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Final Examination

Phys331: Electromagnetic Theory I

2026/01/08

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) all calculations are to be conducted in the notations and conventions of the formulae sheets provided during the exam unless explicitly stated otherwise in the question;

(2) I take *full responsibility* for any ambiguity in my selections in “multiple choice questions”;

(3) incorrect selections will receive $-1/7$ of the question’s points;

(4) I am expected to provide *step-by-step explanation of how I solved the question* and am expected to do so *only within the answer boxes* provided with the questions: the explanation is supposed to be succinct, well-articulated, and correct both scientifically and mathematically;

(5) no partial credit is awarded for the explanations provided in the answer boxes;

(6) some questions of some students will be randomly selected for inspection: *a question (if selected for inspection) might be awarded negative points* if its explanation is incorrect or insufficient to get the correct answer, even if the correct option is selected;

(7) any page which does not contain *both my name and student id* may not be graded;

(8) any extra sheet that I may use are for my own calculations and will not be graded.

Signature: ______________________

Question:	1	2	3	4	Total
Points:	31½	10½	52½	10½	105

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Question: 1: Electric displacement field . . . . *(31½ points)*
(Midterm Three Question Reloaded)

In this question, you may use the Laplacian operator in the spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2 f}{\partial \phi^2} \tag{1}$$

You may also use the information that the Legendre polynomials $P_\ell(x)$ (also equivalent to the notation $P_\ell^0(x)$) satisfy for all non-negative integer ℓ the differential equation $\frac{d}{dx}\left((1-x^2)\frac{dP_\ell(x)}{dx}\right) + \ell(\ell+1)P_\ell(x) = 0$.
Please take $\pi\epsilon_0 \simeq 3.6^{-1} \times 10^{-10}$ and $\pi\epsilon \simeq 1.8^{-1} \times 10^{-10}$ (a good approximation for polytetrafluoroethylene)

Consider an empty space filled with a uniform electric field of magnitude $E_0/3$, whose source will not be relevant for our purposes. Assume that an electrically uncharged polytetrafluoroethylene (colloquially known as *teflon*) droplet, of perfect spherical shape, is put into this otherwise empty space. Set up a coordinate system such that the geometric center of the droplet coincides with the origin. Furthermore, in the parts below, take A and B to be two points that are collinear with the origin such that B is in the middle. Additionally, the length of the line segments, $|AB|$ and $|BO|$, satisfy the relation $|AB| = |BO| = 4R$ where R is the radius of the droplet.

(a) **(10½ points)** Given that E_0 is sufficiently small such that the linear optics is a valid approximation (i.e. $\mathbf{D} = \epsilon\mathbf{E}$), what would be the magnitude of the electric field at the geometric center of the droplet?

☐ $\frac{3E_0}{4}$ ■ $\frac{E_0}{4}$ ☐ $\frac{4E_0}{5}$ ☐ $\frac{E_0}{5}$ ☐ $\frac{5E_0}{6}$ ☐ $\frac{E_0}{6}$ ☐ $\frac{6E_0}{7}$ ☐ $\frac{E_0}{7}$

(b) **(10½ points)** Given that E_0 is sufficiently small such that the linear optics is a valid approximation (i.e. $\mathbf{D} = \epsilon\mathbf{E}$), what would be the ratio V_A/V_B where V_A & V_B are electric potentials evaluated at the points A & B?

☐ $\frac{2^8-2^0}{2^9-2^2}$ ☐ $\frac{2^9-2^0}{2^9-2^2}$ ☐ $\frac{2^{10}-2^0}{2^9-2^2}$ ☐ $\frac{2^{11}-2^0}{2^9-2^2}$ ☐ $\frac{2^8-2^0}{2^{10}-2^2}$ ☐ $\frac{2^9-2^0}{2^{10}-2^2}$ ☐ $\frac{2^{10}-2^0}{2^{10}-2^2}$ ■ $\frac{2^{11}-2^0}{2^{10}-2^2}$

(c) **(10½ points)** Given that E_0 is sufficiently small such that the linear optics is a valid approximation with the caveat that the polytetrafluoroethylene droplet has a constant uniform permanent polarization (i.e. polarization in the absence of an external electric field) in the direction of the external electric field, and with the magnitude same as $\epsilon_0 E_0/3$, what would be the ratio V_A/V_B where V_A & V_B are electric potentials evaluated at the points A & B?

