



Name:	
Student ID:	



# Midterm Examination - 1

## Phys331: Electromagnetic Theory I

2025/11/13

*Please carefully read below before proceeding!*

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

- (1) all calculations are to be conducted in the notations and conventions of the formulae sheets provided during the exam unless explicitly stated otherwise in the question;
- (2) I take *full responsibility* for any ambiguity in my selections in “multiple choice questions”;
- (3) incorrect selections will receive  $-1/7$  of the question's points;
- (4) I am expected to provide *step-by-step explanation of how I solved the question* and am expected to do so *only within the answer boxes* provided with the questions: the explanation is supposed to be succinct, well-articulated, and correct both scientifically and mathematically;
- (5) no partial credit is awarded for the explanations provided in the answer boxes;
- (6) some questions of some students will be randomly selected for inspection: *a question (if selected for inspection) might be awarded negative points* if its explanation is incorrect or insufficient to get the correct answer, even if the correct option is selected;
- (7) any page which does not contain *both my name and student id* may not be graded;
- (8) any extra sheet that I may use are for my own calculations and will not be graded.

Signature: \_\_\_\_\_

This exam has a total of 5 questions, some of which may be for bonus points. You can obtain a maximum grade of 105+0 from this examination.

Question:	1	2	3	4	5	Total
Points:	7	21	14	31½	31½	105

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**Question: 1:** Concept of Units (A simple case study) ..... (7 points)

In cosmology, the kinematics are handled using a different unit-system than SI; indeed, although *meter* and *second* are the standard units for most of the engineering and applied physics, they are too small for cosmological scales so instead we might prefer *lightyear* and *gigayear* which we can take to be  $\text{ly} = 10^{16} \text{ m}$   $\text{Gyr} = 3 \times 10^{16} \text{ s}$ . Then, if absolute value of the angular momentum per units mass for the galaxy Andromeda is approximated as  $|j| = 2 \times 10^{26} \text{ m}^2 \text{ s}^{-1}$  in the SI units, which option below would be the correct expression for it?

- ☐  $|j| = 6^{-1} \times 10^{16} \text{ ly}^2 \text{ Gyr}^{-1}$ 
☐  $|j| = 6 \times 10^{16} \text{ ly}^2 \text{ Gyr}^{-1}$ 
☒  $|j| = 6 \times 10^{10} \text{ ly}^2 \text{ Gyr}^{-1}$
- ☐  $|j| = \frac{2}{3} \times 10^{10} \text{ ly}^2 \text{ Gyr}^{-1}$ 
☐  $|j| = \frac{2}{3} \times 10^{-10} \text{ ly}^2 \text{ Gyr}^{-1}$ 
☐  $|j| = 6 \times 10^{-10} \text{ ly}^2 \text{ Gyr}^{-1}$
- ☐  $|j| = 6 \times 10^{-16} \text{ ly}^2 \text{ Gyr}^{-1}$ 
☐  $|j| = 6^{-1} \times 10^{-16} \text{ ly}^2 \text{ Gyr}^{-1}$

**Solution 1.1** - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

We are given  $\text{m} = 10^{-16} \text{ ly}$  and  $\text{s}^{-1} = 3 \times 10^{16} \text{ Gyr}^{-1}$ , thus

$$|j| = 2 \times 10^{26} \text{ m}^2 \text{ s}^{-1} = 6 \times 10^{10} \text{ ly}^2 \text{ Gyr}^{-1} \quad (1)$$

**Question: 2:** Concept of Units (A complicated case study) ..... (21 points)

On his 1881 paper “*On the physical units of nature*”, G. Johnstone Stoney argues the utility of choosing fundamental units in terms of constants of nature, hence creating a unit system which is named after him. In this so-called *Stoney natural units*, we trade the SI units  $A$ ,  $m$ ,  $\text{kg}$ ,  $s$  for the constants of nature  $c$ ,  $e$ ,  $G$ ,  $\epsilon_0$  which we will take to be defined in this question as

$$\begin{aligned}
 (\text{speed of light}) \quad c &= 3 \times 10^8 \text{ m s}^{-2} \\
 (\text{charge of proton}) \quad e &= 2 \times 10^{-19} \text{ A s} \\
 (\text{gravitational constant}) \quad G &= 7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \\
 (\text{vacuum permittivity}) \quad \epsilon_0 &= 9 \times 10^{-12} \text{ A}^2 \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^4
 \end{aligned} \quad (2)$$

Answer the questions below based on these definitions, along with the facts

$$\sqrt{630} \approx 25, \quad \begin{pmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 3 & -1 & -2 \\ 2 & -3 & -1 & 4 \end{pmatrix}^{-1} = \frac{1}{8} \begin{pmatrix} 6 & 6 & -1 & 1 \\ -4 & 4 & 2 & -2 \\ 0 & 8 & -4 & -4 \\ -6 & 2 & 1 & -1 \end{pmatrix} \quad (3)$$

(a) (7 points) Consider Alice, whose mass is measured to be “ $80 \text{ A}^0 \text{ kg}^1 \text{ s}^0 \text{ m}^0$ ” in SI units. In Stoney natural units, her mass would be “ $X c^{a_1} e^{a_2} G^{a_3} \epsilon_0^{a_4}$ ” for some exponents  $a_i$ . What is  $X$ ?

- ☐  $10^3$ 
☐  $10^4$ 
☐  $10^5$ 
☐  $10^6$ 
☐  $10^7$ 
☐  $10^8$ 
☐  $10^9$ 
☒  $10^{10}$



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(b) (7 points) Consider Alice again: which of below would be her mass in Stoney natural units?

