Phys210: Mathematical Methods in Physics II

Homework 6

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) (if any) are for your own practice!
- Unless stated otherwise, you are expected to show your derivation of the results.
- The homework is due May 17[rd] 2024, 23:59 TSI.

Consider the higher order function $\mathcal{F} :: \mathbb{Z}^{+} \to \left(\mathbb{R}^2 \to \mathbb{C}\right)$ with the information

$$\mathcal{F}(1) = (x, y) \to \frac{1}{3}\left(\frac{2-2x}{(x-1)^2+y^2} - \frac{4(x+2)}{(x+2)^2+y^2} + 3\right) + \frac{2iy\left(x^2+y^2+2\right)}{((x-1)^2+y^2)\,((x+2)^2+y^2)} \tag{0.1a}$$

$$\mathcal{F}(2) = (x, y) \to \frac{x^4 + x^2\left(2y^2-5\right) + y^4 + 5y^2 + 4}{((x-1)^2+y^2)\,((x+2)^2+y^2)} - \frac{6ixy}{((x-1)^2+y^2)\,((x+2)^2+y^2)} \tag{0.1b}$$

$$\mathcal{F}(n) = (x, y) \to -\frac{3xy^{n-3}}{x^2+y^2} + \frac{3iy^{n-2}}{x^2+y^2} + 1 \qquad \text{for } n > 2 \tag{0.1c}$$

(1) Problem One *(4 points)*

Determine the largest set $S \subset \mathbb{Z}^{+}$ such that $(\forall n \in S)\mathcal{F}(n)$ has a non-empty region of analyticity.

Hint 1: The question is equivalently this: for which values of n, there is a region in the complex plane within which $\mathcal{F}(n)$ satisfies Cauchy-Riemann equations (hence analytic)?

(2) Problem Two *(not graded)*

The question above can be solved rather efficiently using Mathematica; for instance:

```
FullSimplify[(
1/3 (3 + (2 - 2 x)/((-1 + x)^2 + y^2) - (4 (2 + x))/((2 + x)^2 + y^2))
+2 I y (2 + x^2 + y^2)/(((-1 + x)^2 + y^2) ((2 + x)^2 + y^2))
)/. {x -> (z + zb)/2, y -> (z - zb)/(2 I)}
] (*zb stands for conjugate of z*)
```

(3) Problem Three *(1 points)*

Consider the functions $\mathcal{F}(m)$ which are analytic in some region in the complex plane. Determine zeroes and poles of these functions.

(4) Problem Four *(1 points)*

Does any of the analytic functions above has an essential singularity?

(5) Problem Five *(not graded)*

The following Mathematica command can be utilized to find the singularities of a function f:

```
FunctionDomain[f[z], z, Complexes]
```

(6) Problem Six *(2 points)*

Consider the functions $\mathcal{F}(m)$ which are analytic in some region in the complex plane. Define new functions $\mathcal{G}(m)$ such that

$$\mathcal{G} :: \mathbb{Z}^{+} \to \left(\mathbb{R}^{2} \to \mathbb{C}\right) \tag{6.1a}$$

$$\mathcal{G}(m)(x, y) = \sqrt{\mathcal{F}(m)(x, y)} \tag{6.1b}$$

Determine the branch points of $\mathcal{G}(m)$.

(7) Problem Seven *(not graded)*

The question above can be solved rather trivially with the following command:

```
ComplexAnalysis`BranchPoints[G[z], z]
```