Conformal Symmetry in Field Theories

Soner Albayrak[1]

Institute of Physics, University of Amsterdam, Amsterdam, 1098 XH, The Netherlands

These are the notes of the lectures prepared for the "Advanced Topics in Theoretical Physics" summer school at *Feza Gürsey Merkezi*, September 2021. These notes are mostly based on other discussions and I provided the sources in relevant places. I may update the notes even after the summer school to keep it up-to-date and self-contained; there are also reminders in blue for me to add further discussion/comments.

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[1] contact@soneralbayrak.com

Contents

Literature Recommendations

One can find many books, reviews, lecture notes, and articles about *conformal symmetry*, *conformal bootstrap program*, *quantum theory of fields*, *renormalization group*, *critical phenomena*, *AdS/CFT correspondence*, *cosmological bootstrap*, *flat space holography*, *string theory* and many other related concepts to conformal symmetry and its usage in theoretical physics. You can go ahead and search these keywords and see how large a literature there exists for each and every one of these topics.

So finding a source for conformal symmetry or its usage is no problem, it is the other way around: there are simply too many resources and one can feel overwhelmed and find it hard to navigate through those articles to learn the subject. So below, I'll list some resources that I would recommend both to naïve younglings who know nothing about these subjects and to those physics-hardened veterans who have heard of/worked on some of these subjects but haven't really got the chance to look at conformal symmetry directly. Please note that the list is *faaaaar* from being complete and in no means is supposed to include good resources and to exclude bad ones: I simply listed what I know (and would recommend) and it is extremely likely that there are reviews out there that would be far more suited yet are unbeknownst to me.

1. The traditional referencing source for conformal symmetry is the so-called *yellow book*, i.e. *Conformal Field Theory* by Francesco, Mathieu, and Sénéchal. It is a rather complete book, i.e. it covers all the basics, however it is published in 1997 and hence does not include recent developments captured by *conformal bootstrap program*. In short, conformal symmetry is a lot easier to solve in two spacetime dimensions and up until the last two decades all the major progress has been in two dimensions, hence naturally most of the book's content is for two dimensional conformal models.

2. The de facto referencing source for the conformal bootstrap program (which aims to constraint or solve conformal models in higher dimensions, i.e. $d \geq 3$) is the review article *The Conformal Bootstrap: Theory, Numerical Techniques, and Applications* by Poland (who happens to be my PhD advisor[2]), Rychkov, and Vichi. The article really does not get into details (and not really pedagogical), so it is not the best source to *learn* stuff. However, it is an excellent collection of sources, so one can use it to find other sources for individual topics.

3. The de facto lecture notes to learn the basics of the conformal symmetry is *TASI Lectures on the Conformal Bootstrap* by Simmons-Duffin (who happens to be my host during my 1-year visit to Caltech[3]). Around similar times, Qualls shared his lecture notes *Lectures on Conformal Field Theory* as well (which is less focused on bootstrap but more on 2d CFTs).[4] A more recent review was given by Osborn, i.e. *Lectures on Conformal Field Theories in more than two dimensions*, however they are somehow more technical and less pedagogical.

[2] Thus I may be biased towards his work due to more exposure.

[3] Thus I may be biased towards his work due to more exposure.

[4] There is also the lecture notes *Applied Conformal Field Theory* by Ginsparg. Honestly, I did not fully read the notes as they are rather focused on $2d$ CFTs which are not really my personal focus; however, some may find them quite useful.

4. Slava Rychkov maintains an active blog where he writes about his papers or his talks among other useful information, see https://sites.google.com/site/slavarychkov. He also wrote lecture notes titled *EPFL Lectures on Conformal Field Theory in D>= 3 Dimensions*, it is a bit older than Simmons-Duffin's lecture notes, but some may prefer the style. He also talked about the *philosophy* of the bootstrap approach as the 27th Ockham Lecture, see https://www.merton.ox.ac.uk/event/27th-ockham-lecture-reductionism-vs-bootstrap-are-things-big-always-made-things-elementary.

5. If you would like a *hardcore* resource, then there is the *holy book*.[5] It is written by Dobrev, Mack, Petkova's, and Todorov in 1977 and is still extremely relevant today. The only problem is that it is rather mathematical (and technical), and you may find it hard — honestly I did not fully read the book myself, I only have used it as a reference to check stuff here and there.

6. If you would like a *layman review* of the conformal bootstrap, then I'd suggest the nature article of Poland and Simmons-Duffin, i.e. https://doi.org/10.1038/nphys3761.

7. If you prefer *watching* to *reading*, then there are recordings of excellent lectures in the summer schools of the conformal bootstrap collaboration.[6] You can also view all videos of the collaboration at their Youtube channel:[7] I personally advice 2017 & 2018 Bootstrap School videos as they were rather introductory and pedagogical!

8. These lectures will be mostly about conformal symmetry in $d \geq 3$ spacetime dimensions, hence $d = 2$ conformal symmetry (and related Virasoro symmetry) are not our focus. For those interested in $d = 2$ conformal symmetry (and its utility in String theory), I already named a few sources, but there are two books I would like to refer for sentimental reasons: *String and Symmetries*, edited by Gülen Aktaş, Cihan Saçlıoğlu, and MeralSerdaroğlu; and *Conformal Field Theory: New Non-perturbative Methods In String And Field Theory*, edited by Yavuz Nutku, Cihan Saçlıoğlu, and Teoman Turgut. The former book consists of the proceedings of the Feza Gürsey Memorial Conference 1 (1994); and the latter book consists of lecture notes of *1998 Summer Research Semester on Conformal Field Theories, M(atrix) Models and Dualities*; both events held at Bogaziçi Üniversitesi-Tübitak Feza Gürsey Institute.

9. Lastly, I can recommend the first two chapters of my thesis as *an earnest attempt at a very pedagogical introduction* to conformal symmetry and the conformal bootstrap program. I tried to give a historical account of the subject, its basics, and recent developments. Even if you don't like it, you may find the references useful as I tried my best to organize and include as many references as possible. You can find it via the publisher's website[8] or via arXiv.[9]

[5]See http://doi.org/10.1007/BFb0009678.
[6]See http://bootstrapcollaboration.com/activities
[7]See https://www.youtube.com/channel/UCgWLG2q2275RuUJ5eNSCCFA/videos.
[8]See https://www.proquest.com/docview/2557212384/2C5A80A2ADB447E7PQ.
[9]See https://arxiv.org/abs/2107.13601.

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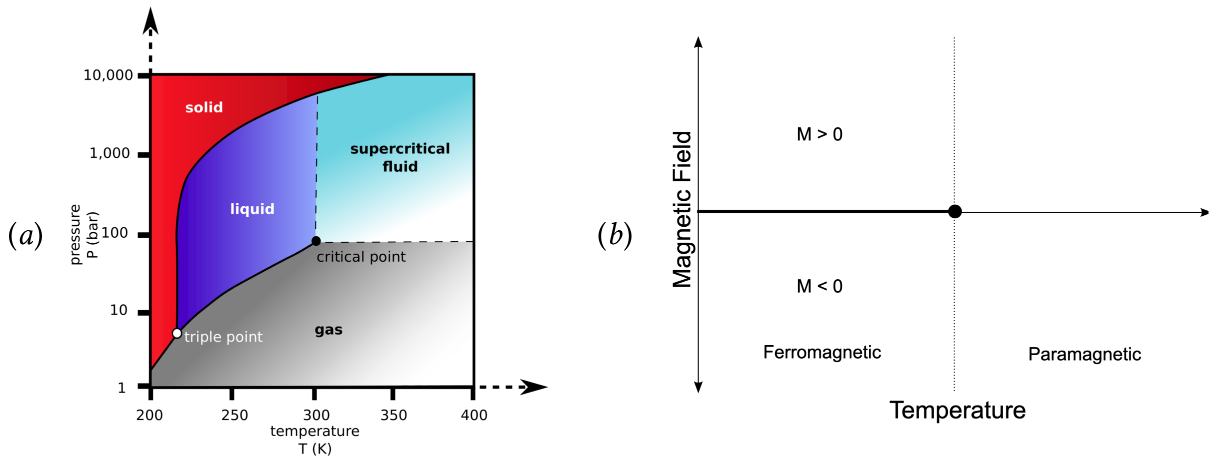


Figure 1: (a) Phase diagram for fluids **(b)** Phase diagram for uniaxial magnets

1 Why conformal symmetry?

1.1 Invitation: Critical phenomena

1.1.1 A brief review of phase diagrams

Let us start with something we observe in everyday life and have been taught in kindergarten: matter exists in different phases. These phases can be differentiated by macroscopic parameters such as viscosity and compressibility; as we are all well aware by our everyday experiences, this translates into fluids having three different phases: *solid* (high viscosity & low compressibility), *liquid* (low viscosity & low compressibility), and *gaseous* (low viscosity & high compressibility). By changing the pressure and the temperature, we can change the phase of a fluid; the landscape of all phases of a given substance is shown in *a phase diagram*, e.g. Fig. (1).

The phase diagram of a system can be n−dimensional: the dimension of the diagram reflects the number of macroscopic parameters to tune with as we move between various phases. In the case of fluids, we have two parameters: *pressure* P and *temperature* T. Similarly, if we consider the phase diagram of a uniaxial magnet, it also has two parameters to tune: temperature and *external magnetic field* h.[10] Other systems may have phase diagrams of different dimensions; for instance, [1] models biological structures as thermodynamic systems and classifies morphological patterns arising on the surface of tissues as various phases; in its application, the resultant phase diagram has three parameters, see Fig. (2).

One can naïvely think that we can always change the dimension of the phase diagram by considering other parameters; for instance, for fluids, we can also consider *molar volume* V as a parameter: the resultant graphical depiction of the phase diagram, along with its projection to various planes, can be seen in Fig. (2). It is true that such depictions can be quite useful, and one might conclude then that the phase diagram is in general of arbitrary dimension, as we can add

[10] A uniaxial magnet is a magnet whose magnetization can point in only one dimension.

[11] See also http://biomodel.uah.es/Jmol/plots/phase-diagrams/inicio.htm.

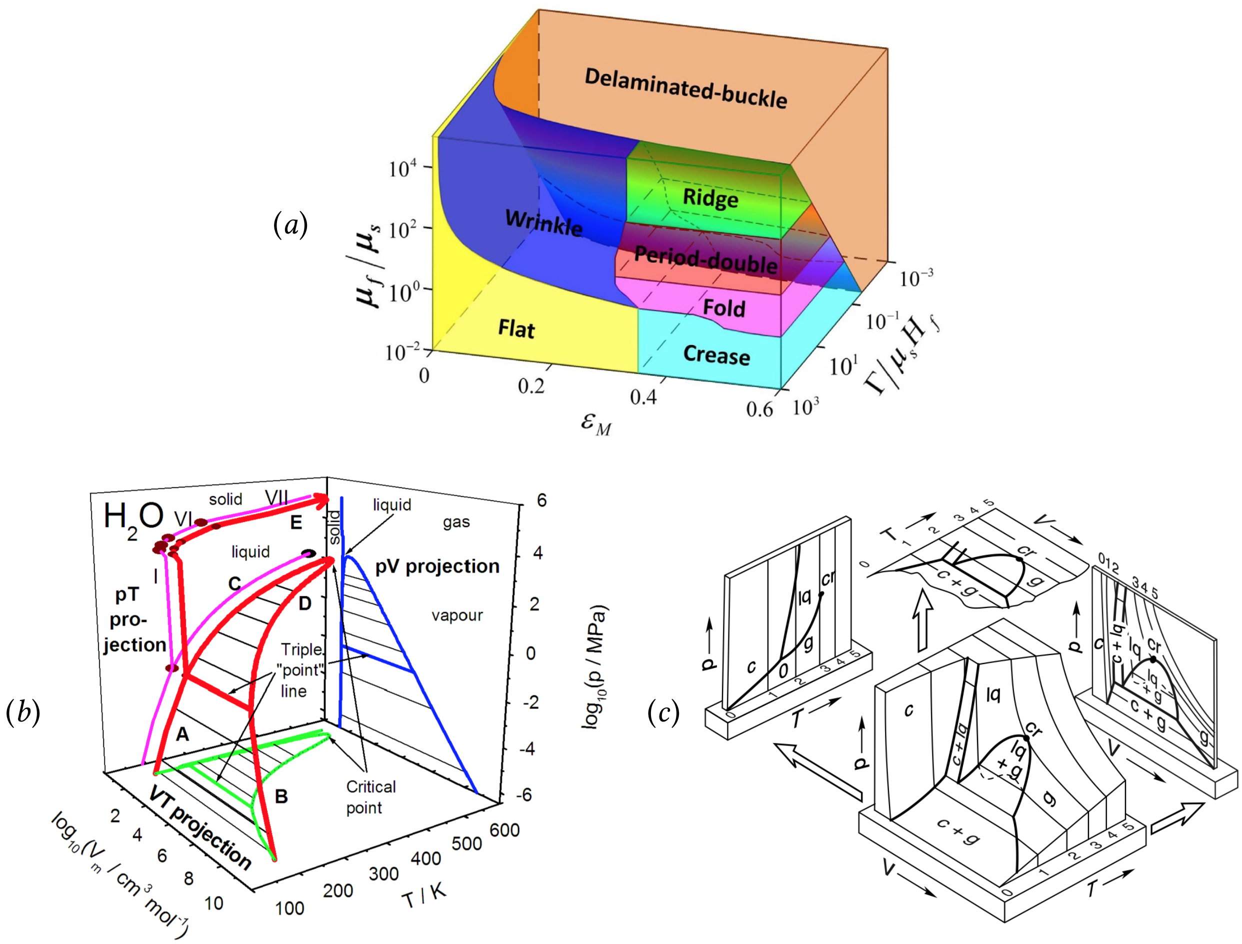


Figure 2: **(a)** A genuine three dimensional phase diagram example: various surface instability patterns induced on biological tissues [1]. One needs to tune three parameter to sweep all phases. **(b)** Three dimensional depiction of the phase diagram of water and its various projections [2]: one can still reach all phases by tuning only two parameters.[11] **(c)** An illustrative orthographic (isometric) three-dimensional pVT diagram of water [2].

more and more directions of arbitrary parameters. However, *what we mean by the dimension of the phase diagram* is the number of parameters *necessary* to tune to sweep all different phases. In the case of fluids, two parameters are sufficient to go over all phases, so we say that the phase diagram of fluids is two-dimensional, so is that of uniaxial magnets.