- ☒  $X c^0 e^1 G^{-1/2} \epsilon_0^{-1/2}$ 
☐  $X c^0 e^1 G^{1/2} \epsilon_0^{-1/2}$ 
☐  $X c^0 e^1 G^{-1/2} \epsilon_0^{1/2}$ 
☐  $X c^0 e^1 G^{1/2} \epsilon_0^{1/2}$
- ☐  $X c^0 e^{-1/2} G^1 \epsilon_0^{-1/2}$ 
☐  $X c^0 e^{-1/2} G^1 \epsilon_0^{1/2}$ 
☐  $X c^0 e^{1/2} G^1 \epsilon_0^{-1/2}$ 
☐  $X c^0 e^{1/2} G^1 \epsilon_0^{1/2}$

(c) (7 points) Now assume that Alice has done some work, say 10 Joules in SI units. What would be this in Stoney natural units for some  $Y \in \mathbb{R}$ ?

- ☐  $Y c^{1/2} e^{3/4} G^{1/4} \epsilon_0^{-3/4}$ 
☐  $Y c^{1/2} e^{3/2} G^{1/4} \epsilon_0^{-3/4}$ 
☐  $Y c^{1/2} e^{3/4} G^{-1/4} \epsilon_0^{-3/4}$ 
☒  $Y c^{1/2} e^{3/2} G^{-1/4} \epsilon_0^{-3/4}$
- ☐  $Y c^{-1/2} e^{3/4} G^{1/4} \epsilon_0^{-3/4}$ 
☐  $Y c^{-1/2} e^{3/2} G^{1/4} \epsilon_0^{-3/4}$ 
☐  $Y c^{-1/2} e^{3/4} G^{-1/4} \epsilon_0^{-3/4}$ 
☐  $Y c^{-1/2} e^{3/2} G^{-1/4} \epsilon_0^{-3/4}$

**Solution 2.1** - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

As mentioned in the cheat sheets explicitly, transformation between different units is actually isomorphic to a basis transformation of a vector space, which can be utilized with a matrix multiplication. So, to rewrite the SI units in terms of  $c, e, G, \epsilon_0$ , we simply need to find that isomorphism.

It is already explicitly mentioned in the cheat sheets that any unit is to be of the form  $m_0 f_1^{m_1} f_2^{m_2} \dots f_n^{m_n}$  on the physical grounds, hence we expect

$$\begin{aligned}
 A &= \alpha_{1,0} c^{\alpha_{1,1}} e^{\alpha_{1,2}} G^{\alpha_{1,3}} \epsilon_0^{\alpha_{1,4}} \\
 m &= \alpha_{2,0} c^{\alpha_{2,1}} e^{\alpha_{2,2}} G^{\alpha_{2,3}} \epsilon_0^{\alpha_{2,4}} \\
 \text{kg} &= \alpha_{3,0} c^{\alpha_{3,1}} e^{\alpha_{3,2}} G^{\alpha_{3,3}} \epsilon_0^{\alpha_{3,4}} \\
 s &= \alpha_{4,0} c^{\alpha_{4,1}} e^{\alpha_{4,2}} G^{\alpha_{4,3}} \epsilon_0^{\alpha_{4,4}}
 \end{aligned} \tag{4}$$

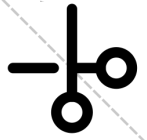
If we now insert these in the right hand side of (2), we get

$$\begin{aligned}
 c &= 3 \times 10^8 \text{ m s}^{-2} = (3 \times 10^8 \alpha_{2,0} \alpha_{4,0}^{-2}) c^{\alpha_{2,1}-2\alpha_{4,1}} e^{\alpha_{2,2}-2\alpha_{4,2}} G^{\alpha_{2,3}-2\alpha_{4,3}} \epsilon_0^{\alpha_{2,4}-2\alpha_{4,4}} \\
 e &= 2 \times 10^{-19} \text{ A s} = (2 \times 10^{-19} \alpha_{1,0} \alpha_{4,0}) c^{\alpha_{1,1}+\alpha_{4,1}} e^{\alpha_{1,2}+\alpha_{4,2}} G^{\alpha_{1,3}+\alpha_{4,3}} \epsilon_0^{\alpha_{1,4}+\alpha_{4,4}} \\
 G &= 7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} = (7 \times 10^{-11} \alpha_{2,0}^3 \alpha_{3,0}^{-1} \alpha_{4,0}^{-2}) c^{3\alpha_{2,1}-\alpha_{3,1}-2\alpha_{4,1}} e^{3\alpha_{2,2}-\alpha_{3,2}-2\alpha_{4,2}} G^{3\alpha_{2,3}-\alpha_{3,3}-2\alpha_{4,3}} \epsilon_0^{3\alpha_{2,4}-\alpha_{3,4}-2\alpha_{4,4}} \\
 \epsilon_0 &= 9 \times 10^{-12} \text{ A}^2 \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^4 = (9 \times 10^{-12} \alpha_{1,0}^2 \alpha_{2,0}^{-3} \alpha_{3,0}^{-1} \alpha_{4,0}^4) c^{2\alpha_{1,1}-3\alpha_{2,1}-\alpha_{3,1}+4\alpha_{4,1}} e^{2\alpha_{1,2}-3\alpha_{2,2}-\alpha_{3,2}+4\alpha_{4,2}} \\
 &\quad \times G^{2\alpha_{1,3}-3\alpha_{2,3}-\alpha_{3,3}+4\alpha_{4,3}} \epsilon_0^{2\alpha_{1,4}-3\alpha_{2,4}-\alpha_{3,4}+4\alpha_{4,4}}
 \end{aligned} \tag{5}$$

