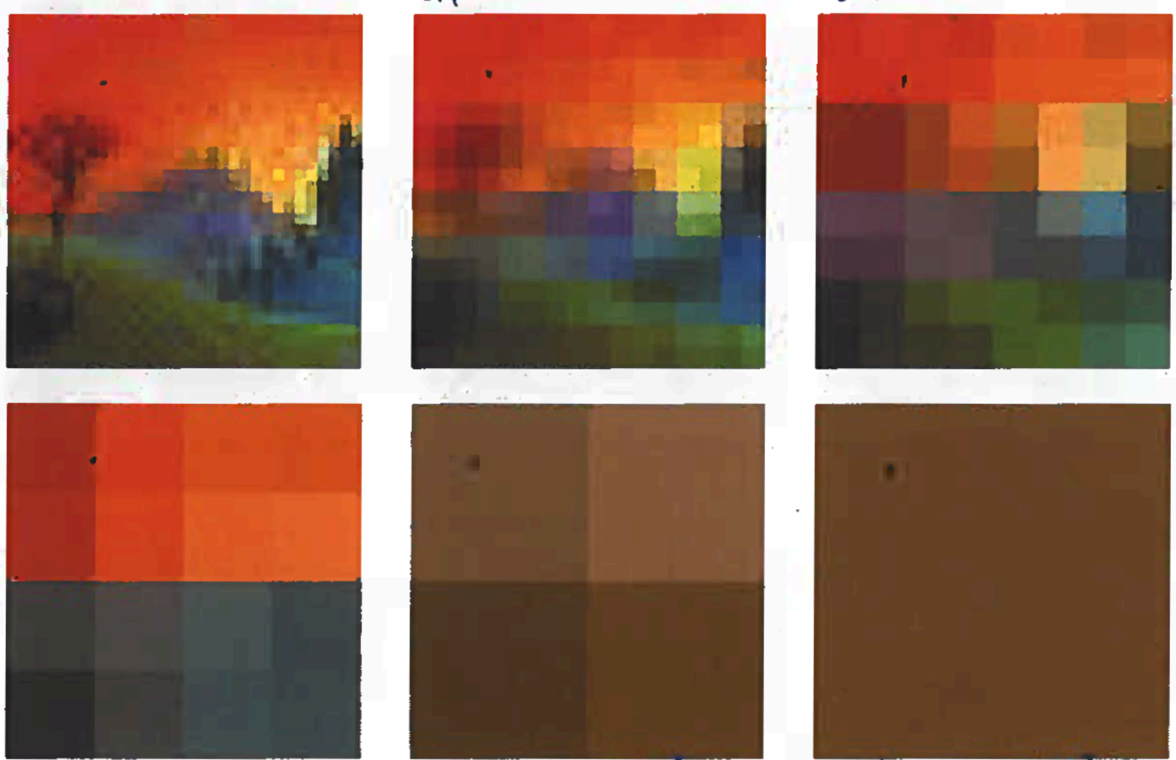


Conditional Expectation

$(\Omega, \mathcal{F}, \mathbb{P})$; $\mathcal{G} \subseteq \mathcal{F}$. We define $\mathbb{E}[X|\mathcal{G}]$ as the $Y \in L^2(\mathcal{G})$ s.t. either

1. $\displaystyle\int_G Y\,d\mathbb{P} = \int_G X\,d\mathbb{P}$ for all $G \in \mathcal{G}$,
2. $\mathbb{E}[(X-Y)Z] = 0$ for all $Z \in L^2(\mathcal{G})$,
3. $Y = \arg\min_{Z\in L^2(\mathcal{G})} \mathbb{E}[(X-Z)^2]$.

t=8 t=4 t=3



t=2 t=1 X(1) t=0 X(0)

$\mathbb{E}[X(4)|\mathcal{F}_1] = X(1)$

Martingale measure

We call $\mathbb{Q}$ a martingale measure if it is equivalent to the real world measure $\mathbb{P}$ (i.e. has the same null sets) and for all $i \in \{0, 1, \ldots, d\}$ the process $\tilde{S}_i = \frac{S_i}{S_0}$ is a martingale.

Remember:

$$C(T) = C(0) + \sum_{i=0}^{d} \int_0^T h_i(t) dS_i(t).$$

Martingale Representation

Theorem 1.10. *Let $W = (W(t))_{t\in[0,T]}$ be a d-dimensional Wiener martingale and let $(\mathcal{F}_t)_{t\in[0,T]}$ be generated by W. Let $M = (M(t))_{t\in[0,T]}$ be a continuous real valued local martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. Then there exists a unique adapted d-dimensional process $\rho = (\rho(t))_{t\in[0,T]}$, $\rho \in \mathcal{S}$, such that for $t \in [0, T]$ we have*

$$M(t) = M(0) + \sum_{i=1}^{d} \int_0^t \rho_i(s) dW_i(s).$$

If the martingale M is square integrable then $\rho \in \mathcal{H}$.

Thus, we can find ρ such that

$$C(T) = C(0) + \sum_{i=0}^{d} \int_0^T h_i(t) dS_i(t) = C(0) + \sum_{i=0}^{d} \sum_{j=0}^{d} \int_0^T h_i(t) \rho_{ij}(t) dW_j(t).$$

Itô Integral and Itô Processes

$\in \mathcal{H} \to$ true mart

$$X(t) = X(0) + \underbrace{\int_0^t \mu(s)ds}_{\in \mathcal{A}} + \underbrace{\int_0^t \sigma(s)dW_s}_{\in \mathcal{S}}$$

Integration Classes

We denote the set of all $\mathbb{R}$-valued and adapted processes Y such that

- $\mathbb{E}\left[\int_0^T |Y(s)|^2 ds\right] < \infty$ by $\mathcal{H}$,

$\subseteq$

- $\mathbb{P}\left[\int_0^T |Y(s)|^2 ds < \infty\right] = 1$ by $\mathcal{S}$ and

$\subseteq$

- $\mathbb{P}\left[\int_0^T |Y(s)| ds < \infty\right] = 1$ by $\mathcal{A}$.

