Phys210: Mathematical Methods in Physics II

Homework 1

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) (if any) are for your own practice!
- Unless stated otherwise, you are expected to show your derivation of the results.
- The homework is due March 8th 2024, 23:59 TSI.

(1) Problem One *(2 points)*

We have seen in class that the sets with infinitely many elements can be unambiguously constructed via set comprehensions and predicates. Let's see this in a few examples.

Consider the predicate

$$\text{isMultipleOfFive} :: \mathbb{Z} \to \texttt{Boolean} \tag{1.1a}$$
$$\text{isMultipleOfFive} = a \to [(\exists n \in \mathbb{Z})\ a = 5n] \tag{1.1b}$$

This function yields the result True if fed with an integer multiple of 5, and False otherwise:

$$\text{isMultipleOfFive}(1) = \mathsf{False} \tag{1.2a}$$
$$\text{isMultipleOfFive}(5) = \mathsf{True} \tag{1.2b}$$
$$\text{isMultipleOfFive}(15) = \mathsf{True} \tag{1.2c}$$

and so on. We can now use this predicate to construct the set of integer multiples of 5: $\boxed{\left\{x \in \mathbb{Z} \mid \text{isMultipleOfFive}(x)\right\}}$.

We will now construct the set of positive integers that are integer multiples of 5 but not integer multiples of 3, i.e. $\{\ldots, 5, 10, 20, 25, 35, \ldots\}$.

1. Define a predicate that yields True if its input is an integer multiple of 3, and False otherwise (call it "isMultipleOfThree").

2. Consider the functions "¬" and "∧":

$$\neg :: \texttt{Boolean} \to \texttt{Boolean} \tag{1.3a}$$
$$\wedge :: (\texttt{Boolean}, \texttt{Boolean}) \to \texttt{Boolean} \tag{1.3b}$$

   which are also called "*not*" and "*and*" respectively. ¬ (∧) is usually used in prefix (infix) form, and they satisfy the following relations:

$$\begin{aligned} \neg\mathsf{True} &= \mathsf{False}\,, & \neg\mathsf{False} &= \mathsf{True}\,, \\ \mathsf{True} \wedge \mathsf{True} &= \mathsf{True}\,, & \mathsf{True} \wedge \mathsf{False} &= \mathsf{False}\,, \\ \mathsf{False} \wedge \mathsf{True} &= \mathsf{False}\,, & \mathsf{False} \wedge \mathsf{False} &= \mathsf{False}\,, \end{aligned} \tag{1.4}$$

   Use the predicates "¬", ∧, "isMultipleOfThree", and "isMultipleOfFive" to construct the set of positive integers that are integer multiples of 5 but not integer multiples of 3.

3. Write down a predicate that yields True if its input is an integer multiple of 5 not not 3.

4. Use the predicate above to construct the set of positive integers that are integer multiples of 5 but not integer multiples of 3.

(2) Problem Two *(1 points)*

In class, we have defined *the group type* as a pair:

$$\mathtt{Group} = (\mathtt{Set}, (\mathtt{Set}, \mathtt{Set}) \to \mathtt{Set}) \tag{2.1}$$

and we stated that a pair (S, o) is a group (i.e. $(S, o) :: \mathtt{Group}$) if the following three statements are correct:

- $(\exists e \in S)(\forall s \in S)\ o(e, s) = o(s, e) = s$
- $(\forall s \in S)\ o(s, i(s)) = o(i(s), s) = e$
- $(\forall a, b, c \in S)\ o(a, o(b, c)) = o(o(a, b), c)$

The function $o :: (S, S) \to S$ is called the *group operation* (given externally), and the function $i :: S \to S$ is called the *inverse*, uniquely fixed by the condition two if (S, o) is indeed a group.

One usually writes down the group operation as an infix function, and group inverse with the form $i(x) \doteq x^{-1}$. For instance, the set $\mathbb{R}\backslash\{0\}$ (real numbers except 0) form a group under arithmetic multiplication with the group inverse being arithmetic inversion.

1. Show that $\left(\mathbb{R}\backslash\{0\},\ (a, b) \to a.b\right)$ indeed forms a group by satisfying the requirements.

2. Show that nonsingular square matrices with complex number entries form a group under matrix addition.

(3) Problem Three *(2 points)*

Show that $(\mathbb{C}, +, \cdot)$ forms a field, where $\mathbb{C}$ is the set of complex numbers, $+$ denotes the arithmetic addition, and $\cdot$ denotes the arithmetic multiplication. Please show that all the necessary conditions for this to be a field (e.g. $(\mathbb{C}, +)$ is a commutative group) is satisfied.

(4) Problem Four *(3 points)*

In the previous question, you show that $(\mathbb{C}, +, \cdot)$ forms a field, and one reason this is possible is because the set $\mathbb{C}$ forms a commutative group under the action of the arithmetic multiplication, i.e. $a \cdot b = b \cdot a$ for two complex numbers a, b. This is a special property of complex numbers: even though they are constructed by a pair of real numbers ($z = x + iy$), they still form a field themselves.

One can try to extend this procedure by considering more copies of real numbers; but **(1)** if we want to do algebra, **(2)** if we want to have inverse of numbers (divisibility), and **(3)** if we want associativity ($(a.b).c = a.(b.c)$), then there is actually only one more possibility: *quaternions* (If you are personally interested why this is the case, you may check out Frobenius theorem). And quaternions do not form a field, but only a *skew field*. You will show it in this question.

Consider the set of quaternions defined as

$$\mathbb{H} = \left\{ a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \,\middle|\, (a, b, c, d \in \mathbb{R}) \right\} \tag{4.1}$$

where we embed this set with the arithmetic addition and the arithmetic multiplication by imposing

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\ ,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}\ ,\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}\ ,\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i} \tag{4.2}$$

For instance, we see that

$$(1 + 3\mathbf{i} + 2\mathbf{j})(2 - \mathbf{i} + \mathbf{k}) = 5 + 7\mathbf{i} + \mathbf{j} + 3\mathbf{k} \tag{4.3}$$

whereas

$$(2 - \mathbf{i} + \mathbf{k})(1 + 3\mathbf{i} + 2\mathbf{j}) = 5 + 3\mathbf{i} + 7\mathbf{j} - \mathbf{k} \tag{4.4}$$

Clearly, multiplication in quaternions is not commutative, unlike the case of complex numbers ($(\forall w, z \in \mathbb{C})\ w \cdot z = z \cdot w$).

Show that $(\mathbb{H}, +, \cdot)$ is a *skew field* by showing that this triple satisfies all the necessary requirements of being a skew field (e.g. $(\mathbb{H}, +)$ is a commutative group, and so on)!

(5) Problem Five *(not graded)*

Quaternions are important for many reasons in fundamental mathematics and physics, but engineers use them all the time as well! The main motivation is that quaternions very efficiently describe rotations in three dimensions (for very good mathematical reasons), hence implementing such rotations using quaternions simplify computation times significantly: check out this Wikipedia page to learn more about it if you are interested!

Despite actually being super useful in many areas, the quaternions are not used by physicists that much, and we will probably not be revisiting them again. Nevertheless, if you happen to choose to play with them to learn more about their properties, I advise you to utilize a computer program. My suggestion would be to use `Mathematica`; for instance, the computations we did in equations (4.3,4.4) can be immediately carried out as follows:

```
Needs["Quaternions`"];
Quaternion[1, 3, 2, 0] ** Quaternion[2, -1, 0, 1]
Quaternion[2, -1, 0, 1] ** Quaternion[1, 3, 2, 0]
```