These equation can be satisfies only if

$$3 \times 10^8 \alpha_{2,0} \alpha_{4,0}^{-2} = 1, \quad 2 \times 10^{-19} \alpha_{1,0} \alpha_{4,0} = 1, \quad 7 \times 10^{-11} \alpha_{2,0}^3 \alpha_{3,0}^{-1} \alpha_{4,0}^{-2} = 1, \quad 9 \times 10^{-12} \alpha_{1,0}^2 \alpha_{2,0}^{-3} \alpha_{3,0}^{-1} \alpha_{4,0}^4 = 1 \tag{6}$$

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and by taking their logarithm of the equations, we can bring them to the form

$$\begin{pmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 3 & -1 & -2 \\ 2 & -3 & -1 & 4 \end{pmatrix} \begin{pmatrix} \log \alpha_{1,0} \\ \log \alpha_{2,0} \\ \log \alpha_{3,0} \\ \log \alpha_{4,0} \end{pmatrix} = - \begin{pmatrix} \log(3 \times 10^8) \\ \log(2 \times 10^{-19}) \\ \log(7 \times 10^{-11}) \\ \log(9 \times 10^{-12}) \end{pmatrix} \quad (7)$$

Likewise, by matching the exponents in (5), we get

$$\begin{pmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 3 & -1 & -2 \\ 2 & -3 & -1 & 4 \end{pmatrix} \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\ \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (8)$$

As we are already provided with

$$\begin{pmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 3 & -1 & -2 \\ 2 & -3 & -1 & 4 \end{pmatrix}^{-1} = \frac{1}{8} \begin{pmatrix} 6 & 6 & -1 & 1 \\ -4 & 4 & 2 & -2 \\ 0 & 8 & -4 & -4 \\ -6 & 2 & 1 & -1 \end{pmatrix} \quad (9)$$

we conclude that

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\ \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 6 & 6 & -1 & 1 \\ -4 & 4 & 2 & -2 \\ 0 & 8 & -4 & -4 \\ -6 & 2 & 1 & -1 \end{pmatrix} \quad (10)$$

and

$$\begin{pmatrix} \log \alpha_{1,0} \\ \log \alpha_{2,0} \\ \log \alpha_{3,0} \\ \log \alpha_{4,0} \end{pmatrix} = -\frac{1}{8} \begin{pmatrix} 6 & 6 & -1 & 1 \\ -4 & 4 & 2 & -2 \\ 0 & 8 & -4 & -4 \\ -6 & 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} \log(3 \times 10^8) \\ \log(2 \times 10^{-19}) \\ \log(7 \times 10^{-11}) \\ \log(9 \times 10^{-12}) \end{pmatrix} \quad (11)$$

indicating that

$$\log \alpha_{3,0} = -\log(2 \times 10^{-19}) + \frac{1}{2}\log(7 \times 10^{-11}) + \frac{1}{2}\log(9 \times 10^{-12}) \quad (12)$$

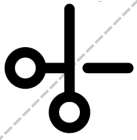
hence

$$\alpha_{3,0} = (2 \times 10^{-19})^{-1} (7 \times 10^{-11})^{1/2} (9 \times 10^{-12})^{1/2} = \frac{\sqrt{630}}{2} 10^7 \approx 1.25 \times 10^8 \quad (13)$$

as we are told to take  $\sqrt{630} \approx 25$ . Therefore, Alice's mass is

$$80 \text{ kg} = 80 \alpha_{3,0} c^{\alpha_{3,1}} e^{\alpha_{3,2}} G^{\alpha_{3,3}} \epsilon_0^{\alpha_{3,4}} = 10^{10} c^0 e^1 G^{-1/2} \epsilon_0^{-1/2} \quad (14)$$

which answers the first two parts.



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To answer the third part, remember that a Joule is a derived unit in SI system, and is equal to  $\text{m}^2 \text{kg s}^{-2}$ . Therefore

$$\text{m}^2 \text{kg s}^{-2} = (\alpha_{2,0}^2 \alpha_{3,0} \alpha_{4,0}^{-2}) c^{\beta_1} e^{\beta_2} G^{\beta_3} \epsilon_0^{\beta_4} \quad (15)$$

where we see that

$$\begin{aligned}
 (\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4) &= (0 \quad 2 \quad 1 \quad -2) \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\ \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} \end{pmatrix} \\
 &= (0 \quad 2 \quad 1 \quad -2) \frac{1}{8} \begin{pmatrix} 6 & 6 & -1 & 1 \\ -4 & 4 & 2 & -2 \\ 0 & 8 & -4 & -4 \\ -6 & 2 & 1 & -1 \end{pmatrix} \\
 &= (1/2 \quad 3/2 \quad -1/4 \quad -3/4)
 \end{aligned} \quad (16)$$

hence

$$10 \text{ Joules} = X c^{1/2} e^{3/2} G^{-1/4} \epsilon_0^{-3/4} \quad (17)$$

for some real number  $X$ .

**Question: 3:** Concept of Charge.....(14 points)

Please take  $\pi \epsilon_0 \approx 3.6^{-1} \times 10^{-10}$ ,  $\pi^2 \epsilon_0^{-1} \approx 10^{12}$ , and  $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$  in this question

(a) (7 points) Consider a point particle of electric charge 1 in an otherwise empty space. With a coordinate system with respect to which the charge is at the origin, which of below would be  $\lim_{x \rightarrow \infty} x^2 |\mathbf{E}(x, 1, x/\sqrt{2})|$ ?