The dimension of the phase diagram, as a global property, may not be as so useful as *the dimension of the fixed points*, i.e. the number of parameters one needs to tune to reach a fixed point in the RG (renormalization group) framework. This is a local property, because it may differ from one fixed point to the other in the same phase diagram. In fact, the number of parameters necessary to reach a *critical fixed point* is same as the number of *relevant operators* (operators whose scaling dimension is less then the spacetime dimension) in the *conformal field theory* which describes that critical fixed point – more on that below! Note that we do not require knowledge of RG framework in these notes, so please remain comfortable if you are unfamiliar with it.

Before we dive into critical fixed points or conformal field theories, let's take a step back and try to understand the phase diagrams in Fig. (1): they consist of some regions (bulk phases), lines dividing those regions (phase transitions), and endpoints of some lines (critical points). From statistical mechanics point of view, the phase diagram consists of *non-analyticities* of the free energy[12] in thermodynamics limit,[13] but actual computation of the free energy is almost always impossible so we do not have a clear handle on the derivation of the phases from the microscopic theory: in fact, the full phase diagram of a substance *can not be computed in general* starting from the microscopic theory! This is not because of our lack of knowledge or of a small technical problem, but of *the undecidable nature of the mathematical problem itself*![14] In short, we have the general formula but it will not do any good for us if we insist on computing the phases from the microscopic theory.

Let us instead construct a *phenomenological* model for the free energy.[15] For that, we consider

[12]For those who have difficulty remembering their stat-mech courses, the free energy is the ultimate function from which all thermodynamic quantities can be derived via differentiation. I'll be rather ambiguous and do not specify *which* free energy I refer to (Helmholtz, Gibbs, Landau, etc.), because it will not matter for the qualitative discussion.

[13]Remember that the free energy F is given as

$$e^{-\beta F} = \mathrm{Tr}\, e^{-\beta H} \tag{1.1}$$

for $\beta \equiv (k_B T)^{-1}$ where k_B is the Boltzmann's constant and T is the temperature. H is the Hamiltonian and the trace denotes a sum over all degrees of freedom mentioned in H. The partition function Z is analytic as the Hamiltonian is usually an analytic function and a *finite* sum of analytic functions is itself analytic, as long as the trace operation is a summation of *finitely many* degrees of freedom. Hence, the phases and phase transitions only appear in the thermodynamic limit, where the degree of freedom becomes infinite and the trace usually turns into a *path integral*.

[14] Here by undecidable we mean that a general algorithm which is guaranteed to solve this problem in finite time cannot be found; see [3] for the case of phase diagrams. I highly recommend "Math Has a Fatal Flaw" video of the Youtube channel *Veritasium* for a broader view of the mathematical issue undecidability, along with incompleteness and inconsistency problems.

[15]The term "phenomenological" needs a little bit explanation. I use it the same way Goldenfeld uses in [4]. He has a beautiful explanation there and I would like to copy/paraphrase/edit some part of it here.

In constructing physical theories, we always use a certain level of description. Thus, in describing the long wavelength behavior of a magnet, we write down equations for the coarse-grained magnetization (here, coarse-grained means that the free energy as a function of various phenomenological variables K_i and write

$$e^{-\beta F(K_i)} = \int \mathcal{D}\phi e^{-\beta L(K_i,\phi)} \tag{1.2}$$

where $\int \mathcal{D}\phi$ denotes a path integral over the field $\phi(x)$ which represents the degrees of freedom in the theory (Knowledge of path integrals is not essential for the discussion so just think of it as an ordinary integral if you are unfamiliar with the concept.). If we started with the microscopic theory, we would instead have trace over degrees of freedom, which would potentially include discrete sets (such as spins on lattice sites), but as we are constructing a *phenomenological model*, we are coarse-graining the microscopic details and representing the degrees of freedom by a continuous field $\phi(x)$.

The $L(K_i, \phi)$ is an arbitrary function for now, which would be the Hamiltonian in the microscopic theory. To leading order, we can assume that the dominant contribution to eqn. (1.2) comes from the form of $\phi(x)$ which minimizes $L(K_i, \phi)$, hence we have

$$e^{-\beta F(K_i)} = e^{-\beta L(K_i,\bar{\phi})} \tag{1.3}$$

where $\bar{\phi}(x) = \bar{\phi}$ is a uniform field that minimizes $L(K_i, \phi)$.

This phenomenological approach can then be summarized as follows:

- Determine a measurable macroscopic variable ϕ which changes over different phases: ϕ is called the *order parameter*.
- Write down an analytic function L of ϕ in terms of the phenomenological parameters K_i. L as a function of ϕ should have the symmetries of the Hamiltonian.
- Minimize L with respect to the variable ϕ: the minimum gives the free energy!

the parameter is smoothed out below a certain detail). In describing the motion of a fluid, we write down equations of motion for the velocity field $v(r, t)$; however, the velocity field is actually a coarse-grained velocity of many particles in a small volume element in the neighborhood of the point r. In these and other examples, the variables of interest are always defined with respect to some coarse-graining process so that phenomena below some scale are subsumed somehow into the equations for the coarse-grained variables of interest.

In a phenomenological description of a system, we try to describe the behavior solely in terms of the coarse-grained variables, without reference to the microscopic physics on scales shorter than the coarse-graining length. Inevitably, the description of the coarse-grained variables introduces other parameters; for example, in a magnet, the parameters of the Landau free energy, or in a fluid, the coefficient of viscosity. These phenomenological parameters are determined by the microscopic physics: for example, the viscosity of a fluid may be calculated, with some approximation, from kinetic theory. A successful phenomenological theory contains only a finite number of such parameters (the smaller the better), and does not attempt to calculate the phenomenological parameters. These are taken to be inputs to the theory.

How does a phenomenological theory change when the microscopic physics is altered? From the above, the only possible change can be in the values of the phenomenological parameters. If new phenomenological parameters have to be introduced whenever the microscopic physics is changed, then the theory is not, by definition, phenomenological! In this sense, a successful phenomenological theory should be insensitive to the changes in the microscopic physics, although the phenomenological parameters may change.

An interesting corollary of this discussion is that all our physical theories are phenomenological, because in principle there may always be more degrees of freedom in finer and finer details!

- If there is a non-analyticity in taking the minimum, that should give us the phase transition!

This approach is called *Landau formalism*, named after *Lev Davidovich Landau*, and $L(K_i, \phi)$ is usually called *the Landau potential.*[16] Physically, Landau formalism is simply an application of the *mean field theory*: we approximate whatever it is of physical interest (such as magnetization field $M(x)$ of a magnet or density field $\rho(x)$ of the fluid) by its mean value and ignore the fluctuations!

Let us illustrate this approach for uniaxial magnets. The relevant parameter here is the magnetization $\vec{M}(x) = M\hat{z}$ of the magnet, which we take to be in the z−direction – note also that it takes its mean value M. Then the simplest form of Landau potential we can write down reads as

$$L(K_i, M) = K_0 + K_1 M^2 + K_2 M^4 - hM \tag{1.4}$$

which is analytic in M and contains the ferromagnetic coupling of the magnetic field M to the external magnetic field h.[17] We note that the truncation of higher order terms (such as M^6) is not warranted in general, however it is valid as long as Landau formalism itself is applicable to the problem at hand.[18]

As K_0 simply shifts the free energy, we can discard it.[19] Likewise we can rescale M to absorb K_2 (which needs to be a positive number for L to have a minimum), hence we have

$$L(a, M) = aM^2 + M^4 - hM \tag{1.5}$$

which gives the free energy by minimization:

$$\{\bar{M}, F(a,h)\} = \begin{cases} \left\{-\dfrac{h}{2a}, -\dfrac{h^2}{4a}\right\} + H.O.T. & a > 0 \\ \left\{\pm\sqrt{\dfrac{-a}{2}}, -\dfrac{a^2}{4}\right\} + H.O.T. & a < 0 \end{cases} \tag{1.6}$$

where we wrote down the result for weak external magnetic field h for simplicity, and $H.O.T.$ refers to terms higher order in h.

It can be checked by looking at the full form of the result that the function is analytic unless $h = 0$. If $h = 0$, then there is a cusp at $a = 0$, i.e. second derivative of $F(a)$ has a discontinuity at $a = 0$. And for $a < 0, h = 0$, there are in fact two minima of the Landau potential, hence two possible choices for the magnetization. We can summarize these findings as follows:

[16] Here I simplified the phenomenological Landau theory, interested reader may consult [4] for an excellent review.

[17] The sign needs to be negative for a ferromagnet so that parallel M and h decreases the energy, as opposed to antiferromagnets coupling $+hM$.

[18] In short, the Landau formalism is a sufficient description only above an *upper critical dimension*; physically, ignored fluctuations become important if the spacetime dimension is not high enough: for uniaxial magnets, the upper critical dimension is 4. If we are above the upper critical dimension, then the truncation of higher order terms, i.e. M^6, is appropriate because their contribution diminishes as we near the critical point. This is actually closely linked to the *renormalization* of quantum fields and suppression of irrelevant terms as the energy scale tends to infinity. Maybe I can put more explanation here or even show how this is true, i.e. t versus Λ^{-1}.

[19] Observable quantities such as susceptibility are derivatives of the free energy.

1. There is no phase transition unless $h = 0$ (F(a,h) is analytic there)

2. There are two minima of the Landau potential for $a < 0$ if $h = 0$: we have a *spontaneous symmetry breaking*. Physically, this means for $h = 0$ that there is no magnetization if a is positive and there is a magnetization $M \neq 0$ if a is negative but the sign of M can be either $\pm$. So we have the *disordered phase* for $a > 0$ and *ordered phase* $a < 0$, and there is a phase transition at $a = 0$ as a is varied at $h = 0$.

3. For $a < 0$, we have a degeneracy among the minimum at $h = 0$, but if we change h the degeneracy is lifted. Hence, if we keep $a < 0$ fixed and vary h from $h > 0$ to $h < 0$, the minimum jumps from $M > 0$ to $M < 0$: there is a non-analyticity for the whole line $a < 0$ at $h = 0$!

4. The first derivative of the free energy is discontinuous in h at $h = 0$ for any value of $a < 0$: the phase transition associated with this whole line is called *first order phase transition*!

5. The free energy and its first derivative are continuous at $a = 0$, $h = 0$: it is the second derivative that brings the non-analyticity when we change a. Hence we call the associated phase transition a *continuous phase transition*![20]

The situation is best understood with the aid of visuals, so please refer to Fig. (3).

We would like to note that Landau formalism is actually incorrect for $d \leq 4$ — see footnote 18 — and we should not use it for anything quantitative. However, the picture it depicts is actually qualitatively correct: if we take that the phenomenological constant a to have a dependence $a \sim T - T_c$, the phases and the phase transitions described in Fig. (3) indeed matches the empirical phase diagram in Fig. (1).

1.1.2 Universality, critical opalescence, and critical exponents

Why did we review the phase diagrams in the previous section: what are we trying to achieve? Well, as the author, I'm trying to motivate conformal field theories but we are still not there yet! But if we go back to our original discussion, our motivation was to understand the nature of the phase diagrams, such as those in Fig. (1). We found out that in the case of uniaxial magnets, there is a continuous phase transition at the so-called *critical point* ($h = 0$, $T = T_c$) and there is a first order phase transition along the line which connects the critical point to the origin ($h = 0$, $0 \leq T < T_c$). We might now ask how important our findings are: after all, uniaxial magnets (magnets whose magnetization can point only in 1 dimension) are highly-idealized and we may naïvely complain that these results do not have a wide range of applicability! However,

[20] In the literature, such a transition is also called a *second-order phase transition*. This follows from Paul Ehrenfest's classification of phase transitions: he proposed that *phase transitions could be classified as n^{th} order if any n^{th} derivative of the free energy with respect to any of its arguments yields a discontinuity at the phase transition.* This classification is incomplete because there are thermodynamic quantities such as the specific heat that actually diverge at some phase transitions rather than exhibiting a simple discontinuity as the Ehrenfest classification implies. Thus, we will take a more modern approach and classify phase transitions as *first-order* or *continuous* as we did above.

