Name:	
Student ID:	

Final Examination

Phys209: Mathematical Methods in Physics I

2025/01/08

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) any of my work *outside the designated areas* in the "fill-in the blank questions" will not be graded;

(2) I take *full responsibility* for any ambiguity in my selection of the correct option in "multiple choice questions";

(3) any of my work *outside the answer boxes* in the "classical questions" will not be graded;

(4) any page which does not contain *both my name and student id* will not be graded;

(5) any extra sheet that I may use are for my own calculations and will *not* be graded.

Signature: ______________________

This exam has a total of 3 questions, some of which may be for bonus points. You can obtain a maximum grade of 34+0 from this examination.

Question	Points	Score
1	17	
2	8	

Question	Points	Score
3	9	
Total:	34	

1 Notations & Conventions

This section contains various useful definitions to refer while solving the problems. Note that it might contain additional information not covered in class, so please do not panick: the questions do not necessarily refer to *everything* in this section.

- **The non-negative integer power** of an object A (denoted A^n) is defined recursively as
$$A^0 = \mathbb{I}\ , \quad A^n = A \cdot A^{n-1}\ \ \forall n \geq 1 \tag{1}$$
with respect to the operation $\cdot$ (such as matrix multiplication or differentiation) and its identity object $\mathbb{I}$.

- **Exponentiation of an object** A (denoted e^A) is
$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{2}$$
where A^n is the $n-$th power of the object A.

- **Logarithm of an object** A (denoted $\log A$) is defined as the inverse of the exponentiation. For objects for which the exponentiation is not a monomorphism (such as complex numbers), logarithm is a *relation* (also called multi-valued function). Conventionally, one imposes restrictions on the domain to ensure that logarithm acts as a function; for instance, for a complex number $z = re^{i\theta} \in \mathbb{C}$ with $(r, \theta) \in (\mathbb{R}^+, \mathbb{R})$, we can define $\log z = i\theta_p + \log r$ where $0 \leq \theta_p < 2\pi$ is called *the principal value of* θ.

- **The generalized power of an object** A (denoted A^α) is defined as
$$A^\alpha = e^{\alpha \log A} \tag{3}$$
If exponentiation is not a monomorphism when acting on the domain of A, A^α is not a function but a relation *unless* a principle domain is selected (similar to the logarithm).

- **Generalized exponentiation of an object** A (denoted α^A) is defined as
$$\alpha^A = e^{A \log \alpha} \tag{4}$$
Depending on the available values for $\log \alpha$, α^A may mean multiple different functions. However, each one is *still* a proper function, not a multi-valued function.

- **Trigonometric functions** cos, sin, tan, cot, csc , sec are defined in terms of the exponential via the equations
$$e^{\pm iA} = \cos(A) \pm i \sin(A)\ ,\ \tan(A) = \frac{1}{\cot(A)} = \frac{\sin(A)}{\cos(A)} \tag{5}$$
$$\csc(A)\sin(A) = 1\ ,\ \sec(A)\cos(A) = 1 \tag{6}$$

- **Hyperbolic functions** cosh, sinh, tanh, coth, csch, sech are defined in terms of the exponential via equations
$$e^{\pm A} = \cosh(A) \pm \sinh(A), \tanh(A) = \frac{1}{\coth(A)} = \frac{\sinh(A)}{\cosh(A)} \tag{7}$$
$$\operatorname{csch}(A)\sinh(A) = 1\ ,\ \operatorname{sech}(A)\cosh(A) = 1 \tag{8}$$

- **Inverse Trigonometric/Hyperbolic functions** are denoted with an *arc* prefix in their naming, i.e. $\arcsin(x) \coloneqq \sin^{-1}(x)$. Like logarithm, these objects are *relations* (not functions) unless their domain is restricted.

