Name:	
Student ID:	

Final Examination

Phys210: Mathematical Methods in Physics II

2024/06/04

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) any of my work *outside the designated areas* in the "fill-in the blank questions" will not be graded;

(2) I take *full responsibility* for any ambiguity in my selection of the correct option in "multiple choice questions";

(3) any of my work *outside the answer boxes* in the "classical questions" will not be graded;

(4) any page which does not contain *both my name and student id* will not be graded;

(5) any extra sheet that I may use are for my own calculations and will *not* be graded.

Signature: ____________________________

This exam has a total of 7 questions, some of which are for bonus points. You can obtain a maximum grade of 22+2 from this examination.

Question	Points	Score
1	9	
2	3	
3	3	
4	2	

Question	Points	Score
5	1	
6	4	
7	0	
Total:	22	

1 Notations & Conventions

- **The non-negative integer power** of an object A (denoted A^n) is defined recursively as follows:

$$A^0 = \mathbb{I}\,, \quad A^n = A \cdot A^{n-1} \;\; \forall n \geq 1 \tag{1}$$

where the operation $\cdot$ is matrix multiplication if A is a matrix, application of differentiation if A is a differential operator (such as $\frac{\mathrm{d}}{\mathrm{d}x}$), or ordinary multiplication if A is simply a scalar number. $\mathbb{I}$ is the identity object with respect to the operation —identity matrix for matrix multiplication, the number 1 for ordinary multiplication, and so on.

- **Exponentiation of an object** A (denoted e^A) is defined as

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{2}$$

where A^n is the $n-$th power of the object A. For instance, we can write down

$$e^{i\frac{\mathrm{d}}{\mathrm{d}x}} = \cos\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) + i\sin\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{3}$$

in accordance with the Euler formula.

- **Logarithm of an object** A (denoted $\log A$) is defined as the inverse of the exponentiation. For objects for which the exponentiation is not a monomorphism (such as complex numbers), logarithm is a *relation* (also called multi-valued function). Conventionally, one imposes restrictions on the domain to ensure that logarithm acts as a function; for instance, for a complex number $z = re^{i\theta} \in \mathbb{C}$ with $(r,\theta) \in (\mathbb{R}^+, \mathbb{R})$, we can define $\log z = i\theta_p + \log r$ where $0 \leq \theta_p < 2\pi$ is called *the principal value of* θ for $\dfrac{\theta - \theta_p}{2\pi} \in \mathbb{Z}$.

- **The generalized power of an object** A (denoted A^α) is defined as

$$A^\alpha = e^{\alpha \log A} \tag{4}$$

If exponentiation is not a homomorphism when acting on the domain of A, A^α is not a function but a relation *unless* a principle domain is selected (similar to the logarithm discussed above).

- **Generalized exponentiation of an object** A (denoted α^A) is defined as

$$\alpha^A = e^{A \log \alpha} \tag{5}$$

Depending on the available values for $\log \alpha$, α^A may mean multiple different functions. However, each one is *still* a proper function, not a multi-valued function.

- **The Kronecker symbol** (also called Kronecker-delta) is defined as

$$\delta :: \{\mathbb{Z}, \mathbb{Z}\} \to \mathbb{Z} \tag{6a}$$

$$\delta = \{i,j\} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{6b}$$

- **The Dirac-delta generalized function** δ is (for all practical purposes of a Physicist) defined via the relation

$$\int_{\mathcal{A}} f(y)\delta(x-y)dy = \begin{cases} f(x) & \text{if } x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

A useful representation of Dirac-delta generalized function is

$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} \tag{8}$$

- **A particular permutation of n objects** is denoted as $(i_1 i_2 \ldots i_n)$ where $i_1 \neq i_2 \neq \ldots \neq i_n \in \{1, \ldots, n\}$. A permutation $(i_1 \ldots i_n)$ is said to be an even (odd) permutation of $(k_1 \ldots k_n)$ if the two are identical after the permutation of an even (odd) number of adjacent indices. For example, (2431) is an even permutation of (2143) and an odd permutation of (2134).

- **Levi-Civita symbol** is denoted as

$$\epsilon :: \{\mathbb{Z}^+, \ldots, \mathbb{Z}^+\} \to \mathbb{Z} \tag{9a}$$

$$\epsilon = \{a_1, \ldots, a_n\} \to \begin{cases} 1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an even permutation of } (12\ldots n) \\ -1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an odd permutation of } (12\ldots n) \\ 0 & \text{otherwise} \end{cases} \tag{9b}$$

- **The determinant function** (denoted det) is defined as

$$\det :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{10a}$$

$$\det = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \to \sum_{i_1,\ldots,i_n} \epsilon_{i_1\ldots i_n} a_{1i_1} \ldots a_{ni_n} \tag{10b}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i,j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

- **The adjugate function** (denoted adj) is defined as

$$\mathrm{adj} :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{11a}$$

$$\mathrm{adj} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & & & \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \tag{11b}$$

$$\text{where } b_{i_n k_n} = \frac{1}{(n-1)!} \epsilon_{i_1 \dots i_n} \epsilon_{k_1 \dots k_n} a_{i_1 k_1} \dots a_{i_{n-1} k_{n-1}} \tag{11c}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

- **Inverse of a matrix** A is to be denoted as A^{-1}: it satisfies the equations $A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{I}$ where $\mathbb{I}$ is the identity matrix. One can prove (which is beyond the scope of this course) that the inverse of a matrix A can be computed through its adjugate and its determinant:

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det A} \tag{12}$$

- **The trace function** (denoted tr) is defined as

$$\mathrm{tr} :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{13a}$$

$$\mathrm{tr} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_i a_{ii} \tag{13b}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

- **Wronskian matrix** of a set of functions $\{f_1(x), \dots, f_n(x)\}$ is defined as a square matrix where the first row is the set of the functions and the $i-$th row is $(i-1)-$th derivative of the functions for all $n \geq i \geq 2$.