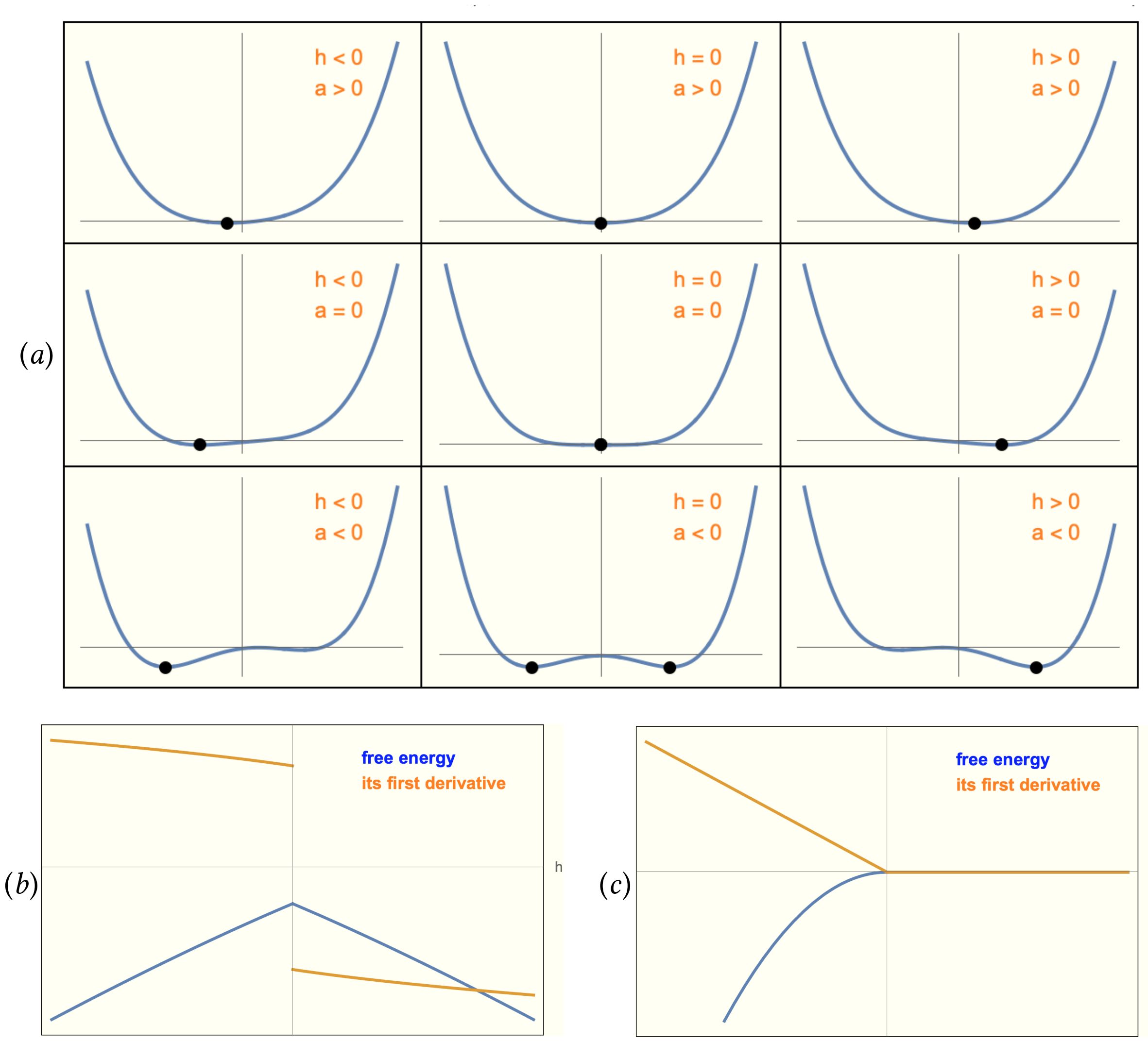


Figure 3: (a) The Landau potential $aM^2 + M^4 - hM$ and its minima with respect to the magnetization M for various values of the phenomenological variable a and the external magnetic field h. We see that there is no ordered phase at $h = 0$ if $a \geq 0$, i.e. the minimum is at $M = 0$. If there is external magnetic field, i.e. $h \neq 0$, then there is a net magnetization M which has the same sign as h as we have the ferromagnetic coupling $-hM$. On the other hand, we still have a net magnetization M even if $h = 0$ as long as $a < 0$; however, the minima are degenerate so we get a *spontaneous symmetry breaking*, i.e. M can take either positive or negative sign. However, even an infinitesimal h lifts the degeneracy so the minima *jumps* from one value to the other if h is changed continuously from $h < 0$ to $h > 0$ at $a < 0$ (this corresponds to the movement over last column): as the minimum moves discontinuously, we have an associated *first order phase transition*. On the contrary, if we move from $a > 0$ to $a < 0$ at $h = 0$ (this corresponds to the movement over middle row), the minimum changes from zero to a nonzero value continuously: there is a *continuous phase transition* at $a = h = 0$. **(b)** The landau potential at constant $a < 0$ as a function of h. We see that its first derivative has a discontinuity, showing that the phase transition is indeed *first-order*. **(c)** The landau potential at constant $h = 0$ as a function of a. We see that both the function and its first derivatives are continuous, hence the phase transition is indeed *continuous*.

this is simply not true! The description of and around the critical point is actually *universal*, and whatever information we can acquire regarding the critical point of the uniaxial magnet immediately applies to the critical point of most fluids! This concept is called *universality* and is related to the fact that *same conformal field theory* describes the critical point of both the uniaxial magnets and the most fluids!

The concept of universality can be best understood in the framework of renormalization group, and we'll review that in the next section. For now, let us focus on the implications of universality instead of deriving it:

- Critical phenomena, i.e. behavior observed at and around the critical points (i.e. continuous phase transitions) are *independent* of the microscopic theory! In other words, *local* properties of different systems around their critical points are same! This is in contrast to the *global* properties, i.e. the whole phase diagram, which varies from system to system!
- Critical phenomena depends on the *dimensionality* of the space and the *symmetries* of the order parameter! In fact, we can define *universality classes* based on these! For example, if the order parameter has $\mathbb{Z}_2$ symmetry,[21] then we call that *Ising universality class* (why Ising, more below). For instance, both the density fluctuation ρ of a fluid and the magnetization M of a magnet has $\mathbb{Z}_2$ symmetry and fall under Ising universality class: they have identical critical behavior!

To summarize, phase diagrams of numerous materials include critical points which has continuous phase transitions, and the behavior of the system at and around the critical points can be categorized into a handful of different universality classes, which means understanding the critical behavior of one system helps us understand the critical behavior of bunch of irrelevant-looking other systems! *But what exactly are we referring to by critical behavior?*

Let us investigate what happens to a fluid near its critical point: if we start with a constant volume of ethane and heat it, it will become opaque at the critical point! If we keep heating it so that it passes the critical point, it will become transparent again, see Fig. (4). This phenomenon is called *critical opalescence*. This becoming-opaque-at-critical-point feature is by no means special to ethane: we experimentally observe that fluids at their critical point become opaque in general!

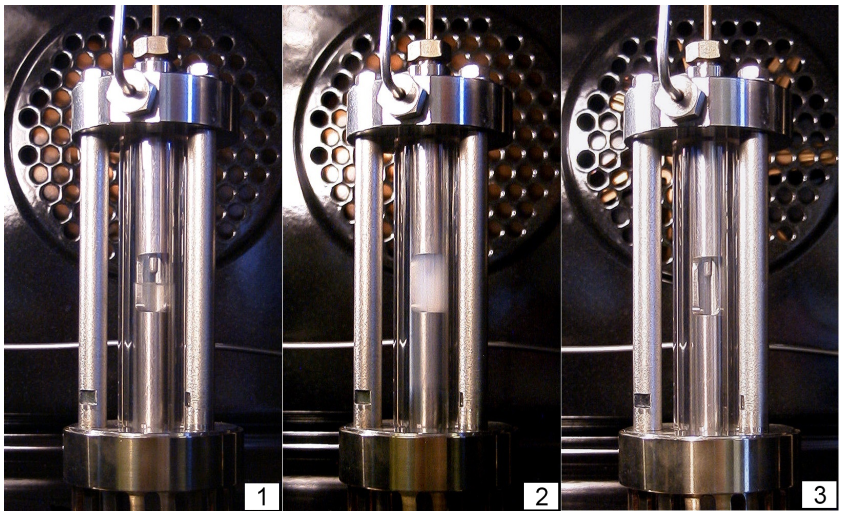
Why is that? At the simplest level, we can understand it intuitively by a phenomenological model of fluids. Let's assume that a scalar field $\rho(x)$ denotes the density of the fluid at position x.[22] By continuity of this field, we expect that the densities $\rho(x)$ and $\rho(y)$ are correlated if y is around x: if the fluid is dense at a point x, it should be dense in its neighborhood as well.[23] As y

[21] $\mathbb{Z}_2$ symmetry refers to *evenness* of a system; for instance, the function x^2 has $\mathbb{Z}_2$ symmetry because it is invariant under $x \longrightarrow -x$.

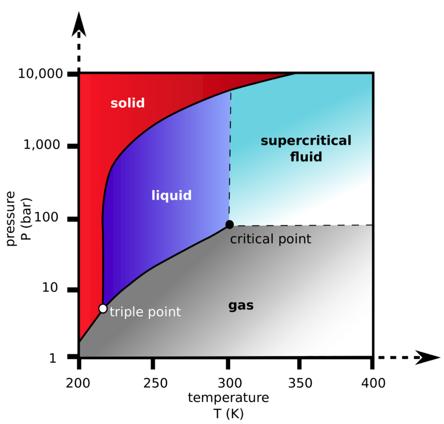
[22] As this is a phenomenological model, we are coarse-graining the microscopic details, so $\rho(x)$ can be thought of as an average density, i.e. number of molecules in the sphere of radius Λ^{-1} centered at x divided by the volume of the sphere. If we take $\Lambda \longrightarrow 0$, $\rho(x)$ becomes a constant and takes its mean value, which gets us back to the mean field theory (i.e. Landau framework). On the other hand if we take $\Lambda \longrightarrow \infty$, we get rid of coarse-graining and get back to the microscopic theory (such as a lattice model). The analysis of the effect of the change in Λ is precisely the RG framework which we will review in section 1.2.1.

[23] Technically, there can be a considerable density difference between a point and its neighboring points: maybe

(a)



(b)



(c)

Table 1. Critical properties of various solvents (Reid et al., 1987)

Solvent	Molecular mass	Critical temperature	Critical pressure	Critical density
	g/mol	K	MPa (atm)	g/cm^3
Carbon dioxide (CO_2)	44.01	304.1	7.38 (72.8)	0.469
Water (H_2O)†	18.015	647.096	22.064 (217.755)	0.322
Methane (CH_4)	16.04	190.4	4.60 (45.4)	0.162
Ethane (C_2H_6)	30.07	305.3	4.87 (48.1)	0.203
Propane (C_3H_8)	44.09	369.8	4.25 (41.9)	0.217
Ethylene (C_2H_4)	28.05	282.4	5.04 (49.7)	0.215
Propylene (C_3H_6)	42.08	364.9	4.60 (45.4)	0.232
Methanol (CH_3OH)	32.04	512.6	8.09 (79.8)	0.272
Ethanol (C_2H_5OH)	46.07	513.9	6.14 (60.6)	0.276
Acetone (C_3H_6O)	58.08	508.1	4.70 (46.4)	0.278
Nitrous oxide (N_2O)	44.013	306.57	7.35 (72.5)	0.452

Figure 4: **(a)** Ethane of constant volume being heated: when it passes the critical point, the critical opalescence is observed, i.e. it becomes *milky*! **(b)** The critical point for liquids lie at the end of the evaporation line between liquid and gaseous phases. **(c)** The critical point of various solvents. Despite the difference in critical temperature and critical pressure, the same phenomenon of critical opalescence is observed in all fluids.

gets further away from x, we expect the correlation to die down if the interactions in the fluid are short-ranged, hence we expect *the correlation function*[24] to have the form

$$\langle \phi(x)\phi(y)\rangle \sim (x-y)^p e^{-(x-y)/\xi} \tag{1.7}$$

where $\phi(x) \equiv \rho(x) - \bar{\rho}$ is the *fluctuation* in the density field and ξ is called the *correlation length*![26]

The correlation length is in general a function of the thermodynamic variables, such as the temperature. As we near a critical point by changing the temperature (or whatever the

a droplet has been spontaneously formed, or a density ripple has just passed. However, such changes depend on time and we are looking at *equilibrium* values of the thermodynamic quantities, hence such differences should be averaged out. Of course, one might ask another question: can we get to the equilibrium in the first place? This is a rather important point because equilibration near a critical point is a non-trivial process. In fact, the time to reach the equilibrium gets larger and larger as we approach the critical temperature: this phenomenon is known as *critical slowing down*! This subtlety is tangential to our main point so we can assume for simplicity that we are *sufficiently close to* the critical point to assume that we are at the critical point for most purposes, yet *sufficiently away from* it so that we could get the equilibrium in a feasible time.

[24]In these notes, we are using Dirac's bra/ket notation in extremely abusive manner! In eqn. (1.7), it refers to the expectation value with respect to Boltzman weight $e^{-\beta H}/Z$ as the probability; hence, $\langle A\rangle = Z^{-1}\int \mathcal{D}\phi A e^{-\beta H[\phi]}$ for the partition function $Z = \int \mathcal{D}\phi e^{-\beta H[\phi]}$. Thus, if $A = \phi\phi$, it becomes a correlation function of ϕ. Note that in the Landau framework we have the Landau potential L instead of the microscopic Hamiltonian H. This path integral form is extremely similar to what we use in high energy physics for quantum fields: there, we use the bra/ket notation for the similar quantity $\langle A\rangle = Z^{-1}\int \mathcal{D}\phi A e^{-S[\phi]}$ for the vacuum generating function[25] $Z = \int \mathcal{D}\phi e^{-S[\phi]}$ and the Euclidean action S — note that we are setting $\hbar = 1$ in these notes. For Lorentzian signature, we define the path integral through a Wick rotation.

We will use Dirac's bra/ket notation to denote conformally invariant structures as well, but there will not be any ambiguity: more on this later!

