Lecture Notes: Self-Financing Trading Strategies and Arbitrage

Mathematical Finance Theory

1. Introduction to Trading Strategies

In mathematical finance, the instructor begins by establishing the fundamental framework for analyzing financial markets. The goal is to understand how investors can construct portfolios of assets and determine whether the market allows for risk-free profit opportunities, known as arbitrage.

The lecture materials present a market model consisting of multiple assets: a risk-free asset (like a bank account or bond) denoted S_0, and risky assets (like stocks) denoted $S_1, S_2, ..., S_m$. The instructor emphasizes that understanding how to trade these assets systematically requires precise mathematical definitions.

2. What is a Trading Strategy?

2.1 Definition of a Trading Strategy

The instructor defines a trading strategy $h = (h_0, h_1, ..., h_m)$ as a collection of adapted stochastic processes. Each component h_i represents the number of units of asset S_i that the investor holds at time t. The term 'adapted' is crucial here—it means that the trading decision at time t can only depend on information available up to that time, not on future information. This captures the realistic constraint that investors cannot see into the future.

For example, if $h_1(t) = 100$, this means the investor holds 100 shares of stock S_1 at time t. If $h_0(t) = 5000$, they have $5000 in the risk-free bank account.

2.2 Portfolio Value

The instructor then defines the portfolio value v(t) at time t as the total market value of all holdings:

$$v(t) = h_0(t)S_0(t) + h_1(t)S_1(t) + ... + h_m(t)S_m(t)$$

This formula simply says: the total wealth equals the sum of (number of units × price per unit) across all assets.

2.3 Proportion Strategy

The instructor introduces an alternative representation using proportions. Instead of tracking the number of shares, one can track what fraction of total wealth is invested in each asset. This is defined as:

$$u_i(t) = h_i(t)S_i(t) / v(t)$$

This representation is convenient because the proportions always sum to 1 (meaning 100% of the wealth is allocated across the assets). For instance, if $u_1 = 0.3$, then 30% of the portfolio is invested in asset S_1.

3. The Crucial Concept: Self-Financing

3.1 What Does Self-Financing Mean?

This is one of the most important concepts in the lecture. The instructor explains that a trading strategy is called 'self-financing' if all changes in portfolio value come solely from capital gains or losses on the existing holdings—not from adding or withdrawing money.

Mathematically, the definition states that for any time interval from t to t', the change in portfolio value equals:

$$v(t') - v(t) = \int_{[t \text{ to } t']} h_0(s)dS_0(s) + \Sigma_i \int_{[t \text{ to } t']} h_i(s)dS_i(s)$$

In differential form, this becomes:

$$dv(t) = \Sigma_i h_i(t)dS_i(t)$$

3.2 Intuitive Understanding

To understand this intuitively, imagine an investor who starts with $100,000. They buy some stocks and bonds. If the strategy is self-financing:

- When they want to buy more stock, they must sell some bonds (or sell other stock)
- They never add external funds
- They never withdraw funds
- All changes in wealth come from price movements of their holdings

This constraint makes the analysis realistic for studying market efficiency and arbitrage, since we want to know if someone can make risk-free profits without continually injecting more capital.

3.3 Alternative Characterization

The instructor presents an equivalent way to express the self-financing condition using the proportion representation:

$$dv(t) = v(t) \Sigma_i u_i(t) [dS_i(t)/S_i(t)]$$

This says: the portfolio return equals the weighted average of individual asset returns, where weights are the portfolio proportions. This is intuitive—if 30% is in an asset that gains 10%, that contributes 3% to the overall portfolio return.

4. Working with Discounted Prices

4.1 Why Discount Prices?

The instructor introduces an important technical tool: working with discounted prices. Since the risk-free asset $S_0(t)$ grows deterministically (like a bank account with interest), it's convenient to measure everything relative to this benchmark. The discounted price of asset i is defined as:

$$\tilde{S}_i(t) = S_i(t) / S_0(t)$$

This represents the value of asset i measured in units of the risk-free asset. For example, if a stock costs $50 and the risk-free asset has grown to $2, then $\tilde{S} = 50/2 = 25$ in discounted terms.

