Originally prepared for *METU - Phys209*

Differential Equations in Physics

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Summary

These are the notes of the lectures prepared for the "PHYS209 MATHEMATICAL METHODS IN PHYSICS I" course at *Middle East Technical University*, 2023-2024 Fall Semester —see https://soneralbayrak.com/teaching/Phys209. These notes are mostly based on other sources and I provided the sources in relevant places. Whenever I get the chance, I will keep updating the notes to keep it up-to-date and self-contained; there are also reminders in blue for me to add further discussion/comments.

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Preface

Remarks to the reader

I would like to make a few points crystal clear:

1. I do not update these notes on a regular basis.

2. The contents are correct to the best of my knowledge, but I have not put the extra effort to make sure that everything is book-level correct; nevertheless, I try to put as many references as possible when relevant, so please make the most of it!

3. I believe that the level of this book is appropriate for an average sophomore, but I would dare say that it should be quite useful even for graduate students of theoretical physics.

4. Lastly, I provide several links as references here and there: I agree that this is a bad practice in academia and one should instead convert them to proper references in the bibliography. Nevertheless, it is faster for me to write this way and faster for the reader to just click on them where they appear, so I'll keep this practice. My upfront apologies if the links get broken in time: hopefully a snapshot will have been available here https://web.archive.org/ (it would be rather amusing if this site itself becomes unavailable).

About the format of the book

This book is publicly available online, and anyone can simply download and read it on their computer. Nevertheless, it is a lot easier on eyes to read a book on paper (or epaper), so I suspect many readers will simply print this book. To make this book convenient for all types of readers, I need to make a few design choices[a] and the most important of it is the format of this book.

[a] For instance, I provide all links in their explicit form (not like google but as https://www.google.com/) so that the book on an ereader without a browser or its printed version is as useful as its digital version.

The most convenient paper that most students have easy access to is A4 (or letterpaper which is very similar in size): so if they choose to print this book, their easiest and cheapest option would be to use A4 paper. However that paper has been historically designed for typewriters which use large monospaced fonts, hence is not really appropriate for a digitally prepared book. Indeed, if you check your favorite-to-read book, you will most likely see that it uses a smaller paper size, with proportionally spaced small fonts.

What is wrong with using a large paper? It is empirically known that a document is properly legible if there are around 60-75 characters on a line: if there are more characters, it becomes harder to read and the reader may end up re-reading same line over and over again (doubling). When used with typewriter, A4 paper indeed has appropriate number of characters one a line, but as stated previously, this is not the optimal setup with the digital fonts, hence making A4 paper *too large for digital books.*

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How to solve the problem that A4 is too large for a digital book? Obviously, we can use a typewriter font for which A4 is historically intended in the first place!  However, such monospaced fonts (Courier being another example) are not aesthetically pleasing and do not belong to modern texts!
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Of course, we can go with a modern proportionally spaced fonts but make the font size large enough such that a line has few enough characters for it to be easily legible. Although a better one than using monospaced fonts, this is still a suboptimal solution to the problem at hand...

Another solution which is somehow popular around the institutions is to use double-spacing among the lines. Indeed, regulations for master and doctorate thesis of various universities include compulsory large spacing among the lines, such as one and a half spacing or double spacing: you can also see this in my master and doctorate thesis: https://arxiv.org/abs/1602.07676, and https://arxiv.org/abs/2107.13601. Although this method can indeed prevent doubling to a degree, it is neither an aesthetic nor an efficient solution.

A somehow better solution than those listed above is to use multiple columns in the document. Indeed, this is the traditional approach in magazines and newspapers, and is immediately applicable in academic papers and manuscripts as well. However, A4 paper is not really big enough to have two columns of text with around 60-75 characters (let alone three or more columns), and although one can go with smaller font sizes to make it more legible digitally, the printed version would still be hard to read either way (proper number of small font characters, or few number of normal font characters).

Common text editors such as Microsoft Word either go with one of the solutions above or do not solve the problem at all. On the contrary, LaTeX templates default to choose another approach: they stick to a modern font with an appropriate spacing and a single column, but they also increase the margins such that a line has proper number of characters. This is an ideal solution if the resultant text will be read digitally, however it leads to a waste of paper when printed.

In this book, we will not follow any of these design choices. Instead, we will go with the rather unorthodox *Tufte style*,[b] an asymmetric allocation of the text in the paper. Indeed, the main text will be in the left of the paper, whereas we have another block of text on the right dedicated to the *sidenotes*,[c] margin figures, and margin tables. We choose a rather narrow font family (*libertine*) and arrange the margins such that the main text is of 26 pica width and side text is of 14 pica width: for the 11pt and 9pt font sizes, this corresponds to roughly 66 and 44 characters of the libertine font for main and side text blocks respectively.[e] Thus we have an ideally-sized main text block and acceptably-sized side text block for a modern proportionally spaced

[b] See https://www.ctan.org/tex-archive/macros/latex/contrib/tufte-latex/

[c] In the traditional layout, one usually uses endnotes, margin notes, or footnotes; in this paper, most "notes" will be sidenotes with occasional footnotes.[d]

[d] Such as this one.

font in an A4 paper, and we do this without unnecessarily wasting the paper.[f]

About the course Phys209

As stated in the front page, this "book" is collection of notes prepared for the course Phys209, with the syllabus provided here: https://soneralbayrak.com/teaching/Phys209. Although the title is somewhat generic, the contents of this book is severely restricted and arranged so as to be an appropriate one-semester-long course for *an average sophomore at the Physics program of Middle East Technical University*. I would also like to acknowledge that the level and approach of this book is based solely on my expectations and projections for such a student, and thus it may not be really appropriate in real life; nevertheless, it is what it is.

I left several sources in the syllabus, including the textbook of the course on which these notes are somewhat based on. I'll also make use of other sources; yet, any error or incorrect information on these notes are entirely my fault and not of any of these sources.

[e] For a nice discussion of these points along with the tools to compute approximate expected number of characters per line, see https://ftp.cc.uoc.gr/mirrors/CTAN/macros/latex/contrib/memoir/memman.pdf

[f] I would like to acknowledge the following nice discussion with which I started to learn more about these *typographical* issues: https://tex.stackexchange.com/questions/71172/why-are-default-latex-margins-so-big.

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Definition and Classification of Differential Equations

1.1 Preliminaries: some basic terminology

1.1.1 Type notation and functions

Consider the number 2. It is an integer, which mathematicians denote as $2 \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers.[g] Computer scientists on the other hand would denote this as

$$2 :: \texttt{Integer} \tag{1.1}$$

where $a :: b$ reads as *"a is of type b"*. Further examples would be

$$3/2 :: \texttt{Rational} \tag{1.2a}$$
$$1.56 :: \texttt{Real} \tag{1.2b}$$
$$1 + i :: \texttt{Complex} \tag{1.2c}$$

Note that an object may be of multiple types. Mathematically, $3 \in \mathbb{Z}$ and $3 \in \mathbb{R}$, meaning

$$3 :: \texttt{Integer} \tag{1.3a}$$
$$3 :: \texttt{Real} \tag{1.3b}$$

A somewhat hybrid notation between mathematicians and computer scientists would be

$$3 :: \mathbb{Z} \tag{1.4a}$$
$$3 :: \mathbb{R} \tag{1.4b}$$

We shall use this notation in the rest of the book.[h]

Just as we do with the explicit numbers above, we can *define* variables with explicit types; for instance,

$$x :: \mathbb{Z} \tag{1.5}$$

It is up to us to choose what we want for the type, we can even left the type unknown; for instance,

$$y :: \mathrm{A} \tag{1.6}$$

means that y is a variable of the type A,[i] where A can be anything.[j]

Unlike the numbers or the variables above, the functions have an input and an output, hence their type actually reads differently.[k] For instance,

$$f :: \mathbb{Z} \to \mathbb{Z} \tag{1.7}$$

denotes *"a function that acts on integers and produces another integer.* An example would be

$$f = \lambda \to \lambda^2 \tag{1.8}$$

[g] $\mathbb{Z}$ is actually an integral domain.