☐ $\frac{2^8-2^0}{2^9-2^2}$ ☐ $\frac{2^9-2^0}{2^9-2^2}$ ■ $\frac{2^{10}-2^0}{2^9-2^2}$ ☐ $\frac{2^{11}-2^0}{2^9-2^2}$ ☐ $\frac{2^8-2^0}{2^{10}-2^2}$ ☐ $\frac{2^9-2^0}{2^{10}-2^2}$ ☐ $\frac{2^{10}-2^0}{2^{10}-2^2}$ ☐ $\frac{2^{11}-2^0}{2^{10}-2^2}$

Solution 1.1 - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

Let us first set our coordinate system such that the electric field is along the $z-$coordinate, i.e. $\mathbf{E} = \frac{E_0}{3}\hat{z}$,

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and choose the radius of the water droplet to be R. Considering the spherical nature of the water and the overall azimuthal symmetry, let us proceed by solving the Laplace equation in the spherical coordinates. We already derived this earlier in the third midterm, hence

$$V(r,\theta)=\sum_{\ell=0}^{\infty}P_\ell^0(\cos\theta)\times\begin{cases}a_\ell r^\ell+b_\ell r^{-1-\ell} & r>R\\ c_\ell r^\ell+d_\ell r^{-1-\ell} & r<R\end{cases}\tag{2}$$

where we are to fix the undetermined coefficients a_ℓ, b_ℓ, c_ℓ, and d_ℓ through the boundary conditions. Indeed,

$$\lim_{r\to\infty}\mathbf{E}(r,\theta)=\frac{E_0}{3}\hat{z}\tag{3}$$

means $\lim_{r\to\infty}V(r,\theta)=-\frac{E_0}{3}z$ hence

$$V(r,\theta)=\begin{cases}-\dfrac{E_0}{3}r\cos\theta+\displaystyle\sum_{\ell=0}^{\infty}P_\ell^0(\cos\theta)b_\ell r^{-1-\ell} & r>R\\ \displaystyle\sum_{\ell=0}^{\infty}P_\ell^0(\cos\theta)c_\ell r^\ell & r<R\end{cases}\tag{4}$$

where we also used the fact that $\lim_{r\to 0}V(r,\theta)<\infty$ (potential being non-singular at the origin) forced $d_\ell=0$.

The next boundary condition is

$$\lim_{r\to R^+}V(r,\theta)=\lim_{r\to R^-}V(r,\theta)\tag{5}$$

This leads to

$$\left[\frac{b_0}{R}-c_0\right]+\left[-\frac{E_0}{3}R+\frac{b_1}{R^2}-c_1R\right]P_1(\cos\theta)+\sum_{\ell=1}^{\infty}\left[\frac{b_\ell}{R^{\ell+1}}-c_\ell R^\ell\right]P_\ell(\cos\theta)=0\tag{6}$$

which can be utilized to fix c_ℓ in terms of b_ℓ through the orthogonality of the Legendre polynomials:

$$V(r,\theta)=\begin{cases}-\dfrac{E_0}{3}r\cos\theta+\displaystyle\sum_{\ell=0}^{\infty}\frac{b_\ell}{r^{\ell+1}}P_\ell^0(\cos\theta) & r>R\\ -\dfrac{E_0}{3}r\cos\theta+\displaystyle\sum_{\ell=0}^{\infty}\frac{b_\ell r^\ell}{R^{2\ell+1}}P_\ell^0(\cos\theta) & r<R\end{cases}\tag{7}$$

The final boundary condition can be derived via the Gauss's law of the electric displacement $\mathbf{D}$, i.e.

$$\lim_{r\to R^+}[\hat{r}\cdot\mathbf{D}]-\lim_{r\to R^-}[\hat{r}\cdot\mathbf{D}]=\sigma_f\tag{8}$$

where σ_f is the surface free charge density which is zero.

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For part (a) & (b), we can proceed further using the linear optics approximation, i.e. the condition in (8) becomes

$$\epsilon_0 \lim_{r\to R^+} [\hat{r}\cdot \mathbf{E}] = \epsilon \lim_{r\to R^-} [\hat{r}\cdot \mathbf{E}] \quad \Rightarrow \quad \epsilon_0 \lim_{r\to R^+} \frac{\partial V}{\partial r} = \epsilon \lim_{r\to R^-} \frac{\partial V}{\partial r} \tag{9}$$

