Name:	
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Make-up Examination

Phys209: Mathematical Methods in Physics I

2025/01/20

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) any of my work *outside the designated areas* in the "fill-in the blank questions" will not be graded;

(2) I take *full responsibility* for any ambiguity in my selections in "multiple choice questions";

(3) any of my work *outside the answer boxes* in the "classical questions" will not be graded;

(4) any page which does not contain *both my name and student id* will not be graded;

(5) any extra sheet that I may use are for my own calculations and will *not* be graded.

Signature: ____________________

This exam has a total of 2 questions, some of which may be for bonus points. You can obtain a maximum grade of 34+0 from this examination.

Question	Points	Score
1	26	
2	8	

Question	Points	Score
Total:	34	

1 Notations & Conventions

This section contains various useful definitions to refer while solving the problems. Note that it might contain additional information not covered in class, so please do not panick: the questions do not necessarily refer to *everything* in this section.

- **The non-negative integer power** of an object A (denoted A^n) is defined recursively as
$$A^0 = \mathbb{I}\ , \quad A^n = A \cdot A^{n-1} \ \ \forall n \geq 1 \tag{1}$$
with respect to the operation $\cdot$ (such as matrix multiplication or differentiation) and its identity object $\mathbb{I}$.

- **Exponentiation of an object** A (denoted e^A) is
$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{2}$$
where A^n is the $n-$th power of the object A.

- **Logarithm of an object** A (denoted $\log A$) is defined as the inverse of the exponentiation. For objects for which the exponentiation is not a monomorphism (such as complex numbers), logarithm is a *relation* (also called multi-valued function). Conventionally, one imposes restrictions on the domain to ensure that logarithm acts as a function; for instance, for a complex number $z = re^{i\theta} \in \mathbb{C}$ with $(r, \theta) \in (\mathbb{R}^+, \mathbb{R})$, we can define $\log z = i\theta_p + \log r$ where $0 \leq \theta_p < 2\pi$ is called *the principal value of* θ.

- **The generalized power of an object** A (denoted A^α) is defined as
$$A^\alpha = e^{\alpha \log A} \tag{3}$$
If exponentiation is not a monomorphism when acting on the domain of A, A^α is not a function but a relation *unless* a principle domain is selected (similar to the logarithm).

- **Generalized exponentiation of an object** A (denoted α^A) is defined as
$$\alpha^A = e^{A \log \alpha} \tag{4}$$
Depending on the available values for $\log \alpha$, α^A may mean multiple different functions. However, each one is *still* a proper function, not a multi-valued function.

- **Trigonometric functions** $\cos, \sin, \tan, \cot, \csc, \sec$ are defined in terms of the exponential via the equations
$$e^{\pm iA} = \cos(A) \pm i \sin(A)\ , \ \tan(A) = \frac{1}{\cot(A)} = \frac{\sin(A)}{\cos(A)} \tag{5}$$
$$\csc(A)\sin(A) = 1\ , \ \sec(A)\cos(A) = 1 \tag{6}$$

- **Hyperbolic functions** cosh, sinh, tanh, coth, csch, sech are defined in terms of the exponential via equations
$$e^{\pm A} = \cosh(A) \pm \sinh(A), \tanh(A) = \frac{1}{\coth(A)} = \frac{\sinh(A)}{\cosh(A)} \tag{7}$$
$$\mathrm{csch}(A)\sinh(A) = 1\ , \ \mathrm{sech}(A)\cosh(A) = 1 \tag{8}$$

- **Inverse Trigonometric/Hyperbolic functions** are denoted with an *arc* prefix in their naming, i.e. $\arcsin(x) \coloneqq \sin^{-1}(x)$. Like logarithm, these objects are *relations* (not functions) unless their domain is restricted.

