Scale vs. Conformal Invariance

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Outline

Questions

- What is the difference between scale and conformal invariance?
- Why are we interested in this problem?
- What have we done so far?
- What could be done in the future?

Introduction

Consider QFTs in d dimensions with symmetry group

$$ISO(d-1,1) \rtimes R^{+} \tag{1}$$

which is the Poincaré group with scale invariance. An OPE example is

$$\langle O_1(x_1) O_2(x_2) \rangle = \frac{c_{12}}{(x_1 - x_2)^{\Delta_1 + \Delta_2}} \tag{2}$$

- Convergence is not proved
- 2-point functions of higher spin operators are not uniquely fixed
- May not be diagonalized with respect to to the scaling dimension
- Bootstrap methods may not be applicable

Introduction

We typically assume the conformal invariance of the fixed points of the β functions. In most studied cases, the symmetry group is enhanced to the conformal group

$$SO(d,2) \tag{3}$$

An OPE example is

$$\langle O_1(x_1) O_2(x_2) \rangle = \frac{c_{12}\delta_{\Delta_1 \Delta_2}}{(x_1 - x_2)^{2\Delta_1}} \tag{4}$$

- Convergence is proved
- 2 and 3-point functions are uniquely fixed
- Diagonal with respect to the conformal dimension
- Bootstrap methods are applicable

Statement of the Problem

A conserved scale (dilatation) current is of the form

$$S_\mu(x) = x^\nu T_\nu{}^\mu(x) + K^\mu(x) \tag{5}$$

where $T_{\mu\nu} = T_{\nu\mu}$ is the stress-energy tensor and K^μ is a local operator. Conservation of S^μ implies

$$T_\mu^{\ \mu} = -\partial_\mu K^\mu \tag{6}$$

A conserved conformal current is of the form

$$j^\mu(x) = v^\nu(x) T_\nu{}^\mu(x) + \partial \cdot v(x) K^\mu(x) + \partial_\nu \partial \cdot v(x) L^{\nu\mu}(x) \tag{7}$$

which implies

$$T_\mu^{\ \mu}(x) = \partial_\nu \partial_\mu L^{\nu\mu}(x) \implies \Theta_\mu^{\ \mu} = 0 \tag{8}$$

The problem is to find a traceless expression for the stress-energy tensor $T_{\mu\nu}$.

Statement of the Problem (RG Perspective)

$c-$theorem

There exists a function $c(g)$ which monotically decreases along the RG flow. [Zamolodchikov(1986)]

Corollary

If $c(g)$ exists, then cyclic or chaotic behaviour in the RG flow is not possible.

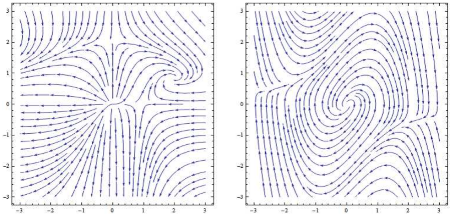

Figure: RG Flow Examples [Nakayama, 2014]

Works in 2D

Two Arguments

1: In 2D, under the conditions (1) unitarity, (2) causality, (3) discrete spectrum in scaling dimension it has been non-perturbatively shown [Polchinski, 1987] that

$$\text{Scale Invariance} \implies \text{Conformal Invariance} \tag{9}$$

Non-unitary counterexamples are known in 2D.
Ex: Non-linear σ model [Hull, Townsend (1986)]

Remark

Generalizing to higher dimensions is not straightforward.

2: $T^{\mu}_{\mu} = \partial_{\mu} K^{\mu}$ requires a vector operator with dimension $d-1$. But, under RG operators get anomalous dimensions which makes the requirement fine-tuned and unlikely and results in a conformal invariant fixed point.

Works in Other Dimensions

A simple counter example is given by [El-Showk, Nakayama, Rychkov, 2011]. Consider $U(1)$ gauge theory in d-dimensions

$$S = \int d^d x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{10}$$

No mass, thus scale invariant. However,

$$T_\mu^{\ \mu} = (d-4) F_{\mu\nu} F^{\mu\nu} = \frac{d-4}{2} \partial^\mu (F_{\mu\nu} A^\nu) \tag{11}$$

Thus, not conformal invariant.
In 4D, [Polchinski et al., 2013] showed that in perturbation theory about a conformal fixed point, the only possible asymptotics is conformally invariant.