with which (7) becomes

$$-\epsilon_0 \frac{E_0}{3}\cos\theta - \sum_{\ell=0}^{\infty} \frac{\epsilon_0 b_\ell (\ell+1)}{R^{\ell+2}} P_\ell^0(\cos\theta) = -\epsilon\frac{E_0}{3}\cos\theta + \sum_{\ell=0}^{\infty}\frac{\epsilon \ell b_\ell}{R^{\ell+2}} P_\ell^0(\cos\theta) \tag{10}$$

indicating

$$\frac{\epsilon_0}{R^2} b_0 + \left[(\epsilon_0-\epsilon)\frac{E_0}{3} + \frac{2\epsilon_0+\epsilon}{R^3} b_1\right]\cos\theta + \sum_{\ell=2}^{\infty}\frac{\epsilon\ell + \epsilon_0(\ell+1)}{R^{\ell+2}} b_\ell P_\ell^0(\cos\theta) = 0 \tag{11}$$

Orthogonality of P_ℓ means

$$\frac{\epsilon_0}{R^2} b_0 = 0\ , \quad (\epsilon_0-\epsilon)\frac{E_0}{3} + \frac{2\epsilon_0+\epsilon}{R^3} b_1 = 0\ , \quad \frac{\epsilon\ell+\epsilon_0(\ell+1)}{R^{\ell+2}} b_\ell = 0 \textit{ for } \ell \geq 2 \tag{12}$$

with which the fact that $\ell, \epsilon_0, \epsilon, R \in \mathbb{R}^+$ leads to

$$b_\ell = \delta_{\ell,1}\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} R^3 \frac{E_0}{3} \tag{13}$$

hence (7) becomes

$$V(r,\theta) = \begin{cases} -\dfrac{E_0}{3} r\cos\theta\left[1 - \dfrac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\left(\dfrac{R}{r}\right)^3\right] & r > R \\ -\dfrac{E_0}{3} r\cos\theta\left[1 - \dfrac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right] & r < R \end{cases} \tag{14}$$

We are told that $\pi\epsilon_0 \simeq 3.6^{-1}\times 10^{-10}$ and $\pi\epsilon \simeq 1.8^{-1}\times 10^{-10}$, hence $\epsilon = 2\epsilon_0$, leading to

$$V(r,\theta) = \begin{cases} -\dfrac{E_0}{3} r\cos\theta\left[1 - \dfrac{1}{4}\left(\dfrac{R}{r}\right)^3\right] & r > R \\ -\dfrac{1}{4}E_0 r\cos\theta & r < R \end{cases} \tag{15}$$

For part (a), the electric field at the origin is simply

$$\lim_{x,y,z\to 0}\mathbf{E}(x,y,z) = -\lim_{x,y,z\to 0}\nabla V(x,y,z) = -\lim_{x,y,z\to 0}\nabla\left(-\frac{1}{4}E_0 z\right) = \frac{E_0}{4}\hat{z} \tag{16}$$

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For part (b), we proceed as follows: since the points A, B, and O are collinear, and since we have the azimuthal symmetry, we can choose the x coordinate such that the line AB lies in the $x-z$ plane, meaning we can choose the points A and B as

$$\begin{aligned} A: &\quad (r,\theta,\phi) = (8R,\alpha,0) \\ B: &\quad (r,\theta,\phi) = (4R,\alpha,0) \end{aligned} \tag{17}$$

for some undetermined angle α where the radial distances are fixed via the given information $|AB| = |BO| = 4R$. Note that we have also ensured that B is between A and the origin.

We can now find the required ratio as

$$\frac{V_A}{V_B} = \frac{V(8R,\alpha,0)}{V(4R,\alpha,0)} = \frac{-\frac{E_0}{3}(8R)\cos\alpha\left[1-\frac{1}{4}\left(\frac{R}{8R}\right)^3\right]}{-\frac{E_0}{3}(4R)\cos\alpha\left[1-\frac{1}{4}\left(\frac{R}{4R}\right)^3\right]} = \frac{2^{11}-2^0}{2^{10}-2^2} \tag{18}$$