- **A complex number** is (for all practical purposes of a Physicist) a pair of two real numbers, i.e. $(z \in \mathbb{C}) \leftrightarrow (x \in \mathbb{R}, y \in \mathbb{R})$ where one can construct z via $z = x + iy$ (i is called *the imaginary unit* with the property $i^2 = -1$); conversely, one can extract x and y via the functions Re and Im: $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

- **Complex conjugation** (denoted $*$) is a function defined to act on complex numbers as

$$* :: \mathbb{C} \to \mathbb{C} \tag{14a}$$

$$* = z \to (z^* = \mathrm{Re}(z) - i\,\mathrm{Im}(z)) \tag{14b}$$

- **Matrix transpose** (denoted T) is a function defined to act on matrices as

$$T :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{15a}$$

$$T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & & & \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \tag{15b}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

- **Hermitian conjugation** (also called *conjugate transpose*, *adjoint*, or *dagger*) is a function to act on matrices of complex entries as

$$\dagger :: \mathfrak{M}_{n\times n}(\mathbb{C}) \to \mathfrak{M}_{n\times n}(\mathbb{C}) \tag{16a}$$

$$\dagger = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a^*_{11} & a^*_{21} & \dots & a^*_{n1} \\ a^*_{12} & a^*_{22} & \dots & a^*_{n2} \\ \dots & & & \\ a^*_{1n} & a^*_{2n} & \dots & a^*_{nn} \end{pmatrix} \tag{16b}$$

- **Characteristic polynomial of** any square matrix A is defined as

$$\det (A - \lambda_i \mathbb{I}) = 0 \tag{17}$$

- **Fourier transforms** are widely-used integral transformations (and are the simplest example of the harmonic analysis) which can be defined with any self-consistent convention. For this examination, please stick to the following conventions for Fourier transformation (and its different versions):

$$f :: \mathbb{R} \to \mathbb{C}\,, \qquad f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \hat{f}(k) \tag{18a}$$

$$\hat{f} :: \mathbb{R} \to \mathbb{C}\,, \qquad \hat{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \tag{18b}$$

$$f :: [a, a+T] \to \mathbb{C}\,, \quad f(x) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{T}x} \hat{f}(n) \tag{19a}$$

$$\hat{f} :: \mathbb{Z} \to \mathbb{C}\,, \qquad \hat{f}(n) = \int_{a}^{a+T} dx e^{-i\frac{2\pi n}{T}x} f(x) \tag{19b}$$

$$f :: \mathbb{Z} \to \mathbb{C}\,, \qquad f(n) = \frac{1}{T} \int_{a}^{a+T} dx e^{i\frac{2\pi n}{T}k} \hat{f}(k) \tag{20a}$$

$$\hat{f} :: [a, a+T] \to \mathbb{C}\,, \quad \hat{f}(k) = \sum_{n=-\infty}^{\infty} e^{-i\frac{2\pi n}{T}k} f(n) \tag{20b}$$

$$f :: \mathbb{Z}_N \to \mathbb{Z}_N\,, \qquad f(n) = \frac{1}{N} \sum_{m=0}^{N-1} e^{i\frac{2\pi nm}{N}} \hat{f}(m) \tag{21a}$$

$$\hat{f} :: \mathbb{Z}_N \to \mathbb{Z}_N\,, \qquad \hat{f}(m) = \sum_{n=0}^{N-1} e^{-i\frac{2\pi nm}{N}} f(n) \tag{21b}$$

where (18), (19), (20), and (21) are called *Fourier Transform*, *Fourier Series*, *Discrete-time Fourier Transform*, and *Discrete Fourier Series* respectively. We will stick to this naming in this examination, but please be reminded that different communities (engineering, math, physics, etc.) use different naming conventions in general.

• **"Even part of" and "odd part of"** (denoted E and O) are higher order functions defined as

$$E :: (\mathcal{A} \to \mathcal{A}) \to (\mathcal{A} \to \mathcal{A}) \tag{22a}$$

$$E = (x \to f(x)) \to \left(x \to f_E(x) = \frac{f(x) + f(-x)}{2}\right) \tag{22b}$$

$$O :: (\mathcal{A} \to \mathcal{A}) \to (\mathcal{A} \to \mathcal{A}) \tag{22c}$$

$$O = (x \to f(x)) \to \left(x \to f_E(x) = \frac{f(x) - f(-x)}{2}\right) \tag{22d}$$

with which any single-argument function satisfies $f = E \cdot f + O \cdot f$, or with a more common notation, $f(x) = f_E(x) + f_O(x)$. Here $\mathcal{A}$ is any field, but we usually take it to be $\mathbb{C}$.

• **Inner product between two functions** f and g shall be denoted in this exam as $\langle f, g\rangle_\omega^{\mathcal{A}}$:

$$\langle , \rangle_\omega^{\mathcal{A}} :: (\mathcal{A} \to \mathbb{C}, \mathcal{A} \to \mathbb{C}) \to \mathbb{C} \tag{23a}$$

$$\langle f, g\rangle_\omega^{\mathcal{A}} = \int\limits_A \Big(f(x)\Big)^* g(x)\omega(x)dx \tag{23b}$$

for $\mathcal{A} \subseteq \mathbb{R}$.

• **Group** is defined as a pair (S, o) where $S :: \texttt{Set}$ and where $o :: (S, S) \to S$ for which the following statements are true:

1. $(\exists e \in S)(\forall s \in S)\ o(e, s) = o(s, e) = s$
2. $(\forall s \in S)\ o(s, i(s)) = o(i(s), s) = e$
3. $(\forall a, b, c \in S)\ o(a, o(b, c)) = o(o(a, b), c)$

for a unique function $i :: S \to S$.

