Itô's Formula An informal idea

Basic Ideas of Pricing in detail later

Arbitrage

No Arbitrage = "There is no such thing as a free lunch."

Arbitrage oportunity = There is the possibility to make money without the risk of loosing any.

Replication

A contingent claim C is replicable if there exists a self financing trading strategy h such that

$$C(T) = C(0) + \sum_{i=0}^{d} \int_0^T h_i(t) dS_i(t).$$

I.e. we can achive the value/payoff of C by investing the initial price and then simply buying and selling assets.

Martingale

Corollary 1.1. *Let*

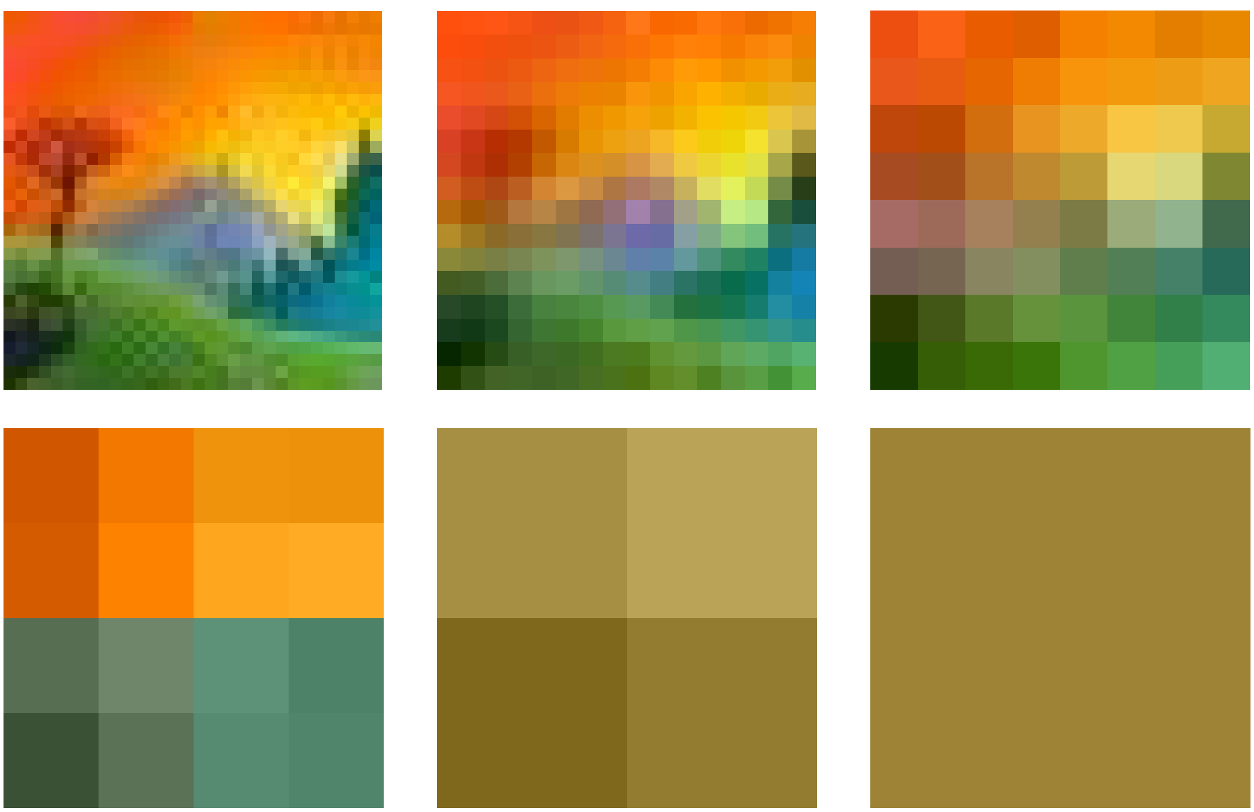
- *M be adapted to $(\mathcal{F}_t)_{t\in[0,T]}$,*
- *$\mathbb{E}[|M_t|] < \infty$ for all $0 \le t \le T$ and*
- *for all $0 \le s \le t \le T$ we have that*

  $\mathbb{E}\left[M(t)\middle|\mathcal{F}_s\right] = M(s)$ then we call M a martingale.