Itô's Isometry

$$\mathbb{E}\Big[\Big|\int_0^T X(s)dW(s)\Big|^2\Big] = \mathbb{E}\Big[\int_0^T |X(s)|^2 ds\Big]$$

Itô's formula

Theorem 1.6. *Let $u \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and X be a d-dimensional Itô process. Then the process given by $u(t, X(t))$ has the stochastic differential*

$$du(t, X(t)) = u_t(t, X(t))dt + \sum_{i=1}^{d} u_{x_i}(t, X(t))dX^i(t)$$
$$+\frac{1}{2}\sum_{i,j=1}^{d} u_{x_i x_j}(t, X(t))dX^i(t)dX^j(t),$$

where for $i, j = 1, \ldots, n$

$$dtdt = dtdW^i(t) = 0, \quad dW^i(t)dW^j(t) = \delta_{ij}dt. \; = \begin{cases} 1\,dt & \text{for } i=j \\ 0\,dt & \text{for } i\neq j \end{cases}$$

Itô's product rule (Corollary 1.7)

$u\left(t,\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = x_1 \cdot x_2 \qquad u_t = 0 \qquad u_{x_1} = x_2,\ u_{x_2} = x_1,\ u_{x_1x_1} = 0 = u_{x_2x_2}$

$u_{x_1x_2} = 1$

$d(Z(t)\cdot Y(t)) = du\left(t,\begin{pmatrix}Z\\Y\end{pmatrix}\right)$

$$= u_t\,dt + u_{x_1}\,dX_1 + u_{x_2}\,dX_2 + \frac{1}{2}\left(u_{x_1x_1}\,dX_1\,dX_1 + u_{x_2x_2}\,dX_2\,dX_2 + 2u_{x_1x_2}\,dX_1\,dX_2\right)$$

$$= 0\,dt + X_2 \cdot dX_1 + X_1\,dX_2 + \frac{1}{2}\left(0 + 0 + 2\cdot 1\cdot dX_1\,dX_2\right)$$

$$= X_2\,dX_1 + X_1\,dX_2 + dX_1\,dX_2$$

$$dS = S\mu\, dt + S\sigma\, dw$$

$$\partial_t h(x) = 0, \quad \partial_x h(x) = \frac{1}{x}, \quad \partial_{xx} h(x) = -\frac{1}{x^2}$$

$$dh(S) = \frac{1}{S} \cdot dS + \frac{1}{2}\left(-\frac{1}{S^2}\right) dS\, dS$$

$$= \frac{1}{S}(S\mu\, dt + S\sigma\, dw) + \frac{1}{2} \cdot \left(-\frac{1}{S^2}\right) \underbrace{(S\mu\, dt + S\sigma\, dw)^2}_{0.0 + S\sigma\, dw\, S\sigma\, dw = S^2\sigma^2 \underbrace{dw\, dw}_{= dt}}$$

$$= \mu\, dt + \sigma\, dw - \frac{1}{2}\sigma^2 dt$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dw$$

$$h(\cancel{X(t)}\ S(t)) = h(S(0)) + \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds + \int_0^t \sigma(s)\, dw(s)$$

$$e^{h(S(t))} = S(t) = e^{h(S(0)) + \int_0^t (\mu - \frac{1}{2}\sigma^2) ds + \int_0^t \sigma\, dw(s)}$$

$$= S(0)\, e^{\underbrace{\int_0^t (\mu_t - \frac{1}{2}\sigma^2) ds + \int_0^t \sigma\, dw(s)}_{= Z(t)}}$$

$$S(t) := S(0) e^{Z(t)} \underset{= u(t, Z(t))}{\rightarrow} u(t, z) = S(0) e^{z} \quad u_t = 0, \ u_z = u, \ u_{zz} = u$$

$$dS(t) = du(t, Z(t)) = 0\, dt + u(t, Z(t))\, dZ(t) + \frac{1}{2} u(t, Z(t))\, dZ(t)\, dZ(t)$$

$$= S(t) \cdot \left(\left(\mu - \cancel{\frac{1}{2}\sigma^2}\right) dt + \sigma\, dw\right)$$

$$+ \cancel{\frac{1}{2} S(t)\, \sigma^2 dt}$$

$$= S(t)\mu\, dt + S(t)\sigma\, dw$$

$$dS = S\mu\, dt + S\sigma\, dW$$

$$d\frac{1}{S} = -\frac{1}{S^2} dS + \frac{1}{2}\left(+\frac{2}{S^3}\right) dS\, dS$$

$$= -\frac{1}{S^2}\left(\mu S\, dt + S\sigma\, dW\right) + \frac{1}{S^3}(S\sigma)^2 dt$$

$$= (\sigma^2 - \mu)\frac{1}{S} dt + \sigma \frac{1}{S} dW$$

$$dB = rB\, dt \quad \Rightarrow d\frac{1}{B} = -r\frac{1}{B} dt$$

$$d\frac{S}{B} = d\left(S\cdot\frac{1}{B}\right) = \frac{1}{B} dS + S\, d\frac{1}{B} + d\frac{1}{B} dS$$

$$= \frac{1}{B}\left(S\mu\, dt + S\sigma\, dW\right) + S(-r)\frac{1}{B} dt + 0$$

$$= \frac{S}{B}(\mu - r) dt + \frac{S}{B}\sigma\, dW$$