4.2 Proposition 2.5: Discounted Portfolio Value

A key result (Proposition 2.5) states that a strategy (h_0, h) is self-financing if and only if the discounted portfolio value satisfies:

$$d[v(t)/S_0(t)] = \Sigma_i\, h_i(t)\, d[S_i(t)/S_0(t)]$$

The proof shown in the images demonstrates this using Itô's lemma and careful manipulation of the differential expressions. The key insight is that when working in discounted terms, the self-financing condition simplifies nicely—the change in discounted wealth depends only on changes in discounted risky asset prices, with the h_0 term (bank account holdings) disappearing from the formula.

5. Local Martingale Measures and Risk-Neutral Pricing

5.1 Equivalent Probability Measures

The instructor now introduces a profound theoretical tool from probability theory. Two probability measures P and Q are called 'equivalent' if they agree on which events are possible and impossible. Formally:

$$Q(E) = 0 \Longleftrightarrow P(E) = 0 \text{ for all events } E$$

This means P and Q describe the same set of possible outcomes but may assign different probabilities to them.

5.2 Definition of Local Martingale Measure

A measure Q equivalent to P is called a 'local martingale measure' (or 'risk-neutral measure') if the discounted prices $\tilde{S}_i = S_i/S_0$ are local martingales under Q for all risky assets i = 1, ..., m.

A martingale is a stochastic process where the expected future value equals the current value—it represents a 'fair game' with no systematic drift. If discounted prices are martingales under Q, it means that under this probability measure, risky assets have no excess return over the risk-free rate—hence the term 'risk-neutral' (investors act as if they don't demand compensation for risk).

5.3 Girsanov's Theorem and Existence

The instructor presents Proposition 2.7, which gives conditions for the existence of a local martingale measure. The key is finding a process φ(t) that satisfies:

$$\mu(t) - r(t) = \sigma(t)\varphi(t)$$

Here, μ(t) is the drift vector (expected returns) of the risky assets, r(t) is the risk-free rate, and σ(t) is the volatility matrix. The process φ(t) is called the 'market price of risk'—it represents how much extra return investors demand per unit of risk.

The instructor then defines the likelihood ratio:

$$L(t) = \exp\left(-\int_0^t \varphi(s)^T dW(s) - (1/2)\int_0^t |\varphi(s)|^2 ds\right)$$

If E[L(T)] = 1 (which is guaranteed under Novikov's condition when $E[\exp((1/2)\int_0^T |\varphi(t)|^2 dt)] < \infty$), then one can define Q via dQ = L(T)dP, and this Q will be a local martingale measure.

5.4 Market Model and Girsanov Application

The lecture materials show the standard market model:

$$dS_0(t) = S_0(t)r(t)dt$$

$$dS_i(t) = S_i(t)\mu_i(t)dt + S_i(t)\Sigma_j \sigma_{ij}(t)dW_j(t)$$

Girsanov's theorem says that $W^Q(t) = W(t) + \int_0^t \varphi(s)ds$ is a Brownian motion under Q when $dQ = L(T)dP$. This change of measure effectively removes the drift $\mu(t)$ and replaces it with $r(t)$, making the discounted prices driftless (martingales).

5.5 Corollary 2.10: Dynamics Under Q

An important consequence (Corollary 2.10) is that if the local martingale measure Q exists, then under Q, the dynamics of risky assets become:

$$dS_i(t) = S_i(t)r(t)dt + S_i(t)\Sigma_j \sigma_{ij}(t)dW^Q_j(t)$$

Notice that the drift $\mu_i(t)$ has been replaced by $r(t)$. Under the risk-neutral measure, all assets grow (in expectation) at the risk-free rate. This is the mathematical foundation for risk-neutral pricing in derivatives.

6. Arbitrage: The Central Problem

6.1 Definition of Arbitrage

The instructor now addresses the fundamental question: can an investor make risk-free profits? A self-financing trading strategy forms an 'arbitrage' if:

- With probability 1, the final wealth is at least as large as the initial investment compounded at the risk-free rate: $P[v(T) \geq v(0)S_0(T)] = 1$
- With positive probability, the final wealth strictly exceeds the risk-free investment: $P[v(T) > v(0)S_0(T)] > 0$

In plain English: an arbitrage is a trading strategy where you cannot lose money (compared to just putting everything in a bank account) and have a genuine chance of making extra profit. This is 'free money' and should not exist in well-functioning markets.