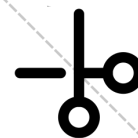
- ☐  $\frac{1}{9} \times 10^{10}$ 
☐  $9 \times 10^{10}$ 
☒  $\frac{3}{5} \times 10^{10}$ 
☐  $\frac{5}{3} \times 10^{10}$ 
☐  $\frac{1}{9} \times 10^{-10}$ 
☐  $9 \times 10^{-10}$ 
☐  $\frac{3}{5} \times 10^{-10}$ 
☐  $\frac{5}{3} \times 10^{-10}$

(b) (7 points) Consider a system of infinitely many point charges, with a charge distribution  $\rho(x, y, z) = \sum_{n=1}^{\infty} x^{-2} \delta(x-n) \delta(y) \delta(z)$ . Let  $\Phi_L$  denote the total electric flux coming out of a sphere of radius  $L$ , whose center is at the origin. What would be  $\lim_{L \rightarrow \infty} \Phi_L$ ?

- ☐  $-\infty$ 
☐  $-\frac{1}{6} \times 10^{12}$ 
☐  $-\frac{1}{6} \times 10^{-12}$ 
☐ 0
 ☐  $\frac{1}{6} \times 10^{-12}$ 
☒  $\frac{1}{6} \times 10^{12}$ 
☐  $6 \times 10^{12}$ 
☐  $\infty$

**Solution 3.1** - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

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In the formulae sheets, we are provided with  $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$  and  $\int_V \nabla \cdot \mathbf{F} dV = \oint_{\partial V} \mathbf{F} \cdot d\mathbf{S}$ : integrating the first equation and using the second for a sphere of radius  $r$ , we get

$$\oint_{S_r^2} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_{B_r^3} \rho dV \quad (18)$$

where  $S_r^2$  and  $B_r^3$  denote a sphere of radius  $r$  and a ball of radius  $r$  respectively (whose center of both is the origin).

For part (a), we consider this equation for a unit point charge at the origin,  $\rho = \delta^3(\mathbf{r})$ , hence we arrive at

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} \quad (19)$$

where we also use the spherical symmetry of a point charge in an otherwise empty space. We then see that

$$\lim_{x \rightarrow \infty} x^2 |\mathbf{E}(x, 1, x/\sqrt{2})| = \frac{1}{4\pi\epsilon_0} \lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 1 + x^2/2} = \frac{1}{6\pi\epsilon_0} \simeq \frac{3.6}{6} \times 10^{10} = \frac{3}{5} \times 10^{10} \quad (20)$$

For part (b), we can realize that the left hand side is the total flux, hence we arrive at

$$\Phi_L = \frac{1}{\epsilon_0} \int_{B_L^3} \rho dV = \frac{1}{\epsilon_0} \int_{B_L^3} \sum_{n=1}^{\infty} x^{-2} \delta(x-n) \delta(y) \delta(z) dV \quad (21)$$

We change the order of integration and the sum to get

$$\Phi_L = \frac{1}{\epsilon_0} \int_{B_L^3} \rho dV = \sum_{n=1}^{\infty} \frac{1}{\epsilon_0} \int_{B_L^3} x^{-2} \delta(x-n) \delta(y) \delta(z) dV \quad (22)$$

Such a change of order is not always warranted and we need to be careful if things are divergent. For this course (or for almost all of the undergraduate physics), we will not need to be that careful and simply proceed by changing orders of integrals and sums without worrying too much, ignoring the blasphemy whispers of our mathematician friends.

The equation above is easy to interpret: the Dirac-delta generalized functions ensure that  $y = z = 0$  and  $x = n$ , and the integration range ensures that  $\delta(x-n)$  gives zero for all  $n > L$  (since the argument of  $\delta(x-n)$  can not be zero for these  $n$  values). Thus, we get

$$\Phi_L = \frac{1}{\epsilon_0} \sum_{n=1}^L n^{-2} \Rightarrow \lim_{L \rightarrow \infty} \Phi_L = \frac{1}{\epsilon_0} \sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6\epsilon_0} \simeq \frac{1}{6} \times 10^{12} \quad (23)$$

**Question: 4:** Electric field in Cartesian coordinates ..... (31½ points)



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In this question, you may use the following Taylor series expansion:

$$x \int_{-1/x}^{1/x} da \int_{-1/x}^{1/x} db (1 + a^2 + b^2)^{-1/2} = \text{constant} - 2\pi x + 2\sqrt{2}x^2 + \dots$$

A square panel of sidelength  $\ell$  and of negligible thickness is of electric charge uniformly distributed with a constant surface charge density  $\sigma > 0$ . Consider this panel in an otherwise empty space and do so in a coordinate system chosen such that the geometric center of the panel is at the origin and that the panel lies on the  $x - y$  plane with its two sides orthogonal to the  $x$ -axis. Below, we consider the electric field  $\mathbf{E}(x, y, z)$  in this coordinate system.

(a) (**10<sup>1/2</sup> points**) For some constant  $c > 0$ , which one of the following statements is true?