[25] If you haven't studied quantum field theory yet, the vacuum generating function Z is the function which can be used to compute any correlation function we like (we do this by taking functional derivatives of Z). By LSZ formalism, one can then extract scattering amplitudes or decay rates from the correlation functions; in short, full knowledge of Z is sufficient to extract most of what we are interested in computing in high energy physics.

[26]It will not be important for the present discussion but let us give some further information on the correlation function $\langle\phi(x)\phi(0)\rangle$ — the correlation function $\langle\phi(x)\phi(y)\rangle$ can be obtained from $\langle\phi(x)\phi(0)\rangle$ through the translation invariance of it, i.e. $\langle\phi(x)\phi(y)\rangle = \langle\phi(x-y)\phi(0)\rangle$.

The exact analytic form of the correlations functions are not known in general (well, blame mathematicians not physicists) and are not really computable unless the model is exactly soluble (which is extremely rare). However, we can compute their asymptotic forms (either in weak coupling or in long distance separation by perturbation theory, or at the critical point by symmetry arguments). For fluids, we have

$$\langle\phi(\mathbf{x})\phi(0)\rangle \sim \frac{1}{\xi^{d-2}}\frac{e^{-|\mathbf{x}|/\xi}}{(|\mathbf{x}|/\xi)^{(d-1)/2}}\left(1+\frac{(d-1)(d-3)}{8(|\mathbf{x}|/\xi)}+\mathcal{O}\left(\frac{1}{(|\mathbf{x}|/\xi)^2}\right)\right) \qquad \text{as } |\mathbf{x}|\longrightarrow\infty \text{ and } \xi \text{ finite} \tag{1.8a}$$

$$\langle\phi(\mathbf{x})\phi(0)\rangle \sim \frac{a^\eta}{|\mathbf{x}|^{d-2+\eta}} \qquad \text{as } \xi\longrightarrow\infty \tag{1.8b}$$

Here d is the dimension of the space and a is a microscopic length scale (such as lattice spaceing). See [5] for details on the first formula. The second formula is the *definition* of the critical exponent η: more on this below.

The Fourier transform of the two point function gives the *structure factor* $S(\mathbf{k})$ which is proportional to the intensity of the light scattered through an angle θ off the fluid where $|\mathbf{k}| = \frac{4\pi}{\lambda}\sin(\theta/2)$ and where λ is the wavelength of the incident light. At the critical point, this then means intesity$\sim |\mathbf{k}|^{\eta-2}$,[27] indicating it diverges as $|\mathbf{k}|\longrightarrow 0$ for $\eta < 2$, explaining the critical opalescence [4]. For a detailed discussion, see [6].

[27]This follows from dimensional analysis: $\int d^d x e^{-i\mathbf{k}\cdot\mathbf{x}}|\mathbf{x}|^{-d+2-\eta} \propto \int\limits_0^\infty |\mathbf{x}|^{1-\eta}e^{-i|\mathbf{k}||\mathbf{x}|\cos\theta}d|\mathbf{x}| \propto |\mathbf{k}|^{\eta-2}\int\limits_0^\infty y^{1-\eta}e^{-iy\cos\theta}dy$

relevant variable is), the correlation length increases: in fact, at the critical point itself, the correlation length becomes infinity. Hence, the interactions in the fluid becomes long-ranged, and the fluctuations become rather strong, scattering various wavelengths of light and turning the substance into an opaque form!

To summarize, while investigating the critical behavior, we found out that the correlation length diverges at the critical point. In fact, we have a whole family of divergent quantities at the critical point, and it would be insightful for us to go over them briefly. However, to understand them better, let us first list what we have defined so far and define further quantities:[28]

- t – reduced temperature[29]
- J – source field (e.g. reduced pressure for fluids and the external magnetic field h for magnets)
- f – free energy
- C – specific heat
- ϕ – order parameter (e.g. reduced density for fluids and magnetization M for magnets)
- χ – response function (e.g. compressibility for fluids and susceptibility for magnets)
- ξ – correlation length

Observable parameters such as specific heat C or compressibility χ of a fluid are in general complicated functions of thermodynamic variables; however, *they show a universal behavior and diverge near a critical point as a power law:*

$$C \propto |t|^{-\alpha} \ , \quad \chi \propto |t|^{-\gamma} \ , \quad \xi \propto |t|^{-\nu} \tag{1.9}$$

where the exponents α, γ, ν are called *critical exponents*![30,31] These exponents form the set of parameters that describe the critical behavior.

Earlier, when we looked at the implications of universality in page 13, we stated that the critical phenomena are independent of the microscopic theory and that it depends only on the *universality class* of the substance. What we meant there is precisely the values of these critical exponents: they *are* universal! For instance, we list examples of several fluids whose critical point lie in *Ising Universality Class* in Table (1). Despite the wild differences in critical temperature, they all exhibit the same exact behavior around the critical point with the critical exponents

$$\alpha \approxeq 0.11 \ , \quad \beta \approxeq 0.33 \ , \quad \gamma \approxeq 1.24 \ , \quad \delta \approxeq 4.79 \ , \quad \eta \approxeq 0.04 \ , \quad \nu \approxeq 0.63 \tag{1.10}$$

Note that these are the critical exponents in three dimensional space, hence are for *3d Ising Universality Class*: universality class does depend on the dimension of space hence these numbers change if we change the dimension of the space.

[28] Wikipedia does a good job at defining and listing them, see https://en.wikipedia.org/wiki/Critical_exponent.

[29] When we say reduced for a thermodynamic quantity x, we mean $\frac{x-x_c}{x_c}$ where x_c is its value at the critical point, e.g. reduced temperature $t \equiv \frac{T-T_c}{T_c}$.

[30] We have actually more critical exponents. If $t < 0$, we can also define $\phi \propto (-t)^{\beta}$. Likewise, if $t = 0$, we define $J \propto \phi^{\delta}$ and $\langle \phi(\mathbf{x})\phi(0) \rangle \propto |\mathbf{x}|^{2-d-\eta}$ (see eqn. (1.8b)).

[31] The critical exponents α, γ, ν have different values depending on $t > 0$ or $t < 0$.

Table 1: Data of various fluids for their critical points at the end of liquid-vapor transition

Substance	T_c (K)	ρ_c (kg/m^3)	**Source**
Deuterium (D_2O)	644	356	D M Sullivan et al, 2000
Sulfur hexafluoride (SF_6)	219	739	Haupt et al, 1999
Carbon dioxyde (CO_2)	304	468	Damay et al, 1998
Difluoromethane (HFC-32)	351	424	Kuwabara et al, 1995
Pentafluoroethane (HFC-125)	339	570	Kuwabara et al, 1995
Dinitrogen (N_2)	126	314	Pestak et al,1983
Neon (Ne)	44	484	Pestak et al,1983

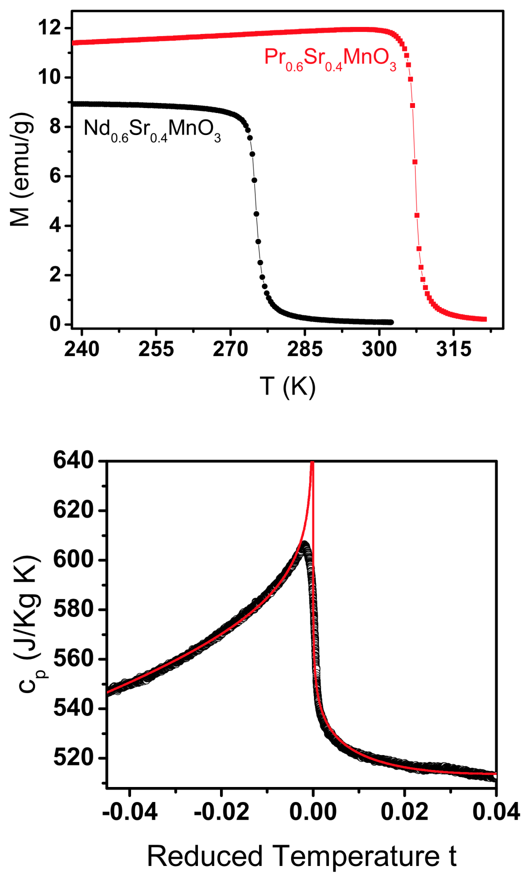
As we stated earlier, the uniaxial magnet at its Curie temperature with zero external magnetic field is also at the critical point described by Ising Universality Class, hence it exhibits exactly the same critical behavior with the same critical exponent. As we show in Fig. (5), various magnets have different Cruie temperatures, but the divergence they exhibit is same.

1.1.3 Summary

We are now at page 17 but haven't really motivated why conformal field theories are worth studying in statistical physics, apart from mentioning its name here and there. We will get to that in the next section, but before that, let's finish setting the scene by summarizing our discussion so far.

- Matter has different phases (duh!) and all phases can be shown in an n–dimensional diagram called *phase diagram* where axes are thermodynamic variables such as temperature, pressure, magnetization, etc.
- Phase diagrams have *non-analyticities* in them, which divide them into different regions (phases). These non-analyticities may (but usually do) include a phase-transition which ends at a critical point.
- Phase transition lines are usually associated with a *first-order phase transition*, i.e. at least one of the first derivatives of the free energy is discontinuous! Such phase transitions can be explained by spontaneous symmetry breaking, but this *does not have to* be the case! For instance, in magnets, this is the case: above and below the transition line are two different ground states of the potential due to breaking of the $\mathbb{Z}_2$ symmetry in the order parameter (magnetization M). In contrast, two sides of the evaporation line in fluids (liquids and gaseous) have the same Euclidean symmetry.[32]

[32] It is interesting to note that all phases of matter are described by the same microscopic Hamiltonian (and the same partition function), but they may have different symmetries. For instance, in solids, the translation symmetry is broken (atoms are locked in specified positions in a lattice), so the symmetries of the fluid changes over the melting line. As a side note, a corollary of this is that we expect the melting line to extend indefinitely (unlike the evaporation line) because solids have a different symmetry group than liquids and that cannot change analytically! In contrast, liquids and gaseous already have the same symmetry group and it is perfectly fine for the evaporation line to end at a point, i.e. critical point.



Material	Curie temperature (K)
Iron (Fe)	1043
Cobalt (Co)	1400
Nickel (Ni)	627
Gadolinium (Gd)	292
Dysprosium (Dy)	88
Manganese bismuthide (MnBi)	630
Manganese antimonide (MnSb)	587
Chromium(IV) oxide (CrO_2)	386
Manganese arsenide (MnAs)	318
Europium oxide (EuO)	69
Iron(III) oxide (Fe_2O_3)	948
Iron(II,III) oxide ($FeOFe_2O_3$)	858
NiO–Fe_2O_3	858
CuO–Fe_2O_3	728
MgO–Fe_2O_3	713
MnO–Fe_2O_3	573
Yttrium iron garnet ($Y_3Fe_5O_{12}$)	560
Neodymium magnets	583–673
Alnico	973–1133
Samarium–cobalt magnets	993–1073
Strontium ferrite	723

Figure 5: The behavior of magnets near their critical point. On the left, we see the transition from ordered to disordered phase (top) and the divergence of the specific heat (bottom) for magnets in $3d$ Ising Universality Class [7]. On the right, Curie temperature (i.e. critical temperature for magnets) for various magnetic substances are listed, see Wikipedia and the references therein: https://en.wikipedia.org/wiki/Curie_temperature.

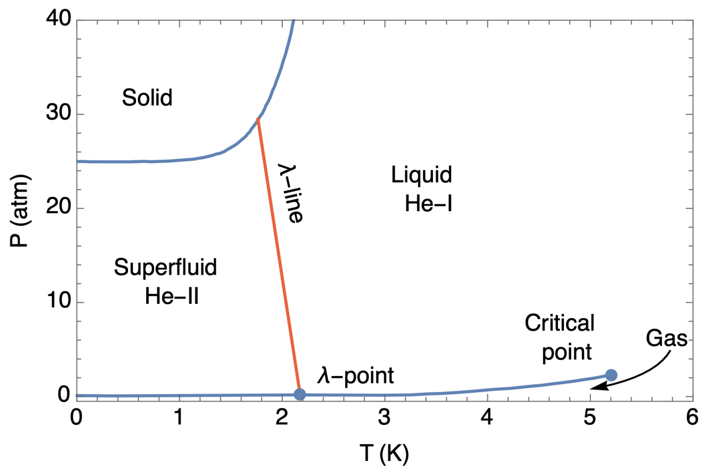


Figure 6: An example of critical line: the phase transition for ^{4}He along its whole λ–line is a continuous phase transition. Figure taken from [8].

- Critical points at the end of phase transition lines have *continuous phase transition*:[33] the correlation length at such points diverge! The critical *surfaces* does not have to be points (despite we have been talking about critical *points* so far), for instance, in ^{4}He, there is a superfluid transition along the so-called λ–line and the phase transition along the whole line is a continuous phase transition, see Fig. (6).

- The behavior of systems around their critical points are independent of the microscopic details of the system but rather on the symmetries and the dimensionality of the space. This in effect divides critical phenemona into *universality classes*: understanding the critical behavior of one system is sufficient to understand the critical behavior of all systems in the same universality class (e.g. understanding the behavior of a uniaxial Nickel magnet at its Curie temperature is sufficient to understand the behavior of compressibility of water or heat capacity of carbon dioxide at their critical points)!

In the next section, we will see how these results relate to conformal field theories!

1.2 Connecting critical phenomena to conformal field theories

1.2.1 A brief review: renormalization group formalism

In the previous section, we concluded with the universality of the critical phenomena and how the correlation length diverges at the critical point. In this section, we will try to see why this is so and how this relates to conformally invariant theories.

As we stated earlier, the concept of universality can be best understood in the framework of renormalization group flows but I'll assume that the reader may not be familiar with it so I'll

[33] I'm not sure if there are exceptions, i.e. if there are endpoints of some transition lines where no continuous phase transitions are associated with those points.

briefly summarize the main points of it qualitatively.

