Review on Spinning Conformal Correlators

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This article uses the embedding formalism to achieve simpler calculations for correlation functions in Conformal Field Theory and the Conformal Bootstrap. Furthermore, to ease the calculation for correlation functions including spins and indices, index-free notation is developed by encoding tensors into polynomials. It is noted that constraints due to tensor conservation are simpler in this formalism, and the number of independent tensor structures of conformal correlators in d dimensions is perfectly equivalent to scattering amplitudes of spinning particles in (d+1)-dimensional Minkowski space.

Introduction - The importance of Conformal Field Theories has been proven in theoretical physics, as it has important applications ranging from critical phenomena [10] to string theory [5]. Two-dimensional models are easily solvable compared to higher-dimensional models. To achieve computations in higher dimensions, the method "conformal bootstrap" [6] [9] has gained significance. The approach of the conformal bootstrap is to establish Operator Product Expansion (OPE) associativity for correlation functions. One main drawback of this method is that it has been limited to functions of scalar operators. Costa et al. [3] aim to extend this method to develop a terminology where tensors' computations can be done in a similar form to scalars.

It is important to note that one of the strong motivations for this paper is the recently found analogy between CFT correlation functions in the Mellin representation and scattering amplitudes [7] [8]. However, since this correlation is beyond the scope of this review, it will not be explained.

To develop the method to compute tensors, firstly, embedding formalism will be reviewed. This formalism raises points in d-dimensional Euclidean space to homogeneous functions on the light-cone of (d+2)-dimensional Minkowski spacetime. The benefit of such a procedure is to convert the conformal group to the Lorentz group, which is easier to compute with.

In the next section, indices of tensors are encoded in polynomials of polarization vectors in $(d+2)$ dimensions, using the introduced embedding formalism. This notation is called index-free formalism. Finally, all methods are going to be summarized and concluded by applications using these methods.

Embedding Formalism - Firstly, the review of embedding formalism begins by considering scalar operators to understand the concept better.

Dirac [4] states that the conformal group $SO(d+1,1)$ resides in the embedding space $\mathbb{M}^{d+2}$, where it is represented as the group of linear isometries. This perspective simplifies constraints emerging from conformal symmetry as simply as if they were Lorentz symmetry. This mapping is similar to stereographic projection in the sense that both conserve conformal properties. A point in d-dimensional space is embedded in a null ray in $\mathbb{M}^{d+2}$ by vectors:

$$P^A = \lambda(1, x^2, x^a), \quad \lambda \in \mathbb{R} \tag{1}$$

where light-cone coordinates are used as:

$$P^A = (P^+, P^-, P^a) \tag{2}$$

and the metric is given by:

$$P \cdot P \equiv \eta_{AB} P^A P^B = -P^+ P^- + \delta_{ab} P^a P^b \tag{3}$$

Note that by Wick rotating, we can convert the calculation from Euclidean space to Minkowski spacetime ($\delta_{ab} \to \eta_{ab}$). Here (and throughout the paper), embedding space quantities are written in capital letters, and physical space quantities are in lowercase. Thus, any linear transformation in the embedded space will map null rays into null rays, which will define a map of physical space into itself. Furthermore, every conformal transformation in physical space (which is a linear transformation in the embedded space) will remain a conformal transformation [4].

Fields in the embedded space $F_{A_1..A_l}(P)$ have the following properties:

1. Null rays: $P^2 = 0$
2. Homogeneity: $F_{A_1..A_l}(\lambda P) = \lambda^{-\Delta} F_{A_1..A_l}(P), \lambda > 0$
3. Symmetric and traceless
4. Transverse: $(P \cdot F)_{A_2...A_l} \equiv P^A F_{AA_2...A_l}$

These properties arise from conformal symmetry, which ease the calculations and enable obtaining physically sensible results. To convert back the final result from the embedded space to the physical space, a simple operation is used as below:

$$f_{a_1 \dots a_l}(x) = \frac{\partial P^{A_1}}{\partial x^{a_1}} \dots \frac{\partial P^{A_l}}{\partial x^{a_l}} F_{A_1 \dots A_l}(P_x) \tag{4}$$

Note that only points (which correspond to scalars) in physical space have been considered until now. In the next section, how spinors (tensors) can be computed will be discussed.

Index-Free Formalism in Physical Space - The authors state that any symmetric tensor can be expressed as a polynomial by the equation:

$$f_{a_1 \dots a_l} \leftrightarrow f(z) \equiv f_{a_1 \dots a_l} z^{a_1} \dots z^{a_l} \tag{5}$$

Furthermore, as primary fields are not only symmetric but also traceless, a submanifold $z^2 = 0$ can be chosen to work with. With the help of this constraint, a polynomial of symmetric traceless tensor (let's call $f_{a_1 \dots a_l}$) and symmetric tensor ($\bar{f}_{a_1 \dots a_l}$) differ by modulo $O(z^2)$. An example can be constructed as below:

$$f_{\mu\nu} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, f(z) = 2az^1 z^2 \tag{6}$$

$$\bar{f}_{\mu\nu} = \begin{pmatrix} b & a \\ a & c \end{pmatrix}, \bar{f}(z) = b(z^1)^2 + 2az^1 z^2 + c(z^2)^2 = f(z) + O(z^2) \tag{7}$$