- |   |   |
|---|---|
| <input type="checkbox"/> $\lim_{\kappa \rightarrow 0^+} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} \rightarrow -\infty$ | <input type="checkbox"/> $\lim_{\kappa \rightarrow 0^-} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} \rightarrow -\infty$ |
| <input type="checkbox"/> $\lim_{\kappa \rightarrow 0^+} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} \rightarrow \infty$  | <input type="checkbox"/> $\lim_{\kappa \rightarrow 0^-} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} \rightarrow \infty$  |
| <input type="checkbox"/> $\lim_{\kappa \rightarrow 0^+} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} = 0$                 | <input type="checkbox"/> $\lim_{\kappa \rightarrow 0^-} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} = 0$                 |
| <input checked="" type="checkbox"/> $\lim_{\kappa \rightarrow 0^+} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} = c$      | <input type="checkbox"/> $\lim_{\kappa \rightarrow 0^-} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} = c$                 |

(b) (**10<sup>1/2</sup> points**) Assume that  $|z| \ll |\ell|$ . Which of the below would give the correct potential energy  $V(0, 0, z)$  under this assumption and a suitable chosen reference point?

- |  |   |   |  |
|--|---|---|--|
| <input type="checkbox"/> $V(0, 0, z) \approx \frac{\sigma}{2\epsilon_0}$   | <input type="checkbox"/> $V(0, 0, z) \approx -\frac{\sigma}{2\epsilon_0}$   | <input type="checkbox"/> $V(0, 0, z) \approx \frac{\sigma}{2\epsilon_0} + \frac{\sigma z}{2\epsilon_0}$   | <input type="checkbox"/> $V(0, 0, z) \approx \frac{\sigma}{2\epsilon_0} - \frac{\sigma z}{2\epsilon_0}$              |
| <input type="checkbox"/> $V(0, 0, z) \approx \frac{\sigma z}{2\epsilon_0}$ | <input type="checkbox"/> $V(0, 0, z) \approx -\frac{\sigma z}{2\epsilon_0}$ | <input type="checkbox"/> $V(0, 0, z) \approx -\frac{\sigma}{2\epsilon_0} + \frac{\sigma z }{2\epsilon_0}$ | <input checked="" type="checkbox"/> $V(0, 0, z) \approx -\frac{\sigma}{2\epsilon_0} - \frac{\sigma z }{2\epsilon_0}$ |

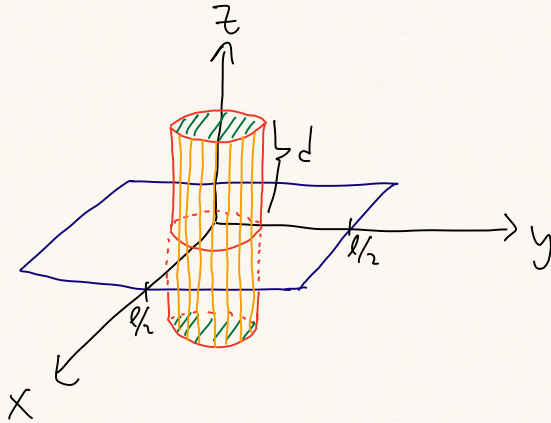
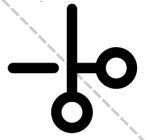
(c) (**10<sup>1/2</sup> points**) In part (a), we considered the asymptotic value of the electric field; in this part, we will consider the asymptotic value of the *rate of change of the electric field*! That is, find

$\lim_{z \rightarrow 0^+} \frac{d(\hat{z} \cdot \mathbf{E}(0, 0, z))}{dz}$  in the region  $z > 0$ , and then evaluate it for a panel whose sidelength  $\ell$  is proportional to its charge density via  $\ell = \frac{2\sigma}{\pi\epsilon_0}$ . What is the result?

- |  |                               |   |                               |   |                              |                                     |                              |
|--|-------------------------------|---|-------------------------------|---|------------------------------|-------------------------------------|------------------------------|
| <input type="checkbox"/> $-\frac{1}{\sqrt{2}}$ | <input type="checkbox"/> $-1$ | <input checked="" type="checkbox"/> $-\sqrt{2}$ | <input type="checkbox"/> $-2$ | <input type="checkbox"/> $\frac{1}{\sqrt{2}}$ | <input type="checkbox"/> $1$ | <input type="checkbox"/> $\sqrt{2}$ | <input type="checkbox"/> $2$ |
|--|-------------------------------|---|-------------------------------|---|------------------------------|-------------------------------------|------------------------------|

**Solution 4.1** - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

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We can schematically show the situation in the image to the left: the panel is on the  $x - y$  plane and oriented as instructed. In part (a), we consider electric field infinitesimally close to the plane, hence we can apply the infinite-plane approximation for which the electric field becomes purely in the  $z$ -direction. One can then consider the Gaussian surface in the schematic image and apply the divergence theorem:

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

The sides of the surface does not contribute, hence the only contribution comes from the top and bottom:

$$\mathbf{E}(0, 0, z) \xrightarrow{|z| \rightarrow 0} \frac{\sigma}{2\epsilon_0} \text{sgn}(z) \hat{z} \quad (24)$$

Clearly, as  $\sigma, \epsilon_0, \ell > 0$ , we see that  $\lim_{\kappa \rightarrow 0^+} \mathbf{E}(0, 0, \kappa \ell) \cdot \hat{z} = \frac{\sigma}{2\epsilon_0} > 0$  and  $\lim_{\kappa \rightarrow 0^-} \mathbf{E}(0, 0, \kappa \ell) \cdot \hat{z} = -\frac{\sigma}{2\epsilon_0} < 0$ .

