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Moduli Space of Superconformal Field Theories

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Outline

Why CFT?

- CFT is one of the essential toolbox in the current research on condensed matter and string theory, giving implications on the classifications of matter and consistent compactifications.
- The construction of CFT's gives an interplay between abstract mathematics and physics, borrowing ideas from complex analysis, algebra, algebraic geometry and find implications on physics.
- Bootstrap approach gives us a way to understand the physical phenomena without knowing the exact Lagrangian of the system, which gives more insights on the phenomenological results.

What CFT gave us?

- In string theory, the worldsheet theory has the conformal symmetry, and the anomaly free CFT gives us the correct number of spacetime dimensionality.
- In string theory compactifications, most of the compactificaitons give rise to a CFT in lower dimensions, and a new kind of symmetry emerges for specific compactifications.
- It has important conclusions on the critical phenomena and gives the exact solution for the Ising model in some specific dimensions.

For $2d$ CFT, condition $T^{\mu}{}_{\mu} = 0$ implies the conformal invariance. On a curved space, this trace is related with the central charge and the Ricci scalar, $T^{\mu}{}_{\mu} = \frac{c}{2}R$.

When we parameterize with the complex coordinates, we get the OPE of $T(z)$ and $T(w)$:

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \dots \quad (1)$$

The Laurent expansion of the field $T(z)$ is given by:

$$T(z) = \sum_{n=0}^{\infty} z^{-n-2} L_n. \tag{2}$$

By using the OPE of the energy momentum tensor, one can find the following algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \tag{3}$$

which is known as the Virasoro algebra. Notice that $n \in \mathbb{Z}$, which means that there are infinitely many elements of algebra generating conformal transformations, and associated to this, we have infinitely many charges that is conserved.

ODTÜ METU

Consider the following action for a scalar field:

$$S = \frac{1}{4\pi\kappa} \int dz d\bar{z} (\partial X \cdot \bar{\partial} X), \tag{4}$$

satisfying the equation of motion:

$$\partial \bar{\partial} X = 0, \tag{5}$$

from which, we can infer:

$$\begin{aligned} j(z) =& i\partial X(z, \bar{z}), \\ \bar{j}(\bar{z}) =& i\bar{\partial} X(z, \bar{z}). \end{aligned} \tag{6}$$

Notice that the first field is chiral, and the second field is an anti-chiral field.

The Green's function associated to this is given by:

$$K(z,\bar{z},w,\bar{w}) = \langle X(z,\bar{z})X(w,\bar{w})\rangle = -\kappa \log|z-w|^2. \tag{7}$$

By taking derivatives with respect to z and w, we can find the following:

$$\langle j(z)j(w)\rangle = \frac{\kappa}{(z-w)^2}. \tag{8}$$

Knowing that $j(z)$ is a primary field with conformal dimension 1, we can Laurent expand it as $j(z) = \sum_{n=0}^{\infty} z^{-n-1} j_n$. Using the OPE above, we can find the following current algebra:

$$[j_m, j_n] = \kappa m \delta_{m+n,0}. \tag{9}$$

Energy Momentum Tensor

The energy momentum tensor can be found by varying the action with respect to the metric. From this, we find

$$T(z) = \frac{1}{2\kappa} N(jj)(z). \tag{10}$$

The following commutation will give us the central charge:

$$\langle 0 | L_{+2} L_{-2} | 0 \rangle = \frac{c}{2}. \tag{11}$$

Using the algebra of the current j and the relation between $T(z)$ and $j(z)$, one can find that the CFT of free boson has $c = 1$.

$$S = \frac{1}{4\pi\kappa} \int dz d\bar{z} \left(\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} \right), \tag{12}$$

with the equations of motion:

$$\partial \bar{\psi} = \bar{\partial} \psi = 0. \tag{13}$$

We see a chiral nature of the fields. Upon investigation, one can find the conformal weight of the associated fields as $[\psi] = (h, \bar{h}) = (\frac{1}{2}, 0)$ and $[\bar{\psi}] = (h, \bar{h}) = (0, \frac{1}{2})$