[h] I personally find this notation clearer when we use it with higher order functions such as derivatives.

[i] This is called a *type variable.*

[j] It could be a simple field such as $\mathbb{Z}$ or $\mathbb{N}$, or it could be a more complex object such as $\mathfrak{M}_{2\times 2}(\mathbb{C})$ which denotes two by two matrices with complex entries.

[k] Physicists tend to refer to multi-valued relations as functions as well: this is a justifiable habit as such relations can always be treated as genuine functions by appropriately restricting their domains.[ℓ] We will stick to this convention in the rest of the book and refer all multi-valued relations (such as an arctan) as functions.

[ℓ] Mathematically, a function yields a unique output for a given input, therefore so-called multi-valued "functions" are not really functions in their full analytic domain. For instance, the relation $\texttt{sqrt} = \lambda \to \sqrt{\lambda}$ is not a function in the complete complex plane, as $\texttt{sqrt}(4) = \pm 2$. One solution is to choose a *restricted domain* so that the relation actually yields a unique solution for a given input from the domain, hence making the relation a genuine function, e.g. choosing the domain $\mathbb{R}^+$ for $\texttt{sqrt}$. In principle, we do not need to make an arbitrary restriction: the strategy would be to analyze the *Riemann surface* of the relation, and then determine the codomain in which the relation yields a unique result; in the case of $\texttt{sqrt}$, we can state $\texttt{sqrt} :: \mathbb{C} \to \mathrm{A}$ where $x \in \mathrm{A}$ if and only if $0 \le \arg(x) < \pi$. This means that $\texttt{sqrt}\left(re^{i\theta}\right) = \sqrt{r}e^{i\theta/2}$ for θ chosen in the range $0 \le \theta < 2\pi$ with $r > 0$; hence, $\texttt{sqrt}(4) = 2$.

The regions of the codomain in which the multi-valued relation becomes a genuine function are called *sheets*; in the example above, we choose a principle sheet (or a first sheet) for the relation $\texttt{sqrt}$: we can move on to the *other sheets* by removing the restriction on θ. Indeed, we have on the second sheet $\texttt{sqrt}(4) = \texttt{sqrt}\left(4e^{i2\pi}\right) = \sqrt{4}e^{i\pi} = -2$, the other solution! One could go on to higher sheets to find even more solutions; in the case of $\texttt{sqrt}$, n–th sheet is actually identified with $(n-2)$–th sheet, hence we have only two so-which gives the integer $f(x) = x^2$ when acted on the integer x.[m] Another example would be

$$g(y) = y/3 \tag{1.9}$$

for which we can write down[n]

$$g :: \mathbb{Z} \to \mathbb{Q} \tag{1.10a}$$
$$y :: \mathbb{Z} \tag{1.10b}$$
$$g(y) :: \mathbb{Q} \tag{1.10c}$$

Of course, we can also extend our interested regime for the input y and simply state

$$g :: \mathbb{C} \to \mathbb{C} \tag{1.11a}$$
$$y :: \mathbb{C} \tag{1.11b}$$
$$g(y) :: \mathbb{C} \tag{1.11c}$$

which is still true for $g(y) = y/3$. In fact, we can even write $g :: \mathrm{A} \to \mathrm{B}$ if we do not care for the explicit types of input and output.[o]

1.1.2 Higher order functions and the derivative

Consider the operation T of *"doubling the output of a function"*. If we apply this operation T to a function f, then it yields a function g such that $g(x) = 2f(x)$. For instance,

$$f = \lambda \to \lambda + 5 \tag{1.12a}$$
$$g = \lambda \to 2\lambda + 10 \tag{1.12b}$$

The question now is this: what is the type of the operation T?

Clearly, $T = f \to g$ as it takes the function f as an input and produces the function g as the output. Thus, we can write it as

$$T :: (\mathrm{A} \to \mathrm{B}) \to (\mathrm{C} \to \mathrm{D}) \tag{1.13}$$

which means if

$$f :: \mathrm{A} \to \mathrm{B} \tag{1.14}$$

then

$$(g = T \cdot f) :: \mathrm{C} \to \mathrm{D} \tag{1.15}$$

T is called *a higher order function*: it acts on a function and produces another function.

The derivative operator *is* a higher order function, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x} :: (\mathrm{A} \to \mathrm{B}) \to (\mathrm{A} \to \mathrm{C}) \tag{1.16}$$

which means[p]

$$x :: \mathrm{A} \tag{1.17a}$$
$$f :: \mathrm{A} \to \mathrm{B} \tag{1.17b}$$
$$f(x) :: \mathrm{B} \tag{1.17c}$$
$$f' :: \mathrm{A} \to \mathrm{C} \tag{1.17d}$$
$$f'(x) :: \mathrm{C} \tag{1.17e}$$

lutions (as expected from a square root operation).

Somewhat more traditional approach to the Riemann surfaces is the *analysis of branch cuts*. We **(1)** take one of the solutions of the relation as the output (called *principal value*), **(2)** determine some lines on the complex plane (branch cuts), **(3)** impose discontinuity on the cuts such that the relation is a true function in the rest of the complex plane! With the insight from Riemann surfaces, we know that moving across such lines actually takes us from one sheet to another —previous (next) sheet if we pass the branch cut (counter)clockwise. For `sqrt`, the conventionally chosen principle value is $\sqrt{r^2} = r$ for $r \in \mathbb{R}^+$, and branch cut is the line $(-\infty, 0)$: $\mathtt{sqrt}(z)$ for any other $z \in \mathbb{C}$ can then be uniquely determined to be consistent with these; for instance $\mathtt{sqrt}\left(-1 \pm i10^{-100}\right) \sim 6 \times 10^{-17} \pm i$ —note the jump!

[m] I'd like to note that there is a common misconception (especially) in the physics community. $f(x)$ is *not* the function, the function is f. f acts on the input x, and produces the output $f(x)$.

[n] If you couldn't remember, $\mathbb{Q}$ denotes the set of rational numbers.

[o] We use different letters for the type variables (A and B) so that the input and output are not necessarily of the same type. On the contrary, the function $h :: \mathrm{A} \to \mathrm{A}$ can only produce integers when acted on integers, reals when acted on reals, and so on.

[p] We are using the common convention $f' := \frac{\mathrm{d}f}{\mathrm{d}x}$ and $f'(x) := \frac{\mathrm{d}f}{\mathrm{d}x}(x)$ for brevity.

For example,

$$f :: \mathbb{R} \to \mathbb{C} \tag{1.18a}$$
$$f = \lambda \to \lambda^2 + 2i \tag{1.18b}$$

leads to

$$f' :: \mathbb{R} \to \mathbb{R} \tag{1.19a}$$
$$f' = \lambda \to 2\lambda \tag{1.19b}$$

where the type variables in eqn. (1.16) are $\mathrm{A} = \mathrm{C} = \mathbb{R}$ and $\mathrm{B} = \mathbb{C}$.