We can similarly answer part (b) as the  $|z| \ll \ell$  assumption is same as the infinite-plane assumption at the leading order. With our  $E$  found above, we can solve the electric potential:

$$\nabla V(x, y, z) = -\mathbf{E}(x, y, z) \Rightarrow \frac{\partial V(x, y, z)}{\partial z} = -\frac{\sigma}{2\epsilon_0} \text{sgn}(z) \Rightarrow V(x, y, z) = -\frac{\sigma|z|}{2\epsilon_0} + f(x, y) \quad (25)$$

for an undetermined function  $f$ . Then, we see that

$$V(0, 0, z) = \text{constant} - \frac{\sigma|z|}{2\epsilon_0} \quad (26)$$

For part (c), we need to consider the more general case, i.e. beyond the infinite-plane approximation. The contribution to the electric potential field at the point  $(0, 0, z)$  due to an infinitesimal rectangle of dimensions  $dx \times dy$  at the point  $(x, y, 0)$  reads via Coulomb law as

$$dV(0, 0, z) = \frac{1}{4\pi\epsilon_0} \frac{\sigma dx dy}{\sqrt{x^2 + y^2 + z^2}} \quad (27)$$

thus the full potential due to the square panel becomes

$$V(0, 0, z) = \frac{\sigma}{4\pi\epsilon_0} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} \frac{dx dy}{\sqrt{x^2 + y^2 + z^2}} \quad (28)$$



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Let us introduce new variables  $a = x/z$  and  $b = y/z$ . The integration then becomes

$$V(0, 0, z) = \frac{\sigma z}{4\pi\epsilon_0} \int_{-\frac{\ell}{2z}}^{\frac{\ell}{2z}} \int_{-\frac{\ell}{2z}}^{\frac{\ell}{2z}} \frac{dad b}{\sqrt{1 + a^2 + b^2}} \quad (29)$$

where we assume that  $z > 0$ : this is fine as we are interested in positive  $z$  values as told in the question. We can now set  $z = \kappa\ell/2$  and then use the given Taylor series expansion to get

$$V(0, 0, \kappa\ell/2) = \frac{\sigma\kappa\ell}{8\pi\epsilon_0} \int_{-\frac{1}{\kappa}}^{\frac{1}{\kappa}} \int_{-\frac{1}{\kappa}}^{\frac{1}{\kappa}} \frac{dad b}{\sqrt{1 + a^2 + b^2}} = \frac{\sigma\ell}{8\pi\epsilon_0} [\text{constant} - 2\pi\kappa + 2\sqrt{2}\kappa^2 + \mathcal{O}(\kappa^3)] \quad (30)$$

where  $\mathcal{O}(\kappa^3)$  denotes terms that depend on  $\kappa$  with cubic or higher orders. By switching back to the parameter  $z$  via  $\kappa = 2z/\ell$ , we get

$$V(0, 0, z) = \text{constant} - \frac{\sigma}{2\epsilon_0} z + \frac{\sqrt{2}\sigma}{\pi\ell\epsilon_0} z^2 + \mathcal{O}(z^3) \quad (31)$$

hence

$$\hat{z} \cdot \mathbf{E}(0, 0, z) = -\hat{z} \cdot \nabla V(0, 0, z) = -\frac{dV(0, 0, z)}{dz} = \frac{\sigma}{2\epsilon_0} - \frac{2\sqrt{2}\sigma}{\pi\epsilon_0} \left(\frac{z}{\ell}\right) + \mathcal{O}(z^2) \quad (32)$$

We see that, as expected, we get back the infinite-plane result when  $z/\ell \ll 1$ , i.e.  $\frac{\sigma}{2\epsilon_0}$ , which we derived using the simpler Gaussian surface method. The current derivation allows us to consider term-by-term corrections to that simple idealization once we take into account the finite nature of the size of the plane!

Finding the answer for part (c) is now straightforward:

$$\frac{d(\hat{z} \cdot \mathbf{E}(0, 0, z))}{dz} = -\frac{2\sqrt{2}\sigma}{\pi\epsilon_0\ell} + \mathcal{O}(z) \quad (33)$$

hence

$$\lim_{z \rightarrow 0} \frac{d(\hat{z} \cdot \mathbf{E}(0, 0, z))}{dz} = -\frac{2\sqrt{2}\sigma}{\pi\epsilon_0\ell} = -\sqrt{2} \quad (34)$$

as we are told to take  $\ell = \frac{2\sigma}{\pi\epsilon_0}$ .

**Question: 5:** Insulators vs Conductors ..... (31½ points)

Please take  $\pi\epsilon_0 \simeq 3.6^{-1} \times 10^{-10}$  in this question

Consider two electrically charged objects of negligible thickness in an otherwise empty space. Their shape is geometrically described by the following two surfaces:

- a hyperboloid segment described by  $x^2 + y^2 - z^2 = a^2$  and  $b < z < b + h$ , with total charge  $Q$
- a cone segment described by  $x^2 + y^2 - z^2 = 0$  and  $b < z < b + h$ , with total charge  $-Q$

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for the parameters  $a, b, h, Q \in \mathbb{R}^+$ . Proceed with the following parts under the assumption that the electric field  $\mathbf{E}(x, y, z)$  can be approximated to satisfy  $\hat{z} \cdot \mathbf{E}(x, y, z) = 0$ .

(a) ( **$10^{1/2}$  points**) Assume that the material of these objects is a perfect insulator and the charge is distributed uniformly. For  $a = 12, b = 1, h = 6, Q = 8$ , what is  $|\mathbf{E}(x, y, z)|$  at  $(x, y, z) = (0, 6, 5)$ ?