Let's remember our phenomenological model for the free energy in eqn. (1.2)

$$e^{-\beta F(K_i)} = \int \mathcal{D}\phi e^{-\beta L[K_i,\phi]} \tag{1.11}$$

This is a phenomenological model, so the field $\phi(x)$ is coarse-grained below a certain length scale, say 1 micrometer. This means we cannot resolve anything below 1 micrometer but do so above it.

We have seen that the critical phenomena (such as critical opalescence in fluids) happen at a rather macroscopic scale, so let's coarse-grain all the fluctuations of the order parameter $\phi(x)$ between 1 micrometer and 1 millimeter. To do so, we separate *all functions* $\phi(x)$ as those which only fluctuate above 1 millimeter (call them $\phi_\ell(x)$) and those which fluctuate between 1 millimeter and 1 micrometer (call them $\phi_h(x)$).[34] As $\int \mathcal{D}\phi$ is a path-integral, it becomes $\int \mathcal{D}\phi_h \int \mathcal{D}\phi_\ell$,[35] hence we now have

$$e^{-\beta F(K_i)} = \int \mathcal{D}\phi_\ell e^{-\beta L'[K_i,\phi_\ell]} \tag{1.12}$$

for

$$e^{-\beta L'[K_i,\phi_\ell]} = \int \mathcal{D}\phi_h e^{-\beta L[K_i,\phi]} \tag{1.13}$$

If all possible terms allowed by the symmetries of the Hamiltonian are included in the effective potential L, then L' should not have any new terms; hence we can write $L'[K_i,\phi_\ell] = L[K_i',\phi_\ell]$, thus we have

$$e^{-\beta F(K_i)} = \int \mathcal{D}\phi e^{-\beta L[K_i,\phi]} = \int \mathcal{D}\phi_\ell e^{-\beta L[K_i',\phi_\ell]} \tag{1.14}$$

where the first integration is coarse-grained over 1 micrometer and second over 1 millimeter. Of course, there is nothing special about these values and we can generally write

$$e^{-\beta F(K_i)} = \int_\Lambda \mathcal{D}\phi e^{-\beta L[K_i(\Lambda),\phi_\Lambda]} \tag{1.15}$$

where the integration is coarse-grained over distances Λ^{-1} and the phenomenological constants in the Landau potential are now functions of the parameter Λ.

What can we do with this parameter Λ? On one hand, we can try to take it to infinity (effectively removing any coarse-graining process): this indicates we are trying to probe the microscopic theory and our phenomenological model will eventually break down; indeed, in statistical physics, we cannot take Λ bigger than the inverse lattice spacing, i.e. a^{-1}. In high energy physics, we do not have a natural microscopic length scale (assuming we are focusing on standard model without gravity) hence we may naively try to take Λ to infinity. In this limit,

[34] Here ℓ and h refer to low and high frequencies respectively. As a side note, it is actually a lot more straightforward to do this separation in Fourier space

[35] For anyone not familiar with the path-integrals, this may seem odd: we divided the whole integration to two parts, so they may expect $\int \mathcal{D}\phi \longrightarrow \int \mathcal{D}\phi_h + \int \mathcal{D}\phi_\ell$. One intuitive (yet not necessarily rigorous) way to see that this isn't the case is to consider the fact that the general function ϕ has both low and high frequency parts so it is not like $\int_a^b dx = \int_a^0 dx + \int_0^b dx$ but rather like $\int dx dy = \int dx \int dy$.

our theory may or may not fail; it is the experiments that ultimately would invalidate (or not) our models.[36] On the other hand, we can take Λ to zero: this means we are *integrating-out* all degrees of freedom, leaving us with the ultimate IR behavior of the theory![37]

The change of Λ by *integrating-out* high frequency modes[38] as we have demonstrated above is called *renormalization group flow*, i.e. RG flow.[39] The main take-away of this argument is that under the change of the *regularization parameter* Λ, the phenomenological parameters K_i change, i.e. $K_i = K_i(\Lambda)$.

The RG flow as we have briefly introduced above does nothing by itself: it does not compute anything, nor does it give any obvious simplification. In practice, one *combines* a computational technique with RG flow to *improve* that technique; for instance, one can combine RG flow with the perturbation theory to yield *RG improved perturbation.* We do not need to know anything about these in these notes.[40] For our purposes, all that matters is that the phenomenological constants K_i in our Landau model changes with the regularization scale Λ.

Let's try to extend this formalism from statistical mechanics to high energy theory: in a d-dimensional Euclidean QFT,[41] the vacuum generating function Z is given as[42]

$$Z[K_i] = \int \mathcal{D}\phi e^{-\int d^dx \mathcal{L}[K_i,\phi(x)]} \tag{1.16}$$

[36] If our model is so-called *nonnormalizable*, it will surely fail as we keep increasing Λ; in fact, we may be able to predict beyond which point our model is unreliable. An example of this is Fermi's four-fermion theory of the weak interaction which breaks down around $\Lambda \sim 300$ GeV. On contrary, if our model is so-called *renormalizable*, then it is self-consistent in the phenomenological sense and hence is insensitive to microscopic theory: we can safely take Λ to ∞. *However*, the resultant theory is *not necessarily the correct model for the phenomena it is written to describe.* In other words, despite being self-consistent at any scale Λ, *the predictions* of a renormalizable theory may diverge from the experimental values after a certain Λ, which indicates a modification of our model is necessary beyond a certain Λ. The fact that renormalizable theories cannot tell their regime of validity, along with the other fact that nonnormalizable theories necessarily break down beyond an energy scale, leave us with one firm conclusion: *we can never write down a physical theory which is guaranteed to work at all length scales!*

[37] As is commonplace in high energy and statistical physics, we will refer the behavior of a a system at very low and very high energies as its IR and UV behavior, where they stand for infrared and ultraviolet (i.e. low and high frequency ligh).

[38] Equivalently coarse-graining low distance details.

[39] In the full RG flow, we actually also rescale after coarse-graining, and do these infinitesimally so that there is *indeed* a continuous flow of coupling constants under RG.

[40] How RG improves perturbation theory is a standard topic, and one can consult their favorite QFT book. In short, instead of having a perturbation expansion with the small parameter $g \log\left(\frac{p^2}{m^2}\right)$, we do a perturbation expansion with the small parameter g_μ which is computed by RG flow of the parameter g. The second expansion is improved because log terms can get large for large momenta p^2.

[41] The argument is same for Lorentzian QFTs as well.

[42] See footnote 25 for a reminder on the vacuum generating function Z.

where $\phi(x)$ collectively denotes all fields, and K_i are usually called *coupling constants*.[43] We do not have a natural regularization parameter here as we did in the Landau theory (or inverse lattice spacing a^{-1} as in statistical mechanics), so the integration is naively over all length scales. However, this is too ambitious: as we stated in footnote 36, we can actually never write down a theory which is *guaranteed* to work at all length scales (according to our current understanding). Thus we have two choices: if our Lagrangian is *nonnormalizable*, we already need a momentum cut-off Λ because our model is guaranteed to fail above a certain threshold. If our Lagrangian is *renormalizable*, then in principle we do not need a cut-off for consistency,[45] though the model may still break down (i.e. cannot explain experiments) after a certain energy threshold Λ. In short, for both cases, it is logical to keep an arbitrary regularization scale Λ in the integration, reminding us that we are not integrating over modes with higher frequency than arbitrary Λ:

$$Z[K_i] = \int_\Lambda \mathcal{D}\phi e^{-\int d^d x \mathcal{L}[K_i(\Lambda), \phi_\Lambda(x)]} \tag{1.18}$$

Such theories are called *effective field theories*, and we just argued that all quantum field theories are in a sense effective field theories![46]

Let's get back to our main take-away both in statistical mechanics and high energy physics: the coupling constants in the Lagrangian (or Hamiltonian in statistical mechanics) change as we change the regularization scale Λ. But what exactly does that mean?

To understand the situation better, let us consider a single scalar field $\phi(x)$ in three space dimensions with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi(x)^2 - \frac{\lambda}{4!}\phi(x)^4 - h\phi(x) \tag{1.19}$$

[43] In general, the Lagrangian[44] has the form

$$\mathcal{L}[K_i, \phi(x)] = \sum_i K_i \mathcal{O}_i(x) \tag{1.17}$$

where $\mathcal{O}_i(x)$ are *operators* in the theory and K_i are coupling constants, i.e. terms that couple the operators to the Laplacian. The operators can be of various forms, simple examples are $\mathcal{O}_1(x) = \phi(x)$ and $\mathcal{O}_2(x) = \phi(x)^2$. If the theory is free or we are perturbing around a free theory, such operators are not really independent (I can imagine some readers already shouting *"one is the square of the other!"*). However, if the theory is strongly coupled, it may not be possible to write down $\mathcal{O}_i(x)$ simply as $\phi(x)$ or $\phi(x)^2$; the very same operators that were $\phi(x)$ and $\phi(x)^2$ in the weak coupling regime can get radically different in the strong coupling regime: we will see an example of that in *the 3d Ising model* below.

[44] In these notes we are abusing the language and refer both to the Lagrangian and the Lagrangian density as Lagrangian for short, which is a fairly common practice in the field.

[45] There are infinities if we don't regularize our integral, but in a normalizable theory these infinities can be removed. So, in effect, even if we put a cut-off Λ, we can consistently take $\Lambda \to \infty$; but we do not need to put a cut-off in the first place (we can regularize the infinities in other ways, such as dimensional regularization).

[46] A related concept here is *the decoupling principle*. It basically states that if we have a very heavy particle in our model, its existence in low energies is only reflected through the modification of the coupling constants of lighter particles. In other words, in low energies, the heavy particles *decouple* from the effective description of the phenomena (in the jargon, their propagators are said to be *frozen*, or that they can be *integrated out*). As a corollary of the decoupling principle, we can never know certainly if there are heavier particles which have not been yet discovered by doing experiments at low energies; because if there are such particles, their contribution to the observables are indistinguishable from their absence with a different set of coupling constants of lighter particles.

We are describing a *massive* scalar field which self-interacts with the interaction strength λ. We observe three things:

1. We could have written $h(x)$ instead of h and treat the magnetic field $h(x)$ as a classical background field (i.e. it does not have a kinetic term).[47] Rather, h is simply a coupling constant in this setting, such as m^2 and λ.
2. The Lagrangian has $\mathbb{Z}_2$ symmetry (same symmetry with Ising Universality Class) if $h = 0$, i.e. it is invariant under $\phi(x) \longrightarrow -\phi(x)$.
3. We could add other terms consistent with the $\mathbb{Z}_2$ symmetry to the Lagrangian, i.e. $\phi(x)^6$. The absence of such terms are yet not warranted but we'll argue later that it is a consistent and reasonable omission.

Let us now apply an RG transformation: as we discussed above, this will change the coupling constants (in this case m^2, λ, and h) as we iterate the RG flow. If we start with *small* m^2, λ, and h for $d > 4$,[48] we can show that λ gets smaller with RG: we call such coupling constants *irrelevant*.[49] On the other hand, m^2 and h are relevant coupling constants, i.e. they get larger under iterations of RG flow.[50] In the coordinate system where the coupling constants are axes, which is actually precisely the phase diagram we have been analyzing in the previous sections, relevant coupling constants form the directions that need to be tuned to reach the critical point; for instance, for $d > 4$, $(h, m^2, \lambda) = (0, 0, 0)$ describes the critical Ising model (m^2 plays the role of the critical temperature, much like a did in eqn. (1.5) for the Landau description of the Ising model) and we need to tune h and m^2 to reach this point ($\lambda \longrightarrow 0$ by itself under RG). On the other hand, if $d < 4$, things get a little bit complicated: λ is also relevant now hence $(h, m^2, \lambda) = (0, 0, 0)$ no longer describes the Ising universality class: it now has 3 relevant directions hence 3 parameters need to be tuned (unlike the fluids or uniaxial magnet).[51] However, if d is *very close to 4* (say

[47] From QFT point of view, that makes much more sense as we can use $h(x)$ as a source term to take functional derivatives of the vacuum generating function Z to construct correlation functions. Our main point here is not that so we are simplifying the situation for a clearer discussion.

[48] In these notes, d always denotes the dimensionality of the space unless stated otherwise.

[49] More correctly, we call a coupling constant (or the direction it forms on the phase diagram) *irrelevant for the point p* if RG flow takes the coupling constant *towards* p. For instance, for the point $\lambda = 0$, the coupling constant λ is irrelevant because RG takes λ closer and closer to 0. On the contrary, if the RG transformation takes the value *away* from point p, then that direction is said to be *relevant for the point p*. For instance, for the point $h = 0$, the coupling constant h is relevant because RG takes any nonzero h further and further away from 0. If RG moves a variable x neither towards to nor away from a point p, then x is said to be *a marginal direction* for point p.

For the concept of being relevant and irrelevant to be mutually exclusive for a point p, the RG transform should not *reach* to that point in finitely many iterations; otherwise, it can move towards the point p in the first n iterations and move away from it in the remaining iterations, making the direction both irrelevant and relevant! Thus, the points p that we referred above should be *fixed-points*: points that RG transformation reaches in *infinitely* many iterations. Also, if we are *exactly at* those fixed points, then we stay at them under RG, i.e. they are *invariant points of RG flow*!

In summary, relevant and irrelevant directions are respectively the unstable and stable directions of the fixed point under perturbation.

[50] See footnote 49.

[51] Remember that we should tune temperature T and external magnetization h for magnets and temperature T and pressure p for fluids.