• **Ring** is defined as a triplet $(S, +, \cdot)$ where $S :: \texttt{Set}$, $+, \cdot :: (S, S) \to S$ for which the following statements are true:

1. $(S, +) :: \texttt{Commutative Group}$
2. $(\forall a, b, c \in S)\ a \cdot (b + c) = a \cdot b + a \cdot c$
3. $(\forall a, b, c \in S)\ (b + c) \cdot a = b \cdot a + c \cdot a$

• **Skew field** is defined as a triplet $(S, +, \cdot)$ where $S :: \texttt{Set}$, $+, \cdot :: (S, S) \to S$ for which the following statements are true:

1. $(S, +, \cdot) :: \texttt{Ring}$
2. $(S\backslash\{0\}, \cdot) :: \texttt{Group}$

where 0 denotes the identity element with respect to $+$.

• **Field** is defined as a triplet $(S, +, \cdot)$ where $S :: \texttt{Set}$, $+, \cdot :: (S, S) \to S$ for which the following statements are true:

1. $(S, +, \cdot) :: \texttt{Ring}$
2. $(S\backslash\{0\}, \cdot) :: \texttt{Commutative Group}$

where 0 denotes the identity element with respect to $+$.

• **Linear space** (also called *vector space*) over a field $F = (S, +, \cdot)$ shall be denoted as $V(F)$ and is defined as a triplet $(V, \oplus, \odot)$ $\big(V :: \texttt{Set}$, $\oplus :: (V, V) \to V$, and $\odot :: (S, V) \to V\big)$ for which the following statements are true:

1. $(V, \oplus) :: \texttt{Commutative Group}$
2. $(\forall v \in V)\ 1 \odot v = v \quad$ (1 is the identity element of $\cdot$)
3. $(\forall v \in V)(\forall s \in S)\ s \odot v \in V$
4. $(\forall v \in V)(\forall a, b \in S)\ (a \cdot b) \odot v = a \odot (b \odot v)$
5. $(\forall v \in V)(\forall a, b \in S)\ (a + b) \odot v = (a \odot v) \oplus (b \odot v)$
6. $(\forall v, w \in V)(\forall s \in S)\ s \odot (v \oplus w) = (s \odot v) \oplus (s \odot w)$

The elements of the set S (V) are called *scalars* (*vectors*).

• **Linear algebra** (also called *vector algebra*) over a field $F = (S, +, \cdot)$ shall be denoted as $L(F)$ and is defined as a quadruple $(V, \oplus, \odot, \otimes)$ $\big(V :: \texttt{Set}$, $\oplus, \otimes :: (V, V) \to V$, and $\odot :: (S, V) \to V\big)$ for which the following statements are true:

1. $(V, \oplus, \odot) :: \texttt{Linear Space}$
2. $(\forall x, y, z \in V)\ x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
3. $(\forall x, y, z \in V)\ (x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$
4. $(\forall x, y \in V)(\forall a, b \in S)\ (a \odot x) \otimes (b \odot y) = (a \cdot b) \odot (x \otimes y)$

• **Lie algebra** is a linear algebra $(V, \oplus, \odot, \otimes)$ with the additional condition that $(\forall x, y \in V)\ x \otimes y = -y \otimes x$.

• **Commutator** is a higher order function which takes two functions $f, g :: \mathcal{A} \to \mathcal{A}$ for any type $\mathcal{A}$, and gives a new function $[f, g] :: \mathcal{A} \to \mathcal{A}$ by cascading their action. It is defined on an object $x \in \mathcal{A}$ as $[f, g](x) = f(g(x)) - g(f(x))$.

• **Basis** B of a vector space V is $(B \supset V) :: \texttt{Set}$ for which following statements are true:

1. $(\forall k \in \{1, 2, \dim B\})(\forall e_1, \ldots, e_k \in B)(\forall c_1, \ldots, c_k \in S)[c_1 = \cdots = c_k = 0] \vee [c_1 e_1 + \cdots + c_k e_k \neq 0]$
2. $(\forall v \in V)(\exists! a_1, \ldots, a_{\dim B} \in S)$ $v = a_1 e_1 + \cdots + a_{\dim B} e_{\dim B}$

• **Normed vector space** over a field F is a vector space $V(F)$ over which a function $\mathsf{norm} :: V \to \mathbb{R}$ exists with the notation $\mathsf{norm} = x \to \|x\|$, for which following statements are true:

1. $(\forall v \in V)[\|v\| \neq 0] \vee [v = 0]$
2. $(\forall v \in V)(\forall s \in F)\|s \odot v\| = |s|.\|v\|$
3. $(\forall v, w \in V)\|v \oplus w\| \leq \|v\| + \|w\|$

• **Inner product vector space** over a field F is a vector space $V(F)$ over which a function $\langle\rangle :: (V, V) \to \mathbb{C}$ exists for which following statements are true:

1. $(\forall v, w \in V)\ \langle v, w\rangle = \langle w, v\rangle^*$
2. $(\forall u, v, w \in V)(\forall a, b \in F)$ $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$
3. $(\forall v \in V\backslash\{0\})\ \langle v, v\rangle > 0$
4. $\langle 0, 0\rangle = 0$

• **Dual of a vector space** $V(F)$ is a vector space denoted as $V^*(F)$ whose elements are linear functions from the vector space $V(F)$ to the underlying field F.

• **Type (r,s) tensor on a vector space V** is an element of vector space $\underbrace{V \otimes V \otimes \cdots V}_{r} \otimes \underbrace{V^* \otimes V^* \cdots \otimes V^*}_{s}$ where $\otimes$ is an associative bilinear map.

• **Tensor algebra $T(V)$** over a vector space V is the direct sum of all possible (r,s) tensor spaces, with the $\otimes$ being the natural product between different tensors.

• **Multivector** (also called $k-$vector) is an element of the vector space whose elements are constructed via the associative antisymmetric *wedge product* $\wedge$ of the underlying vectors; for instance, $u \wedge v \wedge w$ is a 3-vector if u, v, w are vectors.