- **The Kronecker symbol** (Kronecker-delta) is defined
$$\delta : \{\mathbb{Z}, \mathbb{Z}\} \to \mathbb{Z} \tag{9}$$
$$\delta = \{i, j\} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{10}$$

- **The Dirac-delta generalized function** δ is (for all practical purposes of a Physicist) defined via the relation
$$\int_{\mathcal{A}} f(y)\delta(x-y)dy = \begin{cases} f(x) & \text{if } x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$
A useful representation of this generalized function is
$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} \tag{12}$$

- **Heaviside generalized function** θ is (for all practical purposes of a Physicist) defined via the relations
$$\int_a^b \theta(x) f(x) dx = \begin{cases} \int_a^b f(x)dx & \text{if } a \geq 0 \\ \int_0^b f(x)dx & \text{if } a < 0 \end{cases} \tag{13}$$
This definition implies that $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$; however, it *does not fix* $f(0)$. We choose *the convention* $f(0) = 1/2$; this ensures
$$\mathrm{sgn}(x) = 2\theta(x) - 1 = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases} \tag{14}$$

- **A particular permutation of n objects** is denoted as $(i_1 i_2 \ldots i_n)$ where $i_1 \neq i_2 \neq \ldots \neq i_n \in \{1, \ldots, n\}$. A permutation $(i_1 \ldots i_n)$ is said to be an even (odd) permutation of $(k_1 \ldots k_n)$ if the two are identical after the permutation of an even (odd) number of adjacent indices. For example, (2431) is an even permutation of (2143) and an odd permutation of (2134).

- **Levi-Civita symbol** ϵ is defined as
$$\epsilon : \{\mathbb{Z}^+, \ldots, \mathbb{Z}^+\} \to \mathbb{Z} \tag{15}$$
$$\epsilon = \{a_1, \ldots, a_n\} \to \begin{cases} 1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an even permutation of } (12\ldots n) \\ -1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an odd permutation of } (12\ldots n) \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

- **The determinant function** (denoted det) is defined

$$\det : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{17}$$

$$\det = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_{i_1,\dots,i_n} \epsilon_{i_1\dots i_n} a_{1i_1}\dots a_{ni_n} \tag{18}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i,j$.

- **The adjugate function** (denoted adj) is defined as

$$\mathrm{adj} : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{19}$$

$$\mathrm{adj} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \tag{20}$$

for

$$b_{k_n i_n} = \sum_{\substack{i_1,\dots,i_{n-1} \\ k_1,\dots,k_{n-1}}} \frac{\epsilon_{i_1\dots i_n}\epsilon_{k_1\dots k_n} a_{i_1k_1}\dots a_{i_{n-1}k_{n-1}}}{(n-1)!} \tag{21}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i,j$.

- **Inverse of an object** A is denoted as A^{-1} and is defined with respect to an operation "$\cdot$" and its identity element $\mathbb{I}$ via the the equations $A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{I}$. If "$\cdot$" is matrix multiplication, then

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det A} \tag{22}$$

- **The trace function** (denoted tr) is defined as

$$\mathrm{tr} : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{23}$$

$$\mathrm{tr} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_i a_{ii} \tag{24}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i,j$.

- **Wronskian matrix** of a set of functions $\{f_1(x),\dots,f_n(x)\}$ is defined as a square matrix where the first row is the set of the functions and the $i-$th row is $(i-1)-$th derivative of the functions for all $n \geq i \geq 2$.

- **A complex number** z is (for all practical purposes of a Physicist) a pair of two real numbers (x,y) where one can construct z via $z = x + iy$ (i is called *the imaginary unit* with the property $i^2 = -1$); conversely, one can extract x and y via $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

- **Complex conjugation** (denoted $*$) is a function defined to act on complex numbers as

$$* : \mathbb{C} \to \mathbb{C} \tag{25}$$

$$* = z \to (z^* = \mathrm{Re}(z) - i\,\mathrm{Im}(z)) \tag{26}$$

- **Matrix transpose** (denoted T) is a function defined

$$T : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{27}$$

$$T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \tag{28}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i,j$.