The derivatives can shrink the codomain of a function;[q] in the above example, the original codomain (that of f) was $\mathbb{C}$ whereas the new codomain (that of f') is $\mathbb{R}$. Nevertheless, we can always *embed* the smaller codomain into a larger one (e.g. all real numbers can be considered as complex numbers as well), hence we can always take $\frac{\mathrm{d}}{\mathrm{d}x} :: (\mathrm{A} \to \mathrm{B}) \to (\mathrm{A} \to \mathrm{B})$. This shows that the derivative is a higher order function that can be *repeatedly applied*; thus, we say[r]

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} :: (\mathrm{A} \to \mathrm{B}) \to (\mathrm{A} \to \mathrm{B}) \tag{1.20a}$$
$$f :: \mathrm{A} \to \mathrm{B} \tag{1.20b}$$
$$f^{(n)} :: \mathrm{A} \to \mathrm{B} \tag{1.20c}$$

1.1.3 Functionals and the integral

In the previous section, we have seen that the derivative is a higher-order function, i.e. it takes a function to another function. Naturally, its inverse is also a higher-order function:[s]

$$\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}} :: (\mathrm{A} \to \mathrm{B}) \to (\mathrm{A} \to \mathrm{B}) \tag{1.21a}$$
$$g :: \mathrm{A} \to \mathrm{B} \tag{1.21b}$$
$$g^{(-1)} :: \mathrm{A} \to \mathrm{B} \tag{1.21c}$$

where $g^{(-n)} = \left(\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}\right) \cdot g^{(1-n)}$ with $g^{(0)} = g$, in line with the notation for derivatives. Fundamental theorem of calculus then tells us that the output $g^{(-1)}(x) :: \mathrm{B}$ can be written as

$$g^{(-1)}(x) = \int_0^x g(t)dt \tag{1.22}$$

which is compatible with $\frac{\mathrm{d}}{\mathrm{d}x} g^{(-1)}(x) = g(x)$.

We have shown above that the *the indefinite integral* is a higher order function, but how about a definite integral? How do we determine its type?

We can start by writing down a generic definite integral:

$$\int_0^{\pi/2} \cos(x)dx = 1 \tag{1.23}$$

[q] Reminder: if $f = \mathrm{A} \to \mathrm{B}$, we call A (B) the (co)domain of f.

[r] We will use the notation such that $f^{(n)}$ is the $n-$th derivative of the function f.

[s] In principle, the anti-derivative can *extend* the codomain of a function, just as derivative shrinks it. We can see this via *integration constant*, which can be anything as long as it is $x-$independent. We put this subtlety aside as we can always extend the original codomain such that it matches the new one, hence (1.21a).

Clearly, we take a function (cos) and a range over which we do the integration (between 0 and $\pi/2$). We can always specify the integration range via the domain of the function,[t] thus

$$\int :: (\mathrm{A} \to \mathrm{B}) \to \mathrm{C} \tag{1.24}$$

as the integration turns the function cosine into a number 1.

Operations that turn functions into numbers are called *functionals*, and definite integration is a functional. For instance, the operation to compute the area under a curve is a functional: if we call that operation T, we then have

$$\begin{align} T &:: (\mathbb{R} \to \mathbb{R}) \to \mathbb{R} \tag{1.25a}\\ f &:: \mathbb{R} \to \mathbb{R} \tag{1.25b}\\ \left(T \cdot f = \int_{-\infty}^{\infty} f(x)dx\right) &:: \mathbb{R} \tag{1.25c}\end{align}$$

Note that the parentheses in the type definition is important; for instance,

$$\begin{align} T &:: (\mathbb{R} \to \mathbb{R}) \to \mathbb{R} \tag{1.26a}\\ F &:: \mathbb{R} \to (\mathbb{R} \to \mathbb{R}) \tag{1.26b}\end{align}$$

denote different objects: T is a functional, which produces a number if given a function as input. F on the other hand produces a function when fed a number, i.e. $F(x) = f$ is a function, whose output can be written as $F(x)(y) = f(y)$. This show that we can actually interpret F as a function of two variables![u]

1.2 Differential equations

1.2.1 Basics

Very broadly, we could define any relation that contains the derivative higher order function $\frac{\mathrm{d}}{\mathrm{d}x}$ and an unknown function f as a differential equation. For instance,

$$\cos\left(\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)f(x) + \frac{1}{f(x)}\right) = 0 \tag{1.29}$$

is a differential equation: but it is neither real-world motivated nor easy-to-solve, so let's skip it and focus on more relevant and simpler cases.[v]

The simplest differential equation is

$$\begin{align} x &:: \mathbb{R} \tag{1.35a}\\ \left(\frac{\mathrm{d}}{\mathrm{d}x} \cdot f\right) &:: \mathrm{A} \to \mathrm{B} \tag{1.35b}\\ \frac{\mathrm{d}}{\mathrm{d}x} \cdot f &= 0 \tag{1.35c}\end{align}$$

[t] For instance, if we would like to do the integration from 0 to 1, we can restrict the function $f(x)$ `::` A $\to$ B to $f(x)$ `::` `UnitReal` $\to$ B where x `::` `UnitReal` means $x \in [0, 1]$.

[u] This property can be generalized. A higher order function

$$f :: \mathrm{A}_1 \to (\mathrm{A}_2 \to (\cdots \to (\mathrm{A}_{n-1} \to (\mathrm{A}_n \to \mathrm{B})))) \tag{1.27}$$

produces a function of $n-1$ variable once given a variable as input. That function then produces another function of $n-2$ variable once given a variable as input, and so on. Indeed, it means

$$\begin{align} x_i &:: \mathrm{A}_i \tag{1.28a}\\ f(x_1)(x_2)\ldots(x_n) &:: \mathrm{B} \tag{1.28b}\end{align}$$

which can easily be re-interpreted as $f(x_1, \ldots, x_n) :: \mathrm{B}$.

This concept of turning higher-order functions into functions of multiple variables (and vise versa) is called *currying*, see bla bla bla. Put some sources here.

[v] You may be surprised with the expression $\exp(\frac{\mathrm{d}}{\mathrm{d}x})$. To understand it, let's first view the taking-the-n^{th}-power operation as a higher order function:

$$P_n :: (\mathrm{A} \to \mathrm{B}) \to (\mathrm{A} \to \mathrm{B}) \tag{1.30}$$

and

$$\begin{align} (P_0 \cdot f = f) &:: \mathrm{A} \to \mathrm{B} \tag{1.31a}\\ \left(P_n \cdot f = f \cdot (P_{n-1} \cdot f)\right) &:: \mathrm{A} \to \mathrm{B} \tag{1.31b}\end{align}$$

meaning

$$\begin{align} x &:: \mathrm{A} \tag{1.32a}\\ (P_n \cdot f)(x) = f(f(\ldots f(x))) &:: \mathrm{B} \tag{1.32b}\end{align}$$

For instance, $P_2 \cdot \cos = \lambda \to \cos(\cos(\lambda))$.

We can now define *exponentiation* as a higher order operation:

$$\begin{align} \exp &:: (\mathrm{A} \to \mathrm{B}) \to (\mathrm{A} \to \mathrm{B}) \tag{1.33a}\\ \exp &= \sum_{n=0}^{\infty} \frac{1}{n!} P_n \tag{1.33b}\end{align}$$

One can then immediately compute, say,

$$\begin{aligned} \exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)x^3 &= x^3 + 3x^2 + 3x + 1\\ \exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)e^{kx} &= e^k e^{kx} \end{aligned} \tag{1.34}$$

and so on.

which states that *there is an unknown function f such that "the derivative higher order function acting on it" leads to the zero function.*[w] You may hope to formally solve this equation by applying $\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}$ to the both sides and use $\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \cdot f = f$, but this actually leads to a circular argument.[x] Instead, let us proceed to apply this function to a real variable and write

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} \cdot f\right)(x) \equiv \frac{\mathrm{d}f}{\mathrm{d}x}(x) \equiv f'(x) = 0 \tag{1.36}$$

for which one usually writes down the result as

$$f(x) = \text{constant} \tag{1.37}$$

immediately. This makes sense, as the derivative of a constant is always zero.