• **Exterior algebra $\Lambda(V)$** over a vector space V is the direct sum of all possible multivectors, with the wedge product $\wedge$ being the natural product between different multivectors.

• **Covariant & Contravariant indices** in our conventions refer to *downstairs* and *upstairs* indices of a tensor's components, hence are multiplied with basis vectors of V^* and V to yield the full tensor, e.g. $T = T^{ij}_{\ \ k}\ e_i ? e_j ? e^k$ with $T^{ij}_{\ \ k}$ having one covariant and two contravariant indices where ? is the associative binary operation appropriate to the algebra considered $(\otimes, \wedge, \dots)$.

• **Contraction** is the action of applying a dual vector $(V \to S)$ to a vector (V), hence reducing a $(r,s)-$tensor to a $(r-1, s-1)-$tensor. In an orthonormal basis with $e^i(e_k) = \delta^i_k$ (such as Cartesian coordinates), this amounts to summing over a covariant and a contravariant indices.

• **Manifold** is (for our purposes) any space that resembles $\mathbb{R}^d$ near its every point, for instance the sphere S^2.

• **(Co)tangent space** to a manifold M at a point x is $\mathbb{R}^d$ centered at x and is denoted as T_xM (T^*_xM). The (co)tangent space is inhabited by the (co)vectors at $x \in M$, with the basis vectors usually chosen as $\frac{\partial}{\partial x^i}$ (dx_i).

• **(Co)tangent bundle** is the *disjoint union* of all (co)tangent spaces of a manifold M, and is denoted as TM (T^*M).

• **Musical isomorphism** between a tangent and cotangent bundle is initiated with two functions: $\flat :: TM \to T^*M$ and $\sharp :: T^*M \to TM$, hence for instance $(x^i e_i)^\flat = (x_i e^i)$, and $(x_i e^i)^\sharp = (x^i e_i)$

• **Field in Physicist terminology** broadly refers to any map from a manifold M to *something* $(\mathbb{R}, TM, \dots)$. The field is *named* appropriately depending on the output: scalar field $(M \to \mathbb{R})$, vector field $(M \to TM)$, tensor field $(M \to (TM \otimes TM \otimes T^*M \otimes \cdots))$, and so on.

• **Differential forms** (or forms for short) are functions that takes a point x from a Manifold M and yields a multi(co)vector from the exterior algebra of the (co)tangent space of M at x, e.g. $\omega = (x,y) \to dx + y dy$.

• **Hodge dual of a multivector or a form** α is denoted as $\star\alpha$, and their components in $\mathbb{R}^d$ are related to one another for $\alpha = \alpha_{i_1 \dots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$ and $\star\alpha = (\star\alpha)_{i_{k+1} \dots i_d} e^{i_{k+1}} \wedge \cdots \wedge e^{i_d}$ as

$$(\star\alpha)_{i_{k+1} \dots i_d} = \frac{1}{(d-k)!} \alpha_{i_1 \dots i_k} \epsilon^{i_1 \dots i_k \ell_{k+1} \dots \ell_d} \delta_{i_{k+1} \ell_{k+1}} \cdots \delta_{i_d \ell_d}$$

• **Exterior derivative** takes a $p-$form ω to $p+1$ form $d\omega$; with the basis vectors $\{dx^i\}$, it reads as

$$\omega = \omega_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \tag{24a}$$

$$d\omega = \frac{\partial \omega_{i_1 \dots i_p}}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \tag{24b}$$

• **Gradient** (denoted grad) is a function `Scalar Field` $\to$ `Vector Field`, defined as $\mathsf{grad} = f \to (df)^\sharp$. ∇f is also used as a notation for $\mathsf{grad}(f)$. In Cartesian coordinates,

$$\begin{aligned} \mathsf{grad} =& \big((x_1, \dots, x_d) \to f(x_1, \dots, x_d)\big) \\ & \to \left((x_1, \dots, x_d) \to \frac{\partial f(x_1, \dots, x_d)}{\partial x_i} \hat{x_i}\right) \end{aligned} \tag{25}$$

• **Divergence** (denoted div) is a function `Vector Field` $\to$ `Scalar Field`, defined as $\mathsf{grad} = v \to \big(\star\, d \star v^\flat\big)$. $\nabla \cdot v$ is also used as a notation for $\mathsf{div}(v)$. In Cartesian coordinates,

$$\begin{aligned} \mathsf{div} =& \big((x_1, \dots, x_d) \to v^i(x_1, \dots, x_d) \hat{x_i}\big) \\ & \to \left((x_1, \dots, x_d) \to \frac{\partial v^i(x_1, \dots, x_d)}{\partial x^i}\right) \end{aligned} \tag{26}$$

• **Curl** (denoted curl) is a function `Vector Field` $\to$ $(d-2)-$`Vector Field`, defined as $\mathsf{curl} = v \to \big(\star\, dv^\flat\big)^\sharp$. In $d=3$, $\nabla \times v$ is also used as a notation for $\mathsf{curl}(v)$; in Cartesian coordinates,

$$\begin{aligned} \nabla \times v =& \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) \hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) \hat{y} \\ & + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \hat{z} \end{aligned} \tag{27}$$

• **Laplacian** (for our purposes) is a function `Tensor Field` $\to$ `Tensor Field`, denoted as Δ, and is defined as follows for practical purposes:

$$\begin{aligned} R &:: TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M \\ R &= R^{i_1 \dots i_r}_{\qquad k_1 \dots k_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{k_1} \otimes \cdots \otimes dx^{k_s} \\ \Delta R &:: TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M \\ \Delta R &= \frac{\partial^2 R^{i_1 \dots i_r}_{\qquad k_1 \dots k_s}}{\partial x^m \partial x_m} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{k_1} \otimes \cdots \otimes dx^{k_s} \end{aligned} \tag{28}$$

• **A harmonic tensor field** R (a harmonic function being a special case as a harmonic type $(0,0)$ tensor field) is an element of the kernel of the Laplacian, i.e. $\Delta R = 0$.