- **Hermitian conjugation** (also called *conjugate transpose*, *adjoint*, or *dagger*) is a function defined as

$$\dagger : \mathfrak{M}_{n\times n}(\mathbb{C}) \to \mathfrak{M}_{n\times n}(\mathbb{C}) \tag{29}$$

$$\dagger = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a^*_{11} & a^*_{21} & \dots & a^*_{n1} \\ a^*_{12} & a^*_{22} & \dots & a^*_{n2} \\ \dots \\ a^*_{1n} & a^*_{2n} & \dots & a^*_{nn} \end{pmatrix} \tag{30}$$

- **Characteristic polynomial** of any square matrix A:

$$\det(A - \lambda_i \mathbb{I}) = 0 \tag{31}$$

- **Laplace transform** is an integral transform (denote $\mathcal{L}$) which converts a function $f : \mathbb{R} \to \mathbb{R}$ into another function $\hat{f} = \mathcal{L}(f)$ such that

$$\hat{f} : \mathbb{C} \to \mathbb{C}\ , \qquad \hat{f}(s) = \int_0^\infty f(x) e^{-xs} dx \tag{32}$$

For *meromorphic* $\hat{f}$ (i.e. $\frac{\text{polynomial}}{\text{polynomial}}$), the inverse is computed by rewriting $\hat{f}(s)$ as a sum $\sum_i a_i (s + r_i)^{-n_i - 1}$ which is clearly (for some $c_{k,\ell}$) the Laplace transform of $f(x) = \sum_i e^{-r_i x} \left(c_{i,1} + c_{i,2} x + \dots c_{i,n_i} x^{n_i}\right)$. Formally,

$$f : \mathbb{R} \to \mathbb{R}\ , \qquad f(x) = \int_{\gamma - i\infty}^{\gamma + i\infty} \hat{f}(s) e^{xs} \frac{ds}{2\pi i} \tag{33}$$

where the *contour integral* in the complex plane is chosen appropriately based on the convergence.

- **Convolution** of two functions f and g (denote $f * g$) is the operation that becomes multiplication in the Laplace domain, i.e. $\mathcal{L}(f * g) \equiv \mathcal{L}(f)\mathcal{L}(g)$; equivalently,

$$(f * g)(x) = \int_0^x f(y) g(x - y) dy \tag{34}$$

- **Commutator** is a higher order function which takes two functions $f, g : \mathcal{A} \to \mathcal{A}$ for any type $\mathcal{A}$, and gives a new function $[f,g] : \mathcal{A} \to \mathcal{A}$ by cascading their action. It is defined on an object $x \in \mathcal{A}$ as $[f,g](x) = f(g(x)) - g(f(x))$.

- **Polar coordinates in $\mathbb{R}^d$** $(r, \theta_1, \dots, \theta_{d-1})$ are defined in terms of the Cartesian coordinates $(x_1, \dots, x_d)$ as

$$x_1 = r\cos(\theta_1)\ , \quad x_d = x_{d-1}\tan(\theta_{d-1}) \tag{35}$$

$$x_i = x_{i-1}\tan(\theta_{i-1})\cos(\theta_i) \quad \text{for } 1 < i < d \tag{36}$$

In two-dimensions, this reduces to the familiar polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$; in 3 ($> 3$) dimensions, it is also called *(hyper)spherical coordinates.*

- **Cylindrical coordinates in $\mathbb{R}^d$** $(\rho, \theta_1, \dots, \theta_{n-1}, x_n, x_{n+1}, \dots, x_d)$ is a coordinate system such that a subset $\mathbb{R}^n$ of the total space $\mathbb{R}^d$ (for $n < d$) is converted into the polar coordinates. For instance, if we convert $\mathbb{R}^2$ of $\mathbb{R}^3$ into polar coordinates, we obtain the familiar *3d cylindrical coordinates*, i.e. $(x,y,z) = (\rho\cos\theta, \rho\sin\theta, z)$.