- **The Kronecker symbol** (Kronecker-delta) is defined
$$\delta : \{\mathbb{Z}, \mathbb{Z}\} \to \mathbb{Z} \tag{9}$$
$$\delta = \{i, j\} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{10}$$

- **The Dirac-delta generalized function** δ is (for all practical purposes of a Physicist) defined via the relation
$$\int_{\mathcal{A}} f(y)\delta(x-y)dy = \begin{cases} f(x) & \text{if } x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$
A useful representation of this generalized function is
$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} \tag{12}$$

- **Heaviside generalized function** θ is (for all practical purposes of a Physicist) defined via the relations
$$\int_a^b \theta(x) f(x) dx = \begin{cases} \displaystyle\int_a^b f(x)dx & \text{if } a \geq 0 \\ \displaystyle\int_0^b f(x)dx & \text{if } a < 0 \end{cases} \tag{13}$$
This definition implies that $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$; however, it *does not fix* $f(0)$. We choose *the convention* $f(0) = 1/2$; this ensures
$$\operatorname{sgn}(x) = 2\theta(x) - 1 = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases} \tag{14}$$

- **A particular permutation of n objects** is denoted as $(i_1 i_2 \ldots i_n)$ where $i_1 \neq i_2 \neq \ldots \neq i_n \in \{1, \ldots, n\}$. A permutation $(i_1 \ldots i_n)$ is said to be an even (odd) permutation of $(k_1 \ldots k_n)$ if the two are identical after the permutation of an even (odd) number of adjacent indices. For example, (2431) is an even permutation of (2143) and an odd permutation of (2134).

- **Levi-Civita symbol** ϵ is defined as
$$\epsilon : \{\mathbb{Z}^+, \ldots, \mathbb{Z}^+\} \to \mathbb{Z} \tag{15}$$
$$\epsilon = \{a_1, \ldots, a_n\} \to \begin{cases} 1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an even permutation of } (12\ldots n) \\ -1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an odd permutation of } (12\ldots n) \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

- **The determinant function** (denoted det) is defined

$$\det : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{17}$$

$$\det = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_{i_1,\dots,i_n} \epsilon_{i_1\dots i_n} a_{1i_1}\dots a_{ni_n} \tag{18}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

- **The adjugate function** (denoted adj) is defined as

$$\mathrm{adj} : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{19}$$

$$\mathrm{adj} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \tag{20}$$

for

$$b_{k_n i_n} = \sum_{\substack{i_1,\dots,i_{n-1} \\ k_1,\dots,k_{n-1}}} \frac{\epsilon_{i_1\dots i_n}\epsilon_{k_1\dots k_n} a_{i_1k_1}\dots a_{i_{n-1}k_{n-1}}}{(n-1)!} \tag{21}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

- **Inverse of an object** A is denoted as A^{-1} and is defined with respect to an operation "$\cdot$" and its identity element $\mathbb{I}$ via the the equations $A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{I}$. If "$\cdot$" is matrix multiplication, then

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det A} \tag{22}$$

- **The trace function** (denoted tr) is defined as

$$\mathrm{tr} : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{23}$$

$$\mathrm{tr} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_i a_{ii} \tag{24}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

- **Wronskian matrix** of a set of functions $\{f_1(x), \dots, f_n(x)\}$ is defined as a square matrix where the first row is the set of the functions and the $i-$th row is $(i-1)-$th derivative of the functions for all $n \geq i \geq 2$.

- **A complex number** z is (for all practical purposes of a Physicist) a pair of two real numbers (x, y) where one can construct z via $z = x + iy$ (i is called *the imaginary unit* with the property $i^2 = -1$); conversely, one can extract x and y via $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

- **Complex conjugation** (denoted $*$) is a function defined to act on complex numbers as

$$* : \mathbb{C} \to \mathbb{C} \tag{25}$$

$$* = z \to (z^* = \mathrm{Re}(z) - i\,\mathrm{Im}(z)) \tag{26}$$

- **Matrix transpose** (denoted T) is a function defined

$$T : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{27}$$

$$T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \tag{28}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

- **Hermitian conjugation** (also called *conjugate transpose*, *adjoint*, or *dagger*) is a function defined as

$$\dagger : \mathfrak{M}_{n\times n}(\mathbb{C}) \to \mathfrak{M}_{n\times n}(\mathbb{C}) \tag{29}$$

$$\dagger = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a^*_{11} & a^*_{21} & \dots & a^*_{n1} \\ a^*_{12} & a^*_{22} & \dots & a^*_{n2} \\ \dots \\ a^*_{1n} & a^*_{2n} & \dots & a^*_{nn} \end{pmatrix} \tag{30}$$