The next simplest example would be the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} \cdot f = f \tag{1.38}$$

for the unknown function f. Solving this equation is equivalent to answering this question: *what function is equal to its derivative?*

Even though what we know and what we try to solve for are all *functions*, the traditional way of writing down such equations is in terms of *the values of functions*; in other words, we say

$$f'(x) = f(x) \tag{1.39}$$

is the differential equation, and we are trying to find the output $f(x)$ that satisfies this. Indeed, in the rest of the notes, we will mostly stick to this more traditional form.

Let us ask the question again: what is the function that is equal to its derivative? We will provide three equivalent answer.

1. We *define* a function as solution of this equation. Indeed, most of the famous mathematical functions (Hypergeometric, Bessel, Hankel, Gegenbauer, etc.) are *defined* as solutions to various differential equations. Analogously, we define

$$\exp :: \mathbb{C} \to \mathbb{C} \tag{1.40a}$$

$$\exp = x \to \exp(x) \text{ such that } \frac{\mathrm{d}\exp(x)}{\mathrm{d}x} = \exp(x) \tag{1.40b}$$

   We call this function *exponential* and usually denote it as $\exp(x) = e^x$.[y]

2. We first assume that $f(x) \neq 0$, with which we can rewrite eqn. (1.39) as

$$\frac{1}{f'(x)} = \frac{1}{f(x)} \tag{1.41}$$

   By chain rule, we have

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} \frac{\mathrm{d}x}{\mathrm{d}f(x)} = 1 \tag{1.42}$$

[w] We use the convention such that 0 can be of any type that yields the ordinary number zero ($0 :: \mathbb{C}$) as the output. In eqn. (1.35c), 0 has the type $A \to (0 :: \mathbb{C})$, which we call *the zero function.*

[x] Naively applying $\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}$ would lead to the equation $f = \frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}} \cdot 0$ but this equation is not necessarily equivalent to the original one. Indeed, both $f = \frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}} \cdot 0$ and $f = g + \frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}} \cdot 0$ would lead to the original equation if $\frac{\mathrm{d}}{\mathrm{d}x} \cdot g = 0$ as well. burada kernel kavramindan, homojen ve hetetojen denklemlerden bahset.

[y] By using various numerical methods, we can compute the value of this function for arbitrary complex numbers, e.g. $e^0 = 1$, $e^1 \sim 2.72$, $e^{1+i} \sim 1.5 + 2.3i$, and so on.

hence the above equation becomes

$$\frac{\mathrm{d}x}{\mathrm{d}f(x)} = \frac{1}{f(x)} \tag{1.43}$$

If we now replace $x = f^{-1}(y)$ where f^{-1} is the inverse of the function f,[z] we get

$$\frac{\mathrm{d}f^{-1}(y)}{\mathrm{d}y} = \frac{1}{y} \tag{1.45}$$

By integrating this function, we get

$$f^{-1}(y) = \int \frac{dy}{y} \tag{1.46}$$

If we now *define* the function *logarithm* as the right hand side, we arrive at the solution that *the function whose derivative is equal to itself is the inverse of the logarithm function*, which we call the exponential function.[aa]

Summary In the first approach, we *defined* the exponential function as the solution of the differential equation $f'(x) = f(x)$. We can then *derive* that its inverse (logarithmic function) can be given as the integral of $1/x$.[ab] In the second approach, we *defined* the logarithmic function as the integral of $1/x$, and then *derived* that its inverse (exponential function) solves the differential equation. Which one we choose is purely conventional.

What did we learn? In math, we *define* many objects as our initial data, and then *derive* other quantities based on those. What we *choose to define* is purely conventional; however, we cannot afford to define too many things and still remain consistent. For instance, in the example above, we actually show that if we give two of the following three statements, the third one is already fixed by the other two: (1) *exponential and logarithm functions are inverse of each other*, (2) *exponential function is the solution of the differential equation* $f'(x) = f(x)$, and (3) *logarithm function is the integration of* $1/x$.

1.2.2 Classification

In the beginning of the section above, we defined differential equations as any relation that contains the derivative operator $\frac{\mathrm{d}}{\mathrm{d}x}$ and an unknown function $f(x)$.[ac] There is nothing that stops us from generalizing this to multiple variables;[ad] indeed, an expression that contains the partial derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ (along with an unknown function $f(x, y)$) is *also* a differential equation. We then divide all differential equations into two categories:

$$\begin{aligned} &\text{A differential equation is called} \begin{Bmatrix} \text{ordinary} \\ \text{partial} \end{Bmatrix} \text{if there are} \\ &\text{derivatives with respect to} \begin{Bmatrix} \text{one} \\ \text{more than one} \end{Bmatrix} \text{variables.} \end{aligned} \tag{1.48}$$

[z] This means

$$f :: \mathbb{C} \to \mathbb{C} \tag{1.44a}$$
$$f^{-1} :: \mathbb{C} \to \mathbb{C} \tag{1.44b}$$
$$f = x \to f(x) \tag{1.44c}$$
$$f^{-1} = f(x) \to x \tag{1.44d}$$

[aa] We can now check that our very initial assumption $f(x) \neq 0$ is indeed satisfied.

[ab] We can show this by using the fundamental theorem of calculus.

[ac] As stated earlier, $f(x)$ is actually *not* the function but the *output* of the function f. Nevertheless, I'll abuse terminology here and there to remain more familiar to physicists.

[ad] Alternatively, we can generalize to multiple *functions*; for instance,

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = g(x)\,, \quad \frac{\mathrm{d}g(x)}{\mathrm{d}x} = -f(x)\,. \tag{1.47}$$

Such relations are called *systems of differential equations*. We will see more about such systems in § 4.

For instance,

$$\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} + f(x,y) = 0 \tag{1.49}$$

is a partial differential equation. Until the last chapter, we will only focus on *ordinary* differential equations!

We also define the *order* of a differential equation to be the highest number of derivatives in it; for instance,

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) = 0 \tag{1.50}$$

is a third order differential equation,[ae] whereas

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) + f(x) \frac{\mathrm{d}^4}{\mathrm{d}x^4} f(x) + x \frac{\mathrm{d}}{\mathrm{d}x} f(x) = 0 \tag{1.52}$$

is a fourth order one. Note that not all differential equations have to have a finite order.[af]

The differential equations are also grouped according to the *linearity* of the unknown function f. For instance, the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + f(x) = 1 \tag{1.55}$$

is called a *linear differential equation*, whereas

$$f(x) \frac{\mathrm{d}}{\mathrm{d}x} f(x) = x^3 \tag{1.56}$$

is a *nonlinear* differential equation.[ag] An easy way to check if a differential equation is linear or nonlinear is to apply the transformation $f(x) \to \lambda f(x)$ for the constant λ: if the differential equation is linear in λ (i.e. it can be written as $\lambda(\dots) + (\dots) = 0$), then the differential equation is a linear differential equation; otherwise, it is a nonlinear differential equation.