• **Helmholtz decomposition of a $3d$ vector field** E is a way of rewriting it in terms of its *scalar potential* Φ (related to the divergence of the vector field) and its *vector potential* V (related to the curl of the vector field): $E = \text{constant } -\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\times V$ where

$$\begin{aligned}\Phi(r) =& \frac{1}{4\pi}\int\limits_{\text{manifold}} \frac{\boldsymbol{\nabla}'\cdot E(r')}{|r-r'|}dV' \\ &- \frac{1}{4\pi}\oint\limits_{\text{boundary}} \frac{\hat{n}'\cdot E(r')}{|r-r'|}dS' \end{aligned}\tag{29a}$$

$$\begin{aligned}V(r) =& \frac{1}{4\pi}\int\limits_{\text{manifold}} \frac{\boldsymbol{\nabla}'\times E(r')}{|r-r'|}dV' \\ &- \frac{1}{4\pi}\oint\limits_{\text{boundary}} \frac{\hat{n}'\times E(r')}{|r-r'|}dS' \end{aligned}\tag{29b}$$

• **Arc-length** is the length of a curve (denoted by s), which satisfies $s = \int\limits_{t_0}^{t}\left|\frac{d\mathbf{x}(t')}{dt'}\right|dt'$. In this equation, $\mathbf{x}(t)$ is the position of a point on the curve, t is the parametrization parameter, and t_0 is the value of t at the starting point of the curve. The arc-length itself can be used to parametrize the curve.

• **Tangent vector to a curve** in the arc-length parametrization is the function $\mathbf{t}(s) = \frac{d\mathbf{x}(s)}{ds}$. It has unit norm, and can be likened to the ratio velocity per speed.

• **Curvature of a curve** κ is a function of the arc-length whose value is $\kappa(s) = \left|\frac{d\mathbf{t}(s)}{ds}\right|$.

• **Principle normal of a curve** $\mathbf{n}$ is a function of the arc-length whose value is $\mathbf{n}(s) = \frac{1}{\kappa(s)}\frac{d\mathbf{t}(s)}{ds}$. It has unit norm, and can be likened to the acceleration unit vector.

• **Binormal vector of a curve** $\mathbf{b}$ is a function of the arc-length whose value is $\mathbf{b}(s) = \mathbf{t}(s)\times\mathbf{n}(s)$ ($|\mathbf{b}(s)| = 1$).

• **Torsion of a curve** τ is a function of the arc-length whose value is $\tau(s) = -\mathbf{n}(s)\cdot\frac{d\mathbf{b}(s)}{ds}$.

• **The Frenet-Serret equations** is a closed system of equations which completely determine the properties of a curve as a function of the curvature and torsion functions. They read as

$$\begin{aligned}\frac{d\mathbf{t}(s)}{ds} = \kappa(s)\mathbf{n}(s)\ , \quad \frac{d\mathbf{b}(s)}{ds} = -\tau(s)\mathbf{n}(s)\ ,\\ \frac{d\mathbf{n}(s)}{ds} = \tau(s)\mathbf{b}(s) - \kappa(s)\mathbf{t}(s)\end{aligned}\tag{30}$$

• **Generalized Stokes theorem** equates the integration of a $p-$form ω over the boundary of a manifold ∂M to the integration of the exterior derivative of the $p-$form $d\omega$ over the manifold M: $\int_{\partial M}\omega = \int_M d\omega$.

• **Integral theorems** are special cases of the generalized Stokes theorem. For a volume $\mathbf{V}\in\mathbb{R}^3$, a surface $\mathbf{S}\in\mathbb{R}^3$, a curve $\gamma\in\mathbb{R}^3$, and a region $\mathbf{D}\in\mathbb{R}^2$ (and for the notation ∂A being boundary of A), we have

$$\begin{aligned}&\int\limits_{\mathbf{V}}\boldsymbol{\nabla}\cdot F dV = \oint\limits_{\partial\mathbf{V}} F\cdot dS\ , \quad \int\limits_{\mathbf{S}}\boldsymbol{\nabla}\times F\cdot dS = \oint\limits_{\partial\mathbf{S}} F\cdot d\Gamma\\ &\int\limits_{\gamma}\boldsymbol{\nabla}f\cdot dr = f\Big|_{\text{initial}}^{\text{final}}\ , \quad \int\limits_{\mathbf{D}}\left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y}\right)dxdy\\ &\qquad = \oint\limits_{\partial\mathbf{D}}\left(L(x,y)dx + M(x,y)dy\right)\end{aligned}\tag{31}$$

• **Polar coordinates in $\mathbb{R}^d$** $(r,\theta_1,\ldots,\theta_{d-1})$ can be defined in terms of the Cartesian coordinates $(x_1,\ldots,x_d)$ as

$$x_1 = r\cos(\theta_1)\ , \quad x_d = x_{d-1}\tan(\theta_{d-1})\tag{32a}$$
$$x_i = x_{i-1}\tan(\theta_{i-1})\cos(\theta_i) \quad \text{for } 1 < i < d\tag{32b}$$

In two-dimensions, this reduces to the familiar polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$; in 3 ($>3$) dimensions, it is also called *(hyper)spherical coordinates.*

• **Cylindrical coordinates in $\mathbb{R}^d$** $(\rho,\theta_1,\ldots,\theta_{n-1},x_n,x_{n+1},\ldots,x_d)$ is a coordinate system such that a subset $\mathbb{R}^n$ of the total space $\mathbb{R}^d$ (for $n<d$) is converted into the polar coordinates. For instance, if we convert $\mathbb{R}^2$ of $\mathbb{R}^3$ into polar coordinates, we obtain the familiar *3d cylindrical coordinates*, i.e. $(x,y,z) = (\rho\cos\theta, \rho\sin\theta, z)$.