For part (c), given information translates into

$$\mathbf{D} = \begin{cases} \epsilon_0 \frac{E_0}{3}\hat{z} + \epsilon \mathbf{E} & r < R \\ \epsilon_0 \mathbf{E} & r > R \end{cases} \tag{19}$$

in our coordinate system. The fourth boundary condition in (8) then leads to

$$-\epsilon_0 \lim_{r\to R^+} \frac{\partial V}{\partial r} = \epsilon_0 \frac{E_0}{3}\cos\theta - \epsilon \lim_{r\to R^-} \frac{\partial V}{\partial r} \tag{20}$$

with which (7) becomes

$$\epsilon_0 \frac{E_0}{3}\cos\theta + \sum_{\ell=0}^{\infty} \frac{\epsilon_0 b_\ell(\ell+1)}{R^{\ell+2}} P_\ell^0(\cos\theta) = (\epsilon+\epsilon_0)\frac{E_0}{3}\cos\theta - \sum_{\ell=0}^{\infty} \frac{\epsilon \ell b_\ell}{R^{\ell+2}} P_\ell^0(\cos\theta) \tag{21}$$

indicating

$$\frac{\epsilon_0}{R^2} b_0 + \left[-\epsilon\frac{E_0}{3} + \frac{2\epsilon_0+\epsilon}{R^3} b_1\right]\cos\theta + \sum_{\ell=2}^{\infty} \frac{\epsilon\ell + \epsilon_0(\ell+1)}{R^{\ell+2}} b_\ell P_\ell^0(\cos\theta) = 0 \tag{22}$$

Orthogonality of P_ℓ means

$$\frac{\epsilon_0}{R^2} b_0 = 0\,, \quad -\epsilon\frac{E_0}{3} + \frac{2\epsilon_0+\epsilon}{R^3} b_1 = 0\,, \quad \frac{\epsilon\ell+\epsilon_0(\ell+1)}{R^{\ell+2}} b_\ell = 0 \text{ for } \ell \geq 2 \tag{23}$$

with which the fact that $\ell, \epsilon_0, \epsilon, R \in \mathbb{R}^+$ leads to

$$b_\ell = \delta_{\ell,1} \frac{\epsilon}{\epsilon+2\epsilon_0} R^3 \frac{E_0}{3} \tag{24}$$

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hence (7) *becomes*

$$V(r,\theta) = \begin{cases} -\dfrac{E_0}{3} r\cos\theta \left[1 - \dfrac{\epsilon}{\epsilon + 2\epsilon_0}\left(\dfrac{R}{r}\right)^3\right] & r > R \\ -\dfrac{E_0}{3} r\cos\theta \left[1 - \dfrac{\epsilon}{\epsilon + 2\epsilon_0}\right] & r < R \end{cases} \tag{25}$$

We are told that $\pi\epsilon_0 \simeq 3.6^{-1}\times 10^{-10}$ *and* $\pi\epsilon \simeq 1.8^{-1}\times 10^{-10}$, *hence* $\epsilon = 2\epsilon_0$, *leading to*

$$V(r,\theta) = \begin{cases} -\dfrac{E_0}{3} r\cos\theta \left[1 - \dfrac{1}{2}\left(\dfrac{R}{r}\right)^3\right] & r > R \\ -\dfrac{1}{6}E_0 r\cos\theta & r < R \end{cases} \tag{26}$$

As before, we have $\dfrac{V_A}{V_B} = \dfrac{V(8R,\alpha,0)}{V(4R,\alpha,0)}$, *hence*

$$\frac{V_A}{V_B} = \frac{V(8R,\alpha,0)}{V(4R,\alpha,0)} = \frac{-\frac{E_0}{3}(8R)\cos\alpha\left[1-\frac{1}{2}\left(\frac{R}{8R}\right)^3\right]}{-\frac{E_0}{3}(4R)\cos\alpha\left[1-\frac{1}{2}\left(\frac{R}{4R}\right)^3\right]} = \frac{2^{10}-2^0}{2^9-2^2} \tag{27}$$

Question: 2: Nature of magnetic field . . . . *(10½ points)*

(Midterm Three Question Reloaded)

Assume that a big press release tomorrow announces the discovery of the magnetic monopoles and we begin to use the equations

$$\nabla\cdot\mathbf{B} = \rho_B/\epsilon_0 \quad , \quad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \tag{28}$$

for magnetostatics. Here, $\mathbf{J}$, ϵ_0, and μ_0 are the current density, the vacuum permittivity, and the vacuum permeability as usual; and, ρ_B is the newly discovered magnetic monopole: the press release basically tells that ρ_B can be nonzero. In this fictitious scenario, what would be the SI unit for ρ_B?