Nonlinear equations are way harder to solve than the linear equations; in fact, we actually do not know how to solve most of the nonlinear equations! In practice, one usually handles them numerically, which is beyond of the scope of this course. If you are only interested in a particular regime, you can also *linearize* a nonlinear equation around that regime, which is what most physicists do in practice. For instance, consider the nonlinear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) + \sin(f(x)) = 0 \tag{1.57}$$

If we say that we are only interested in the results $f(x) \ll 1$, then we can linearize this equation as

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) + f(x) = 0 \tag{1.58}$$

which has the solution

$$f(x) = ce^{-x} \tag{1.59}$$

[ae] See if you can convince yourself that

$$f(x) = c_0 + c_1 x + c_2 x^2 \tag{1.51}$$

for the coefficients c_i is the solution to this equation.

[af] It is perfectly possible to define the differential equation

$$\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = f(x) + 3x^2 + 3x + 1 \tag{1.53}$$

for which

$$f(x) = x^3 \tag{1.54}$$

is a solution (see the footnote v). However, clearly, this differential equation has arbitrarily high numbers of derivatives, hence it is of infinite order.

[ag] One important feature of linear differential equations is that their solutions obey *the principle of supersposition*; that is, if $f(x)$ and $g(x)$ are two solutions to the linear differential equation, then $c_1 f(x) + c_2 g(x)$ is also a solution for arbitrary constants $c_{1,2}$.

Table 1.1: Illustration of various differential equations

Example differential equation	ordinary?	linear?	homogeneous?
$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + f(x) = 0$	✓	✓	✓
$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + f(x) = x^2$	✓	✓	✗
$f(x)\frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3} + \left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2 = 0$	✓	✗	✓
$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + \sin(f(x)) = 0$	✓	✗	✗
$\frac{\partial^2 f(x,y)}{\partial x \partial y} + f(x,y) = 0$	✗	✓	✓
$\frac{\partial^2 f(x,y)}{\partial x^2} + f(x,y) = x^2$	✗	✓	✗
$f(x)\frac{\partial^3 f(x,y)}{\partial x^3} + \left(\frac{\partial f(x,y)}{\partial y}\right)^2 = 0$	✗	✗	✓
$\frac{\partial^2 f(x,y)}{\partial x \partial y} + \sin(f(x,y)) = 0$	✗	✗	✗

which satisfies our necessary condition for $x \gg 1$.[ah]

A last classification we can do with our differential equations is their *homogeneity*: a differential equation is said to be *homogeneous* if it is invariant under the scaling of the unknown function. This is just a fancy way of saying that the differential equation does not change even if we replace $f(x)$ with $\lambda f(x)$ for an unknown constant λ.

We can summarize the classification of all differential equations with examples as given in Table 1.1

[ah] We can actually solve the full nonlinear differential equation eqn. (1.57); the result is

$$f(x) = 2\,\mathrm{arccot}\left(\frac{2e^x}{c}\right) \tag{1.60}$$

which matches the linearized result in the regime it is valid, i.e.

$$\lim_{x\to\infty} 2\,\mathrm{arccot}\left(\frac{2e^x}{c}\right) = \lim_{x\to\infty} ce^{-x} \tag{1.61}$$

2

Linear equations with constant coefficients

2.1 Linear mappings and kernels

Formally, we could write down the most generic linear ordinary differential equation for the unknown function f as

$$g\left(x, \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = h(x) \tag{2.1}$$

for arbitrary known functions g and h. Indeed, this is a linear equation in the function f, and it has only one kind of derivative, $\frac{\mathrm{d}}{\mathrm{d}x}$, hence it is an ordinary differential equation.

Let's assume that we are given such an equation for known g and h, and we are trying to solve for f. A naive attempt would be to write down

$$f(x) = \frac{1}{g\left(x, \frac{\mathrm{d}}{\mathrm{d}x}\right)} h(x) \tag{2.2}$$

which looks like a total nonsense! Nevertheless, we cannot help but realize that it does somewhat work in some cases; for instance, for

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = x^2 \tag{2.3}$$

we can write down

$$f(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} x^2 \tag{2.4}$$

which we can rewrite as

$$f(x) = \int dx x^2 = \frac{x^3}{3} + \text{constant} \tag{2.5}$$

by observing that integral is *the inverse of derivative.*[ai]

We need to be careful with such manipulations, but physicists *tend to* define things *formally*, which allows such expressions. For instance, we could say that *the formal solution* to the differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + c^2\right) f(x) = 0 \tag{2.6}$$

is

$$f(x) = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + c^2\right)^{-1} 0 \tag{2.7}$$

For a physicist, there is nothing wrong with writing things like the equation above *as long as we are careful with what we mean*! To spell out what we really mean with such an equation, we need to set up some terminology.

[ai] Rigorously speaking, we are referring to indefinite integrals (also known as antiderivatives or Newton integrals).

Remember how we defined the derivative higher order function (or its integer powers) in eqn. (1.20a):

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} :: (\mathrm{A} \to \mathrm{B}) \to (\mathrm{A} \to \mathrm{B}) \tag{2.8a}$$

$$f :: \mathrm{A} \to \mathrm{B} \tag{2.8b}$$

$$f^{(n)} :: \mathrm{A} \to \mathrm{B} \tag{2.8c}$$

The operation of taking derivatives is *a map of functions to functions*; in fact, it is a *linear map*![aj] Linear maps are really useful when we work with vectors, but we will see below that an important notion called *kernel* can be extended from vector spaces to the functions as well.[ak]

In vector spaces linear transformations are implemented by matrices; for instance, the transformation "clockwise rotation by $\pi/4$" on $2d$ vectors can be implemented by the matrix

$$R(-\pi/4) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{2.12}$$

which indeed rotates any vector $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ to its rotated version $R(-\pi/4)\cdot\vec{v}$; for instance, the unit vector pointing to NorthEast direction on a map —i.e. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$— gets rotated to the vector pointing to the East by this 45 degrees of clockwise rotation:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.13}$$

In fact *a general counterclockwise rotation by an angle θ* can be implemented by the matrix

$$R(-\pi/4) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{2.14}$$

The *kernel of a map* (or equivalently the kernel of the matrix that implements that map) is the set of vectors that are mapped to *zero vector*; for instance, we can show that the only such vector for the rotation matrix is the zero vector itself; in other words,

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.15}$$

is true only if $a = b = 0$; thus, we write

$$\ker\left[R(\theta)\right] = \{\vec{0}\} \tag{2.16}$$

which means *the only vector that can be rotated to the zero vector is the zero vector itself.* When said this way, it clearly makes sense!

Let's look at another example: we define the matrix S as

$$S = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \tag{2.17}$$

[aj] We can easily see the linearity by noting the relation

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(c_1 f(x) + c_2 g(x)\right) = c_1\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x) + c_2\frac{\mathrm{d}^n}{\mathrm{d}x^n}g(x) \tag{2.9}$$

for arbitrary coefficients c_1 and c_2.

[ak] The analogy is as follows: functions are like vectors, and linear transformations due to derivatives are like matrix multiplications. Indeed, a matrix M (say $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$) acting on a vector v (say $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$) is a linear mapping, just as the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ turning the function x^2 into $2x$.

The analogy extends to the equations. We could solve $M\cdot w = v$ for the unknown vector w, similar to how we solve $\frac{\mathrm{d}}{\mathrm{d}x}f(x) = x^2$ for the function f. In fact such analogies can be made more precise if we realize that a function f is in some sense an infinite dimensional vector. Indeed, in a neighborhood containing the point c in which the function f is analytic, we can just do a Taylor expansion and rewrite $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} f_n x^n \tag{2.10}$$

where f_n can be viewed as an infinite-dimensional vector $f_n = (f_0, f_1, \ldots)$.[aℓ]

[aℓ] If we take a step back, we can actually realize that the converse is also true (in fact, it is *generically* true): *any vector v is simply a function from integers to the domain of the components of the vector.*

What do we mean by that? Consider the vector $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$. This vector is equivalent to the set of relations $v_1 = 1$, $v_2 = 0$, and $v_3 = -3$. But that is simply a function

$$v :: \mathbb{N} \to \mathbb{R} \tag{2.11a}$$

$$v = n \to \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n = 2 \\ -3 \text{ if } n = 3 \\ \text{undefined otherwise} \end{cases} \tag{2.11b}$$

This process can be generalized to any finite or infinite dimensional vector.