- **Characteristic polynomial** of any square matrix A:

$$\det(A - \lambda_i \mathbb{I}) = 0 \tag{31}$$

- **Laplace transform** is an integral transform (denote $\mathcal{L}$) which converts a function $f : \mathbb{R} \to \mathbb{R}$ into another function $\hat{f} = \mathcal{L}(f)$ such that

$$\hat{f} : \mathbb{C} \to \mathbb{C}\ , \qquad \hat{f}(s) = \int_0^\infty f(x) e^{-xs} dx \tag{32}$$

For *meromorphic* $\hat{f}$ (i.e. $\frac{\text{polynomial}}{\text{polynomial}}$), the inverse is computed by rewriting $\hat{f}(s)$ as a sum $\sum_i a_i (s + r_i)^{-n_i-1}$ which is clearly (for some $c_{k,\ell}$) the Laplace transform of $f(x) = \sum_i e^{-r_i x} \left(c_{i,1} + c_{i,2}x + \dots c_{i,n_i} x^{n_i}\right)$. Formally,

$$f : \mathbb{R} \to \mathbb{R}\ , \qquad f(x) = \int_{\gamma - i\infty}^{\gamma + i\infty} \hat{f}(s) e^{xs} \frac{ds}{2\pi i} \tag{33}$$

where the *contour integral* in the complex plane is chosen appropriately based on the convergence.

- **Convolution** of two functions f and g (denote $f * g$) is the operation that becomes multiplication in the Laplace domain, i.e. $\mathcal{L}(f * g) \equiv \mathcal{L}(f)\mathcal{L}(g)$; equivalently,

$$(f * g)(x) = \int_0^x f(y) g(x - y) dy \tag{34}$$

- **Commutator** is a higher order function which takes two functions $f, g : \mathcal{A} \to \mathcal{A}$ for any type $\mathcal{A}$, and gives a new function $[f, g] : \mathcal{A} \to \mathcal{A}$ by cascading their action. It is defined on an object $x \in \mathcal{A}$ as $[f, g](x) = f(g(x)) - g(f(x))$.

- **Polar coordinates in $\mathbb{R}^d$** $(r, \theta_1, \dots, \theta_{d-1})$ are defined in terms of the Cartesian coordinates $(x_1, \dots, x_d)$ as

$$x_1 = r\cos(\theta_1)\ , \quad x_d = x_{d-1}\tan(\theta_{d-1}) \tag{35}$$

$$x_i = x_{i-1}\tan(\theta_{i-1})\cos(\theta_i) \quad \text{for } 1 < i < d \tag{36}$$

In two-dimensions, this reduces to the familiar polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$; in 3 ($> 3$) dimensions, it is also called *(hyper)spherical coordinates.*

- **Cylindrical coordinates in $\mathbb{R}^d$** $(\rho, \theta_1, \dots, \theta_{n-1}, x_n, x_{n+1}, \dots, x_d)$ is a coordinate system such that a subset $\mathbb{R}^n$ of the total space $\mathbb{R}^d$ (for $n < d$) is converted into the polar coordinates. For instance, if we convert $\mathbb{R}^2$ of $\mathbb{R}^3$ into polar coordinates, we obtain the familiar *3d cylindrical coordinates*, i.e. $(x, y, z) = (\rho\cos\theta, \rho\sin\theta, z)$.

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2 Fill-in the blanks

Each correct answer is worth 1.7 *point.*

Question: 1 . . . . *(17 points)*

With this final exam, you will have finished your basic education in the differential equations. So let us review the whole semester to make sure that you are ready to move on!

We tentatively defined a differential equation as any equation that involves differentiation; for instance, we can call "$\cos\left(1+\frac{d}{dx}\right)\sin(2+f(x)) = 3$" a differential equation. In practice though, we focus on a special subset called *finite-order* differential equations: such differential equations do not involve derivatives of arbitrarily high orders; indeed, when expanded as a series, any term with $\frac{d^k}{dx^k}$ obeys $k \leq n$ for a finite n: n is called the *order* of the differential equation.