Due to the fermionic nature of the fields, one can write two different periodicity conditions for the fields:

$$\begin{aligned} &\psi(e^{2\pi i} z) = +\psi(z) && \text{Neveu-Schwarz Sector (NS).} \\ &\psi(e^{2\pi i} z) = -\psi(z) && \text{Ramond Sector (R).} \end{aligned} \tag{14}$$

The Green's function for the fields is given by:

$$\psi(z)\psi(w) = \frac{\kappa}{z-w} + ... \tag{15}$$

as we know that ψ is a conformal field with weight $(h, \bar{h}) = (\frac{1}{2}, 0)$, we can Laurent expand this function as $\psi(z) = \sum_r z^{-r-\frac{1}{2}}\psi_r$, with $r \in \mathbb{Z}$ corresponds to Ramond sector, $r \in \mathbb{Z} + \frac{1}{2}$ is NS sector. One can find the algebra of the ψ_r as:

$$\{\psi_r, \psi_s\} = \kappa\delta_{r+s,0}. \tag{16}$$

By the same approach, one can find the energy momentum tensor of the corresponding action as:

$$T(z) = N(\psi\partial\psi), \tag{17}$$

and by taking the relevant expansions and commutations, we get the following result:

$$[L_m, \psi_r] = \left(-m - \frac{r}{2}\right)\psi_{m+r}. \tag{18}$$

By looking at the commutation $\langle 0|L_2 L_{-2}|0\rangle = \frac{c}{2}$, we get the central charge of the free fermion CFT as $c = \frac{1}{2}$.

Extension to supersymmetric conformal field theory, one needs to include a primary field, namely $G(z)$, constructed from $j(z)$ and $\psi(z)$ as

$$G(z) = N(j\psi)(z). \tag{19}$$

Observe that the conformal weight of this field is $3/2$, and it satisfies the following OPEs:

$$\begin{aligned} T(z)G(w) &= \frac{3/2}{(z-w)^2}G(w) + \frac{\partial_w G(w)}{z-w} + \ldots, \\ G(z)G(w) &= \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w} + \ldots. \end{aligned} \tag{20}$$

To construct $\mathcal{N} = 2$ SCFT, one needs to introduce two partners of operators, bosonic part of these will be $T(z), J(z)$ where $J(z)$ is a $U(1)$ current, and to fermionic partners will be $G^1(z)$ and $G^2(z)$. Out of these fermionic operators, the following two operators defined, that will make OPE's, or the algebra, more neat:

$$\begin{aligned} G^+(z) &= \frac{1}{\sqrt{2}}(G^1(z) + iG^2(z)), \\ G^-(z) &= \frac{1}{\sqrt{2}}(G^1(z) - iG^2(z)). \end{aligned} \tag{21}$$

The following list gives the full algebra of the $\mathcal{N} = 2$ SCFT:

$$
\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \ldots, \\
T(z)G^{\pm}(w) &= \frac{3/2}{(z-w)^2} G^{\pm}(w) + \frac{\partial_w G^{\pm}(w)}{(z-w)} + \ldots, \\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)} + \ldots, \\
G^{+}(z)G^{-}(w) &= \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial_w J(w)}{z-w} + \ldots, \\
J(z)G^{\pm}(w) &= \pm\frac{G^{\pm}}{z-w} + \ldots, \qquad J(z)J(w) = \frac{c/3}{(z-w)^2} + \ldots.
\end{aligned}
\tag{22}
$$

In 2d CFT for operator with conformal weight $(h, \bar{h})$:

- irrelevant if $h + \bar{h} > 2$
- relevant if $h + \bar{h} < 2$
- marginal if $h + \bar{h} = 2$

Terminology arises from acting these operators and letting RG flow act on the theory and drive it towards IR.

- Irrelevant operators will have no effect on the theory, RG drives its coefficient to zero. They are suppressed.
- Relevant operators can be dominant and it can make the theory trivial in the IR limit.

Our aim is to understand the marginal operators and their effect on the theories.