Conditional Expectation

$(\Omega, \mathcal{F}, \mathbb{P})$; $\mathcal{G} \subseteq \mathcal{F}$. We define $\mathbb{E}[X|\mathcal{G}]$ as the $Y \in L^2(\mathcal{G})$ s.t. either

1. $$\int_G Y d\mathbb{P} = \int_G X d\mathbb{P} \qquad \text{for all } G \in \mathcal{G}.$$
2. $$\mathbb{E}[(X - Y)Z] = 0 \qquad \text{for all } Z \in L^2(\mathcal{G}).$$
3. $$Y = \arg\min_{Z \in L^2(\mathcal{G})} \mathbb{E}[(X - Y)^2].$$



Martingale measure

We call $\mathbb{Q}$ a martingale measure if it is equivalent to the real world measure $\mathbb{P}$ (i.e. has the same null sets) and for all $i \in \{0, 1, \ldots, d\}$ the process $\tilde{S}_i = \frac{S_i}{S_0}$ is a martingale.

Remember:

$$C(T) = C(0) + \sum_{i=0}^{d} \int_0^T h_i(t) dS_i(t).$$

Martingale Representation

Theorem 1.10. *Let $W = (W(t))_{t\in[0,T]}$ be a d-dimensional Wiener martingale and let $(\mathcal{F}_t)_{t\in[0,T]}$ be generated by W. Let $M = (M(t))_{t\in[0,T]}$ be a continuous real valued local martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. Then there exists a unique adapted d-dimensional process $\rho = (\rho(t))_{t\in[0,T]}$, $\rho \in \mathcal{S}$, such that for $t \in [0, T]$ we have*

$$M(t) = M(0) + \sum_{i=1}^{d} \int_0^t \rho_i(s) dW_i(s).$$

If the martingale M is square integrable then $\rho \in \mathcal{H}$.

Thus, we can find ρ such that

$$C(T) = C(0) + \sum_{i=0}^{d} \int_0^T h_i(t) dS_i(t) = C(0) + \sum_{i=0}^{d} \sum_{j=0}^{d} \int_0^T h_i(t)\rho_{ij}(t) dW_j(t).$$

Itô Integral and Itô Processes

$$X(t) \quad = \quad X(0) \quad + \quad \int_0^t \mu(s)ds \quad + \quad \int_0^t \sigma(s)dW_s$$

Integration Classes

We denote the set of all $\mathbb{R}$-valued and adapted processes Y such that

- $\mathbb{E}\left[\int_0^T |Y(s)|^2 ds\right] < \infty$ by $\mathcal{H}$,
- $\mathbb{P}\left[\int_0^T |Y(s)|^2 ds < \infty\right] = 1$ by $\mathcal{S}$ and
- $\mathbb{P}\left[\int_0^T |Y(s)| ds < \infty\right] = 1$ by $\mathcal{A}$.

Itô's Isometry

$$\mathbb{E}\Big[\Big|\int_0^T X(s)dW(s)\Big|^2\Big] = \mathbb{E}\Big[\int_0^T \big|X(s)\big|^2 ds\Big]$$

Itô's formula

Theorem 1.6. *Let $u \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and X be a d-dimensional Itô process. Then the process given by $u(t, X(t))$ has the stochastic differential*

$$\begin{aligned} du(t, X(t)) = u_t(t, X(t))dt &+ \sum_{i=1}^d u_{x_i}(t, X(t))dX^i(t) \\ &+ \frac{1}{2}\sum_{i,j=1}^d u_{x_i x_j}(t, X(t))dX^i(t)dX^j(t), \end{aligned}$$

where for $i, j = 1, \ldots, n$

$$dtdt = dtdW^i(t) = 0, \quad dW^i(t)dW^j(t) = \delta_{ij}dt.$$

Itô's product rule (Corollary 1.7)

$$d\big(Z(t) \cdot Y(t)\big) =$$