Choosing this subset is justified as majority of the equations we encounter in real life are of this type. However, we specialized further by focusing our attention on the differential equations in which the unknown function appears at most as a first order term. In other words, if our differential equation is $\mathcal{F}\left(f(x), g(x), x, \frac{d}{dx}\right) = 0$ for a known external function $g(x)$ and the unknown function $f(x)$, then it satisfies $\frac{d^2}{d\lambda^2}\mathcal{F}\left(\lambda f(x), g(x), x, \frac{d}{dx}\right) = 0$ for any λ. Such differential equations are called *linear*; for instance, consider the differential equation $\mathcal{F}\left(f(x), g(x), x, \frac{d}{dx}\right) = f'''(x) + xf''(x) + f(x)^2 + \cos(x) = 0$; clearly, $\frac{d^2}{d\lambda^2}\mathcal{F}\left(\lambda f(x), g(x), x, \frac{d}{dx}\right) = 2f(x)^2 \neq 0$. Therefore, this is not in the subclass we focused our main attention throughout the semester!

Let us remember the first weeks of the semester: we focused on *linear, ordinary, finite-order differential equations with constant coefficients*. We learned that such differential equations can be neatly analyzed with *Laplace* transform; namely, it is a lot easier to work with the function $\hat{f}(s) := \int\limits_0^\infty f(x)e^{-xs}dx$ instead of $f(x)$ for such differential equations! Even though this transformation is well defined for a broad set of functions, we only used this method for differential equations with *constant coefficients* because only for such differential equations $\hat{f}(s)$ satisfies an algebraic equation! If we use this transformation, say, to solve $f''(x) = xf(x)$, the equation $\hat{f}(s)$ satisfies actually will not be an algebraic equation!

Converting differential equations to algebraic equation is the dream, and we learned other methods as well to achieve this goal! For instance, *Frobenius method* ensures that we can solve infinitely many algebraic equations for infinitely many unknown coefficients if we consider a series expansion around a regular singular point (remember that we do not have a series expansion method to work around *essential/irregular singular* points!) Although solving infinitely many equations seemed impossible at first, we quickly discovered that the equations are actually interrelated (e.g. $a_n = 2a_{n-1}+1$ for all unknown coefficients $a_{n>1}$), and can be recursively solved.

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Such equations (e.g. $a_n = 2a_{n-1} + 1$) are called ___*recursion/incidence*___ equations and solving them gives the series solution.

Throughout the semester we have learned several methods and techniques such as these, but we should never lose the forest for the trees: at the end of the day, we are trying to understand the phenomena described by the differential equations, and obtaining the final result may not be the best approach to achieve this goal. This is why sometimes it is beneficial to analyze the same system in a different formalism, and we advocated this throughout the matrix reformulation of differential equations: you will appreciate this better when you learn phase space in your junior and senior years, but we nevertheless discussed some conceptual points such as normal modes of a system, appearance of time-ordering as a result of Volterra integral equation (and how that will be related to time dependent Hamiltonians in quantum mechanics later), and how concept of self-adjoint operators and Sturm–Liouville theory is the natural framework for operators with real eigenvalues. Ultimately, we are bounded by the available time and had to make some sacrifices to discuss whatever we chose to cover in a sufficient depth. This is why we only went over ___*partial*___ differential equations (e.g. equations with derivatives with respect to more than one variable) very briefly; nevertheless, we solved in class ___*the wave equations*___, i.e. differential equations of the form $\dfrac{\partial^2 f(x,t)}{\partial t^2} = c^2 \dfrac{\partial^2 f(x,t)}{\partial x^2}$.

This being the final exam, I think you all deserve a small bonus; please enter the name *Durmuş Ali Demir* to this empty space to get your bonus: ___*Durmuş Ali Demir*___. If you do not know, he was an important Turkish Physicist whom we have lost in 2024. Being a scientist is a hard life, so small things (such as a young scientist candidate learning the name of a late scientist) means a lot to us all!

3 Choose the correct option

You do not need to show your derivation in this part.
Incorrect answer for a question of X point is worth $-X/4$ points: this ensures that the randomly given answer has an expectation value of 0 point.

Question: 2 . . . . *(8 points)*
Wave equation is a partial differential equation, a rather important one. It is the differential description of the *plane wave*, a crude yet surprisingly effective description of oscillating waves originated far away from the observer. Since *aurora borealis* (also known as northern lights) is a natural phenomena due to the electromagnetic shock waves coming from the sun, a distant object for us observers at Earth, we can use the wave equation to understand that electromagnetic transmission during their journey to the Earth! You can more fancifully state to your non-physicist friends that you basically explored some part of the theoretical background of northern lights in your final exam (not a too incorrect statement).