☐ $A^1\ m^{-4}\ s^4$ ■ $A^1\ m^{-4}\ s^2$ ☐ $A^1\ m^{-2}\ s^4$ ☐ $A^1\ m^{-2}\ s^2$

☐ $A^1\ m^{-2}\ s^0$ ☐ $A^1\ m^0\ s^0$ ☐ $A^1\ m^2\ s^2$ ☐ $A^1\ m^4\ s^4$

Solution 2.1 - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

The first equation of (28) *indicates that*

$$[\rho_B] = [\epsilon_0][\nabla][\mathbf{B}] \tag{29}$$

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It is not uncommon for people to know the units of ϵ and $\mathbf{B}$ but let's assume that we have not memorized them. This means, we should derive them from their respective equations, i.e. the Coulomb's law

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3} \quad \Rightarrow \quad [\epsilon_0] = \frac{[q]^2}{[\mathbf{r}]^2[\mathbf{F}]} \tag{30}$$

and Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \quad \Rightarrow \quad [\mathbf{B}] = \frac{[\mathbf{F}]}{[q][\mathbf{v}]} \tag{31}$$

hence

$$[\rho_B] = \frac{[q]^2}{[\mathbf{r}]^2[\mathbf{F}]}[\nabla]\frac{[\mathbf{F}]}{[q][\mathbf{v}]} = [q][\mathbf{r}]^{-2}[\nabla][\mathbf{v}]^{-1} = (\mathrm{A}^1\ \mathrm{s}^1)(\mathrm{m}^1)^{-2}(\mathrm{m}^{-1})(\mathrm{m}^1\ \mathrm{s}^{-1})^{-1} \tag{32}$$

simplifying to

$$[\rho_B] = \mathrm{A}^1\ \mathrm{m}^{-4}\ \mathrm{s}^2 \tag{33}$$

Note that we can rewrite this as

$$[\rho_B] = \frac{\mathrm{A}^1}{\mathrm{m}^2}\frac{1}{\mathrm{m}^2\ \mathrm{s}^{-2}} \tag{34}$$

where the first term is the units for current density, $\mathbf{J}$, and the second piece is a velocity squared in the denominator. It is not really clear in SI units but we might suspect that there is a relation between ρ_B and $\mathbf{J}/c^2$ for the speed of light c: to investigate this better, we should actually work in Gaussian units as any self-respecting theoretician would do.

Question: 3: Magnetic field of rotating electric charge . . . . *($52\frac{1}{2}$ points)*

(Modified version of Example 11 of §5.4 of Griffiths, which we partially solved in class)

In this question, you may use the following equation

$$\mathbf{v}_{\textit{arbitrary point}} - \mathbf{v}_{\textit{geometric center}} = \vec{\omega}\times\left(\mathbf{r}_{\textit{arbitrary point}} - \mathbf{r}_{\textit{geometric center}}\right) \tag{35}$$

which is valid for a rigid body rotating with angular velocity $\vec{\omega}$ about an axis through its geometric center

A spherical porcelain ball, a perfect insulator for practical purposes, is endowed with a uniform surface charge density σ. Below, we will assume **(1)** this ball is rotating with a constant angular velocity $\vec{\omega}$ and **(2)** the angular speed ω is sufficiently small such that we can ignore the equatorial bulging, i.e. the object has a constant radius R.

We will measure the magnetic effects of this ball at an arbitrary point $\mathbf{r}$; to do this, set up a coordinate system such that $\mathbf{r} = r\hat{z}$, that $\vec{\omega}\cdot\hat{y} = 0$, and that the origin coincides with the geometric center of the ball. Furthermore, define $\psi := \arcsin(\hat{x}\cdot\hat{\omega})$.

(a) **($10\frac{1}{2}$ points)** Which one below is the correct expression for $\vec{\omega}$?

■ $\omega(\sin\psi\,\hat{x} + \cos\psi\,\hat{z})$ ☐ $\omega(\cos\psi\,\hat{x} + \sin\psi\,\hat{z})$ ☐ $\omega(\sin\psi\,\hat{x} - \cos\psi\,\hat{z})$ ☐ $\omega(\cos\psi\,\hat{x} - \sin\psi\,\hat{z})$

☐ $\sin\psi\,\hat{x} + \cos\psi\,\hat{z}$ ☐ $\cos\psi\,\hat{x} + \sin\psi\,\hat{z}$ ☐ $\sin\psi\,\hat{x} - \cos\psi\,\hat{z}$ ☐ $\cos\psi\,\hat{x} - \sin\psi\,\hat{z}$

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(b) **(10½ points)** Consider a point $\mathbf{r}'$ with the standard spherical coordinate designation $(r', \theta', \phi') = (R, \pi/3, \pi/6)$. What is the velocity of this point for $\psi = \pi/4$?