• **(Anti)holomorphic function** of a complex variable is a function f for which the derivative with respect to z ($\bar{z}$) is uniquely defined, i.e.

$$\begin{aligned}\frac{df(x,y)}{dz} &:= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x + i\Delta y}\\ \left(\frac{df(x,y)}{d\bar{z}}\right. &:= \left.\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x - i\Delta y}\right)\end{aligned}\tag{33}$$

is well-defined and independent of the order of limits (this condition leads to *Cauchy-Riemann equations*). As any antiholomorphic function can be written as *complex conjugate of a holomorphic function*, one usually focuses on the analysis of holomorphic functions alone.

• **An analytic function** is a function that can be expanded as a convergent power series. Cauchy's integral formula ensures that *a complex analytic function* (with a series expansion in z) is equivalent to a *holomorphic function* and visa versa.

• **Cauchy's integral formula** for a function f complex-analytic in the region $D\subset\mathbb{C}$ can be written as $f(z) = \oint\limits_{\partial D}\frac{f(\omega)}{\omega - z}\frac{d\omega}{2\pi i}$.

• **Laurent series** of a function complex analytic for $R_1 < |z-a| < R_2$ is the convergent series expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$.

• **A pole of a complex analytic function** f is a point $a \in \mathbb{C}$ such that $f(a)$ is singular and S is a non-empty set for $S = \{m \in \mathbb{Z} \mid (z-a)^m f(z) \text{ is analytic at } a\}$. $\min(S)$ is called the order of the pole.

• **A zero of a function** f is the value a such that $f(a) = 0$. $\min\{m \in \mathbb{Z} \mid \lim_{z \to a} (z-a)^{-m} f(z) \neq 0\}$ is called the order of the zero.

• **A meromorphic function** f in a domain D is a holomorphic function in D except a set of points at which f has a pole. For example, $z \to \frac{1}{\sin(z)}$, $z \to \frac{e^z}{z}$ are meromorphic functions in $D = \mathbb{C}$. If we also include infinity ($D = \mathbb{C} \cup \{\infty\}$), they are no longer meromorphic as they are singular at infinity but that singularity is not a pole. In fact, *the only meromorphic functions in the Riemann sphere (i.e. $D = \mathbb{C} \cup \{\infty\}$) are rational functions*, e.g. $z \to \frac{(z-1)(z+i)}{(2z+\pi)(z-i+1)}$.

• **Residue** of a complex function at an isolated singularity a is defined as

$$\text{Res} :: (\mathbb{C} \to \mathbb{C}\ ,\ \mathbb{C}) \to \mathbb{C} \tag{34a}$$

$$\text{Res}(f, a) = \frac{1}{2\pi i} \oint_{C_a} f(z) dz \tag{34b}$$

for an infinitesimal closed contour C_a centered at a.

• **Cauchy's principal value** (denoted p.v.) is for our purposes defined via the relation

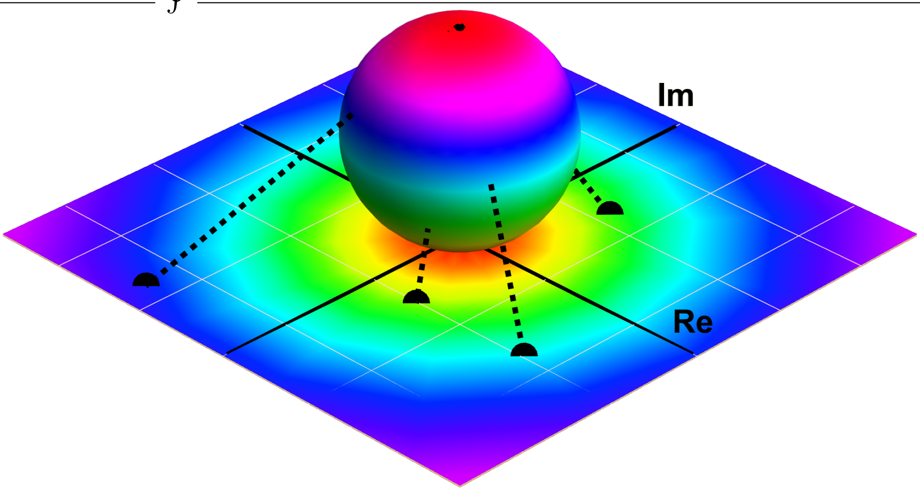
$$\text{p.v.} \int_a^c f(x) dx = \lim_{\epsilon \to 0} \left[\int_a^{b-\epsilon} f(x) dx + \int_{b+\epsilon}^{c} f(x) dx \right] \tag{35}$$

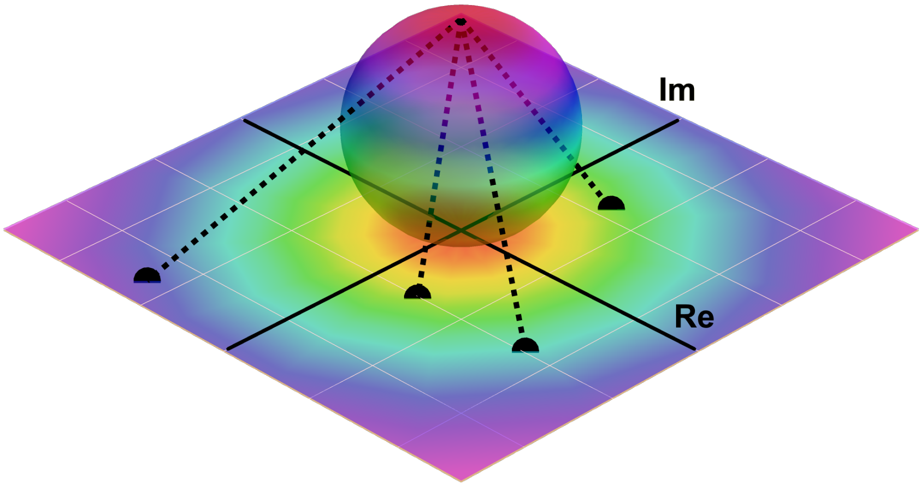
for $a < b < c$, where $f(x)$ is assumed to be analytic in $[a, c] \backslash \{b\}$. If $f(x)$ is analytic at b, the principle value gives the same result with the ordinary integral; on the other hand, if $f(x)$ is not analytic at b, the principle value assigns a well-defined value to the integral which would be otherwise ill-defined as a function.