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2 Classical questions

You need to show your derivation in this part.

Question: 1 . . . . *(26 points)*

Consider the following differential equation for an unknown real function $f(x)$

$$\exp\left(a\frac{d}{dx}\right)f(x) = g(x) \tag{37}$$

(a) **(6 points)** Let's say we are given the information that $a \in \mathbb{R}$ is a small constant, i.e. $a \ll 1$. Use the Taylor series definition of exponential and truncate in the second order in a, (i.e. impose $a^n = 0$ for $n \geq 3$). What is the homogeneous solution of this differential equation?

Solution 1.1 *As* $\exp\left(a\frac{d}{dx}\right) = \sum_{i=0}^{\infty}\frac{a^i}{i!}\frac{d^i}{dx^i}$ *and we impose* $a^n = 0$ *for* $n \geq 3$*, the differential equation becomes*

$$\left(1 + a\frac{d}{dx} + \frac{a^2}{2}\frac{d^2}{dx^2}\right)f(x) = g(x) \tag{38}$$

The homogeneous solution to this differential equation can be found by solving the characteristic equation $r^2 + \frac{2}{a}r + \frac{2}{a^2} = 0$ *(which has the roots* $r_\pm = -\frac{1\pm i}{a}$*), hence the homogeneous solution* $f_h(x)$ *is*

$$f_h(x) = c_+\exp\left(-\frac{1+i}{a}x\right) + c_-\exp\left(-\frac{1-i}{a}x\right) \tag{39}$$

for arbitrary complex numbers $c_\pm$*. We can equivalently rewrite this using the Euler equation as*

$$f_h(x) = c_1e^{-x/a}\cos(x/a) + c_2e^{-x/a}\sin(x/a) \tag{40}$$

for $c_{1,2} \in \mathbb{R}$ *as* $f(x)$ *is real.*

(b) **(4 points)** Is the behavior of the homogeneous solution in the limit $a \to 0$ consistent with the expectations from the form of the differential equation in that limit?

Solution 1.2 *Clearly, we observe the behavior* $\lim_{a\to 0} f_h(x) = 0$*, which we can check with the sandwich theorem, i.e.*

$$-\lim_{a\to 0}|f_h(x)| \leq \lim_{a\to 0} f_h(x) \leq \lim_{a\to 0}|f_h(x)| \tag{41}$$

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with $\lim_{a\to 0}|f_h(x)| < \lim_{a\to 0}(|c_1|+|c_2|)e^{-x/a} = 0$. *This is consistent with the differential equation, i.e.*

$$\lim_{a\to 0}\left[\exp\left(a\frac{d}{dx}\right)f(x)=0\right] = [1.f(x)=0] \tag{42}$$

(c) **(10 points)** Let's keep focusing on small a case analyzed in the parts above. With $g(x) = \begin{cases} h(x) \text{ if } x>0 \\ 0 \text{ if } x<0 \end{cases}$ for an arbitrary function $h(x)$, what is the formal expression for the particular solution in terms of $h(x)$?

Solution 1.3 *We know that the particular solution for a linear, ordinary differential equation with constant coefficients such as*

$$\left(1+a\frac{d}{dx}+\frac{a^2}{2}\frac{d^2}{dx^2}\right)f(x)=g(x) \tag{43}$$

takes the form of a convolution. Since we are given the nonhomogeneous piece is zero for $x<0$, *we can immediately write down*

$$f_p(x) = \int_0^x i(y)h(x-y)dy \tag{44}$$

for the impulse response $i(x)$ *whose Laplace transform is one over characteristic equation, that is*

$$\begin{aligned}\mathcal{L}\{i\}(s) =& \frac{1}{\frac{a^2}{2}s^2+as+1} = \frac{2}{a^2}\frac{1}{s^2+\frac{2}{a}s+\frac{2}{a^2}} = \frac{2}{a^2}\frac{1}{\left(s+\frac{1+i}{a}\right)\left(s+\frac{1-i}{a}\right)} \\ =& \frac{i/a}{s+\frac{1+i}{a}} - \frac{i/a}{s+\frac{1-i}{a}}\end{aligned} \tag{45}$$

