A Review: Spinning Conformal Correlators

Mustafa Ateş[1]

[1]*Middle East Technical University (METU), Ankara, Turkiye*

(Dated: December 26, 2024)

In this letter, the article "Spinning Conformal Correlators"[1] has been reviewed. The article introduces a simplified framework for studying symmetric traceless tensor operators in Conformal Field Theory (CFT), extending techniques like conformal bootstrap beyond scalar operators. Using an embedding formalism with index-free polynomials, computations are simplified, and tensor structures in n-point correlation functions are classified. This approach bridges CFT correlators with scattering amplitudes in higher-dimensional Minkowski space, offering new insights into their connection via AdS/CFT duality. The work enhances the computational toolkit for understanding the structure and constraints of higher-dimensional CFTs.

Introduction - In two dimensions, CFTs are well understood, thanks to the infinite-dimensional Virasoro Algebra. However, in higher dimensions, exact solutions are rare. Although scalar operators in CFTs have been widely explored, the study of higher-spin tensor operators remains a more complex but highly promising area of research.

The conformal bootstrap[2, 3] is one of the approaches that enables us to solve or constrain a higher-dimensional CFT by using the operator product expansion (OPE) associativity. However, in this article, another formalism, the so-called embedding space formalism with index-free notation is developed. This formalism provides a powerful toolkit for handling spinning operators, such as those describing conserved currents or the stress-energy tensor.

The key question this paper addresses is: How can we efficiently compute correlation functions of spinning operators in a way that respects conformal symmetry? Furthermore, can we extend bootstrap methods to include these more complex cases?

Embedding Formalism - In the reviewed paper, the authors considered CFT in $d \geq 3$ Euclidean dimensions, which implies the conformal group is $SO(d+1,1)$. It is beneficial to state that the equations can be Wick-rotated to the Lorentz signature. It is widely recognized that conformal symmetry places significant restrictions on the correlation functions of primary operators within a theory. While these restrictions are relatively straightforward to determine for primary scalar fields, they become more complex and less intuitive for primary fields with nonzero spin. To tackle this challenge, 'embedding formalism' is developed. In this framework, conformal symmetry in d-dimensions is reinterpreted as Lorentz symmetry in a (d+2)-dimensional Minkowski space, denoted $\mathbb{M}^{d+2}$.

A point in physical space corresponds to a null ray in $\mathbb{M}^{d+2}$ consisting of vectors

$$P^A = \lambda\left(1, x^2, x^a\right), \quad \lambda \in \mathbb{R}, \tag{1}$$

This correspondence is constructed by using the following properties on a tensor field $F_{A_1 \ldots A_l}$:

1. Defined on the cone $P^2 = 0$.
2. Homogeneous of degree $-\Delta$:
$$F_{A_1 \ldots A_l}(\lambda P) = \lambda^{-\Delta} F_{A_1 \ldots A_l}(P), \quad \lambda > 0.$$
3. Symmetric and traceless.
4. Transverse:
$$(P \cdot F)_{A_2 \ldots A_l} \equiv P^A F_{A A_2 \ldots A_l} = 0.$$

All of these conditions imply SO(d+1,1) is invariant. Projecting the tensor field $F_{A_1 \ldots A_l}$ to the Poincaré section defines a symmetric tensor field on $\mathbb{R}^d$:

$$f_{a_1 \ldots a_l}(x) = \frac{\partial P^{A_1}}{\partial x^{a_1}} \cdots \frac{\partial P^{A_l}}{\partial x^{a_l}} F_{A_1 \ldots A_l}(P_x). \tag{2}$$

Briefly, rather than performing calculations with primary tensor fields directly in the physical space, we can work with tensor fields in $\mathbb{M}^{d+2}$, where $SO(d+1,1)$ symmetry is manifest. The results can then be projected onto $\mathbb{R}^d$. This approach ensures the conformal invariance of the final outcome automatically.

Index Free Formalism - The key insight is that every symmetric tensor can be encoded by a d-dimensional polynomial:

$$f_{a_1 \ldots a_l} \text{ symmetric} \leftrightarrow f(z) \equiv f_{a_1 \ldots a_l} z^{a_1} \cdots z^{a_l}. \tag{3}$$

It is well-known that spin l primary fields are symmetric traceless tensors. These tensors can be completely represented by restricting the corresponding polynomial $f(z)$ to the submanifold $z^2 = 0$:

$$f_{a_1 \ldots a_l} \text{ symmetric traceless} \leftrightarrow f(z)|_{z^2=0}. \tag{4}$$

Here, z stands for a complex variable.

