Quantum Hall Effect and Conformal Field Theory Through Moore-Read States

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The realization that Chern-Simons theories were related to conformal blocks, has reflected the same fact for the Fractional Quantum Hall Effect (FQHE), which has an effective theory description in the form of Chern-Simons theory. The relation between FQHE and Conformal Field Theory (CFT) has been examined through Laughlin and Pfaffian states within the Moore-Read conjecture. It is found that FQHE wave functions can be written as conformal blocks and abelian/non-abelian statistics can be obtained from these constructions. In this review, Laughlin wave functions have been written as correlators of free boson (1+1)d CFT. Additionally, Pfaffian states have been written as correlators of free fermion (Majorana) (1+1)d CFT.

Keywords: Laughlin states, Moore-Read/Pfaffian states, Conformal Block

I. OUTLINE

In Witten's pioneering work on QFT and the Jones polynomial[1], he showed that Chern-Simons theory correlators can be demonstrated as conformal blocks. A natural following question was that, directed by Moore and Read[2][3], is there any correspondence between CFT and FQHE, which can be described as effectively Chern-Simons theory. The question led that the blocks in certain rational conformal field theories could be interpreted as the wave functions of Quantum Hall quasiparticles, and one way to understand the Moore-Read conjecture is as a generalization of this idea to the wave function of the electrons themselves. Of course, this was in a sense natural also because the Laughlin and Pfaffian wave functions were indeed, up to an exponential normalization factors, conformal blocks.

It was shown later by Hansson, et al.[4] later that how to generalize the Moore-Read picture to cases where the wave functions were not single conformal blocks, but linear combinations of such blocks, and they also showed that the result was identical to the states obtained from the standard conformal field construction. This is far from expected because the result is identical instead of lying in the same universality class.

While using CFT to obtain FQHE wave functions, also statistics of related particles can be deduced directly[10][11]. Again this can be seen in the papers of Moore-Read[2][3]. In addition to the Moore-Read papers, Fröhlich, et.al.[8] has investigated non-abelian braid statistics generally.

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II. STATISTICS

Particle exchange statistics in 2D are totally different than 3D or higher dimensions. The topology governs the realm of 2D. In old fashion perspective, when particles are interchanged with each other in 3D, the representation of the symmetric group (S_n) is enough to describe physics. For bosons, the trivial representation and for fermions the alternating representation are used to describe statistics. The more modern approach is to interchange particles by following some definite paths in space. This description reduces the action of S_n in $d > 2$. But in $d = 2$, the related group is the braid group, B_n, which is generated by a set of elementary exchanges. These exchanges σ_i, exchange particles i and $i+1$ along a path not enclosing any other particles. This exchange can be seen as the following,

$$\Psi_{p;i_1..i_s}(z_1..z_{i_s}..z_{i_r}..z_n) = \sum_q B_{pq}[i_1,..,i_n]\Psi_{q;i_1..i_n}(z_1..z_n) \tag{1}$$

Where B_{pq} is the braid matrix. This reduction is due to the non-trivial path taken by particles in the exchange. The most common encounter is fractional statistics, where related excitations are anyons. In this context, B_n is represented as $e^{i\theta}$ where θ is real and so is a one-dimensional, abelian representation. Bose and Fermi statistics are special cases of this representation. The particle statistics are called non-Abelian statistics when particles exchange as a non-Abelian representation of the B_n.

Additionally, when the braid group approach is considered, fusion rules have to also be taken into account. In the case of more than one particle species, when a particle species is moved around another kind of particle species there is a matrix effect similar to B^2. When different species of particles are exchanged, they may lead to another particle or simply annihilate. Related physics can be exactly described well by Fusion rules which can be seen in the papers[7][8].

III. CFT CONSTRUCTION

The QHE can be simply obtained by considering electron gas in the 2D plane with a perpendicular strong background magnetic field, near to zero temperature, and random potential to mimic disorder[14][15]. The simplicity of the problem appears to be the strength of the effect. The system has the Hamiltonian,

$$\mathbf{H} = \frac{1}{2m}(\vec{\mathbf{p}} + e\vec{\mathbf{A}})^2 + \frac{e^2}{\mathbf{r}} \tag{2}$$

When symmetric gauge, $\vec{A} = \frac{B}{2}(y, -x, 0)$, imposed on the free Hamiltonian, the Lowest Landau level wave functions take the following form,

$$\Psi_{LLL}(z_i, \bar{z_i}) = f(z_1, .., z_N)e^{-\sum_{i=1}^{N}|z_i|^2/4\ell_B^2} \tag{3}$$

where $f(z_i)$ is any holomorphic function. This holomorphic function can be modified to describe particular physics. In the interacting case, it is not possible to write exact wave functions, but Laughlin made a bold move and wrote directly his celebrated wave function, which greatly overlaps with the ground state, in 1981[9],

$$\Psi_L(z_i) = \prod_{i<j}^{N}(z_i - z_j)^m e^{-\sum_{i=1}^{N}|z_i|^2/4\ell_B^2} \tag{4}$$

To imply anti-symmetry in electron wave functions, filling fraction $\nu = \frac{1}{m}$ imposed where m is odd.