Therefore

$$i(x) = \frac{i}{a}e^{-\frac{1+i}{a}x} - \frac{i}{a}e^{-\frac{1-i}{a}x} = \frac{2e^{-x/a}}{a}\sin(x/a) \tag{46}$$

Thus the particular solution is

$$f_p(x) = \frac{2}{a}\int_0^x e^{-y/a}\sin(y/a)h(x-y)dy \tag{47}$$

(d) **(6 points)** With the homogeneous and particular solutions found above, compute $f(x)$ if we are given $h(x)=e^{-x/a}$, $f(0)=2$, and $f\left(\frac{a\pi}{2}\right)=e^{-\pi/2}$.

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Solution 1.4 *For $h(x) = e^{-x/a}$, the particular solution becomes*

$$f_p(x) = \frac{2e^{-x/a}}{a} \int_0^x \sin(y/a) dy = \frac{2e^{-x/a}}{a} \left(-a\cos(y/a)\right) \Big|_0^x = 2e^{-x/a} \left(1 - \cos(x/a)\right) \tag{48}$$

therefore

$$f(x) = c_1 e^{-x/a} \cos(x/a) + c_2 e^{-x/a} \sin(x/a) + 2e^{-x/a} \left(1 - \cos(x/a)\right) \tag{49}$$

We see that $f(0) = c_1$, hence the given information fixes $c_1 = 2$, reducing the expression to the form

$$f(x) = c_2 e^{-x/a} \sin(x/a) + 2e^{-x/a} \tag{50}$$

This then implies $f(a\pi/2) = e^{-\pi/2}(c_2 + 2)$, which then fixes $c_2 = -1$ with the other boundary condition. We are then left with the full solution:

$$f(x) = e^{-x/a} \left(2 - \sin(x/a)\right) \tag{51}$$

Question: 2 . . . . *(8 points)*

Consider the same equation as the previous question, i.e.

$$\exp\left(a\frac{d}{dx}\right) f(x) = g(x) \tag{52}$$

This time however, we are not allowed to take a to be small constant: we only know that $a \in \mathbb{R}$.

(a) **(6 points)** Use the infinite Taylor series definition of the exponential function to solve the differential equation for the special case $g(x) = 0$.

Solution 2.1 *We know that, by definition*

$$\exp\left(a\frac{d}{dx}\right) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n}{dx^n} \tag{53}$$

which then implies

$$\exp\left(a\frac{d}{dx}\right) f(x) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n f(x)}{dx^n} = \sum_{n=0}^{\infty} \frac{a^n f^{(n)}(x)}{n!} \tag{54}$$

This is actually the Taylor series expansion for $f(x + a)$ around the point x! For instance, note that

$$f(x+3) = \sum_{n=0}^{\infty} \frac{3^n f^{(n)}(x)}{n!} \tag{55}$$

where the above expression is simply a generalization of this. We therefore conclude that the

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differential equation

$$\exp\left(a\frac{d}{dx}\right)f(x) = 0 \tag{56}$$

is actually equivalent to

$$f(x+a) = 0 \tag{57}$$

Since this is true for any x, there is no nontrivial solution, the only solution is $f(x) = 0$.

(b) **(2 points)** Solve the general case where $g(x)$ is any arbitrary function!

Solution 2.2 *In the previous part, we have shown that*

$$\exp\left(a\frac{d}{dx}\right)f(x) = f(x+a) \tag{58}$$

therefore the differential equation becomes

$$f(x+a) = g(x) \tag{59}$$

which means that the most general solution $f(x)$ for any arbitrary function $g(x)$ is

$$f(x) = g(x-a) \tag{60}$$

« « « Congratulations, you have made it to the end! » » »