- ☐ 0   
 ☐  $2 \times 10^9$    
 ☐  $3 \times 10^9$    
☒  $5 \times 10^9$    
☐  $6 \times 10^9$    
☐  $10^{10}$    
☐  $1.2 \times 10^{10}$    
☐  $1.5 \times 10^{10}$

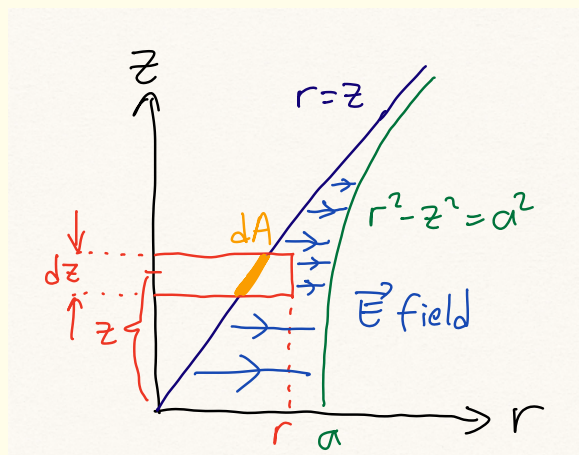
(b) ( **$10^{1/2}$  points**) Assume that the material of these objects is a perfect insulator and the charge is distributed uniformly. For  $a = 3, b = 4, h = 6, Q = 8$ , what is  $|\mathbf{E}(x, y, z)|$  at  $(x, y, z) = (5, 0, 6)$ ?

- ☒ 0   
 ☐  $2 \times 10^9$    
 ☐  $3 \times 10^9$    
☐  $5 \times 10^9$    
☐  $6 \times 10^9$    
☐  $10^{10}$    
☐  $1.2 \times 10^{10}$    
☐  $1.5 \times 10^{10}$

(c) ( **$10^{1/2}$  points**) Assume that the material of these objects is a perfect conductor and the charge distributes itself through the surface accordingly. What would be the ratio  $\frac{\sigma(z=a)}{\sigma(z=2a)}$  where  $\sigma(z)$  denotes the surface charge density of the cone as a function of the position  $z$ ?

- ☐  $\frac{\log(3)}{\log(5/3)}$    
☐  $\frac{\log(5/3)}{\log(3)}$    
☐  $2 \frac{\log(3)}{\log(5/3)}$    
☐  $2 \frac{\log(5/3)}{\log(3)}$
- ☐  $\frac{\log(2)}{\log(5/4)}$    
☐  $\frac{\log(5/4)}{\log(2)}$    
☐  $2 \frac{\log(2)}{\log(5/4)}$    
☒  $2 \frac{\log(5/4)}{\log(2)}$

**Solution 5.1** - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.



Any given information in the question remains invariant under a rotation in the  $x - y$  plane, hence we can leverage this by working with cylindrical coordinates, i.e.

$$x = r \cos(\theta) \quad , \quad y = r \sin(\theta) \quad (35)$$

for  $r \in \mathbb{R}^+$  and  $\theta \in [0, 2\pi)$ . Then, our surfaces are described as

$$r = z \quad , \quad r = \sqrt{z^2 + a^2} \quad (36)$$

as we are interested in  $z > 0$  region since the relevant segment is defined there. We can see a schematic drawing of the cross-section of these surfaces in the left.

In reality, we have no right to expect to have an  $\mathbf{E}$ -field that is orthogonal to the  $z$ -axis; however, we are given this piece of information since we are told that  $\hat{z} \cdot \mathbf{E}(x, y, z) = 0$ : this is only possible if  $\mathbf{E}(x, y, z)$  does not have any  $z$  component! Therefore we can find the  $\mathbf{E}$ -field between the surfaces by considering a Gaussian surface of cylindrical geometry (the schematic red drawing in the image) of infinitesimal thick-



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ness  $dz$  and of radius  $r$ : the only flux that passes this closed volume is through its infinitesimal vertical side, hence

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \Rightarrow |\mathbf{E}(r, z)|(2\pi r dz) = \frac{|\sigma(z)|dA}{\epsilon_0} \quad (37)$$

where  $dA$  is the infinitesimal area segment in this infinitesimal Gaussian surface, drawn in orange in the schematic above. It is a cylindrical shell of length  $2\pi z$  and of width  $\sqrt{2}dz$ , hence  $dA = 2\sqrt{2}\pi z dz$ , leading to

$$|\mathbf{E}(r, z)| = \frac{\sqrt{2}|\sigma(z)|}{\epsilon_0} \frac{z}{r} \quad (38)$$

One might get confused regarding the appearance of  $\sqrt{2}$ : it is because of the fact that we are considering an area element on the surface  $r = z$ . In general situations, one needs to consider the tangent plane to the surface; in our case, our surface is already planar so we do not need to do any complicated computations: by simple geometry, we see that the “length” of the orange line is  $\sqrt{2}$  times  $dz$ .

Another point of possible confusion might be our choice of  $\sigma(z)$ : in principle, the surface charge density depends on two variables that parametrize our surface; in the case of the cone, these can be chosen as  $z$  and  $\theta$  (as  $r = z$ ). However, the polar symmetry ensures that  $\sigma$  does not depend on  $\theta$  hence the most general surface charge density for our setup is simply  $\sigma(z)$ .