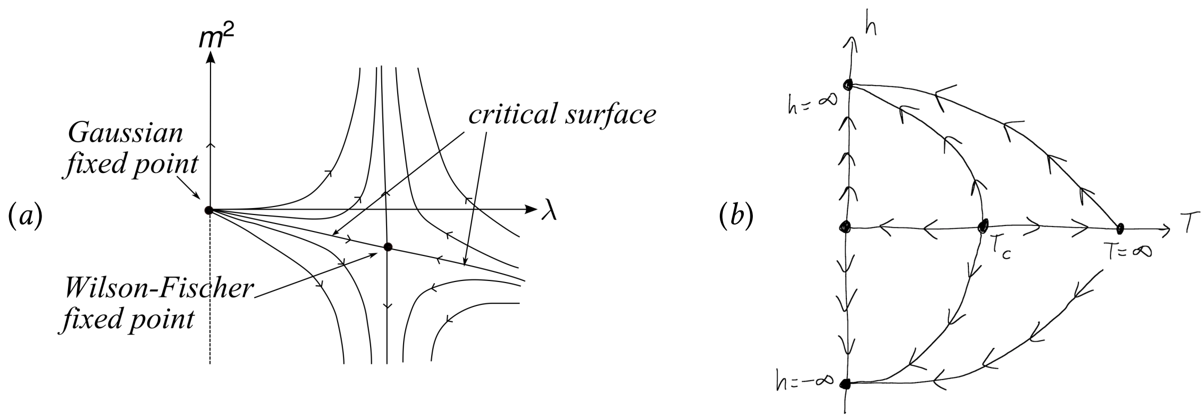


Figure 7: (a) Phase diagram of the $3d$ scalar field with $\mathbb{Z}_2$ symmetry, with the RG trajectories shown. We see that the coupling constants move towards four possible fixed points under RG transformation: two sinks at $m^2 = \pm\infty$, one Gaussian fixed point, and one interacting fixed point called *Wilson-Fischer fixed point.* **(b)** Phase diagram for uniaxial magnet , with the RG trajectories shown (see also Fig. (1)). There are five fixed points: one Gaussian fixed point at $h = T = 0$, two sinks at $h = \pm\infty$, one interacting fixed point at $\{T, h\} = \{T_c, 0\}$, and one non-interacting fixed point at $T = \infty$.

$d = 3.99$), then λ is barely relevant and we can expect a new critical point close to $(0, 0, 0)$, which corresponds to Ising Universality class.[52] In fact, in their 1971 paper titled *Critical exponents in 3.99 dimensions* [10], Wilson and Fisher found such a point and computed the critical exponents as expansion of the small parameter $\epsilon = 4 - d$. One can reach this critical point (1) by tuning m^2/λ[53] so that one is on the *critical surface* and then (2) by applying the RG flow which will take the system away from the Gaussian point and towards this new point: see Fig. (7) for the diagrammatic illustration of this procedure along with the RG flows on the phase diagram of the $3d$ scalar.

Thus, in summary, the description of a system (the values of coupling constants) change by at which scale we look at the system (different regularization Λ), and the modification in the description of the system as we change our perspective can be shown as a flow in the phase diagram for the system: this is called *RG flow* and Fig. (7) illustrate this for a scalar field and uniaxial magnet.

Let's comment more on Fig. (7a). We observe that there are four fixed points for the RG flow; *the Gaussian fixed point*, i.e. $\lambda = h = m^2 = 0$,[54] two sinks[55] at $m^2 = \pm\infty$, and *the Wilson-Fisher fixed point* at $(m^2, \lambda) = (m_c^2, \lambda_c)$. In comparison, Fig. (7b) has five fixed points: one Gaussian, two sinks, one critical fixed point (at Curie temperature), and one so-called discontinuity fixed point

[52] One motivation for this is called *cross-over* phenomena. Basically it says that despite λ is relevant, we still spend a lot of time around the critical point during RG (as λ is barely relevant), hence we can effectively treat λ as an irrelevant parameter. The end of section 5.4 of [9] touches on this, in fact simply see the whole chapter for a thorough discussion.

[53] The precise value of this ratio to reach the critical point is not unique and depends on the chosen UV regularization of the theory [11].

[54] It is called so because the Lagrangian is then Gaussian.

[55] These fixed points are called sinks because they do not have any relevant direction: a system never moves away from sink under RG flow!

at $T = \infty$. We review types of different fixed points in Table (2) at the end of this section.

Let's take a step back and review our situation: we stated above that the $3d$ scalar theory at Wilson-Fisher fixed point is in $3d$ Ising Universality Class, but what exactly is the Lagrangian for this theory? Can we compute the critical exponents analytically and exactly? Even before all these, what exactly do we know about this point?

Our questions are rather valid and to the point (pun intended), but the answers will be disappointing. We remind the reader that we started *around* the Gaussian fixed point $m^2 = \lambda = h = 0$, and said that Wilson and Fisher found a new fixed point nearby. This procedure is in fact no coincidence: we have no way to map out the RG flow exactly and completely, and we have no way of doing actual computations (like literal computation of the path integrals and such) unless we are nearby the Gaussian fixed point. Thus, necessarily, we start around the Gaussian fixed point and look for new fixed points as Wilson and Fisher did.

The previous paragraph is rather disturbing: it basically means we somehow stumbled upon the Wilson-Fisher fixed point, and maybe there are a lot more fixed points that are unbeknownst to us! Indeed, by the conventional way of doing things (which I'll list below), there is no way we can guarantee that we do not have new fixed points, and hence there is no guarantee that Wilson-Fisher fixed point is the correct one to describe the $3d$ Ising Universality Class. The more modern[56] way of mapping the fixed points of a phase diagram is in fact to *look for* conformal field theories with correct number of relevant operators and the correct symmetry group (more on this below). For instance, it is shown in [12] by conformal bootstrap techniques that *"...3D Ising CFT is the only $\mathbb{Z}_2$−symmetric CFT in 3 dimensions with exactly two relevant operators..."*, which indicates that if the Wilson-Fisher fixes point is describing the $3d$ Ising CFT, then there can be no other fixed points with two relevant operators which describe a $\mathbb{Z}_2$−symmetric CFT in the whole phase diagram![57]

Before detailing this conformal bootstrap approach, we need to fully appreciate why this approach is powerful *and necessary* in the first place! To do so, let's give some general remarks and comment on a few points:

- Strongly coupled quantum field theories (or statistical models) are not soluble![58] Not only do we not have the required mathematical tools to solve them, but also do we not have the mathematical framework to *hope* to solve them in future![59]

- If we cannot analytically solve the problem, we can try numerically solving it on a computer! For that, we put the theory on a lattice (we need a discretization to employ the computer) and simulate its behavior. The branch of the physics that adopts this approach

[56] This adjective may be unfair to other methods, but with my limited knowledge this is how I see it.

[57] There is a always a chance that there is a fixed point with scale symmetry without conformal symmetry, but this is quite unlikely: we'll elaborate on this later.

[58] Of course, this is a generic statement and there are exceptions. However, to my knowledge, they are either in low dimensions (such as 1d or 2d models) or have additional symmetries (such as supersymmetry of conformal symmetry). Put references here

[59] We are talking about the impossibility of finding an algorithm to solve them *generically* because of the undecidable nature of the mathematical problem (see footnote 14).

is called *lattice simulations*.[60] Despite its relative success, this approach is limited because (a) the computing power is limited and (b) the complexity of the problem can be rather high to utilize this approach![61]

- We can solve free theories.[62] As we cannot solve strongly coupled theories, we can assume weak-coupling and do perturbation theory: this is basically what we do for most of the standard model![63]

- If we cannot take the coupling constants to be small, we can try to vary other parameters and try to find a free theory this way. For instance, instead of considering a scalar field $\phi(x)$, we can consider N copies of it: most of the time the theory is free if $N \longrightarrow \infty$; we can then (1) do an expansion around that free theory, (2) compute corrections order by order in $1/N$, (3) assume that truncation at some order is fine, (4) take $N = 1$ (or whatever the original number of copies) at the end. This procedure is called *large N expansion*, see [14] for a 200-page review of this concept!

- Another parameter that we can vary to approach a free theory is the dimension of the space itself! As we noted above, there exists a fixed point (Wilson-Fisher fixed point) if $d < 4$ and this fixed point merges with the Gaussian fixed point as d approaches 4. As we know how to deal with the Gaussian fixed point, we can take $\epsilon = d - 4$ to be small so that the Wilson-Fisher fixed point is close enough for us to use our Gaussian fixed point knowledge to analyze that critical point (and this is how that critical point was discovered in the first place)! This approach is called *epsilon expansion*, and in a sense similar to large N expansion: we expand in ϵ, compute first few terms, truncate the expansion, and take $\epsilon \longrightarrow 1$ to get the critical point of $3d$ Ising model.[64]

The items above show how desperate our situation actually is as theoretical physicists! Nevertheless, do not despair! We can still make progress in the realm of strongly coupled theories by realizing that fixed points are actually scale-invariant, hence they enjoy a higher symmetry than the rest of the phase diagram!

Let me expand on the scale invariance. Under RG transformation, the correlation length ξ scales linearly, i.e. it behaves as $\xi \longrightarrow \ell\xi$. Thus, at the fixed points of the RG transformation, ξ is either 0 or ∞. As a corollary, *any point at the phase diagram where the correlation length diverges is a fixed point of the RG flow!* Hence, by our discussion in the previous section we can conclude that the critical phenomena happen at some of the fixed points of the RG flow! *In short, understanding the nature of fixed points and the emergent scale invariance is invaluable to understanding critical phenomena!*

[60] Add some sources here

[61] For instance, the problem of solving $3d$ Ising model on the lattice is shown to be an *NP-complete problem* in 2000 [13]; basically meaning that it is not feasible in practice to try to solve this model on a lattice!

[62] Gaussian path-integrals are the only path-integrals we can compute exactly. This is certainly true in standard statistical mechanics and QFT context but I wonder if there are exotic functions which can be analytically computed as well, maybe I can look into that!

[63] We are rather lucky that the couplings of the electroweak theory (describing, electric, magnetic, and weak interactions) are rather small at the typical energies we observe the universe, so that we can use perturbation theory to describe the world successfully!

[64] There is a whole bunch of literature on the validity of this approach and how to improve it. Add references

Table 2: Classification of fixed points

# of relevant directions	ξ	Type of fixed point	Physical Domain	Explanation
0	0	Sink	Bulk phase	No relevant directions, hence trajectories only flow to them: they correspond to *stable bulk phases*, and the nature of the coupling constant characterize the phase; e.g., in $3d$ Ising model with nearest neighbor ferromagnet interaction in external field h, there are two sinks at $h = \pm\infty \& T = 0$, indicating the stable bulk phase is a net magnetization in all temperatures, whose sign is determined by sign of h.
1	0	Discontinuity FP	Plane of coexistence	Correspond to points on a phase boundary and describe *a first order phase transition* where an order parameters exhibits discontinuous behavior (*We note that the interpretation of RG transformations near first order phase transformations is delicate* and this explanation is rather simplified, see [15] for a careful discussion). E.g., all points on the line $h = 0, T < T_c$ in a ferromagnet flow to a discontinuity FP at $h = 0, T = 0$. This FP is unstable towards the sinks. Lastly, this is the only fixed point where one of the eigenvalues of the real space RG flow is ℓ^d.[65]
1	0	Continuity FP	Bulk phase	Represents *a phase of the system*, e.g. $h = 0, T = \infty$ which attracts all points on the line $h = 0, T > T_c$ in a ferromagnet. Phases described by this FP are unstable towards a sink.
2	0	Triple point	Triple point	Nothing interesting.
2	∞	Critical FP	Critical manifold	We explained this whole section.
Greater than 2	∞	Multicritical point	Multicritical point	Higher-dimensional generalization of what we have explained this whole section.
Greater than 2	0	Multiple coexistence FP	Multiple coexistence	Nothing interesting.

Let us end our scandalously lightening review of RG flow and turn to the relation to the conformal field theories in the next section; but before that, we present Table (2) as a general reference: it contains a brief categorization of fixed points according to their dimensionality and the value of correlation length, see Table 9.1 of [4] for the original/better version of it.

1.2.2 Dimensional counting and scale invariance

Let us consider the most generic form of the Lagrangian we gave in eqn. (1.17), i.e. $\mathcal{L}[K_i(\Lambda), \phi(\Lambda, x)] = \sum_i K_i(\Lambda)\mathcal{O}_i(\Lambda, x)$. The action then reads

$$S = \sum_i \int_\Lambda d^d x K_i(\Lambda)\mathcal{O}_i(\Lambda, x) \tag{1.20}$$

where Λ is *the momentum cut-off*, i.e. the regularization parameter.[66] If we scale $x \longrightarrow \ell^{-1}x$, then the lower limit Λ^{-1} gets scaled as well, hence we get

$$\int_\Lambda d^d x K_i(\Lambda)\mathcal{O}_i(x) = \int_{\ell^{-1}\Lambda} d^d(\ell^{-1}x) K_i(\Lambda)\mathcal{O}_i(\Lambda, \ell^{-1}x) \tag{1.21}$$

[65] Let ℓ be a coarse-graining factor and d is the dimension of the system. If one of the eigenvalues of the RG flow near a critical point is ℓ^d, then that critical point is either a first-order or a discontinuity fixed point! All other fixed points have $y < d$ for the eigenvalue ℓ^y, see Table 2. This is called *Nienhuis-Nauenberg criterion*, see [16] for further details.

[66] As we stated earlier, we need this parameter in statistical mechanics and in nonnormalizable quantum field theories. For renormalizable qft models, we can take it to infinity self-consistently but let's keep it for the sake of argument.