Next, the fact that correlation of a collection of fields can be written as conformal blocks.

$$\left\langle \prod_{r=1}^{n} \Psi_{i_r}(x_r) \right\rangle = \sum_{q} |\mathscr{F}_{p;i_1\ldots i_n}(z_1, ..., z_n)|^2 \tag{5}$$

Where $z_r = x_r + iy_r$ defined and $\mathscr{F}_p$ denote conformal blocks while p-label denotes different functions run over a finite set in RCFT. To construct FQHE wavefunctions in a particular CFT correlator, consider 1+1 dimensional free boson theory. The two-point correlator of this theory is,

$$\langle\varphi_1(z)\varphi_1(w)\rangle = -ln(z - w) \tag{6}$$

and its vertex operator is defined as,

$$V_1(z) = :e^{i\sqrt{m}\varphi_1(z)}: \tag{7}$$

Now, the Laughlin wave function can be written as,

$$\begin{aligned} \Psi_L(z_i) &= \langle 0|\,\mathbf{R}\Big\{V_1(z_1)\cdots V_1(z_N)e^{-i\sqrt{m}\rho_m\int d^2z'\varphi_1(z')}\Big\}\,|0\rangle \\ &\equiv \langle V_1(z_1)V_1(z_2)\cdots V_1(z_N)\rangle_{\frac{1}{m}} \\ &= \prod_{i<j}^{N}(z_i - z_j)^m e^{-\sum_{i=1}^{N}|z_i|^2/4\ell_B^2} \end{aligned} \tag{8}$$

where $\mathbf{R}$ is radial ordering, ρ_m is the constant background particle density, and $|z_1| \geq |z_2| \geq \cdots \geq |z_N|$.

With operator insertion of $\Psi_{qh}(\eta_i) = \exp\Big\{\frac{i}{\sqrt{q}}\varphi(z)\Big\}$ in the correlator, quasiparticles that have non-trivial statistics can be obtained. In Laughlin's case, this exactly yields fractional statistics by $\theta/\pi = 1/q$, which is an example of abelian statistics.

For the even denominator plateaus, $\nu = \frac{1}{m}$ filling fraction, where m is even, Pfaffian state is one candidate[1],

$$\Psi_{Pf}(z_i) = Pf(\frac{1}{z_i - z_j})\prod_{i<j}^{N}(z_i - z_j)^m e^{-\sum_{i=1}^{N}|z_i|^2/4\ell_B^2} \tag{9}$$

To construct this in a particular CFT correlator, this time consider 1+1 dimensional free Majorana fermion theory. The two-point correlator of this theory is,

$$\langle\chi_1(z)\chi_1(w)\rangle = \frac{1}{z - w} \tag{10}$$

and its vertex operator is defined as, operator as

$$V_1(z) = :\chi_1(z)e^{i\sqrt{m}\varphi_1(z)}: \tag{11}$$

Now, the Pfaffian wave function can be written as,

$$\begin{aligned} \Psi_{Pf}(z_i) &= \langle 0|\,\mathbf{R}\Big\{V_1(z_1)\cdots V_1(z_N)e^{-i\sqrt{m}\bar{\rho}\int d^2z'\varphi_1(z')}\Big\}\,|0\rangle \\ &\equiv \langle V_1(z_1)V_1(z_2)\cdots V_1(z_N)\rangle_{\frac{1}{m}} \\ &= Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}^{N}(z_i - z_j)^m e^{-\sum_{i=1}^{N}|z_i|^2/4\ell_B^2} \end{aligned} \tag{12}$$

where again $\mathbf{R}$ is radial ordering, ρ is the constant background particle density, and again $|z_1| \geq |z_2| \geq \cdots \geq |z_N|$.

Hence, unlike plasma analogy construction, CFT constructions directly yield wave functions but not their modulus squared! Both cases result in ground-state wave functions. Like Laughlin's case, with operator insertion of $\Psi_{qh}(\eta_i)$ in the correlator, quasiparticles can be obtained. In the FQHE context, these are anyons with fractional statistics and fractional charges. However, in the Pfaffian state, non-Abelian statistics have been observed in the excitations, more explicitly quasi-holes. This result is a direct result of the non-commutative nature of the monodromy of related conformal blocks that were introduced[13]. Hence, braiding matrices do not commute, unlike Laughlin's case.

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[1] Others are anti-Pfaffian, Bonderson-Slingerland and non-abelian spin singlet (NASS) states.

IV. CONCLUSIONS

It has been shown that FQHE wave functions can be written as linear combinations of conformal blocks. With this relation, non-trivial statistics of 2D can be extracted from these CFT constructions. Additionally, with particular conformal fields, FQHE wave functions can be seen as abstract mathematical equalities without using much more "physical" microscopic approaches such as Ginzburg-Landau theories.

The CFT used for generating the wave functions, should not be thought of as Hamiltonians or Lagrangians for the electrons, but an auxiliary theory to generate wave functions. It is, however, natural to ask whether there is any “derivation” of the CFT construction of the QH wave functions from a microscopic (non-relativistic) theory of the electrons. It turns out there is in the paper[6]. It seems non-abelian statistics realization are much more implicit when monodromy and holonomy is considered.

Overall, by inserting the CFT picture into the QHE problem, some remarkable progress can be understood more comprehensibly and completely such as conformal blocks/FQHE wave functions, (non)abelian statistics, and hierarchical QH states and bulk-boundary correspondence in both pictures.

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