• **Conformal transformation** (for our purposes) is any mapping $x \to x'$ of the coordinates for which the angles between (co)tangent vectors do not change, e.g. $\frac{\langle dx, dy \rangle}{\sqrt{|dx||dy|}} = \frac{\langle dx', dy' \rangle}{\sqrt{|dx'||dy'|}}$; for instance, translation $x' = x + a$, rotation $x' = e^{i\theta} x$ or scaling $x' = \lambda x$ are so.

$\oint$

• **Riemann sphere** (denoted $\widehat{\mathbb{C}}$) is the *compactification* of the complex plane $\mathbb{C}$. More simply, it is the inclusion of *infinity* as a single point to the complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, such that we get a complete symmetry between large numbers and small numbers: large numbers are points close to the *north pole* (by convention), whereas small numbers are points close to the *south pole*; the map $z \to \frac{1}{z}$ sends such numbers to each other and switches north and south poles (hence $\infty \leftrightarrow 0$).





• **Stereographic projection** is a conformal mapping between $\mathbb{R}^d$ ($d-$dimensional plane) and S^d ($d-$sphere); however, we are only interested in the map between $\mathbb{R}^2$ (the complex plane) and S^2 (the Riemann sphere). Geometrically, the mapping can be readily applied as follows: **(1)** embed S^2 and $\mathbb{R}^2$ into $\mathbb{R}^3$ such that the origin of $\mathbb{R}^2$ and south pole of S^2 coincide; **(2)** draw a line from north pole to a point $z \in \mathbb{R}^2$; **(3)** the intersection of the line with S^2 gets mapped to z.

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2 Fill-in the blanks

Each correct answer is worth 0.75 *point.*

Question: 1 . . . . *(9 points)*

To extend the tools of calculus from functions with the domain $\mathbb{R}$ to functions with more general domains, we need to establish a few notations, which is indeed what we have done immediately after discussing the general properties of linear spaces and linear algebras. We could have avoided the terminology and simply have focused on explicit domains such as $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{C}$, but these are merely examples of a ___*manifold*___ which we tentatively define as *a space that resembles a Euclidean space near its each point.* A nontrivial example for such a space is the ordinary *sphere* (denoted S^2), which without a doubt looks like a Euclidean space locally (hence the flat-Earth believers). If we consider the *tangent space* at each point of this sphere, their disjoint union gives us the ___*tangent bundle*___ which is schematically the codomain of a *vector field*, as any vector at any point of the sphere belongs to it.

Differentiation of vector fields come in different flavors: for instance, the operation ___*divergence*___ converts the vector field into a scalar field while differentiating it, whereas ___*curl*___ keeps a $3d$ vector field as a vector field. However, the latter preserves a vector field only in three dimensions; the operation ___*laplacian*___ on the other hand takes a vector field in any dimension to another vector field albeit being a second-order differential operation.

All of the differential operators mentioned above can be constructed in a coordinate-system independent way, making them manifestly observer-covariant. To achieve this, one needs the tools of the differential geometry, i.e. the exterior derivative and the hodge duality. For example, the gradient operation acting on a scalar field f is written down as $(df)^\sharp$, where $\sharp$ utilizes the correspondence between tangent and cotangent spaces (a correspondence also called ___*musical isomorphism*___).

With the differentiation taken care of, we next turned to the integration: we discussed in class that higher order integration is actually fundamentally different from the one-dimensional integration as *indefinite integration*, *signed definite integration*, and *unsigned definite integration* start to diverge, i.e. fundamental theorem of calculus is no longer applicable. Of these three, we focused on ***signed definite integration*** which is essentially pairings between *integration surfaces* and *forms*. The simplest such integration is the *line integration in* $\mathbb{R}^d$: explicit computation of such an integration would require *curve parametrization* which is efficiently done by using arc-length and the intrinsic properties of the curve (called *curvature* and ___*torsion*___). These two scalar functions are sufficient to determine the tangent vector to the curve via the coupled differential equations called ***Frenet-Serret equations***.

Although we have the sufficient tools to carry out explicit computations most of the time, we should *delay* it as much as possible, seeking ways to reduce the complexity of the computation first. One way to achieve this is to choose *the correct coordinate system.* For instance, to model the attraction between two Hydrogen atoms, we would rather work with ***spherical/polar coordinates*** where one coordinate is the distance between the atoms and the other coordinate(s) are simply angles. Such curvilinear coordinate transformations become

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rather straightforward when we use partial derivatives as the unit vectors (i.e. $\vec{v} = x^i \frac{\partial}{\partial x^i}$) since the coordinate transformation becomes a mere application of the ***chain rule***, i.e. $\frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}$. Similarly, we can convert *integration measures* between different coordinate systems; indeed, the volume measure $dxdydz$ in Cartesian coordinates becomes $rdrdzd\theta$ in ***cylindrical coordinates***.

3 Choose the correct option

You do not need to show your derivation in this part.
Incorrect answer for a question of X point is worth $-X/4$ points: this ensures that the randomly given answer has an expectation value of 0 point.

