Phys209: Mathematical Methods in Physics I

Homework 2

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) are for your own practice!
- The homework is due October 20th 2023, 23:59 TSI.

(1) Problem One

We have seen in class that exponential functions form the homogeneous solutions to the *linear ordinary differential equations with constant coefficients*. Let us expand more on that in this question.

(1.1) (0.8pt)

We can define a *higher order differential operator* $\mathcal{D}$ as

$$\mathcal{D} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.1a}$$

$$\mathcal{D} = a_0 \mathcal{I} + a_1 \frac{\mathrm{d}}{\mathrm{d}x} + \cdots + a_n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \tag{1.1b}$$

where $\mathcal{I}$ is the identity higher order function

$$\mathcal{I} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.2a}$$

$$\mathcal{I} = \left(x \to f(x) \right) \to \left(x \to f(x) \right) \tag{1.2b}$$

We will drop $\mathcal{I}$ in the rest of the homework as it does not create any ambiguity (see also homework 1 for more on this operator).

The operator $\mathcal{D}$ acts on a function and creates another function; for instance, if we define *the square function* sqr as

$$\mathrm{sqr} :: \mathbb{C} \to \mathbb{C} \tag{1.3a}$$

$$\mathrm{sqr} = x \to x^2 \tag{1.3b}$$

we immediately see that

$$\mathcal{D} \cdot \mathrm{sqr} :: \mathbb{C} \to \mathbb{C} \tag{1.4a}$$

$$\mathcal{D} \cdot \mathrm{sqr} = x \to \left(a_0 x^2 + 2a_1 x + 2a_2 \right) \tag{1.4b}$$

To show this, we need to know how $\frac{\mathrm{d}^n}{\mathrm{d}x^n}$ acts: what should be the ??? below?

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} :: (\mathbb{C} \to \mathbb{C}) \to ??? \tag{1.5a}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} = ??? \tag{1.5b}$$

Hint: see eqn. 1.2 of homework 1

(1.2)   (0.8pt)

Please find out ??? below:

$$\mathcal{D} \cdot \cos = ??? \tag{1.6a}$$
$$\mathcal{D} \cdot \sin = ??? \tag{1.6b}$$
$$\mathcal{D} \cdot \exp = ??? \tag{1.6c}$$

(1.3)   (1.6pt)

Consider the simple function

$$\text{egFunc1} :: \mathbb{C} \to \mathbb{C} \tag{1.7a}$$
$$\text{egFunc1} = x \to x(x-1) \tag{1.7b}$$

which has the abbreviated name for *exempli gratia function one*. We observe that this function maps the set $\{0, 1\}$ to the set $\{0\}$:

$$\{0, 1\} \xrightarrow{\text{egFunc1}} \{0\} \tag{1.8}$$

Likewise, the following function

$$\text{egFunc2} :: \mathbb{C} \to \mathbb{C} \tag{1.9a}$$
$$\text{egFunc2} = x \to \left(x^3 - \pi x^2 - 6x^2 + 6\pi x + 8x - 8\pi \right) \tag{1.9b}$$

has the map

$$\{2, 4, \pi\} \xrightarrow{\text{egFunc2}} \{0\} \tag{1.10}$$

Both of these functions map a particular set to $\{0\}$; in fact, the first set needs not be a finite set. For example, it is actually an infinite set for the *cosine function*, please fill in ??? below:

$$\{???\} \xrightarrow{\cos} \{0\} \tag{1.11}$$

The input set that gets mapped to $\{0\}$ by a transformation is called *the kernel of the transformation*. In vector spaces, the linear transformations are implemented by matrices, and the kernel of such linear transformations are equivalent to the *nullspace of the transformation matrix*. In contrast, the analogous transformations are implemented by differential operators on the space of functions, and the kernel of the differential operator actually forms the *homogeneous solution*. We can summarize this paragraph with the following table:

Input	Transformation	Output	Kernel
Sets	Function application	{0}	Zeros of function
Vectors	Matrix multiplication	$\vec{0}$	Null space of the matrix
Functions	Differential operator	zero	Homogeneous solutions

In the table above, zero is the *zero function* defined as follows:

$$\text{zero} :: \mathbb{C} \to \mathbb{C} \tag{1.12a}$$

$$\text{zero} = x \to 0 \tag{1.12b}$$

Clearly, this function takes anything to 0, i.e.

$$\text{zero}(2) = 0 \,, \quad \text{zero}(\pi) = 0 \,, \quad \text{zero}(i) = 0 \,, \quad \text{zero}(-1.2) = 0 \,, \quad \dots \tag{1.13}$$

To find the *kernel* of a given differential operator $\mathcal{D}$, we simply solve for the function f which satisfies

$$\mathcal{D} \cdot f :: \mathbb{C} \to \mathbb{C} \tag{1.14a}$$

$$\mathcal{D} \cdot f = \text{zero} \tag{1.14b}$$

What would be the type and result for the kernel of the differential operator $\left(\frac{\mathrm{d}}{\mathrm{d}x} - a \right)$:

$$\ker \left(\frac{\mathrm{d}}{\mathrm{d}x} - a \right) :: \text{???} \tag{1.15a}$$

$$\ker \left(\frac{\mathrm{d}}{\mathrm{d}x} - a \right) = \text{???} \tag{1.15b}$$

where ker denotes the kernel; for example,

$$\ker \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) :: \{\mathbb{C} \to \mathbb{C}\} \tag{1.16a}$$

$$\ker \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) = \{x \to \text{constant}\} \tag{1.16b}$$

Hint: Note that the kernel of a differential operator is a list of functions, hence the kernel above has the type $\{\mathbb{C} \to \mathbb{C}\}$, not $\mathbb{C} \to \mathbb{C}$.

