Phys209: Mathematical Methods in Physics I

Homework 1

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Only the first question will be graded; other question(s) are for your own practice!
- The homework is due October 13th 2023, 23:59 TSI.

(1) Problem One *(only graded question, worths 6pts)*

We know that an ordinary function f takes a value as the input and spits out another value as the output, e.g.

$$f :: \mathbb{R} \to \mathbb{R} \tag{1.1a}$$
$$f = x \to 2\cos(3x) \tag{1.1b}$$

On the contrary, we have seen that *higher order functions* can take functions as input as well, e.g.

$$\frac{\mathrm{d}}{\mathrm{d}x} :: (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}) \tag{1.2a}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} = \big(x \to f(x)\big) \to \big(x \to f'(x)\big) \tag{1.2b}$$

In the rest of this question, we will talk more about higher order functions and the derivative operator.

(1.1) (1.5pt)

Consider the square function defined over complex numbers as follows:

$$\mathrm{sqr} :: \mathbb{C} \to \mathbb{C} \tag{1.3a}$$
$$\mathrm{sqrt} = x \to x^2 \tag{1.3b}$$

How would you define a similar *higher order function* (let's call it sqr2), which takes a function f and outputs another function g such that $g(x) = 2f(x)$?

$$\mathrm{sqr2} :: ??? \tag{1.4a}$$
$$\mathrm{sqrt2} = ??? \tag{1.4b}$$

(1.2) (1.5pt)

Consider the identity function $\mathcal{I}$

$$\mathcal{I} :: \mathbb{C} \to \mathbb{C} \tag{1.5a}$$
$$\mathcal{I} = x \to x \tag{1.5b}$$

which naturally generalizes as a higher order function

$$\mathcal{I} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.6a}$$

$$\mathcal{I} = \big(x \to f(x)\big) \to \big(x \to f(x)\big) \tag{1.6b}$$

which takes a function to itself. Note that we are abusing the notation here by denoting two different objects with the same letter $\mathcal{I}$.

We can take linear combinations of higher order functions; for instance, we can say

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{I}\right) :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.7a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{I}\right) = \big(x \to f(x)\big) \to \big(x \to f(x) + f'(x)\big) \tag{1.7b}$$

What would be the type and definition of this object when it acts on the cosine function (denoted as cos):

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{I}\right) \cdot \cos :: ??? \tag{1.8a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{I}\right) \cdot \cos = ??? \tag{1.8b}$$

(1.3) (1.5pt)

We usually abuse notation and use the numbers themselves as higher order functions, and we also replace *"application of a higher order function"* by "multiplication from left". As examples, we write

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + 1\right) f(x)\,, \quad \left(\frac{\mathrm{d}}{\mathrm{d}x} - 2\right) f(x) \tag{1.9}$$

where we actually mean

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{I}\right) \cdot f(x)\,, \quad \left(\frac{\mathrm{d}}{\mathrm{d}x} - 2\mathcal{I}\right) \cdot f(x) \tag{1.10}$$

In the rest of this homework (and in the lectures), I will keep abusing the language as long as it is clear what we mean.

Another place we abuse the notation is where we chain applications of derivatives. For instance, the expression

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + 3\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - 2\right) f(x) \tag{1.11}$$

only makes sense when we interpret it as

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + 3\right) \cdot \left(\left(\frac{\mathrm{d}}{\mathrm{d}x} - 2\right) \cdot f(x) \right) \tag{1.12}$$

whereas

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}x} + 3\right) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x} - 2\right) \right) \cdot f(x) \tag{1.13}$$

does not make sense: $\left(\frac{\mathrm{d}}{\mathrm{d}x} + 3\right)$ as we defined takes a type $\mathbb{R} \to \mathbb{R}$ as the input; it cannot act on $\left(\frac{\mathrm{d}}{\mathrm{d}x} - 2\right)$ which has the type $(\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R})$.

There are actually objects that can act on higher order functions (these are themselves called higher order too); for instance, consider the following operator:

$$\mathcal{C} = \Big[\big(x \to f(x)\big) \to \big(x \to f'(x)\big)\Big] \to \Big[\big(x \to f(x)\big) \to \big(x \to f''(x)\big)\Big] \tag{1.14a}$$

which turns a *first derivative higher order function* to a *second derivative higher order function*, i.e.

$$\mathcal{C} \cdot \frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} \tag{1.15}$$

What is the type of this operator $\mathcal{C}$ when acting on real variables and real functions, i.e

$$\mathcal{C} ::??? \tag{1.16}$$

(1.4) (1.5pt)

Remember the exponential function

$$\exp :: \mathbb{C} \to \mathbb{C} \tag{1.17a}$$

$$\exp = x \to \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1.17b}$$

How would we generalize this to higher order functions?