Question: 2 . . . . *(3 points)*

Consider a curve $\gamma \in \mathbb{R}^2$ such that any point on it has the position vector $\mathbf{x} :: \mathbb{R}^2 \to T\mathbb{R}^2$ for $\mathbf{x}(x, y) = (x + y\sqrt{1-x^2})\mathbf{i} + (\sqrt{1-x^2} - xy)\mathbf{j}$ with the restrictions $x = \cos(y)$ and $\pi \geq y \geq 0$.

(a) **(1 point)** What is the total length of this curve?

■ $\frac{\pi^2}{2}$ ☐ $\frac{\pi^3}{3}$ ☐ $\frac{\pi^4}{4}$ ☐ $\frac{\pi^5}{5}$ ☐ None

(b) **(1 point)** What is the position vector $\mathbf{x}$ at $s = 1/2$ in the arc-length parametrization?

☐ $\mathbf{x} = \cos(1)(\mathbf{i}+\mathbf{j}) + \sin(1)(\mathbf{i}+\mathbf{j})$
■ $\mathbf{x} = \cos(1)(\mathbf{i}-\mathbf{j}) + \sin(1)(\mathbf{i}+\mathbf{j})$
☐ $\mathbf{x} = \cos(1)(\mathbf{i}+\mathbf{j}) + \sin(1)(\mathbf{i}-\mathbf{j})$
☐ $\mathbf{x} = \cos(1)(\mathbf{i}-\mathbf{j}) + \sin(1)(\mathbf{i}-\mathbf{j})$
☐ None

(c) **(1 point)** What is the curvature function $\kappa(s)$ in the arc-length parametrization?

☐ $\kappa(s) = \frac{1}{2s}$ ☐ $\kappa(s) = \frac{1}{s}$ ☐ $\kappa(s) = s$ ☐ $\kappa(s) = 2s$ ■ **None**

Question: 3 . . . . *(3 points)*

Consider the function $f :: \mathbb{R}^3 \to \mathbb{R}$ whose value in polar coordinates reads as $f(r, \theta_1, \theta_2) = \frac{\cot(\theta_1) + r\sin(\theta_1)\sin^2(\theta_2)}{r^2\sin^2(\theta_1)\cos^3(\theta_2) + \sin(\theta_2)}$.

(a) **(1 point)** What is the value of this function at the Cartesian point $(x, y, z) = (0, 1, 0)$?

☐ -1 ☐ 0 ■ 1 ☐ 2 ☐ None

(b) **(1 point)** What is the value of this function at the Cartesian point $(x, y, z) = (0, 1, 1)$?

☐ -1 ☐ 0 ☐ 1 ■ 2 ☐ None

(c) **(1 point)** What is the value of this function at the Cartesian point $(x, y, z) = (1, 0, 0)$?

☐ -1 ■ 0 ☐ 1 ☐ 2 ☐ None

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Question: 4 . . . . *(2 points)*

Consider the function $f :: \mathbb{R}^3 \to T\mathbb{R}^3$ whose value in polar coordinates reads as $f(r, \theta_1, \theta_2) = r \sin^2(\theta_1)\cos^2(\theta_2)\frac{\partial}{\partial r} + \sin(\theta_1)\cos(\theta_1)\cos^2(\theta_2)\frac{\partial}{\partial \theta_1} - \sin(\theta_2)\cos(\theta_2)\frac{\partial}{\partial \theta_2}$.

(a) **(1 point)** What is the value of this function at the Cartesian coordinates?

■ $x\frac{\partial}{\partial x}$ ☐ $y\frac{\partial}{\partial x}$ ☐ $z\frac{\partial}{\partial x}$ ☐ $x\frac{\partial}{\partial y}$ ☐ None

(b) **(1 point)** What is the projection of this vector field along the radial direction at the equator of the unit sphere for $\theta_2 = \pi/4$?

☐ $-\frac{1}{2}$ ☐ $-\frac{1}{\sqrt{2}}$ ☐ $\frac{1}{\sqrt{2}}$ ■ $\frac{1}{2}$ ☐ None

Question: 5 . . . . *(1 points)*

Consider the bivector field $A(x, y, z) = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ given in Cartesian coordinates. Which of the following objects (given in polar coordinates) can coincide with this function on a unit sphere?

☐ $\frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \theta_1}$ ☐ $\cot(\theta_1)\frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \theta_1}$ ☐ $\frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \theta_2}$ ☐ $\cot(\theta_2)\frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \theta_2}$ ■ **None**

Question: 6 . . . . *(4 points)*

Compute the Fourier coefficient $\hat{f}(k)$ for the function

(a) **(1 point)** $f(x) = (1 + x^2)^{-1}$ at $k = \log(1)$.

☐ $\frac{1}{2}$ ☐ $\frac{1}{\pi}$ ☐ 2 ■ π ☐ None

(b) **(1½ points)** $f(x) = \cos(x)(1 + x^2)^{-1}$ at $k = 0$.

☐ $e^1\pi^1$ ☐ $e^1\pi^{-1}$ ■ $e^{-1}\pi^1$ ☐ $e^{-1}\pi^{-1}$ ☐ None

(c) **(1½ points)** $f(x) = x(1 + x^2)^{-1}$ at $k = -\log(2)$.

■ $\frac{i\pi}{2}$ ☐ $\frac{\pi}{2}$ ☐ $i\pi$ ☐ π ☐ None

Bonus Question: 7 . . . . *(2 points)*

Which `Mathematica` code below would determine the branch points of $f = x \to \log\left(\sqrt{1-x}\right)$?

☐ `BranchPoints[Log[Sqrt[1 - x]], x]`

■ `ComplexAnalysis‘BranchPoints[Log[Sqrt[1 - x]], x]`

☐ `Log[Sqrt[1 - x]] // FindBranchPoints`

☐ `BranchPoints[Log[Sqrt[1 - x]]]`

☐ None

« « « Congratulations, you have made it to the end! » » »