(1.4) (0.8pt)

We have seen in class that the way to solve for the homogeneous equation for the *linear ordinary differential equations with constant coefficients* is to solve the *characteristic equation*, and then exponentiate the roots. For instance,

$$\left(\frac{d}{dx} - a\right)\left(\frac{d}{dx} - b\right)f(x) = 0 \quad \Rightarrow \quad f(x) = c_1 e^{ax} + c_2 e^{bx} \qquad (1.17)$$

where the roots $r_{1,2}$ of the characteristic equation $(r - a)(r - b) = 0$ are a, b.

With this same method, solve the differential equation:

$$\left(\frac{d^5}{dx^5} - 3\frac{d^4}{dx^4} - 23\frac{d^3}{dx^3} + 51\frac{d^2}{dx^2} + 94\frac{d}{dx} - 120\right)f(x) = ??? \qquad (1.18)$$

Hint: When you construct the characteristic equation, you will observe that it is a 5th order polynomial. Abel–Ruffini theorem guarantees that we cannot find an algebraic solution for such a polynomial for arbitrary coefficients; however, this particular polynomial will have nice roots as I choose the coefficients deliberately. Nevertheless, it would be hard for you to find the roots by pen and paper, so let me give you two of the roots: $r = 5$ and $r = -4$. You can now in principle solve for the other three roots by using cubic equation (see [wikipedia](#)). But even that is extremely hard to do with pen and paper, so I'll give you one more root: $r = 3$. Now you can find the other two roots easily :)

(1.5) Not graded

Following two-line code solves the previous problem:

```
DSolve[f'''''[x] − 3 f''''[x] − 23 f'''[x] + 51 f''[x] + 94 f'[x] −
120 f[x] == 0, f[x], x]
```

Likewise, we can factorize any given polynomial, which vastly simplifies our computations in daily life:

```
r^5 − 3 r^4 − 23 r^3 + 51 r^2 + 94 r − 120 // Factor
```

(2) Problem Two   *(2 points)*

In class, we have talked about the integral transformations. They are higher order functions that transforms a function from "one function

domain" to another function in "another function domain", i.e.

$$\text{IT} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{2.1a}$$

$$\text{IT} = (x \to f(x)) \to \left(s \to \int_\alpha^\beta K(x,s)f(x)dx \right) \tag{2.1b}$$

where the integration range and the kernel of the integral transformation (i.e. $K(x,s)$) are fixed by the chosen transformation. For instance, a special integral transformation called *Laplace transformation* (about which we have talked a lot this week) takes the form

$$\mathcal{L} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{2.2a}$$

$$\mathcal{L} = (x \to f(x)) \to \left(s \to \int_0^\infty e^{-xs}f(x)dx \right) \tag{2.2b}$$

Let us play with this transformation a little bit.

(2.1)   (2pt)

Consider the constant function defined as follows:

$$\text{const}_a :: \mathbb{C} \to \mathbb{C} \tag{2.3a}$$
$$\text{const}_a = x \to a \tag{2.3b}$$

which satisfies $\text{const}_a(x) = a$ for any x. The Laplace transform of this function is given as

$$\mathcal{L} \cdot \text{const}_a :: \mathbb{C} \to \mathbb{C} \tag{2.4a}$$
$$\mathcal{L} \cdot \text{const}_a = s \to \frac{a}{s} \tag{2.4b}$$

hence $(\mathcal{L} \cdot \text{const}_a)(s) = \frac{a}{s}$. We derived this by using the identity $\int_0^\infty e^{-xs}a\,dx = \frac{a}{s}$.

Let us consider a more general function:

$$f_{n,a} :: \mathbb{C} \to \mathbb{C} \tag{2.5a}$$
$$f_{n,a} = x \to x^n e^{ax} \tag{2.5b}$$

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What is its Laplace transform, i.e.

$$\mathcal{L} \cdot f_{n,a} :: \mathbb{C} \to \mathbb{C} \tag{2.6a}$$

$$\mathcal{L} \cdot f_{n,a} = \text{???} \tag{2.6b}$$

Derive the result step by step. You may assume $s > 0$ in the Laplace domain, but do not assume $n \in \mathbb{Z}$. *Hint: See the definition of Gamma function.*

(2.2) Not graded

Following code solves the previous problem:

```
LaplaceTransform[x^n Exp[a x], x, s]
```