Briefly, we will express the results for physical-space correlators using polynomials instead of tensors. Furthermore, we will eliminate any polynomial terms explicitly proportional to z^2. This approach results in a polynomial that encodes the original symmetric traceless tensor as described in Eq.(5):

$$f(z) = \tilde{f}(z) + O(z^2). \tag{5}$$

The next stage is to move the discussion to the embedding space. What would happen when physical points and null rays correspondence is used to pursue the authors' mathematical construction?

$$F_{A_1\ldots A_l}(P) \text{ symmetric} \leftrightarrow F(P,Z) \equiv F_{A_1\ldots A_l}(P) Z^{A_1}\cdots Z^{A_l}. \tag{6}$$

The equation above implies tensors will in general depend on P. The following diagram will be helpful to figure the steps of encoding tensors with polynomials.

$$\begin{tikzcd} F_{A_1\ldots A_l}(P) \arrow[r, "(6)"] \arrow[d, "(2)"'] & F(P,Z) \arrow[d, dashed] \\ f_{a_1\ldots a_l}(x) \arrow[r, "(3)"'] & f(x,z) \end{tikzcd}$$

We now focus on tensors that are symmetric, traceless, and transverse (STT). For these tensors, the associated polynomial can be constrained to the subset of Z's satisfying $Z^2 = 0$ and $Z \cdot P = 0$:

$$F_{A_1\ldots A_l}(P)\,(\text{STT}) \leftrightarrow F(P,Z)\big|_{Z^2=0,\, Z\cdot P=0}. \tag{7}$$

To clarify, we are referring to the following. Let $F_{A_1\ldots A_l}(P)$ represent an STT tensor, and let $\tilde{F}_{A_1\ldots A_l}(P)$ be any tensor whose polynomial matches $F(P,Z)$ modulo terms proportional to Z^2 and $Z \cdot P$:

$$F(P,Z) = \tilde{F}(P,Z) + \mathcal{O}(Z^2, Z\cdot P). \tag{8}$$

In this case, $F_{A_1\ldots A_l}(P)$ can be reconstructed from $\tilde{F}_{A_1\ldots A_l}(P)$, up to terms involving gauge freedom.
This principle is particularly useful since it preserves the transversality condition.

In the paper, Costa et al. (2011) mainly focus on tensors constructed from metrics and components of M^{d+2} vectors. For such tensors, the canonical approach to derive the encoding polynomial $\tilde{F}(P,Z)$ involves removing all terms in $F(P,Z)$ that are proportional to Z^2 and $Z\cdot P$. This method is particularly useful since it not only preserves the transversality condition but also strengthens it.

Referring back to the encoding polynomials, the transversality condition can be expressed as:

$$P \cdot \frac{\partial}{\partial Z} F(P,Z) = 0, \tag{9}$$

Alternatively, this condition can also be written as:

$$F(P, Z + \alpha P) = F(P,Z), \quad \forall \alpha. \tag{10}$$

These conditions are satisfied modulo P^2 in general and exactly when the tensor is inherently transverse. According to this discussion, the inherently transverse polynomial $\tilde{F}(P,Z)$ is constructed from $F(P,Z)$ by excluding all terms involving Z^2 and $Z \cdot P$. This process follows the "canonical approach" previously described.

This framework is particularly advantageous since inherently transverse polynomials are simple to identify. A polynomial $\tilde{F}(P,Z)$ is inherently transverse if and only if the variable Z_A appears exclusively through the antisymmetric tensor:

$$C_{AB} \equiv Z_A P_B - Z_B P_A. \tag{11}$$

Example - As an example, let's consider two-point correlator of vector fields $\langle v_a(x_1) v_b(x_2)\rangle$. Describing in the embedding formalism:

$$G_{AB}(P_1,P_2) \equiv \langle V_A(P_1) V_B(P_2)\rangle. \tag{12}$$

Homogeneity and transversality properties imply the following equations.