6.2 Lemma 2.14: The Key Arbitrage Detection Result

Lemma 2.14 provides a powerful tool for detecting arbitrage opportunities. It states: Suppose there exists a self-financing strategy h such that the portfolio value satisfies:

$$dv(t) = v(t)\rho(t)dt$$

for some adapted process $\rho(t)$. Then either $\rho(t) = r(t)$ almost surely for all t, or the model admits arbitrage.

The intuition is clear: if a portfolio grows deterministically at rate $\rho(t)$, and if $\rho(t) > r(t)$ on some set of positive measure, then this portfolio outperforms the risk-free rate without risk—which is arbitrage. Conversely, if $\rho(t) < r(t)$ somewhere, one can reverse the strategy (short the portfolio, invest proceeds at rate r) to create arbitrage.

6.3 Example: Finding Arbitrage

The lecture presents an example with a market consisting of:

$dS_0(t) = rS_0(t)dt \; dS(t) = S(t)\mu dt + S(t)\sigma dW(t)$

The instructor then adds an artificial asset $\xi(t) = S(t)^2$ to demonstrate how arbitrage can arise. Using Itô's lemma to compute $d\xi(t)$, the lecturer shows that one can construct a self-financing portfolio involving S_0, S, and ξ whose return $\rho(t)$ differs from $r(t)$. By Lemma 2.14, this creates an arbitrage opportunity.

The key calculation uses Itô's formula for $d\xi = d(S^2)$:

$$d\xi(t) = 2S(t)dS(t) + (dS(t))^2$$

After substituting the dynamics of S and expanding, one finds that ξ grows at a rate involving both μ and σ^2, which generically differs from r. This demonstrates that introducing derivative securities (like squared payoffs) can create arbitrage if not priced correctly.

6.4 Proof Strategy of Lemma 2.14

The proof shown in the images is constructive and elegant. The lecturer defines the sets:

$A_+ = \{(\omega,t) : \rho(t) > r(t)\}$ $A_- = \{(\omega,t) : \rho(t) < r(t)\}$ $A = A_+ \cup A_-$

The idea is to construct a new strategy ($\bar{h}_0$, $\bar{h}$) that exploits any deviation of ρ from r. Specifically:

- When ρ(t) > r(t), go long the strategy h (take the same position)
- When ρ(t) < r(t), go short the strategy h (take the opposite position)
- When ρ(t) = r(t), do nothing (invest in the risk-free asset)

The modified strategy is defined as:

$$\bar{h}_i(t) = (\mathbb{1}\_\{A_+\}(t)h_i(t) - \mathbb{1}\_\{A_-\}(t)h_i(t))\ \bar{v}(t)/v(t)$$

The factor $\bar{v}(t)/v(t)$ ensures the new strategy is self-financing. The lecturer carefully computes $d\bar{v}(t)$ on each of the regions A_+, A_-, and A^c.

On A_+, where ρ > r, the calculation shows:

$$d\bar{v}(t) = \bar{v}(t)\rho(t)dt$$

On A_-, where ρ < r, the short position combined with investing at the risk-free rate yields:

$$d\bar{v}(t) = \bar{v}(t)(2r(t) - \rho(t))dt$$

Since ρ < r on A_-, we have 2r - ρ > r, so the portfolio still grows faster than the risk-free rate.

Defining $x(t) = r(t)\mathbb{1}\_\{A^c\} + \rho(t)\mathbb{1}\_\{A_+\} + (2r(t) - \rho(t))\mathbb{1}\_\{A_-\}$, the instructor notes that x(t) > r(t) on A (the set where ρ ≠ r). Solving the ODE:

$$\bar{v}(T) = \bar{v}(0) \exp(\textstyle\int_0^T x(t)dt)$$

Since $\int_0^T x(t)dt > \int_0^T r(t)dt$ whenever A has positive measure, this strategy guarantees $\bar{v}(T) > \bar{v}(0)S_0(T)$ with positive probability, establishing arbitrage.

This proof is constructive—it doesn't just show arbitrage exists, but explicitly constructs the arbitrage strategy. The key insight is that any deterministic deviation in return from the risk-free rate can be exploited by either going long (when ρ > r) or short (when ρ < r).