With the magnitude of electric field between the surfaces computed, we can immediately write down the  $\mathbf{E}$ -field as follows:

$$\mathbf{E}(r, z) = \frac{\sqrt{2}\sigma(z)}{\epsilon_0} \frac{z}{r} \hat{\mathbf{r}} \quad (39)$$

Note that we take  $\mathbf{E}$ -field to be in  $\hat{\mathbf{r}}$  direction: this is the only direction that **(1)** has polar symmetry in the  $x - y$  plane and **(2)** also orthogonal to  $z$ -direction. Also, we choose  $\hat{\mathbf{r}}$  but not  $-\hat{\mathbf{r}}$  as we want the electric field away from the cone for positive  $\sigma$ .

It is important to emphasize that this is actually not the full electric field, it has to have  $\hat{\mathbf{z}}$  component! (otherwise we can not actually satisfy  $\nabla \times \mathbf{E} = 0$ !) However, we are simply ignoring that component in our computations as requested in the question!

Let's now focus on part (a): as the charge density is uniform, we can take  $\sigma(r, z) = \frac{-Q}{\text{area of cone segment}}$ .

We already see geometrically that the area element is simply  $\sqrt{2}dxdy$ ; however, we can also derive this explicitly using our calculus knowledge; indeed, we know that the area element for a surface described

by  $z = f(x, y)$  is simply  $dA = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}dxdy$ , which becomes  $\sqrt{2}dxdy$  for  $z = r$  as expected. Thus, in cylindrical coordinates

$$\text{area of cone segment} = \int_0^{2\pi} \int_b^{b+h} \sqrt{2}r dr d\theta = \sqrt{2}\pi [(b+h)^2 - b^2] = \sqrt{2}\pi h(2b+h) \quad (40)$$

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hence

$$\sigma(r, z) = -\frac{Q}{\sqrt{2}\pi h(2b + h)} \quad (41)$$

meaning

$$\mathbf{E}(r, z) = -\frac{Q}{\epsilon_0 \pi h(2b + h)} \frac{z}{r} \hat{r} \quad (42)$$

We can now immediately answer part (a): observe that  $a = 12$  with  $z = 5$  implies that any point with  $5 < r < 13$  is between the surfaces, hence at  $r = 6$ , we have

$$|\mathbf{E}(r = 6, z = 5)| = \left| \frac{8}{3.6^{-1} \times 10^{-10} \times 6 \times 8} \frac{5}{6} \right| = 5 \times 10^9 \quad (43)$$

where we are also given  $a = 12$ ,  $b = 1$ ,  $h = 6$ ,  $Q = 8$  and  $\pi\epsilon_0 \approx 3.6^{-1} \times 10^{-10}$ .

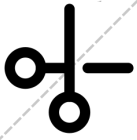
Let us move on to part (b). We actually do not need to do much here as we are asked for the magnitude of the  $\mathbf{E}$ -field not between the surfaces but inside of the cone itself as  $r < z$  for  $\mathbf{E}(x = 5, y = 0, z = 6)$ . Normally, we expect a non-zero electric field here however we are told to take  $\hat{z} \cdot \mathbf{E}(x, y, z) = 0$  within the question, meaning we can easily use our Gaussian surface approach of part (a) here as well (since we have the full polar symmetry in  $x - y$  plane). But this time, there is no enclosed charge hence we simply have  $\mathbf{E}(5, 0, 6) = 0$ .

In part (c), we are told to assume that the material is a perfect conductor: this means that all points on the cone has a unique electric potential (say  $V_c$ ) and all points on the hyperboloid has a unique electric potential (say  $V_h$ ): note that this follows from the fact that there can not be electric field inside these conductors hence the electric potential is constant throughout these surfaces. As electric field is a conservative vector field in electrostatics, we then immediately have

$$V_h - V_c = - \int_{\text{any point on the cone}}^{\text{any point on the hyperboloid}} \mathbf{E} \cdot d\mathbf{l} \quad (44)$$

Since the electric field is radial in the given approximation, we can choose  $d\mathbf{l} = \hat{r} dr$ , hence if we choose to integrate over a radial line at some  $z$ , (39) becomes

$$\begin{aligned} V_h - V_c &= - \int_{r=z}^{r=\sqrt{a^2+z^2}} \frac{\sqrt{2}\sigma(z)}{\epsilon_0} \frac{z}{r} dr = - \frac{\sqrt{2}\sigma(z)z}{\epsilon_0} \log(r) \Big|_{r=z}^{r=\sqrt{a^2+z^2}} \\ &= - \frac{\sqrt{2}}{\epsilon_0} \left[ \sigma(z)z \log \left( \frac{\sqrt{a^2+z^2}}{z} \right) \right] \\ &= - \frac{1}{\sqrt{2}\epsilon_0} \left[ \sigma(z)z \log \left( 1 + \frac{a^2}{z^2} \right) \right] \end{aligned} \quad (45)$$



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Since the left hand side is a constant, the right hand side has to be independent of the variable  $z$ , which means

$$\sigma(z) = \frac{\alpha}{z \log\left(1 + \frac{a^2}{z^2}\right)} \quad (46)$$

for some undetermined coefficient  $\alpha$ . We can actually find this constant as well by integrating  $\sigma(z)$  over the whole cone and equating it to  $-Q$ ; however, we need not to: for part (c), we only need the ratio of surface charge densities which can be immediately written down as

$$\frac{\sigma(z = a)}{\sigma(z = 2a)} = \frac{\frac{\alpha}{a \log\left(1 + \frac{a^2}{a^2}\right)}}{\frac{\alpha}{2a \log\left(1 + \frac{a^2}{4a^2}\right)}} = 2 \frac{\log(5/4)}{\log(2)} \quad (47)$$

« « « Congratulations, you have made it to the end! » » »