$$G_{AB}(\lambda P_1, P_2) = G_{AB}(P_1, \lambda P_2) = \lambda^{-\Delta} G_{AB}(P_1,P_2) \tag{13}$$

$$P_1^A G_{AB}(P_1,P_2) = 0, \quad P_2^B G_{AB}(P_1,P_2) = 0 \tag{14}$$

The following equation is constructed by the conditions mentioned earlier.

$$G_{AB}(P_1,P_2) = \frac{1}{(P_{12})^{\Delta}}\left[c_1 \tilde{W}_{AB} + c_2 \frac{P_{1A}P_{2B}}{P_1\cdot P_2}\right] \tag{15}$$

where

$$\tilde{W}_{AB} = \eta_{AB} - \frac{P_{1B}P_{2A}}{P_1 \cdot P_2} \tag{16}$$

The second term in G_{AB} represents a pure gauge contribution and vanishes under projection. $\tilde{W}_{AB}$ will project to the following identity after performing a few calculations.

$$w_{ab} = \delta_{ab} - 2\frac{(x_{12})_a (x_{12})_b}{x_{12}^2} \tag{17}$$

It is beneficial to state that P_{ij} is defined:

$$P_{ij} \equiv -2 P_i \cdot P_j \tag{18}$$

also

$$-2P_{x_i}\cdot P_{x_j} = x_{ij}^2 \quad (x_{ij} \equiv x_i - x_j)\,. \tag{19}$$

Eventually, we obtain the well-known function.

$$\langle v_a(x_1) v_b(x_2)\rangle = c_1 \frac{w_{ab}}{(x_{12}^2)^{\Delta}} \tag{20}$$

Summary and Conclusion - This paper developed a new method for calculating the correlation functions of operators with arbitrary spin. Thanks to the formalism in this paper, calculating the correlation functions of symmetric traceless tensors imposed by the symmetries of CFT has become as straightforward as analyzing scalar functions. Since this paper was primarily written to develop a computational method, the authors did not present any applications. However, in the 13 years since its publication, the formalism constructed in this paper has been applied in numerous fields.

In the introduction section, I mentioned the questions this paper aimed to address. One of these questions was whether we could extend bootstrap methods. The goal of the conformal bootstrap is to use operator product associativity to constrain or solve higher-dimensional CFTs. In another paper written by the same authors in the same year, this formalism was used to advance bootstrap methods[4].

These two papers are widely used in various fields because they simplify calculations significantly. Among these applications, the most notable ones are in the areas of AdS/CFT correspondence and holography[5, 6]. We know that research in these areas is conducted to develop a theory of quantum gravity. Therefore, the fact that this paper provides a mathematical toolkit to address one of the most enigmatic questions in theoretical physics makes this work, as well as future research in this field, highly impactful.

[1] Miguel S. Costa, João Penedones, David Poland, and Slava Rychkov, "Spinning Conformal Correlators," *Journal of High Energy Physics*, **2011**, 71 (2011). DOI: 10.1007/JHEP11(2011)071

[2] S. Ferrara, A. F. Grillo, and R. Gatto, "Tensor Representations of Conformal Algebra and Conformally Covariant Operator Product Expansion," *Annals of Physics*, vol. 76, 1973, pp. 161.

[3] A. M. Polyakov, "Nonhamiltonian approach to conformal quantum field theory," *Zh. Eksp. Teor. Fiz.*, vol. 66, 1974, p. 23. Translated in *JETP*, vol. 39, 1974, p. 1018.

[4] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, "Spinning Conformal Blocks," *JHEP*, vol. 11, p. 154, 2011, doi: 10.1007/JHEP11(2011)154, arXiv:1109.6321 [hep-th].

[5] A. L. Fitzpatrick, J. Kaplan, D. Poland, and D. Simmons-Duffin, "The Analytic Bootstrap and AdS Superhorizon Locality," *JHEP*, vol. 12, p. 004, 2013, doi: 10.1007/JHEP12(2013)004, arXiv:1212.3616 [hep-th].

[6] M. S. Costa, V. Gonçalves, and J. Penedones, "Spinning AdS Propagators," *JHEP*, vol. 09, p. 064, 2014, doi: 10.1007/JHEP09(2014)064, arXiv:1404.5625 [hep-th].