Application (Critical Phenomena)

- Second order phase transition of the water shows the universality

$$\delta\rho(T) \sim (T - T_c)^{\beta} \quad C \sim (T - T_c)^{-\alpha} \tag{12}$$

  where $\beta \sim 0.325$ and $\alpha \sim 0.11$ are critical exponents and they also occur in 3D Ising model.
- At the critical point, thermodynamics quantities show the scaling behaviour (Scaling Hypothesis) [Widom, 1965]
- Scaling itself is not enoung to determine the critical exponents
- Idea: Critical exponents for $d > 2$ are controlled by dynamics of CFTs in 2D.
- Aim: Show Scale Invariance $\implies$ Conformal Invariance to understand the critical phenomena in higher dimensions.

Application (QCD)

- High energy properties of QCD are asymptotic to the Gaussian scale invariant fixed point in the UV limit. [Gross, Wilczek, Politzer, 1973]
- Idea for Hierarcy Problem: Use the non-trivial scale invariant fixed points to rely on the "anomalous dimensions" of various operators to deviate from the naive engineering dimensional counting.
- Aim(?): If we further find conformal invariance the amount of anomalous dimensions can be reduced.

Other Applications

- **Cosmology and Gravity:** Primordial Fluctuations in CMB, Quantum Gravity
- **String Theory:** AdS/CFT correspondence, Holography
- **Other Disciplines:** Biology, Economy, Human Behaviour

Future Directions

Scale without conformal invariance: theoretical foundations

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ABSTRACT: We present the theoretical underpinnings of scale without conformal invariance in quantum field theory. In light of our results the gradient-flow interpretation of renormalization-group (RG) flow is challenged, due to deep connections between scale-invariant theories and recurrent behaviors in the RG. We show that, on scale-invariant trajectories, there is a redefinition of the dilatation current that leads to generators of dilatations that generate dilatations. Finally, we develop a systematic algorithm for the search of scale-invariant trajectories in perturbation theory.

KEYWORDS: Space-Time Symmetries, Renormalization Group

Future Directions

Parisi-Sourlas Supertranslation and Scale without Conformal symmetry

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Inspired by the possibility of emergent supersymmetry in critical random systems, we study a field theory model with a quartic potential of one superfield, possessing the Parisi-Sourlas supertranslation symmetry. Within perturbative ϵ expansion, we find nine non-trivial scale invariant renormalization group fixed points, but only one of them is conformal. We, however, believe scale invariance without conformal invariance cannot occur without a sophisticated mechanism because it predicts the existence of a non-conserved but non-renormalized vector operator called virial current, whose existence must be non-generic. We show that the virial current in this model is related to the supercurrent by supertranslation. The supertranslation Ward-Takahashi identity circumvents the genericity argument, explaining its non-renormalization property.

Future Directions

Scale And Conformal Invariance in 2d σ-Models, with an Application to $\mathcal{N} = 4$ Supersymmetry

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Abstract: By adapting previously known arguments concerning Ricci flow and the c-theorem, we give a direct proof that in a two-dimensional sigma-model with compact target space, scale invariance implies conformal invariance in perturbation theory. This argument, which applies to a general sigma-model constructed with a target space metric and B-field, is in accord with a more general proof in the literature that applies to arbitrary two-dimensional quantum field theories. Models with extended supersymmetry and a B-field are known to provide interesting test cases for the relation between scale invariance and conformal invariance in sigma-model perturbation theory. We give examples showing that in such models, the obstructions to conformal invariance suggested by general arguments can actually occur in models with target spaces that are not compact or complete. Thus compactness of the target space, or at least a suitable condition of completeness, is necessary as well as sufficient to ensure that scale invariance implies conformal invariance in models of this type.

Figure: On 23 May 2024

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