To achieve this smugness (in an harmless & playful way of course), let us first set up our coordinates: call the line joining the centers of Earth and Sun on 2025/01/08 *the x-axis*, choose $x = 0$ to be the center of the sun with Earth lying on $x > 0$, and let t denote the time with $t = 0$ being the starting time of this exam. Any solar activity occuring during this exam will generate electromagnetic shock waves that will reach to the Earth in approximately 8 minutes.

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Around the half time before they reach to the Earth, they will look like plane waves to us, so let us model these electromagnetic waves as solutions to the differential equation

$$\frac{\partial^2\phi(x,t)}{\partial t^2} = c^2\frac{\partial^2\phi(x,t)}{\partial x^2} \tag{37}$$

where $\phi(x,t)$ denotes *the amplitude* of the light (such that your eyes are sensitive to its square, *the intensity* of the light).

(a) **(1 point)** Let us first consider a single variable example where y is a fixed function of time t, i.e. $y = f(t)$. Which one of the below is the correct chain rule?

■ $\frac{d}{dy} = \frac{1}{f'(t)}\frac{d}{dt}$ ☐ $\frac{d}{dy} = f'(t)\frac{d}{dt}$ ☐ $\frac{d}{dy} = \frac{1}{f^{-1}(t)}\frac{d}{dt}$ ☐ $\frac{d}{dy} = f^{-1}(t)\frac{d}{dt}$
☐ None

Solution 2.1 *The chain rule is*

$$\frac{d}{dt} = \frac{dy}{dt}\frac{d}{dy} = \frac{df(t)}{dt}\frac{d}{dy} = f'(t)\frac{d}{dy} \tag{38}$$

hence the first equation is correct.

(b) **(2 points)** Now instead, consider the multi-variable case for the functions $A = f(a,b)$ and $B = g(a,b)$; again, which one of the below is the correct chain rule?

■ $\frac{\partial}{\partial a} = \frac{\partial f(a,b)}{\partial a}\frac{\partial}{\partial A} + \frac{\partial g(a,b)}{\partial a}\frac{\partial}{\partial B}$, $\frac{\partial}{\partial b} = \frac{\partial f(a,b)}{\partial b}\frac{\partial}{\partial A} + \frac{\partial g(a,b)}{\partial b}\frac{\partial}{\partial B}$

☐ $\frac{\partial}{\partial a} = \frac{\partial f^{-1}(a,b)}{\partial a}\frac{\partial}{\partial A} + \frac{\partial g^{-1}(a,b)}{\partial a}\frac{\partial}{\partial B}$, $\frac{\partial}{\partial b} = \frac{\partial f^{-1}(a,b)}{\partial b}\frac{\partial}{\partial A} + \frac{\partial g^{-1}(a,b)}{\partial b}\frac{\partial}{\partial B}$

☐ $\frac{\partial}{\partial a} = \frac{\partial f(a,b)}{\partial a}\frac{\partial}{\partial B} + \frac{\partial g(a,b)}{\partial a}\frac{\partial}{\partial A}$, $\frac{\partial}{\partial b} = \frac{\partial f(a,b)}{\partial b}\frac{\partial}{\partial B} + \frac{\partial g(a,b)}{\partial b}\frac{\partial}{\partial A}$

☐ $\frac{\partial}{\partial a} = \frac{\partial f^{-1}(a,b)}{\partial a}\frac{\partial}{\partial B} + \frac{\partial g^{-1}(a,b)}{\partial a}\frac{\partial}{\partial A}$, $\frac{\partial}{\partial b} = \frac{\partial f^{-1}(a,b)}{\partial b}\frac{\partial}{\partial B} + \frac{\partial g^{-1}(a,b)}{\partial b}\frac{\partial}{\partial A}$

☐ None

Solution 2.2 *There is not much to derive: if you cannot recognize the correct chain rule, please consult to your calculus book to review the details!*

(c) **(3 points)** We will now solve this differential equation by converting it to a simpler one —we actually did solve this precise equation in this precise method in the very last class of the semester! For that, we use our physical intuition: if this equation describes a plane wave propagating at a constant speed, the value of the function ϕ should be same at the