If we now look at the *kernel of this linear transformation*, we find a non-trivial result; in fact, we can immediately write down

$$\ker[S] = \left\{ \vec{0}, \begin{pmatrix} 2a \\ -a \end{pmatrix} \right\} \tag{2.18}$$

which means not only the zero vector gets mapped to zero vector, but also any vector of the form $\begin{pmatrix} 2a \\ -a \end{pmatrix}$ becomes zero under the action of this matrix. Indeed, we see that

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2a \\ -a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.19}$$

What does this mean? And what is the action of this linear transformation? Just like the rotation matrix rotates any input vector, this S matrix also transforms the input vectors, but it actually *squeezes* them. Indeed, we see that for any vector pointing in any direction, the action of this transformation squeezes them into the $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ direction. We can see this explicitly:

$$S \cdot \vec{v}_{\text{input}} = \vec{v}_{\text{output}} \tag{2.20a}$$

$$\vec{v}_{\text{input}} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.20b}$$

$$\vec{v}_{\text{output}} = (a + 2b) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.20c}$$

Summary: We have seen with examples that some linear transformations (such as rotation) has a *trivial kernel*,[am] whereas other transformations (such as squeezing) may have a nontrivial kernel.

[am] We say that a kernel is trivial if it only includes the identity element ($\vec{0}$ vector in the case of vector spaces).

Quick check in vector spaces: Whether a linear transformation has a trivial kernel or not can immediately be checked in the case of vector spaces by computing the *determinant* of the matrix that implements that transformation. If the determinant is zero (e.g. $\det S = 0$), then the kernel is nontrivial; otherwise ($\det R = 1$) the kernel is trivial.

The importance of nontrivial kernel: If the kernel is nontrivial, then the transformation is not uniquely invertible. For instance, if we have

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.21}$$

Then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a solution, but so is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. In fact, the full family of solutions is given as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2a \\ -a \end{pmatrix} \tag{2.22}$$

On the other hand, the rotation having a trivial kernel makes sure that we have a unique answer; for instance,

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 2\end{pmatrix} \tag{2.23}$$

has the unique answer

$$\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 3\end{pmatrix} \tag{2.24}$$

Back to the differential equations: The story with the matrices immediately carries over to the differential equations: the differential operators have nontrivial kernels, and these results are called *homogeneous solutions*. For the general differential equation

$$g\left(x,\frac{\mathrm{d}}{\mathrm{d}x}\right)f(x)=h(x) \tag{2.25}$$

the solution then becomes

$$f(x)=p(x)+\ker\left[g\left(x,\frac{\mathrm{d}}{\mathrm{d}x}\right)\right] \tag{2.26}$$

where $p(x)$ is called the *particular solution*, and the elements of the kernel are the homogeneous solutions.

2.2 Homogeneous solutions

2.2.1 Basics

We have discussed in § 1.2.1 that the solution to the differential equation $f'(x)=f(x)$ is given as[an]

$$f(x)=e^x \tag{2.27}$$

We can actually generalize this to[ao]

$$\left(\frac{\mathrm{d}}{\mathrm{d}y}-\lambda\right)e^{\lambda y}=0 \tag{2.29}$$

which says the *linear ordinary differential equation with constant coefficient*

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-\lambda\right)f(x)=0 \tag{2.30}$$

has the *solution*

$$f(x)=e^{\lambda x} \tag{2.31}$$

In the fancy language, we can now write this result as

$$\ker\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}-\lambda\right)\right]=\left\{0,e^{\lambda x}\right\} \tag{2.32}$$

[an] As we discussed in that section, this result is either a definition or a derived result depending our conventions.

[ao] One way to show this is the judicious use of the chain rule as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^x=e^x &\xrightarrow{\text{define } x=\lambda y}\frac{\mathrm{d}}{\mathrm{d}x}e^{\lambda y}=e^{\lambda y}\\ &\xrightarrow{\text{use chain rule}}\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}e^{\lambda y}=e^{\lambda y}\\ &\xrightarrow{\text{use } y=x/\lambda}\frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}y}e^{\lambda y}=e^{\lambda y}\\ &\xrightarrow{\text{rewrite}}\left(\frac{\mathrm{d}}{\mathrm{d}y}-\lambda\right)e^{\lambda y}=0\end{aligned} \tag{2.28}$$

which means that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - \lambda\right) f(x) = h(x) \quad \Rightarrow \quad f(x) = p(x) + ce^{\lambda x} \tag{2.33}$$

for the arbitrary variable c, where we will discuss the computation of particular solution $p(x)$ later.

One immediate observation we can make is that $e^{\lambda x}$ would still be a solution if there were more terms to the left of the equation; in other words,

$$g\left(x, \frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \lambda\right) f(x) = 0 \tag{2.34}$$

is still satisfied for $f(x) = e^{\lambda x}$. This becomes particularly interesting if $g\left(x, \frac{\mathrm{d}}{\mathrm{d}x}\right)$ is a product of $\left(\frac{\mathrm{d}}{\mathrm{d}x} - a\right)$, i.e.

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - r_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - r_2\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}x} - r_n\right) f(x) = 0 \tag{2.35}$$

Clearly $e^{r_n x}$ is a solution, but as these terms commute with each other, we can immediately write down the full solution as[ap]

$$f :: \mathbb{C} \to \mathbb{C} \tag{2.36a}$$

$$f = x \to \sum_{i=1}^{n} c_i e^{r_i x} \tag{2.36b}$$

for arbitrary constants c_i.

Differential equations are usually given in the form

$$\left(a_0 + a_1\frac{\mathrm{d}}{\mathrm{d}x} + a_2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \cdots + a_n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\right) f(x) = 0 \tag{2.37}$$

which can be brought to the form eqn. (2.35) by simply finding the roots of the equation[aq]

$$a_0 + a_1 r + a_2 r^2 + \cdots + a_n r^n = 0 \tag{2.38}$$

If the coefficients a_i are simply complex numbers (or real numbers as a special case of complex numbers), we can always find n complex roots r_i![ar]

2.2.2 Repeated roots

Consider the differential equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - r_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - r_2\right) f(x) = 0 \tag{2.39}$$

which has the solution $f(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. If we now do a change of parameters as

$$r_1 = r\,, \quad r_2 = r + \delta\,, \quad c_1 = a - \frac{b}{\delta}\,, \quad c_2 = \frac{b}{\delta} \tag{2.40}$$

[ap] This follows from the principle of superposition, see footnote ag.

[aq] This equation is called *characteristic equation* of the given system.