A natural definition for a higher order function

$$T :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.18}$$

would be

$$\begin{aligned}\exp(T) &:: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \\ \exp(T) \cdot f &= f + (T \cdot f) + \frac{1}{2} T \cdot (T \cdot f) + \frac{1}{3!} T \cdot (T \cdot (T \cdot f)) + \dots\end{aligned} \tag{1.19}$$

for any function $f :: \mathbb{C} \to \mathbb{C}$.

As an example, consider $f = \cos$. We see that

$$\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\cos(x) = \cos(x) - \sin(x) - \frac{1}{2}\cos(x) + \frac{1}{3!}\sin(x) + \dots \tag{1.20a}$$

$$= \cos(1)\cos(x) - \sin(1)\sin(x) \tag{1.20b}$$

$$\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\cos(x) = \cos(x+1) \tag{1.20c}$$

We see that the effect of $\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ on $\cos(x)$ is to shift its argument! This is actually no coincidence: we will see why this is important and what $\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ means from the *group theory perspective* in later sections!

By doing an analogous computation, show what the action of $\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ on x^4 is:

$$\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)x^4 = ??? \tag{1.21}$$

(2) Problem Two *(not graded)*

It is straightforward to define $\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ and check its action in `Mathematica`! By running this code, see what you get if you act $\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ on various functions!

```
(* Define n^th derivative*)
der[n_] := D[#, {x, n}] &;
(* Define exponentiated derivative*)
ExpDer := Sum[1/n! der[n][#], {n, 0, \[Infinity]}] &;
(*We can now check its action on various functions:*)
ExpDer /@ {Cos[x], x^3, Exp[2 x]}
```

(3) Problem Three *(not graded)*

In the class, we have shown that the solution to *order−n constant-coefficient linear homogeneous ordinary differential equations* are in the form

$$f(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x} \tag{3.1}$$

where c_i are arbitrary coefficients and r_n are the roots of the characteristic equation. Reproduce this result via Mathematica with the following code:

```
(*Most general form of order-n constant-coefficient linear \
homogeneous ordinary differential equation is of the following form:*)
eqn [n_] := (Sum[a[i] D[f[x], {x, i}], {i, 0, n}] == 0);
(*We can solve this equation immediately for any given $n$; for \
instance, :*)
solutions = {DSolve[eqn[3], f[x], x], DSolve[eqn[4], f[x], x],
        DSolve[eqn[5], f[x], x]};
(*Better formatting:*)
Flatten[solutions] /. Root[__, i_] :> Subscript[r, i] // Column
```

(4) Problem Four *(not graded)*

The notation that we have been using have had extensive use of double colon, i.e. ::. Indeed, we write

$$2 :: \mathbb{Z} \tag{4.1a}$$

$$x :: \mathbb{R} \tag{4.1b}$$

$$f :: \mathbb{R} \to \mathbb{R} \tag{4.1c}$$

and so on. Basically, $a :: b$ means *"a is of the type b"*, so above expressions read as

Number 2 is of the integer type.
The variable x is of the real type.
The function f is of the "real to real" type.

This is not really a common notation in math and physics; because it is relatively new (less than a century old). The field of math that deals with such objects is called *type theory*, and some of the notations that we are using for the functions (i.e. $f = x \to f(x)$) is actually related to the even newer field *typed lambda calculus*.

These fields are really important in computer science, and especially for functional programming languages such as *Haskell*. In fact, the reason we are using :: instead of : is so that we can reserve : for its traditional role in functional programming!

The language Haskell, the lambda calculus, and the related concepts such as Curry–Howard correspondence are actually smaller pieces of a grand field of Math: **category theory**! There is so much about this field, and you may google about it if you like (I also started writing some notes but I proceed extremely slowly!)

Now comes the problem:

1. If you find any one of these interesting, I suggest that you do a little bit reading. Learning more about the cutting edge things in math, computer science, and their potential applications can be quite rewarding.

2. If you are not necessarily interested, then ***do nothing!*** None of these stuff is needed for this course, and you may never need them unless you study mathematical physics in the future.