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(position,time)= (x, t) and at the (position,time)= $(x \pm v\Delta t, t + \Delta t)$ where v is the speed of the wave and we consider waves propagating both in positive and negative x directions! Obviously, the only constant in the differential equation is c, and since we expect the velocity to be constant, we need $c = v$ (dimensional analysis fixes this up to an immaterial overall factor that can be chosen 1). This then implies $\phi(x, t) = \phi(x \pm c\Delta t, t + \Delta t)$, suggesting that the differential equation can be simplified if we work with a *better* set of parameters:

$$A = x + ct \ , \quad B = x - ct \tag{39}$$

where $A = 0$ and $B = 0$ describes a motion with constant speed towards negative and positive x direction respectively. Using the chain rule, one can rewrite original equation as

$$\lambda \frac{\partial^2}{\partial A \partial B} \phi(A, B) = 0 \tag{40}$$

What is λ?

☐ $-2c$ ☐ $+2c$ ■ $-4c^2$ ☐ $4c^2$ ☐ None

Solution 2.3 *Note that*

$$\begin{aligned}\frac{\partial^2}{\partial t^2} =& \frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t}\frac{\partial}{\partial A} + \frac{\partial B}{\partial t}\frac{\partial}{\partial B}\right) = \frac{\partial}{\partial t}\left(c\frac{\partial}{\partial A} - c\frac{\partial}{\partial B}\right) = c\frac{\partial}{\partial A}\frac{\partial}{\partial t} - c\frac{\partial}{\partial B}\frac{\partial}{\partial t} \\ =& c\frac{\partial}{\partial A}\left(c\frac{\partial}{\partial A} - c\frac{\partial}{\partial B}\right) - c\frac{\partial}{\partial B}\left(c\frac{\partial}{\partial A} - c\frac{\partial}{\partial B}\right) \\ =& c^2\left(\frac{\partial^2}{\partial A^2} - 2\frac{\partial^2}{\partial A \partial B} + \frac{\partial^2}{\partial B^2}\right)\end{aligned} \tag{41}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x^2} =& \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x}\frac{\partial}{\partial A} + \frac{\partial B}{\partial x}\frac{\partial}{\partial B}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial A} + \frac{\partial}{\partial B}\right) = \frac{\partial}{\partial A}\frac{\partial}{\partial x} + \frac{\partial}{\partial B}\frac{\partial}{\partial x} \\ =& \frac{\partial}{\partial A}\left(\frac{\partial}{\partial A} + \frac{\partial}{\partial B}\right) + \frac{\partial}{\partial B}\left(\frac{\partial}{\partial A} + \frac{\partial}{\partial B}\right) \\ =& \left(\frac{\partial^2}{\partial A^2} + 2\frac{\partial^2}{\partial A \partial B} + \frac{\partial^2}{\partial B^2}\right)\end{aligned} \tag{42}$$

Therefore

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)\phi(x, t) = 0 \quad \Rightarrow \quad -4c^2\frac{\partial^2}{\partial A \partial B}\phi(A, B) = 0 \tag{43}$$

However, *one could in principle collect all the terms on the right hand side in the original equation, hence start with* $0 = \left(c^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\phi(x, t)$*, leading to* $\lambda = 4c^2$*. Because of this ambiguity, I'll consider this question ill-defined, and award all students the full credit.*

(d) **(2 points)** It is now easy to solve this differential equation: note that

$$\lambda \frac{\partial^2}{\partial A \partial B} \phi(A, B) = 0 \tag{44}$$

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implies for nonzero λ that $\frac{\partial}{\partial A}\left(\frac{\partial \phi(A,B)}{\partial B}\right) = 0$, hence

$$\frac{\partial \phi(A,B)}{\partial B} = g(B) \tag{45}$$

for an arbitrary function g. This equation then implies

$$\phi(A,B) = c_1(A) + c_2(B) \tag{46}$$

for $\frac{\partial c_2(B)}{\partial B} = g(B)$ and an arbitrary function $c_1(A)$. Since $g(B)$ was also arbitrary, we can simply say that the equation above is the solution for arbitrary functions $c_{1,2}$. Which one of the below options is then a solution to the wave equation and that it might describe a burst of *oscillating* electromagnetic wave originated at the sun and is *approaching* to Earth?