To obtain the tensor back from the polynomial, we use a simple approach:

$$f_{a_1 \dots a_l} = \frac{\partial}{\partial z^{a_1}} \dots \frac{\partial}{\partial z^{a_l}} f(z) \tag{8}$$

Index-Free Formalism in the Embedded Space -
Similarly to what we did in the physical space, the tensor can be encoded as follows:

$$F_{A_1 \dots A_l}(P) \leftrightarrow F(P, Z) \equiv F_{A_1 \dots A_l}(P) Z^{A_1} \dots Z^{A_l} \tag{9}$$

In addition to the constraints we have for physical space, we can also make use of the transverse property. Thus, any tensor (let's call $\bar{F}_{A_1 \dots Al}(P)$ and a symmetric, traceless, and transverse (STT) tensor ($F_{A_1 \dots Al}(P)$) contain the same information if their encoded polynomials differ by a modulo term proportional to Z^2 and $Z \cdot P$.

$$F(P, Z) = \bar{F}(P, Z) + O(Z^2, P \cdot Z) \tag{10}$$

Let us consider two different forms of transversality given in the main article and see how both are connected to each other. The authors state that these two equations are equivalent:

$$P \cdot \frac{\partial}{\partial Z} F(P, Z) = 0$$

$$F(P, Z + \alpha P) = F(P, Z), \forall \alpha$$

Without using the transverse condition, one can see that two polynomials given in the second equation are connected to the same tensor:

$$F(P, Z) = F_{A_1 \dots A_l}(P) Z^{A_1} \dots Z^{A_l}$$

$$F(P, Z + \alpha P) = \\ F_{A_1 \dots A_l}(P) Z^{A_1} \dots Z^{A_l} + \alpha^l F_{A_1 \dots A_l}(P) P^{A_1} \dots P^{A_l}$$

$$\frac{\partial}{\partial Z^{A_1}} \dots \frac{\partial}{\partial Z^{A_l}} F(P, Z + \alpha P) = F_{A_1 \dots A_l}(P) = \\ \frac{\partial}{\partial Z^{A_1}} \dots \frac{\partial}{\partial Z^{A_l}} F(P, Z)$$

Thus, we understand that the transverse condition limits the polynomial we use to ease the calculations further. To be more specific, the authors state that Z vectors can exist in the polynomial without breaking transversality if and only if they exist in such a form:

$$C_{AB} \equiv Z_A P_B - Z_B P_A \tag{11}$$

Let us substitute $Z \to Z + \alpha P$ to prove C_{AB} conserves transversality:

$$C_{AB} = Z_A P_B + \alpha P_A P_B - Z_B P_A - \alpha P_B P_A = C_{AB}$$

Here we already know that P_A and P_B commute since they belong to the same vectors.

Example of a Two-Point Correlation Function - Consider the correlation function of two vectors in the embedded space:

$$G_{AB} \equiv \langle V_A(P_1) V_B(P_2) \rangle \tag{12}$$

As the authors state, all four conditions given in Section 2.1 are satisfied in such a function:

$$G_{AB}(P_1, P_2) = \frac{1}{(P_{12})^{\Delta}} [c_1 \tilde{W}_{AB} + c_2 \frac{P_{1A} P_{2B}}{P_1 \cdot P_2}] \tag{13}$$

where

$$\tilde{W}_{AB} = \eta_{AB} - \frac{P_{1B}P_{2A}}{P_1 \cdot P_2} \tag{14}$$

Now, to convert the tensor into a polynomial, use index-free notation as:

$$G(P_1, Z_1, P_2, Z_2) = Z_1^A Z_2^B G_{AB}(P_1, P_2) \tag{15}$$

Using the formula for G_{AB} in Eq.(13) and using $C_1^{AB} \cdot C_{2AB}$ in the polynomial (since this is the only way to preserve the transverse constraint and other terms are in modulo $O(Z^2, P \cdot Z)$) the result will be in the form of:

$$G(P_1, Z_1, P_2, Z_2) = constant \frac{C_1 \cdot C_2}{(P_1 \cdot P_2)^{\Delta+1}} \tag{16}$$

Summary and Conclusion - To sum up, the methods introduced by Costa et al. [3] have been reviewed in this paper. Using embedding formalism and index-free notation, calculations of spins and tensors are simplified to calculations in polynomials and scalars. Constraints and properties such as transversality and symmetric traceless tensors in newly introduced notations are recalculated to observe their impact. Furthermore, a simple example of two vectors' correlation function is calculated to show the strong strength of this method. Using such methods, the authors have computed "Spinning Conformal Blocks" [2] shortly after the paper's publication. Moreover, as mentioned in the introduction part, these methods played a key role in computing AdS propagators [1].

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