If we further scale $\Lambda \longrightarrow \ell\Lambda$, we then obtain the relation

$$K_i(\Lambda)\mathcal{O}_i(\Lambda, x) = \ell^{-d}K_i(\ell\Lambda)O_i(\ell\Lambda, \ell^{-1}x) \tag{1.22}$$

In general, the coupling constants' functional form can be rather complicated; after all, this is what determines the flows in Fig. (7). However, near critical point, we can *linearize* the action of RG flow to leading order, hence the coupling constants takes a homogeneous form:[67]

$$K_i(\ell\Lambda) = \ell^{[K_i]}K_i(\Lambda) \tag{1.23}$$

where $[K_i]$ is called the *mass dimension* of K_i.[68] This then indicates $\mathcal{O}_i(\Lambda, x) = \ell^{[K_i]-d}O_i(\ell\Lambda, \ell^{-1}x)$. This equation suggests that $\mathcal{O}_i$ also transforms homogeneously if we write it in terms of right variables. Instead of writing it as $\mathcal{O}(\Lambda, x)$, let us write it as $\mathcal{O}(\Lambda, \Lambda x)$ for the energy scale Λ and the dimensionless parameter Λx. Thus

$$\mathcal{O}_i(\ell\Lambda, \Lambda x) = \ell^{d-[K_i]}\mathcal{O}_i(\Lambda, \Lambda x) \tag{1.25}$$

Clearly $\mathcal{O}_i(\Lambda, \Lambda x)$ is a homogeneous function in its first variable: $\mathcal{O}_i(\ell\Lambda, \Lambda x) = \ell^{[\mathcal{O}_i]}\mathcal{O}_i(\Lambda, \Lambda x)$ for the *mass dimension* of the operator $\mathcal{O}_i$. We then conclude

$$[K_i] + [\mathcal{O}_i] = d \tag{1.26}$$

In footnote 49, we categorized coupling constants by their relevance around the critical point, i.e. whether they need to be tuned or not. There, we said that an irrelevant coupling constant vanishes by RG flow aroung the Gaussian fixed point. We see from eqn. (1.23) and eqn. (1.26) that we can determine the relevance of coupling constants by the scaling dimension of the operator that accompanies them:

$$\mathcal{O} \text{ is called } \left\{\begin{array}{r} \text{a relevant} \\ \text{an irrelevant} \\ \text{a marginal} \end{array}\right\} \text{ operator if } \left\{\begin{array}{l} [\mathcal{O}_i] < d \\ [\mathcal{O}_i] > d \\ [\mathcal{O}_i] = d \end{array}\right\}. \tag{1.27}$$

Therefore, we can ignore all irrelevant operators $\mathcal{O}$ in the Lagrangian if we are interested in studying the critical behavior, because their contribution goes to zero as we near the critical

[67]We call a function homogeneous in its n^{th} argument if it satisfies $f(\dots, ax_n, \dots) = a^k f(\dots, x_n, \dots)$.

[68]Actually, we are glossing over an important detail here. When we linearize RG flow, it takes the form of a matrix equation, i.e.

$$\begin{pmatrix} K_1 \\ K_2 \\ \vdots \end{pmatrix}' = M \begin{pmatrix} K_1 \\ K_2 \\ \vdots \end{pmatrix} \tag{1.24}$$

The coupling constants themselves do not simply scale as in eqn. (1.23) individually, but rather their combinations as eigenvectors of M does so. In fact eigenvalues of M determines how they'll scale as well, i.e. their mass-dimensions! However, as this detail is not really necessary for the main discussion, I'll assume in the text that M is diagonal hence K_i can satisfy eqn. (1.23).

point. This is precisely why we truncated higher order terms, i.e. ϕ^6, in eqn. (1.19).[69]

Around the Gaussian fixed point (i.e. the free theory), we can write down operators in terms of the fundamental field in the Lagrangian; for instance, we can take[70]

$$\mathcal{O}_1 = \phi\,, \quad \mathcal{O}_2 = \phi^2\,, \quad \mathcal{O}_3 = \phi^4\,, \quad \mathcal{O}_4 = \phi^6\,, \quad \mathcal{O}_5 = T_{\mu\nu} = g_{\mu\nu}\mathcal{L} + \partial_\mu\phi\partial_\nu\phi \tag{1.29}$$

where $T_{\mu\nu}$ is the stress tensor of the theory and $\mathcal{L}$ refers to the Lagrangian in eqn. (1.19).[71] As the Lagrangian includes a term $(\partial\phi)^2$ without a coupling constant in the front,[72] eqn. (1.26) indicates $[\partial\phi] = d/2$, hence $[\phi] = \frac{d-2}{2}$, thus this gives us

$$[\mathcal{O}_1] = \frac{d-2}{2}\,, \quad [\mathcal{O}_2] = d-2\,, \quad [\mathcal{O}_3] = 2d-4\,, \quad [\mathcal{O}_4] = 3d-6\,, \quad [\mathcal{O}_5] = d \tag{1.30}$$

We see that $\mathcal{O}_1$ and $\mathcal{O}_2$ are always relevant and that the stress tensor is always marginal. On the contrary, $\mathcal{O}_3$ ($\mathcal{O}_4$) is relevant only if $d < 4$ ($d < 3$), indicating that we were right to drop $\mathcal{O}_4$ around $d = 4$ to get the Wilson-Fisher fixed point at page 24.

The mass dimensions of the operators play critical role (pun intended) in the explanation of the critical phenomena. The critical exponents given in eqn. (1.10) can in fact be extracted from the mass dimension of the two coupling constants, hence from the mass dimensions of the accompanying operators by eqn. (1.26). Indeed, for Ising Universality Class, we can write down[73]

$$\alpha = \frac{d-2[\epsilon]}{d-[\epsilon]}\,, \quad \beta = \frac{[\sigma]}{d-[\epsilon]}\,, \quad \gamma = \frac{d-2[\sigma]}{d-[\epsilon]}\,, \quad \delta = \frac{d-[\sigma]}{[\sigma]}\,, \quad \nu = \frac{1}{d-[\epsilon]}\,, \quad \eta = 2[\sigma]-d+2 \tag{1.31}$$

where σ and ϵ are standard symbols used for the relevant operators in the conformal field theory that describes the critical point of Ising Universality Class.

A careful reader may have noticed a tension between the two statements we have made:

[69] There is an important detail that we are glossing over: *dangerously irrelevant operators.* Some operators cannot be just dropped even though their contribution vanishes as we approach the critical point. This is simply because the partition function Z (or vaccum generating function in high energy physics) may not be analytic in that limit. To understand this, let's consider $Z = Z[K_r, K_i, K_d]$ for relevant K_r, irrelevant K_i, and dangerously irrelevant K_d coupling constants. Under RG, K_r gets bigger so we keep it there. On the contrary, $K_i \longrightarrow 0$, hence we can replace it with 0: $Z = Z[K_r, 0, K_d]$. Similarly, K_d goes to zero as well, but we cannot replace it because the function Z is *not analytic* around $K_d = 0$. Instead, it behaves like this:

$$\lim_{K_d\longrightarrow 0} Z[K_r, 0, K_d] = \lim_{K_d\longrightarrow 0} \frac{1}{K_d^{\alpha}} Z'[K_r] \tag{1.28}$$

for some parameter $a > 0$. In short, an irrelevant coupling constant is called *dangerously irrelevant* if the partition function is singular as that term vanishes.

This detail is not really important for our current discussion so we will omit any subtlety related to this.

[70] In QFT, we should actually define our operators after normal ordering, i.e. $\mathcal{O}_2(x) =: \phi(x)\phi(x) :$, $\mathcal{O}_3 =: \phi(x)\phi(x)\phi(x) :$, etc. This is to ensure that we can put multiple operators at the same spacetime point without any divergence (by normal ordering, we are actually removing the divergence). For instance, the correlator $\langle\phi(x)\phi(x)\rangle = \infty$ whereas $\langle : \phi(x)\phi(x) : \rangle = 0$. This detail is not important for the main discussion in the text.

[71] The computation of the stress tensor is most conveniently done by changing the Euclidean/Lorentzian metric to an arbitrary one $g_{\mu\nu}$ and then variate the Lagrangian with respect to it, i.e. $T^{\mu\nu} \equiv \frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}\mathcal{L})}{\delta g_{\mu\nu}}$ for $g = \left|\det(g_{\mu\nu})\right|$.

[72] This is how we normalized our field ϕ in the first place, which is rather conventional.

[73] Add references for scaling relations.

Table 3: The behavior of the Ising Universality Class as we change the dimension of the space from 4 to 3. As we described in section 1.2.1, the Gaussian fixed point describes the Ising Universality Class for $d \geq 4$, and Wilson Fisher fixed point emerges as we continuously decrease d, which takes over the description of the Ising critical point. As this fixed point is strongly coupled (unlike Gaussian), the relevant operators σ and ϵ get anomalous dimensions γ_σ and γ_ϵ, and can no longer be related as ϕ^2 and ϕ were, i.e. $[\epsilon] \neq 2[\sigma]$.

Spacetime dimension d	:	$4 \longrightarrow 3$
Fixed point	:	Gaussian $\longrightarrow$ Wilson-Fisher
$\mathbb{Z}_2$-odd operator $\mathcal{O}_1$	:	$\phi(x) \longrightarrow \sigma(x)$
$\mathbb{Z}_2$-even operator $\mathcal{O}_2$	:	$\phi(x)^2 \longrightarrow \epsilon(x)$
Mass dimension of $\mathcal{O}_1$, i.e. $[\mathcal{O}_1]$	:	$\frac{d-2}{2} \longrightarrow \frac{d-2}{2} + \gamma_\sigma$
Mass dimension of $\mathcal{O}_2$, i.e. $[\mathcal{O}_2]$	:	$d-2 \longrightarrow d-2+\gamma_\epsilon$

1. The critical exponents listed in eqn. (1.10) ($\alpha \approxeq 0.11$, $\delta \approxeq 4.79, \dots$) can be extracted from the scaling dimensions of the operators via eqn. (1.31), i.e. $\alpha = \frac{d-2[\epsilon]}{d-[\epsilon]}$, $\delta = \frac{d-[\sigma]}{[\sigma]}$, etc.

2. The mass dimensions of the operators are given by eqn. (1.30), i.e. $[\mathcal{O}_1] = \frac{d-2}{2}$ and $[\mathcal{O}_2] = d-2$.

Clearly, the second point ensures that the critical exponents computed from them are *rational* numbers; for instance, if we take $\sigma = \mathcal{O}_1$ and $\epsilon = \mathcal{O}_2$, we compute $\alpha = \frac{4-d}{2}$ and $\delta = \frac{d+2}{d-2}$, which do not match the experimental values!

Well, we can resolve the problem by noting that Gaussian fixed point describes Ising universality class only if $d > 4$ and the experimental values are computed at $d = 3$. Indeed, if we were to do experiments at higher dimensions, we would confirm that the critical exponents are rational values! However, for $d = 3$, the correct critical point is Wilson-Fisher critical point, and the experiments indicate that the mass dimensions of the operator should become irrational: *we say that the operator gets an anomalous dimension if its mass dimension do not match its value at free theory*, see Table (3).

Let's try to understand *anomalous dimensions*. To do that, we need to go back to eqn. (1.25) and try to understand it physically. We have

$$\{x, \Lambda\} \longrightarrow \{\ell^{-1}x, \ell\Lambda\} \quad \Longrightarrow \quad \mathcal{O}_i(\Lambda, \Lambda x) \longrightarrow \ell^{d-[K_i]}\mathcal{O}_i(\Lambda, \Lambda x) \tag{1.32}$$

Let's understand this for $\ell = 10$, $x = 10$mm, and $\Lambda^{-1} = 1\mu$m: we are redefining our rulers such that we now call a millimeter what we used to call 10 millimeters, and we are also dropping all the details between 1μm and 10μm in the original scale. With the new rulers, our coarse-graining cut-off is exactly same $\Lambda^{-1} = 1\mu$m; so effectively, we zoomed-out from the system (lost some fine-details) but rescaled our rulers such that the system measures same in the new setting. So we should *see* the system same, albeit a little-bit blurred.

There is no guarantee that the operators would look same under such a blurring change; in fact, we do not expect them to look so! However, at the critical point, we have *scale-invariance*, i.e. the system look *exactly same* if you zoom-out, i.e. we do not get blurring with the above prescription. And as we linearized the action of RG *around a fixed point*, it makes sense that the operator $\mathcal{O}(\Lambda, \Lambda x)$ remain same upto an overall factor in this prescription, at least in the leading order.

How about the fixed point itself, i.e. what happens exactly at the fixed point? Well, we stated that we reach the fix point only after *an infinite number of RG iterations*, so it should not depend on Λ. Another way to see this is that Λ describes the scale at which we look at the system, and the system is *scale-invariant* at the critical point, so it does not depend on Λ: the operator $\mathcal{O}(a, b)$ should beacome $\mathcal{O}(b/a)$ at the critical point for $\mathcal{O}(\Lambda, \Lambda x)$. But this means, we have

$$\mathcal{O}(\lambda x) = \frac{1}{\lambda^{\Delta_{\mathcal{O}}}}\mathcal{O}(x) \tag{1.33}$$

via eqn. (1.25). Here, Δ is called *the scaling dimension* of the field.