- ☐ $\phi(t,x) = 10 \exp\left(|x+ct|^2\right)$
- ☐ $\phi(t,x) = -10 \exp\left(|x+ct|^2\right)$
- ☐ $\phi(t,x) = 10 \cos\left(|x+ct|^2\right)$
- ☐ $\phi(t,x) = -10 \cos\left(|x+ct|^2\right)$
- ■ **None**

Solution 2.4 *We would like a solution which moves* towards *Earth. As explicitly hinted in the previous part, this then implies $\phi(x,t) = \phi(x+c\Delta t, t+\Delta t)$ —if the wave was moving away (hence moving in negative x direction), then we would need $\phi(x,t) = \phi(x-c\Delta t, t+\Delta t)$. We also discussed this precise difference (which sign describes in which direction the motion is) in class as well.*

The condition $\phi(x,t) = \phi(x+c\Delta t, t+\Delta t)$ is satisfied only if $\phi(x,t)$ is a function of $x-ct$, i.e. $\phi(x,t) = c_2(B) = c_2(x-ct)$. None of the options above satisfies this, hence they all describe burst of electromagnetic radiation moving away *from the Earth!*

Question: 3 . . . . *(9 points)*

In the previous question, we analyzed the wave equation: crude description of a wave far away from the observer (hence plane approximation holds) and which is at constant speed. In this question, we will solve a different partial wave equation, and it will have a solution that describes a wave which does not have constant speed but constant *acceleration*!

Consider the differential equation

$$\frac{\partial^2 \psi(x,t)}{\partial t \partial x} = -at\frac{\partial^2 \psi(x,t)}{\partial x^2} \tag{47}$$

for a constant $a > 0$. As we will see below, a will be the acceleration of the wave (note that it has the correct dimensions, i.e. length/time^2).

(a) **(2 points)** Just like the previous question, we will find out a new set of parameters where the differential equation will be trivial to solve! We choose them as

$$A = x - \frac{a}{2}t^2\ , \quad B = x + \frac{a}{2}t^2 \tag{48}$$

Note that $A = 0$ describes a motion accelerated towards positive x axis with a constant acceleration a, and $B = 0$ describes a similar motion towards the negative x axis. With these new parameters, which one of the chain rules below is the correct one?

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☐ $\frac{1}{at}\frac{\partial}{\partial x} = \frac{\partial}{\partial B} + \frac{\partial}{\partial A} \quad , \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial B} - \frac{\partial}{\partial A}$

■ $\frac{\partial}{\partial x} = \frac{\partial}{\partial B} + \frac{\partial}{\partial A} \quad , \quad \frac{1}{at}\frac{\partial}{\partial t} = \frac{\partial}{\partial B} - \frac{\partial}{\partial A}$

☐ $\frac{1}{at}\frac{\partial}{\partial x} = \frac{\partial}{\partial B} - \frac{\partial}{\partial A} \quad , \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial B} + \frac{\partial}{\partial A}$

☐ $\frac{\partial}{\partial x} = \frac{\partial}{\partial B} - \frac{\partial}{\partial A} \quad , \quad \frac{1}{at}\frac{\partial}{\partial t} = \frac{\partial}{\partial B} + \frac{\partial}{\partial A}$

☐ None

Solution 3.1 *The chain rule is straightforward:*

$$\frac{\partial}{\partial x} = \frac{\partial A}{\partial x}\frac{\partial}{\partial A} + \frac{\partial B}{\partial x}\frac{\partial}{\partial B} = \frac{\partial}{\partial A} + \frac{\partial}{\partial B} \tag{49}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial A}{\partial t}\frac{\partial}{\partial A} + \frac{\partial B}{\partial t}\frac{\partial}{\partial B} = -at\frac{\partial}{\partial A} + at\frac{\partial}{\partial B} \tag{50}$$

(b) **(3 points)** Define the operator $\mathcal{D} = \frac{\partial^2}{\partial t \partial x} + at\frac{\partial^2}{\partial x^2}$. Which one below is correct?