☐ $|\mathbf{v}'| = 0$ ■ $|\mathbf{v}'| = \sqrt{\frac{7}{32}}\omega R$ ☐ $|\mathbf{v}'| = \sqrt{\frac{7}{16}}\omega R$ ☐ $|\mathbf{v}'| = \sqrt{\frac{9}{32}}\omega R$

☐ $|\mathbf{v}'| = \sqrt{\frac{9}{16}}\omega R$ ☐ $|\mathbf{v}'| = \sqrt{\frac{11}{32}}\omega R$ ☐ $|\mathbf{v}'| = \sqrt{\frac{11}{16}}\omega R$ ☐ $|\mathbf{v}'| = \sqrt{\frac{1}{2026}}\omega R$

(c) **(10½ points)** What is the magnitude of the surface current density, i.e. $|\mathbf{K}|$, at the point $(r', \theta', \phi') = (R, \pi/3, \pi/6)$ for $\psi = \pi/4$?

☐ $|\mathbf{K}| = 0$ ■ $|\mathbf{K}| = \sqrt{\frac{7}{32}}\omega R\sigma$ ☐ $|\mathbf{K}| = \sqrt{\frac{7}{16}}\omega R\sigma$ ☐ $|\mathbf{K}| = \sqrt{\frac{9}{32}}\omega R\sigma$

☐ $|\mathbf{K}| = \sqrt{\frac{9}{16}}\omega R\sigma$ ☐ $|\mathbf{K}| = \sqrt{\frac{11}{32}}\omega R\sigma$ ☐ $|\mathbf{K}| = \sqrt{\frac{11}{16}}\omega R\sigma$ ☐ $|\mathbf{K}| = \sqrt{\frac{1}{2026}}\omega R\sigma$

(d) **(10½ points)** What is the magnitude of the surface current density, i.e. $|\mathbf{K}|$, at the point $(r', \theta', \phi') = (R/2, \pi/3, \pi/6)$ for $\psi = \pi/4$?

■ $|\mathbf{K}| = 0$ ☐ $|\mathbf{K}| = \sqrt{\frac{7}{32}}\omega R\sigma$ ☐ $|\mathbf{K}| = \sqrt{\frac{7}{16}}\omega R\sigma$ ☐ $|\mathbf{K}| = \sqrt{\frac{9}{32}}\omega R\sigma$

☐ $|\mathbf{K}| = \sqrt{\frac{9}{16}}\omega R\sigma$ ☐ $|\mathbf{K}| = \sqrt{\frac{11}{32}}\omega R\sigma$ ☐ $|\mathbf{K}| = \sqrt{\frac{11}{16}}\omega R\sigma$ ☐ $|\mathbf{K}| = \sqrt{\frac{1}{2026}}\omega R\sigma$

(e) **(10½ points)** By integrating the surface charge density with the appropriate kernel (as is explicitly derived in the book), one arrives at the expression

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 R^4 \sigma}{3r^3}\vec{\omega} \times \mathbf{r} \tag{36}$$

for the vector potential outside the ball. What is the magnitude of the vector potential at the point $\mathbf{r} = 3R\hat{x} + 4R\hat{y}$ for $\sigma = 25/(\mu_0 R)$ and $\vec{\omega} = 3\hat{y}$?

☐ $|\mathbf{A}(3R\hat{x} + 4R\hat{y})| = 0$ ☐ $|\mathbf{A}(3R\hat{x} + 4R\hat{y})| = \frac{R}{5}$ ☐ $|\mathbf{A}(3R\hat{x} + 4R\hat{y})| = \frac{2R}{10}$ ■ $|\mathbf{A}(3R\hat{x} + 4R\hat{y})| = \frac{3R}{5}$

☐ $|\mathbf{A}(3R\hat{x} + 4R\hat{y})| = \frac{4R}{20}$ ☐ $|\mathbf{A}(3R\hat{x} + 4R\hat{y})| = \frac{5R}{25}$ ☐ $|\mathbf{A}(3R\hat{x} + 4R\hat{y})| = \frac{6R}{30}$ ☐ $|\mathbf{A}(3R\hat{x} + 4R\hat{y})| = \frac{7R}{35}$

Solution 3.1 - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

Let us start with part (a): we have two vectors $\vec{\omega}$ and $\mathbf{r}$ and we are told to orient the coordinate system

