Phys209: Mathematical Methods in Physics I

Homework 5

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) are for your own practice!
- Unless stated otherwise, you are expected to show your derivation of the results.
- The homework is due November 10th 2023, 23:59 TSI.

(1) Problem One *(6 points)*

We have discussed several times in class that most differential equations are not solvable. In this question, we review one by one several classes of solvable differential equations. Please show your steps in each question.

(1.1) (0.5pt)

Linear ordinary differential equations with constant coefficients form the most easy-to-solve class of differential equations: we have seen several examples of this; let us solve one more such an equation. Consider $f''(x) + 2f'(x) + f(x) = 0$. What is the function $f(x)$ that satisfies this relation?

Answer: The characteristic equation for this differential equation is $r^2 + 2r + 1 = 0$ with the double root $r = -1$, hence

$$f(x) = c_1 e^{-x} + c_2 x e^{-x} \tag{1.1}$$

for the undetermined coefficients c_1 and c_2.

(1.2) (1pt)

Next class of equations are those that can be rewritten in the form $\mathcal{D}_1.\mathcal{D}_2 \ldots \mathcal{D}_n.f(x) = 0$ for first order differential operators $\mathcal{D}_i$. Show that $x^6 f''(x) + 3x^5 f'(x) - f(x) = 0$ can be rewritten in this form.
Hint: Assume $\mathcal{D}_i$ is of the form $\left(x^{a_i}\frac{d}{dx} + b_i\right)$ for constants a_i, b_i.

(1.3) (1pt)

Solve the differential equation in question 1.2 by making use of two facts: 1) we can solve first order differential equations of the form $x^a f'(x) + bf(x) = 0$, and 2) the differential operators $\mathcal{D}_i$ in question 1.2 commute, i.e. $\mathcal{D}_i.\mathcal{D}_j.f(x) - \mathcal{D}_j.\mathcal{D}_i.f(x) = 0$

(1.4) (1pt)

Another type of differential equation that we have discussed in class is that which can be solved via reparameterization, i.e. by writing everything in terms of a new parameter $u(x)$ instead of x. Solve the differential equation $f''(x) + f'(x) + e^{-2x} f(x) = e^{-2x}$ this way.

(1.5) (0.5pt)

The fourth type of differential equation is the one in which only the derivatives of the unknown function appear; for instance, $g^{(4)}(x) + 2g^{(3)}(x) + g^{(2)}(x) = 0$ does not have the function $g(x)$ (or even $g'(x)$): solve this differential equation for the function $g(x)$ by first solving it for the function $h(x) = g''(x)$.

(1.6) (1pt)

The fifth class of differential equations that can be solved more easily form *the exact differential equations*; in other words, they can be rewritten as $\frac{d}{dx}(\cdots) = 0$. For example, show that the differential equation $xf^{(3)}(x) + f^{(2)}(x) + x^{-1} f^{(1)}(x) - x^{-2} f(x) = 0$ is exact. Only show that it is exact, do not try to solve it!

(1.7) (1pt)

The last class of differential equations that we have discussed is that for which we know some of the solutions. We introduced the concept *reduction of order*, and stated that "an order n differential equation with k known solutions" is *equivalent to* "an order $k - n$ differential equation with no known solutions". We'll go over that concept in this question: for this, show that $(x-1)f''(x) - xf'(x) + f(x) = 0$ can be rewritten as a first order differential equation if we know that $f_1(x) = e^x$ is a solution. *Hint: Rewrite the differential equation for the unknown function* $g(x)$ *for* $f(x) = g(x)e^x$.