☐ $\mathcal{D} = -2at\left(\frac{\partial}{\partial A} - \frac{\partial}{\partial B}\right)\frac{\partial}{\partial B}$ ☐ $\mathcal{D} = 2at\left(\frac{\partial}{\partial A} - \frac{\partial}{\partial B}\right)\frac{\partial}{\partial B}$

☐ $\mathcal{D} = -2at\left(\frac{\partial}{\partial A} + \frac{\partial}{\partial B}\right)\frac{\partial}{\partial B}$ ■ $\mathcal{D} = 2at\left(\frac{\partial}{\partial A} + \frac{\partial}{\partial B}\right)\frac{\partial}{\partial B}$ ☐ None

Solution 3.2 *As we have*

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial A} + \frac{\partial}{\partial B} \quad , \quad \frac{\partial}{\partial t} = -at\frac{\partial}{\partial A} + at\frac{\partial}{\partial B} \tag{51}$$

we immediately get

$$\frac{\partial}{\partial t} + at\frac{\partial}{\partial x} = 2at\frac{\partial}{\partial B} \tag{52}$$

hence

$$\mathcal{D} = \frac{\partial^2}{\partial t\partial x} + at\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + at\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x} = 2at\frac{\partial}{\partial B}\left(\frac{\partial}{\partial A} + \frac{\partial}{\partial B}\right) = 2at\left(\frac{\partial}{\partial A} + \frac{\partial}{\partial B}\right)\frac{\partial}{\partial B} \tag{53}$$

(c) **(4 points)** We can now immediately see the solution we wanted to obtain: a wave that accelerates at a constant rate! To see that, note that any function that satisfies $\frac{\partial}{\partial B}f(A,B) = 0$ *is* a solution to the original equation, i.e.

$$\frac{\partial}{\partial B}f(A,B) = 0 \quad \Rightarrow \quad \pm 2at\left(\frac{\partial}{\partial A} \pm \frac{\partial}{\partial B}\right)\frac{\partial}{\partial B}f(A,B) = 0 \quad \Rightarrow \quad \frac{\partial^2 f(x,t)}{\partial t\partial x} = -at\frac{\partial^2 f(x,t)}{\partial x^2} \tag{54}$$

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and obviously $f(A,B) = c(A)$ is a solution to $\frac{\partial}{\partial B} f(A,B) = 0$ for any arbitrary function c. Therefore, we conclude that

$$\psi(x,t) = c\left(x - \frac{a}{2}t^2\right) \tag{55}$$

is a solution to the original differential equation for any function c!

Let us now revisit *aurora borealis*, as x and t defined in the previous question. For a burst of charged particles modeled as a wave getting accelerated under the electromagnetic fields of the sun's plasma, the wave approximation of the particles satisfy the following equation around the halfway to Earth (hence at $t = 4$ minutes)

$$\psi(x,4) = e^{-(x-16)^2} \cos(x) \tag{56}$$

What is $\psi(x,t)$ for $a = 2$?

- ☐ $\psi(x,t) = e^{-(x-4t)^2} \cos(x - 4t + 16)$
- ☐ $\psi(x,t) = e^{-(x+4t-32)^2} \cos(x + 4t - 16)$
- ■ $\psi(x,t) = e^{-\left(x-t^2\right)^2} \cos\left(x - t^2 + 16\right)$
- ☐ $\psi(x,t) = e^{-\left(x+t^2-32\right)^2} \cos\left(x + t^2 - 16\right)$
- ☐ None

Solution 3.3 *We are given that*

$$\psi(x,t) = c\left(x - \frac{a}{2}t^2\right) \tag{57}$$

for an undetermined function. We are also given that

$$\psi(x,4) = e^{-(x-16)^2} \cos(x) \tag{58}$$

This then implies

$$c\,(x - 8a) = e^{-(x-16)^2} \cos(x) \tag{59}$$

Setting $x = z + 8a$ for the variable z, we then fix the undetermined function c:

$$c(z) = e^{-(z+8a-16)^2} \cos(z + 8a) \tag{60}$$

which then gives

$$\psi(x,t) = c\left(x - \frac{a}{2}t^2\right) = e^{-\left(x-\frac{a}{2}t^2+8a-16\right)^2} \cos\left(x - \frac{a}{2}t^2 + 8a\right) \tag{61}$$

leading to

$$\psi(x,t) = e^{-\left(x-t^2\right)^2} \cos\left(x - t^2 + 16\right) \quad \textit{for } a = 2 \tag{62}$$

« « « Congratulations, you have made it to the end! » » »