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such that $\mathbf{r}$ *points along the* $\hat{z}$ *axis and* $\vec{\omega}$ *lies in the* $x - z$ *plane (it is orthogonal to* $\hat{y}$*). Therefore*

$$\begin{aligned}\vec{\omega} =& \omega(\hat{\omega}\cdot\hat{x})\hat{x} + \omega(\hat{\omega}\cdot\hat{z})\hat{z} \\ =& \omega(\hat{\omega}\cdot\hat{x})\hat{x} + \omega\sqrt{1-(\hat{\omega}\cdot\hat{x})^2}\hat{z} \\ =& \omega\left(\sin\psi\hat{x} + \cos\psi\hat{z}\right)\end{aligned} \tag{37}$$

For part (b), we start with the following observation: the coordinate system is such that the geometric center of the ball is the origin (it does not move with respect to the coordinate system), hence (35) reads as

$$\mathbf{v}_{arbitrary\ point} = \vec{\omega}\times\mathbf{r}_{arbitrary\ point} \tag{38}$$

For $\mathbf{r}' = r'\sin\theta'\cos\phi'\hat{x} + r'\sin\theta'\sin\phi'\hat{y} + r'\cos\theta'\hat{z}$, *we then have*

$$\begin{aligned}\mathbf{v}' =& \omega r'\left(\sin\psi\hat{x} + \cos\psi\hat{z}\right)\times\left(\sin\theta'\cos\phi'\hat{x} + \sin\theta'\sin\phi'\hat{y} + \cos\theta'\hat{z}\right) \\ =& \omega r'\left[-\cos\psi\sin\theta'\sin\phi'\hat{x} + \left(\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta'\right)\hat{y} + \sin\psi\sin\theta'\sin\phi'\hat{z}\right]\end{aligned} \tag{39}$$

which means

$$\left|\mathbf{v}'\right| = \omega r'\sqrt{\sin^2\psi\cos^2\theta' + \sin^2\theta'\left(\cos^2\psi + \sin^2\psi\sin^2\phi'\right) - 2\sin\psi\cos\psi\sin\theta'\cos\theta'\cos\phi'} \tag{40}$$

For $\left(\theta',\phi',\psi\right) = \left(\frac{\pi}{3},\frac{\pi}{6},\frac{\pi}{4}\right)$, *we have*

$$\cos\left(\theta'\right) = \sin\left(\phi'\right) = \frac{1}{2}\,, \quad \cos\left(\phi'\right) = \sin\left(\theta'\right) = \frac{\sqrt{3}}{2}\,, \quad \cos(\psi) = \sin(\psi) = \frac{\sqrt{2}}{2} \tag{41}$$

which leads to

$$\left|\mathbf{v}'\right| = \sqrt{\frac{7}{32}}\omega R \tag{42}$$

For part (c), we simply use the fact that $\mathbf{K} = \sigma\mathbf{v}'$. *In comparison, in part (d),* $\mathbf{K} = 0$ *as there is zero charge density at* $r' = R/2$.

In part (e), we are given $\mathbf{r} = 3R\hat{x} + 4R\hat{y}$ *hence* $r = 5R$ *which is outside the sphere, meaning we can use*

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 R^4\sigma}{3r^3}\vec{\omega}\times\mathbf{r} \tag{43}$$

which becomes

$$\mathbf{A}(3R\hat{x} + 4R\hat{y}) = \frac{\mu_0 R^4\frac{25}{\mu_0 R}}{3(5R)^3}(3\hat{y})\times(3R\hat{x} + 4R\hat{y}) \tag{44}$$

hence

$$\left|\mathbf{A}(3R\hat{x} + 4R\hat{y})\right| = \frac{3R}{5} \tag{45}$$

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Question: 4: Concept of magnetization .... *(10½ points)*

(Modified version of Example 1 of §6.2 of Griffiths, which we partially solved in class)

Assume that you are working with ferromagnetic materials, such as various cobalt alloys that are used in modern hard disk drives, and you are tasked to derive a formula for the magnetic field $\mathbf{B}$ generated by a uniformly magnetized substance. As a starting point, you are allowed to approximate the substance as a perfect ball of radius R. For the uniform magnetization $\mathbf{M} = 2\hat{x}$ (within the material), what is the quantity $\hat{x} \cdot \mathbf{B}(\mathbf{r} = 2\hat{z})$ for $\mu_0 R^3 = 6$?