7. Local Martingales vs True Martingales

7.1 The Example from the Lecture

The final part of the lecture materials presents an important technical example distinguishing local martingales from true martingales. The instructor defines:

$$X(t) = \int_0^t (1/(1-s))\, dW(s) \text{ for } t \in [0,1)$$

This is a stochastic integral with an integrand that blows up as t approaches 1. The instructor then defines a stopping time $\tau = \inf\{s \in [0,1) : X(s) = 1\}$ and a stopped process:

$$Y(t) = X(t) \text{ if } t < \tau, \text{ and } Y(t) = 1 \text{ if } t \geq \tau$$

7.2 Questions About Martingale Properties

The instructor poses three questions:

- Is X a local martingale?
- Is X a true martingale?
- Is Y a local martingale?

7.3 Analysis of the Example

X is indeed a local martingale because it's a stochastic integral with respect to a Brownian motion. By definition, such integrals are local martingales. One can verify this by choosing stopping times $\tau_n = \inf\{t : |X(t)| \geq n\} \wedge (1 - 1/n)$, which localize X to a true martingale X^{τ_n}.

However, X is NOT a true martingale. To be a martingale, we need $E[\int_0^t (1/(1-s))^2 ds] < \infty$, but this integral diverges as $t \to 1$. Specifically:

$$\int_0^t (1/(1-s))^2 ds = [1/(1-s)]_0^t = 1/(1-t) - 1 \to \infty \text{ as } t \to 1$$

This violates the integrability condition required for X to be a true martingale.

Y, being a stopped version of X, is also a local martingale. The stopping preserves the local martingale property. However, Y is also not a true martingale because stopping doesn't fix the integrability problem—the process still has the potential to behave badly near the boundary.

7.4 Why This Matters

This example is crucial because it shows that 'local martingale measure' is genuinely weaker than 'equivalent martingale measure.' In the fundamental theorem of asset pricing:

- No arbitrage is equivalent to the existence of a local martingale measure
- No arbitrage + a certain boundedness condition (like 'no free lunch with vanishing risk') is equivalent to the existence of a true equivalent martingale measure

The distinction matters in practice when dealing with markets that have boundary behavior or where asset prices can explode to infinity in finite time. In such cases, local martingale measures exist but true martingale measures may not.

8. Summary and Key Takeaways

8.1 Main Concepts

The lecture materials present a comprehensive framework for analyzing financial markets through the lens of arbitrage theory. The key concepts are:

- Trading strategies represent how investors allocate wealth across assets over time
- Self-financing strategies capture realistic trading where wealth changes come only from price movements, not external cash flows
- Local martingale measures provide the theoretical foundation for risk-neutral pricing
- Arbitrage represents risk-free profit opportunities that should not exist in efficient markets
- Lemma 2.14 provides a practical tool for detecting arbitrage by checking whether portfolio returns can systematically exceed the risk-free rate

8.2 Practical Implications

For practitioners, these theoretical results have concrete implications:

- Derivative securities must be priced consistently with underlying assets to avoid arbitrage
- Risk-neutral valuation (pricing derivatives by computing expected payoffs under Q and discounting at the risk-free rate) is justified by the existence of the martingale measure
- Market completeness depends on whether every contingent claim can be replicated by a self-financing strategy
- In incomplete markets, there may be multiple martingale measures, leading to a range of arbitrage-free prices

8.3 Mathematical Sophistication

The mathematical machinery employed—stochastic calculus, Itô's lemma, Girsanov's theorem, and martingale theory—may seem abstract, but it provides precise answers to fundamental questions:

- When can arbitrage exist?
- How should derivatives be priced?
- What trading strategies are feasible without external funding?
- How do we handle the difference between local and true martingales in practical applications?

The instructor's presentation moves systematically from definitions to theorems to examples, building a complete picture of how modern financial mathematics analyzes market efficiency and derivatives pricing. The proof of Lemma 2.14, in particular, demonstrates how abstract probability theory translates into concrete trading strategies that exploit market inefficiencies.

8.4 Further Study

These lecture materials form part of a larger course on mathematical finance. Students should supplement this material with:

- The fundamental theorems of asset pricing (relating no-arbitrage to martingale measures)
- Market completeness and derivative pricing
- The Black-Scholes-Merton model as a special case
- American options and optimal stopping
- Extensions to jump processes and incomplete markets

— *End of Lecture Notes* —