[ar] The field of complex numbers is algebraically closed, hence such polynomials *always* have solutions. In contrast, the field of real numbers is *not* algebraically closed; for instance, $x^2+1 = 0$ has no real root. For more information on these, see *fundamental theorem of algebra.*

our statement becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-r\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}-(r+\delta)\right)f(x)=0 \quad f(x)=ae^{rx}+b\frac{e^{(r+\delta)x}-e^{rx}}{\delta} \tag{2.41}$$

If we now take the limit $\delta \to 0$ and recognize the definition of derivative, we arrive at

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-r\right)^2 f(x)=0 \quad \to \quad f(x)=ae^{rx}+bxe^{rx} \tag{2.42}$$

The way we arrived at this curious result is not satisfactory: we did a particular transformation in eqn. (2.40) and we do not have a strong reason to choose that transformation. For instance, if we instead choose

$$r_1=r\,,\quad r_2=r+\delta\,,\quad c_1=a-b\,,\quad c_2=b \tag{2.43}$$

and then take the limit $\delta \to 0$, we end up with

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-r\right)^2 f(x)=0 \quad \xrightarrow{???} \quad f(x)=ae^{rx} \tag{2.44}$$

We missed the second piece of $f(x)$ in eqn. (2.42).

What is the resolution of this discrepancy? We have two potential scenarios: **(a)** xe^{ax} is *a spurious solution*,[as] or **(b)** eqn. (2.44) misses one of the solutions.

We can check it straightforwardly that the option **(b)** is the correct case,[at] indicating that out choice of reparametrization of the variables in terms of infinitesimal variable δ affects which solutions we obtain. This then begs the question: *can we potentially have more solutions?*

We have mathematical arguments why a second order differential equation should have two solutions,[au] so we can already infer that eqn. (2.42) is the full solution; however, let's see another method to derive why this is the case.

Define a new function g such that $f(x)=g(x)e^{rx}$.[av] If we insert this into the original differential equation, we immediately see that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-r\right)^2 f(x)=0 \quad \xrightarrow{f(x)=g(x)e^{rx}} \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x)=0 \tag{2.46}$$

which tells us that *the most general result* is $f(x)=(ax+b)e^{rx}$. In fact, this derivation generalizes, i.e.

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-r\right)^n f(x)=0 \quad \xrightarrow{f(x)=g(x)e^{rx}} \quad \frac{\mathrm{d}^n}{\mathrm{d}x^n}g(x)=0 \tag{2.47}$$

yielding $f(x)=(a_1+a_2x+\ldots a_nx^{n-1})e^{rx}$.

With the discussion above, we can now write down the most general homogeneous solution to a linear ordinary differential equation with constant coefficients:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-r_1\right)^{m_1+1}\left(\frac{\mathrm{d}}{\mathrm{d}x}-r_2\right)^{m_2+1}\cdots\left(\frac{\mathrm{d}}{\mathrm{d}x}-r_n\right)^{m_n+1} f(x)=0 \tag{2.48a}$$

$$\Rightarrow \qquad f(x)=\sum_{i=1}^{n}\left[\left(\sum_{k=0}^{m}c_{ik}x^k\right)e^{r_ix}\right] \tag{2.48b}$$

[as] Spurious solutions are fake results that emerge as solutions even though they actually do not solve the problem.

[at] We only need to check

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-r\right)^2 (xe^{rx})=0 \tag{2.45}$$

[au] Maybe expand on this more.

[av] Note that we can do this *without a loss of generality*!

for arbitrary coefficients c_{ij}.

2.2.3 Examples

RLC circuits, damper-spring systems, traffic models, etc.

2.3 Laplace transform

Consider the following higher order function:[aw]

$$\mathfrak{IT} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{2.50a}$$

$$\mathfrak{IT} = (x \to f(x)) \to \left| s \to \int_{\alpha}^{\beta} K(x,s) f(x) dx \right| \tag{2.50b}$$

where $\mathfrak{IT}$ is an *integral transform*, i.e. it maps a function to another one by using the integration operation. The function K above is called *the kernel of the tranformation*: different kernels (along with different integration ranges) lead to different integral transforms.

The Laplace transform is a special kind of an integral transformation:

$$\mathcal{L} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{2.51a}$$

$$\mathcal{L} = (x \to f(x)) \to \left| s \to \int_{0}^{\infty} e^{-xs} f(x) dx \right| \tag{2.51b}$$

which plays a immense role in the analysis of linear ordinary differential equations with constant coefficients because such equations become algebraic under this transformation. To see this, consider how the laplace transform interacts with the derivative operation: replace f with g' above, and integrate by parts

$$\mathcal{L} = (x \to g'(x)) \to \left(s \to \left[s \int_{0}^{\infty} e^{-xs} g(x) dx - g(0) + \lim_{x\to\infty} e^{-xs} g(x) \right] \right) \tag{2.52}$$

We will assume that the last piece is zero, which is a necessary condition for the Laplace transform to be well-defined in the first place.[ax] Thus

$$\mathcal{L} \cdot g' = s \to (s(\mathcal{L} \cdot g)(s) - g(0)) \tag{2.53}$$

or in a more conventional notation, we state

$$\frac{\mathrm{d}g(x)}{\mathrm{d}x} \xrightarrow{\text{Laplace transform}} sG(s) - g(0) \tag{2.54}$$

where $G(s)$ is the laplace transform of $g(x)$.

One can repeat this process iteratively for higher numbers of derivative; in fact, we can immediately write down the Laplace transform of $n-$the derivative of a function:

$$\left(\mathcal{L} \cdot g^{(n)}\right)(s) = s^n \left(\mathcal{L} \cdot g\right)(s) - \sum_{i=0}^{n-1} s^{n-i-1} g^{(i)}(0) \tag{2.55}$$

[aw] Note that the letters on the left hand side of an arrow are *placeholders*, i.e. they do not inherently carry information. Such parameters are called dummy variables in math (or scooping variables in computer science) and they are ubiquitous in math and physics; for instance, the integrals $\int dx f(x)$ and $\int dy f(y)$ are the same expression as x and y are dummy variables. Similarly, the expressions $x \to f(x)$ and $y \to f(y)$ are equivalent.

It gets complicated with the higher order functions as they include multiple arrows; in such cases, the left hand side of *each arrow* contains only placeholders for *the right hand side of that particular arrow*. For example, let us rewrite eqn. (2.50) in a colorful way:

$$\mathfrak{IT} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{2.49a}$$

$$\mathfrak{IT} = (x \to f(x)) \to \left(s \to \int_{\alpha}^{\beta} K(x,s) f(x) dx \right) \tag{2.49b}$$

Variables of the same color can be changed as they are dummy variables for the same color arrow (in the case of the color magenta, the variables are dummy variables of the integration operation). For instance, following expressions are all equivalent:

$$\mathfrak{IT} = (x \to f(x)) \to \left(s \to \int_{\alpha}^{\beta} K(x,s) f(x) dx \right)$$

$$\mathfrak{IT} = (y \to f(y)) \to \left(z \to \int_{\alpha}^{\beta} K(x,z) f(x) dx \right)$$

$$\mathfrak{IT} = (y \to g(y)) \to \left(z \to \int_{\alpha}^{\beta} K(s,z) g(s) ds \right)$$

Note that the letters K, α, β are not dummy variables as they are externally fixed. Nevertheless, we *can* turn them into dummy variables of the equal sign $=$ by defining them in the left hand side of $=$; e.g.

$$\mathfrak{IT}_{K,\alpha,\beta} = (y \to g(y)) \to \left(z \to \int_{\alpha}^{\beta} K(s,z) g(s) ds \right)$$

$$\mathfrak{IT}_{T,\gamma,\lambda} = (y \to g(y)) \to \left(z \to \int_{\gamma}^{\lambda} T(s,z) g(s) ds \right)$$

are equivalent expressions —just like $f(x) = x^2$ and $f(y) = y^2$ being equivalent expressions.

[ax] Otherwise, the integral in the definition does not converge.