Hint: volume bound current density $\mathbf{J}_\text{b}$ and surface bound current density $\mathbf{K}_\text{b}$ can be derived from the magnetization via the relations $\mathbf{J}_\text{b} = \nabla \times \mathbf{M}$ and $\mathbf{K}_\text{b} = \mathbf{M} \times \hat{n}$.

Hint 2: the identity $\nabla \cdot \frac{\mathbf{r}}{r^3} = 0$ is valid for $\mathbf{r} \in \mathbb{R}^3 \backslash \{0\}$.

☐ ∞ ☐ 1 ☐ $\frac{1}{2}$ ☐ $\frac{1}{4}$ ☐ 0 ☐ $-\frac{1}{4}$ ■ $-\frac{1}{2}$ ☐ -1

Solution 4.1 - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

For the uniform magnetization $\mathbf{M} = M\hat{M}$, we immediately see that

$$\mathbf{J}_\text{b} = 0\,, \quad \mathbf{K}_\text{b} = M(\hat{M} \times \hat{r}) \tag{46}$$

In the previous question, we have dealt with a rotating charged sphere which has the surface charge density

$$\mathbf{K} = \sigma \mathbf{v} = \omega R \sigma (\hat{\omega} \times \hat{r}) \tag{47}$$

This means that the magnetic field generated by a uniformly magnetized sphere is equivalent to the magnetic field generated by a uniformly rotating sphere of constant surface charge density via the identification $\mathbf{M} \leftrightarrow R\sigma\vec{\omega}$. Therefore, we can immediately use (36) to conclude

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 R^3}{3} \mathbf{M} \times \frac{\mathbf{r}}{r^3} \tag{48}$$

for $|\vec{r}| > R$. We can now use equation (bj) in the formula sheet to conclude

$$\begin{aligned} \mathbf{B}(\mathbf{r}) =& \nabla \times \mathbf{A} = \frac{\mu_0 R^3}{3} \nabla \times \left(\mathbf{M} \times \frac{\mathbf{r}}{r^3} \right) \\ =& \frac{\mu_0 R^3}{3} \left[\mathbf{M} (\nabla \cdot \frac{\mathbf{r}}{r^3}) - (\mathbf{M} \cdot \nabla) \frac{\mathbf{r}}{r^3} \right] \end{aligned} \tag{49}$$

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By using the second hint and $\mathbf{M} = 2\hat{x}$*, we arrive at*

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= -\frac{2\mu_0 R^3}{3}\frac{\partial}{\partial x}\frac{\mathbf{r}}{r^3} = -\frac{2\mu_0 R^3}{3r^3}\frac{\partial \mathbf{r}}{\partial x} - \frac{2\mu_0 R^3 \mathbf{r}}{3}\frac{\partial (x^2+y^2+z^2)^{-3/2}}{\partial x} \\ &= -\frac{2\mu_0 R^3}{3r^3}\hat{x} - \frac{2\mu_0 R^3 \mathbf{r}}{3}(-\frac{3}{2})\frac{2x}{r^5} = -\frac{2\mu_0 R^3}{3r^5}\left[r^2\hat{x} - 3x\mathbf{r}\right] \\ &= -\frac{2\mu_0 R^3}{3r^5}\left[(y^2+z^2-2x^2)\hat{x} - 3xy\hat{y} - 3xz\hat{z}\right]\end{aligned} \tag{50}$$

hence

$$\mathbf{B}(2\hat{z}) = -\frac{2\mu_0 R^3}{3 * 2^5}4\hat{x} = -\frac{\mu_0 R^3}{12}\hat{x} \quad \Rightarrow \quad \hat{x}\cdot\mathbf{B}(\mathbf{r}=2\hat{z}) = -\frac{1}{2} \quad \textit{for} \quad \mu_0 R^3 = 6 \tag{51}$$

One key point in our derivation was to recognize that this question is equivalent to the previous one upon the identification of magnetization $\mathbf{M}$ *with* $R\sigma\vec{\omega}$*. Indeed, this is explicitly mentioned in the solution of this problem (i.e. example 1 of §6.2) in Griffiths, and we explicitly discussed this in class as well. However, it is not essential to recognize this to solve this problem. One can basically simply integrate the surface charge density to find the vector potential explicitly (without using the readily given formula (36)): this is a tedious yet doable computation, which was explicitly carried out in the solution of example 11 of §5.4 (we also discussed this in class). In any case, whatever approach the student chooses would not affect the final answer.*

« « « Congratulations, you have made it to the end! » » »