ODTÜ METU

Consider the case of a free boson on a circle of radius R_0

$$S_{R_0} = \int d^2 z \partial X \bar{\partial} X, \tag{23}$$

where $X \sim X + 2\pi R_0$. Consider the operator $\mathcal{O} = \partial X \bar{\partial} X$ and deform the theory by it:

$$S_{R0} \to S_R = S_{R0} + \epsilon \int d^2 z \mathcal{O}(z, \bar{z}) = (1+\epsilon) \int d^2 z \partial X \bar{\partial} X. \tag{24}$$

By redefining $\tilde{X} = \sqrt{1+\epsilon} X$, the periodicity conditions becomes $\tilde{X} \sim \tilde{X} + 2\pi R_0(\sqrt{1+\epsilon})$, and the action becomes:

$$S_R = \int d^2 z \partial X \bar{\partial} X. \tag{25}$$

- The marginal operator changed the radius on which the scalar field target space. So, the family of CFTs defined on a circle can be parameterized by the radius of the circle.
- As we know the partition function is invariant under the redefinition of $R \to 1/R$, and the moduli space of the free boson on a circle is restricted to values $(1, \infty)$. This is the known T-duality.
- Not all $(1, 1)$ operators can be used to deform theory. After perturbation, they may remain as $(1, 1)$ operators. Such operators may not be used to move around the moduli space as they spoil the conformal symmetry. These are called *truly marginal* operators.

Define the following operators, where ϕ is a primary field with conformal dimension $(\frac{1}{2}, \frac{1}{2})$:

$$
\begin{aligned}
\hat{\phi}(w,\bar{w}) &= \oint dz G^{-}(z)\phi(w,\bar{w}), \\
\Phi_{(1,1)}(w,\bar{w}) &= \oint d\bar{z}\bar{G}^{-}(\bar{z})\hat{\phi}(w,\bar{w}), \\
\tilde{\phi}(w,\bar{w}) &= \oint dz\bar{G}^{-}(\bar{z})\phi(w,\bar{w}), \\
\Phi_{(-1,1)}(w,\bar{w}) &= \oint d\bar{z}G^{+}(z)\tilde{\phi}(w,\bar{w}).
\end{aligned} \tag{26}
$$

These are truly marginal operators for $\mathcal{N}=2$ SCFT.

Locally, the CFT Zamolodchikov metric is block diagonal between the operators $\Phi_{(1,1)}, \Phi_{(-1,1)}$. When we consider a Calabi-Yau target space, these will have geometrical implications.

- These fields are also elements of rings, from which we can identify them as differential forms of $(2,1)$ and $(1,1)$.

$$\Phi_{(-1,1)} \sim b_{i\bar{j}}\lambda^i\psi^{\bar{j}}, \quad (27)$$

with $b_{i\bar{j}}$ is a $(1,1)$ harmonic form. To the lowest order when we consider the map between rings, this operator takes the form $b_{i\bar{j}}\partial X\bar{\partial}X^{\bar{j}}$ which is nothing but the Kähler form, and changes the size of the Kähler structure, 'size' of the Calabi-Yau.

- $\Phi_{(1,1)}$ operators give pure-index type metric perturbations, thus deforming the 'shape' of the target space manifold, meaning the complex structure of the Calabi-Yau, while preserving the Calabi-Yau conditions, a direct manifestation on the conformal invariance.

Thus, we have two types of operators that can deform $\mathcal{N} = 2$ SCFT to a nearby theory by changing the geometry of the associated Calabi-Yau manifold, without spoiling it.

Conclusion

- CFT constraints gives us correct form of compactifications on a Calabi-Yau manifolds.
- Marginal operators gives us different theories, meaning that they move us to different points in the moduli space of theories.
- Upon further restriction from dualities, the moduli space is reduced to a certain degree.

- Can bootstrap method find other consistency conditions on Calabi-Yau compactifications and remove some part of moduli space? Do these results will be consistent with the Swampland Program?
- Can bootstrap method find some other CFT's that can be defined on 6-dimensions and associated geometry on which string theory can be compactified?
- Can bootstrap method find associated phenomenological results (cosmology) such that we can predict the parent theory and the space on which parent theory is compactified?