We can now justify our previous statement of ***Laplace transform converts linear ordinary differential equations with constant coefficients into algebraic ones***! Start with the most generic such differential equation:

$$\sum_{i=0}^{n} a_i f^{(i)}(x) = g(x) \tag{2.56}$$

which is *homogeneous* if $g(x) = 0$ and nonhomogeneous otherwise. If we take the Laplace transform of this equation, we end up with

$$\sum_{i=0}^{n} a_i \left[s^i F(s) - \sum_{k=0}^{i-1} s^{i-k-1} f^{(k)}(0) \right] = G(s) \tag{2.57}$$

where $F(s) := (\mathcal{L} \cdot f)(s)$ and $G(s) := (\mathcal{L} \cdot g)(s)$ are defined for brevity. By using algebra, we can rewrite this equation in the form

$$F(s) = \frac{\sum_{i=0}^{n-1} f^{(i)}(0) \left[\sum_{k=1+i}^{n} a_k s^{k-i-1} \right]}{\sum_{i=0}^{n} a_i s^i} + \frac{G(s)}{\sum_{i=0}^{n} a_i s^i} \tag{2.58}$$

Let us comment on this result a little bit. **Firstly**, we can immediately state that the solution $f(x)$ to the differential equation in eqn. (2.56) is simply the *inverse Laplace transform* of $F(s)$. Even though this is a well-defined transformation that we can introduce, we actually do not need it: we will discuss other methods to obtain $f(x)$ from $F(s)$. **Secondly**, we can actually see that the first piece is the homogeneous solution to the differential equation, and the second piece is the particular solution: Laplace transform allowed us to solve both of them at once!

Consider the simple case of $n = 2$:

$$F(s) = \frac{f(0)\,(a_1 + a_2 s) + f^{(1)}(0) a_2}{a_0 + a_1 s + a_2 s^2} + \frac{G(s)}{a_0 + a_1 s + a_2 s^2} \tag{2.59}$$

If r_1 and r_2 are two distinct roots of $a_0 + a_1 s + a_2 s^2 = 0$, we can simply write down this expression as *bla bla bla bla*to be written later, probably next year.

We have covered several topics in class but I will not be able to type them in time. So I'm postponing that to next year; after all, all of those topics are already in the textbook — chapter 6 of Elementary Differential Equations and Boundary Value Problems" by Boyce and Diprima (10th edition). The summary is as follows:

1. Derive laplace transforms of common functions

2. Solving homogeneous differential equations using Laplace transformation, and "proof" of why characteristic equation method works

3. Discussion of particular solutions in Laplace domain; example: RLC circuit with an AC input

4. Discussion of how it is tiresome to repeat the computations for each nonhomogeneous piece and why we need a universal solution true for any non-homogeneous piece. For this, we need to find an operation that maps to multiplication in Laplace domain

5. Derive $H(s) = F(s)G(s)$ means $h(x)$ =convolution of $f(s)$ and $g(x)$, i.e. derive convolution

6. Introduce impulse response: it is the solution to the same differential equation with nonhomogeneous part $H(s) = 1$ in laplace domain (with $i(0) = i'(0) = \cdots = 0$).

7. Introduce Dirac-delta distribution, discuss why it is a generalized function but not a well defined function. Show that it maps to 1 as needed.

8. Write down the most generic solution to a linear ordinary differential equation with constant coefficients. Show examples.

3 Linear homogeneous equations with functional coefficients

Summary of what we have discussed in class (to be typed later, maybe next year):

1. Differential equations with functional coefficients *do not necessarily have generic analytic solutions*: we do not know how to solve them except the isolated cases!

2. Explicit solutions of $\left(x\frac{\mathrm{d}}{\mathrm{d}x}+a\right)f(x)=0$ and $\left(x^k\frac{\mathrm{d}}{\mathrm{d}x}+a\right)f(x)=0$ for $k\neq 1$.

3. How these differential operators can be chained, and how it leads to the following form:
$$\left(x^k\frac{\mathrm{d}}{\mathrm{d}x}+a_1\right)\cdots\left(x^k\frac{\mathrm{d}}{\mathrm{d}x}+a_n\right)f(x)=0 \tag{3.1}$$
for both $k=1$ & $k\neq 1$.

4. How these equations are secretly related to the differential equations with constants coefficients through a reparametrization, i.e.
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}&=\frac{1}{1-n}x^n\frac{\mathrm{d}}{\mathrm{d}x}\text{ for }y=x^{-n+1}\\ \frac{\mathrm{d}}{\mathrm{d}y}&=x\frac{\mathrm{d}}{\mathrm{d}x}\text{ for }y=\log(x)\end{aligned} \tag{3.2}$$

5. Lesson: The differential equations with constant coefficients are nice guys with straightforward general solutions: do your best to check if a given differential equation can be brought to that form! Usually, the physics of the problem gives us insight as to whether that would be possible.

6. Discuss why $k=1$ case above is treated differently, show a little bit about scale invariance, and informally mention the Mellin transform: we do not need to know this for this course (but it is super relevant in modern physics)!

7. Introduce Euler equations: show how this is solvable simple because of the scale invariance and how it is actually a subset of similar higher order equations.

8. Discuss change of variables to make equations constant coefficients:
$$\begin{gathered}f''(x)+p(x)f'(x)+q(x)f(x)=0\\ \Rightarrow\\ (u')^2f''(u)+(u''(x)+u'(x)p(x))y'(u)+q(x)y(u)=0\end{gathered} \tag{3.3}$$
for a change of parameter $x\to u(x)$. For this to be constant coefficient, we need $u(x)=\int\sqrt{q(x)}dx$ and $\frac{u''(x)+u'(x)p(x)}{(u')^2}$ =constant.

9. An example equation where change of variables would work: $ty'' + (t^2 - 1)y' + t^3 y = 0$.

10. Reparametrization is harder to do generically for higher orders! For those equations (also for second order equations without reparametrization), check if the function $f(x)$ is missing. If that is missing, we can lower the order of differential equation by writing it in terms of a new function $g(x) = f'(x)$. If both $f(x)$ and $f'(x)$ are missing, use $g(x) = f''(x)$, and so on!

11. Examples (page 135 of textbook): $xf''(x) + f'(x) = 0$, $x^2 f''(x) + 2xf'(x) = 2$. Solve these!

12. Another thing to check is if the equation is *exact*, i.e. if it can be rewritten as a total derivative:

$$\left[p_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n} + \cdots + p_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + p_0(x)\right] f(x) = 0$$

$$\overset{???}{\Longrightarrow}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left[q_{n-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \cdots + q_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + q_0(x)\right] f(x)\right) = 0$$

If that is the case, then we can turn an order-n (non)homogeneous differential equation into an order-$(n-1)$ nonhomogeneous one.

13. For a second order differential equation $p(x)f''(x) + q(x)f'(x) + r(x)f(x) = 0$, the condition $p''(x) - q'(x) + r(x) = 0$ is sufficient for it to be exact.

14. Examples (page 157 of textbook): $f''(x) + xf'(x) + f(x) = 0$, $x^2 f''(x) + xf'(x) - f(x) = 0$.

15. **Reduction of order:** if we already know k solutions of an order n differential equation, we can use that information to transform the system into an order $n - k$ differential equation with no known solutions. This is rather useful as lower differential equations are easier to solve, and it becomes extremely useful if we know one solution of a second order differential equation as first order differential equations are always formally solvable (we will discuss this in more detail later).

16. Examples (page 174 of textbook): $xf''(x) - f'(x) + 4x^3 f(x) = 0$ with $f_1(x) = \sin(x^2)$, $(x-1)f''(x) - xf'(x) + f(x) = 0$ with $f_1(x) = e^x$.

4

Linear nonhomogeneous equations with functional coefficients

5

First order differential equations and their formal solutions

6

Partial differential equations

A

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