Phys209: Mathematical Methods in Physics I

Homework 6

Soner Albayrak[†]

[†]*Middle East Technical University, Ankara 06800, Turkey*

Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) are for your own practice!
- Unless stated otherwise, you are expected to show your derivation of the results.
- The homework is due November 17th 2023, 23:59 TSI.

(1) Solving a cubic equation *(6 points)*

In this problem, we are going to test our skills that we have developed up until this point to solve the following cubic differential equation: $x^2 f'''(x) + \frac{(4+x)x}{2} f''(x) - \frac{4-x}{2} f'(x) - \frac{1}{2} f(x) = 0$

(1.1) Reduction of order (1pt)

As we are all well aware, generic differential equations are hard to solve! But if we know one of the solutions already, then we can actually reduce the order of the differential equation. Let us assume that *we physically expect a solution that decreases as a power law*, that is one of the solutions should be

$$f_1(x) = \frac{1}{x^k}$$

for some k. Use the *method of reduction of order* to reduce the order of the given differential equation by using the given information.
Hint: Take

$$f(x) = \frac{g(x)}{x^k}$$

for the unknown function $g(x)$ and unknown parameter k, and find k such that $g(x)$ does not appear in the resultant differential equation, but only its derivatives.

(1.2) Reduction of order (continued) (1pt)

The new differential equation can be brought to the form

$$a(x)g'''(x) + b(x)g''(x) - g'(x) = 0$$

What are the values of $a(x)$ and $b(x)$?

(1.3) Solving the second order equation (1pt)

Insert $h(x) = g'(x)$ in the differential equation you have found, and bring it to the form

$$\Big(\alpha(x)h'(x) + h(x)\Big) + \beta(x)\frac{d}{dx}\Big(\alpha(x)h'(x) + h(x)\Big) = 0$$

for some $\alpha(x)$ and $\beta(x)$.

What are the values of $\alpha(x)$ and $\beta(x)$?

(1.4) Solving the first order equation (1pt)

Solve $\alpha(x)h'(x) + h(x) = 0$. What is the value $h(x)$?

(1.5) Finding the other solution to the second order equation (0.5pt)

Use reduction of order to find the other solution to the differential equation in question 1.3. Indeed, that is a second order differential equation for $h(x)$, and we have found one of the solutions for $h(x)$ in question 1.4 (call that $h_1(x)$). Therefore, you only need to insert $h(x) = h_1(x)j(x)$ for the unknown function j, and find $j(x)$. Then, insert it back to find $h(x)$: it should be of the form

$$h(x) = c_1(x + a) + c_2 h_1(x)$$

for arbitrary constants c_1, c_2. What is the value of a?

(1.6) Final result (1pt)

Integrating $h(x)$, we can get $g(x)$ from which $f(x)$ can be obtained. We then arrive at the final result that the differential equation

$$x^2 f'''(x) + \frac{(4+x)x}{2} f''(x) - \frac{4-x}{2} f'(x) - \frac{1}{2} f(x) = 0$$

has the solution

$$f(x) = c_1 \frac{1}{x^k} + c_2 \frac{e^{\gamma x}}{x^k} + c_3 \frac{x^2 + \rho x + \lambda}{x^k}$$

for arbitrary coefficients c_1, c_2, c_3. What are these parameters γ, ρ, λ?

(1.7) Bonus question *(not graded)*

The question above can be solved by the following Mathematica code:

```
DSolve[x^2 f'''[x] + ((4 + x) x)/2 f''[x] - (4 - x)/2 f'[x] -
1/2 f[x] == 0, f[x], x]
```