As is customary for scale-invariant and conformal invariant theories, we will use the Greek letter Δ to denote the scaling dimension. The free theory (which is itself scale-invariant) has the so-called *engineering scaling dimensions*, these were what we presented in eqn. (1.30). For a scalar field, as we saw there, we have

$$\Delta_{\text{free scalar}} = \frac{d-2}{2} \tag{1.34a}$$

which follows from setting the mass dimension of its kinematic term to d, i.e. $[(\partial\phi)^2] = d$. In contrast, the kinematic term for a *fermionic field* and gauge fields are $\overline{\psi}\partial\!\!\!/\psi$ and $\left(\partial_{[\mu}A_{\nu]}\right)^2$ hence[74]

$$\Delta_{\text{free fermion}} = \frac{d-1}{2}\,, \qquad \Delta_{\text{gauge vector}} = \frac{d-2}{2} \tag{1.34b}$$

We can also define the antisymmetric tensor $F \equiv \partial_{[\mu}A_{\nu]}$, for which we have

$$\Delta_{\text{antisymmetric tensor}} = \frac{d}{2} \tag{1.34c}$$

These *engineering scaling dimensions* are valid at Gaussian fixed point, but they get modified when we move to another fixed point, especially if the theory at that fixed point is strongly coupled! Then we define *anomalous dimension* (denoted by γ) as

$$\Delta_{\mathcal{O}} = \Delta_{\mathcal{O}_{\text{free}}} + \gamma_{\mathcal{O}} \tag{1.35}$$

For instance, the operators σ and ϵ of $3d$ Ising model have the values[75]

$$\Delta_\sigma \sim \frac{1}{2} + 0.018\,, \qquad \Delta_\epsilon = 1 + 0.413 \tag{1.36}$$

where the former two values are engineering scaling dimensions[76] and the latter two are the anomalous dimensions. As a corollary of the above discussion, we can immediately say that *the*

[74]In these notes we follow the conventions $A_{[x}B_{y]} = \frac{1}{2}A_xB_y - \frac{1}{2}A_yB_x$ and $A_{\{x}B_{y\}} = \frac{1}{2}A_xB_y + \frac{1}{2}A_yB_x$.

[75]See [12] for the computations of these values by conformal bootstrap.

[76]As $\sigma \sim \phi$ and $\epsilon \sim \phi^2$ around the free theory, their engineering scaling dimensions are $[\sigma] = \frac{d-2}{2}$ and $[\epsilon] = d-2$.

fact that the anomalous dimension of the operators σ and ϵ are large indicates that the 3*d Ising Model is strongly coupled and hence we cannot hope to describe it by traditional methods such as perturbation theory.*

How come do we get anomalous dimensions γ? I am not asking the dynamics how γ emerges, but rather how come it is consistent that we can have irrational values for scaling dimensions of operators $\mathcal{O}$, yet we can have dimensionless action, i.e. $[\mathcal{L}] = d$? Indeed, if we consider the following relation at the critical point[77]

$$\begin{aligned} \int d^d x \left(\frac{\partial \mathcal{O}(x)}{\partial x}\right)^2 &= \int d^d(\lambda y)\left(\frac{\partial \mathcal{O}(\lambda y)}{\partial(\lambda y)}\right)^2 = \int d^d \lambda^{d-2}\left(\frac{\partial \mathcal{O}(\lambda y)}{\partial y}\right)^2 \\ &= \int d^d y \lambda^{d-2-2\Delta}\left(\frac{\partial \mathcal{O}(y)}{\partial y}\right)^2 \end{aligned} \tag{1.37}$$

which forces $\Delta = [\mathcal{O}]$, i.e. that the scaling dimension is equal to the mass dimension, which is always rational. What went wrong?

Scaling dimension should not be equal to mass dimension. To see that, consider the two-point correlation function $\langle \phi(x)\phi(0)\rangle$. By dimensional analysis, it has the mass dimension $[\langle \phi(x)\phi(0)\rangle] = d-2$; at the critical point, we then expect the scaling behavior

$$\langle \phi(\mathbf{x})\phi(0)\rangle \sim \frac{1}{|\mathbf{x}|^{d-2}} \tag{1.38}$$

But the definition of the critical exponent η follows from $\langle \phi(\mathbf{x})\phi(0)\rangle \sim |\mathbf{x}|^{-d+2-\eta}$ hence this naïvely indicates that we always have $\eta = 0$, which clearly contradicts the empirical evidence!

The resolution to this problem is related to a common misconception. It is often stated that *there is no length scale at the fixed point of RG flow, hence there is no length scale at the critical point as well.* This statement is incorrect! The correct statement is this: *the physics near and at the critical point is governed by the long-range physics and is independent of the microscopic details; nevertheless, we still need the microscopic length scale for consistency.* Therefore, the operator $\mathcal{O}(x)$ actually has a a dependence on the microscopic scale a: $\mathcal{O} = \mathcal{O}_a(x)$. If we *scale all our rulers by* λ, so that we call λ meters what we used to call 1 meter, then both a and x get scaled and we obtain $\mathcal{O}_{\lambda a}(\lambda x) = \lambda^{-[\mathcal{O}]}\mathcal{O}_a(x)$. On the contrary, if we simply scale x as a parameter, then we get $\mathcal{O}_a(\lambda x) = \lambda^{-\Delta_{\mathcal{O}}}\mathcal{O}_a(x)$. Thus, we can consistently have[78]

$$\langle \phi_a(x)\phi_a(0)\rangle = \frac{a^\eta}{x^{2[\phi]+\eta}}\,, \quad \langle \phi(x)\phi(0)\rangle = \frac{1}{x^{2\Delta_\phi}} \tag{1.39}$$

for

$$\phi_a(x) = a^{\Delta_\phi - [\phi]}\phi(x)\,, \quad \eta = 2(\Delta_\phi - [\phi]) \tag{1.40}$$

With $\Delta_{\mathcal{O}} = \Delta_{\mathcal{O}_{\text{free}}} + \gamma_{\mathcal{O}}$ and $[\mathcal{O}] = \Delta_{\mathcal{O}_{\text{free}}}$, we see that η is simply twice the anomalous dimension γ.

What is this microscopic length scale a? We really do not need to know! Indeed, a does not show up anywhere in our equations and affects nothing about the computations: this is why it is

[77] It is important that we are precisely at the critical point so that neither the integral limits nor the operator $\mathcal{O}(x)$ does have any regulator Λ dependence.

[78] See eqn. (1.8).

stated that the physics at critical point is independent of the microscopic details. We detailed this subtlety just to make sure that the concept of anomalous dimension makes sense to the reader: indeed, if we did not have a–dependence at the critical point, we could not have any anomalous dimension. Nevertheless, as a is only there to make sure dimension counting checks out,[79] we'll forget about it in the rest of these notes: we will stick to $\phi(x)$ with its scaling dimension Δ_ϕ.

This concludes our discussion of dimensional counting and scale-invariance. We'll summarize these findings in the next section.

1.2.3 Summary

- Renormalization group flow formalism allows us to look at a system from different scales, and compute how its behavior change as we go from UV scales to IR scales.
- With the RG, we observe that the behavior around fixed points are solely determined by the relevant directions, hence we expect a *universal scaling* around the critical point![80]

[79]It is worth stressing the importance of the remarkable conclusion of this discussion. The critical phenomena is observed at lengths of ξ, say of the order of microns or larger, and the microscopic length scale a is of the order of an atom. So naively, the ratio $\frac{a}{\xi}$ is sufficiently small that we can replace it by 0. However, if we do this, a does not appear in any final formula hence we cannot get any anomalous dimension: *so the critical phenomena is an area of physics where we cannot get rid of an extremely small parameter ($\frac{a}{\xi}$) if we would like to describe the phenomena correctly!* This is rather remarkable, and shows how incorrect the often-heard statement *"the only important length scale near the critical point is the correlation length"* really is!

There is contradiction between *physics being independent of microscopic details* and *we cannot get rid of microscopic length scale a.* Mathematically speaking, if we have a function $f(a,\xi)$ and we would like to take $a \longrightarrow 0$, we need the function to be regular around $a \sim 0$. If it has the form $\lim_{a\to 0} a^{-\sigma} g(\xi)$, then we cannot take $a = 0$ in $f(a,\xi)$. The situation is simply this: we only need a as an overall factor to fix dimensions; it does not interfere with the rest of the expression!

The first explicit recognition of such a case (that a function has the form $f(a,\xi) \lim_{a\to 0} a^{-\sigma} g(\xi)$ so that one cannot discard a) is probably the calculation of how a converging shock wave is focused (see the footnote in page 197 of [4]). In [17], there is an extensive study of such problems and their solutions, accomplished by making the explicit hypothesis that anomalous dimensions exist combined with numerical methods. *Historically, however, the problem of anomalous dimensions in critical phenomena was discovered and solved apparently without knowledge or recognition of these other phenomena: this is rather interesting because Landau had the first-hand knowledge of anomalous dimensions in both the critical phenomena and the problem of the converging shock wave but apparently he made no reference to a connection between them!*

Finally, we would like to note that, rather ironically in a sense , we do not need the microscopic length scale a if we are away from the critical point: as the correlation length ξ is finite there, that does the work to fix the dimensions, see eqn. (1.8a) for an example!

[80]For a qualitative understanding, let us consider a system close to criticality. The trajectories of the systems on the critical manifold remain on the critical manifold and flow to the fixed point. On the other hand, if the system is slightly off the critical manifold, it will approach to the critical point during part of the RG, but will ultimately repelled from the critical manifold and flow away because of the unstable directions due to relevant operators.

Independent of the initial position off the critical manifold, the system will first flow toward the critical point because the only singularities of the flow field are the fixed points themselves! Likewise, independent of where they are off the critical manifold, they will be eventually repelled with the same eigenvalues because it is the same unstable directions that cause the behavior! *The fact that it is the same eigenvalues which drive all slightly off-critical systems away from the fixed point is the origin of universality!* [4]

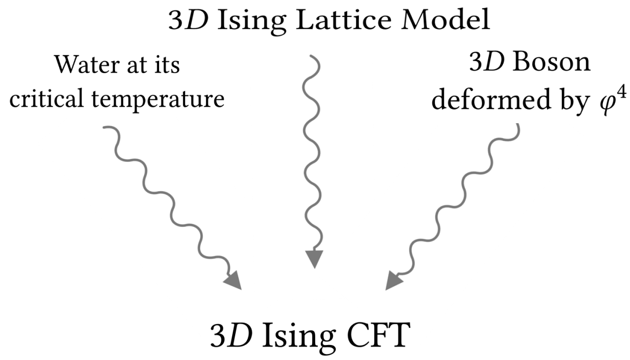


Figure 8: Many different phenomena at microscopic scales are described by the same conformal field theory at macroscopic scales; i.e. they fall in the same universality class! This is called *IR equivalence.*

- Fixed points of RG have scale-invariance, hence operators at that point satisfy the equation

$$\mathcal{O}(\lambda x) = \lambda^{-\Delta_{\mathcal{O}}}\mathcal{O}(x) \tag{1.41}$$

  where $\Delta_{\mathcal{O}}$ is called *scaling dimension* of the operator $\mathcal{O}$.

- Scaling dimensions of *relevant operators*, whose scaling dimension is less than d, determine the critical exponents if that fixed point is associated with a critical phenomena. In Ising model, we have two relevant operators σ and ϵ and we can write critical exponents as $\alpha = \frac{d-2\Delta_\epsilon}{d-\Delta_\epsilon}$, $\beta = \frac{\Delta_\sigma}{d-\Delta_\epsilon}$, etc.

- We can compute scaling dimensions of the operators if we have a free theory, i.e. the scalar operator ϕ has the scaling dimension $\Delta_{\text{free scalar}} = \frac{d-2}{2}$.

- If the theory is strongly coupled, then scaling dimension of operators diverge from their values at free theory; the difference is called *anomalous dimension*: $\Delta_{\mathcal{O}} = \Delta_{\mathcal{O}_{\text{free}}} + \gamma_{\mathcal{O}}$.

- There is no analytic and exact way to solve strongly-coupled theories. We usually do several approximations or expansions: *discretization on a lattice*, *large N expansion*, *$\epsilon-$expansion*, etc.[81] These methods may or may not yield reliable and robust results.

- The alternative (and relatively recently developed) approach is to use *conformal bootstrap program*. It uses rigorous self-consistency and symmetry conditions to solve the landscape of CFT's for a given universality class. We'll detail this program in the following chapters.

- Why use conformal symmetry for critical phenomena? This follows from IR equivalence: many CFTs are in the same universality class as a lot of physically-relevant critical phenomena, and solving the CFT is sufficient to understand, say, the divergence of the heat capacity of water near its critical point: see Fig. (8) for illustration of this for Ising Universality Class.

- How do we know that we have *conformal symmetry* at fixed points? Technically, all we know is that we have *scale symmetry* at the fixed points and *not all scale invariant theories*

[81] See the bullets listed in page 25 and the next one for further details. Maybe put some resources here

are conformally invariant! There are strong motivations to anticipate conformal invariance (we'll discuss these), but for now, we can take a simpler approach: we *will assume* that the critical point is described by a conformal theory, compute the critical data, and check with the experiments to see if they agree!

I have partially finished up until this point!

1.3 Using CFTs beyond statistical physics

- local and scale-invariant theories are usually CFTs
- String theory and 2d CFTs
- AdS/CFT correspondence
- CFTs in cosmology
- flat space holography

2 What is conformal symmetry?

2.1 Invitation: understanding Gallileon transformations in a mathematical framework

- what is a group?
- galilean transformations as a group
- Relativity and Lorentzian transformations
- Poincare group

2.2 Scale and conformal algebras

- discussion of algebras from a pure mathematical point of view

2.3 Conformal algebra in the framework of QFTs

- Killing equations and relevant discussion

3 Review: Quantum Field Theory

3.1 some subsections

- quantum mechanics, ward identities, path integral, quantization, LSZ formalism, correlation functions

3.2 Wigner's classication of operators: little group formalism

4 Conformal field theories

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