Name:	
Student ID:	

Make-up Examination

Phys210: Mathematical Methods in Physics II

2024/06/11

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) any of my work *outside the designated areas* in the "fill-in the blank questions" will not be graded;

(2) I take *full responsibility* for any ambiguity in my selection of the correct option in "multiple choice questions";

(3) any of my work *outside the answer boxes* in the "classical questions" will not be graded;

(4) any page which does not contain *both my name and student id* will not be graded;

(5) any extra sheet that I may use are for my own calculations and will *not* be graded.

Signature: ______________________

This exam has a total of 3 questions, some of which are for bonus points. You can obtain a maximum grade of 22+2 from this examination.

Question	Points	Score
1	18	
2	4	

Question	Points	Score
3	0	
Total:	22	

Name:	
Student ID:	

Question: 1 .... *(18 points)*

Find a non-trivial solution of the partial differential equation

$$\left(\frac{\partial^4}{\partial x \partial y^3} - x^2 \frac{\partial^2}{\partial x \partial y} - 2x \frac{\partial}{\partial y}\right) f(x, y) = 0 \tag{1}$$

which goes to zero as $y \to \infty$.

Hint: use Fourier transform to convert this to an ordinary differential equation (derivatives only with respect to a single independent variable), solve it there, and use residue theorem to compute a solution for $f(x, y)$.

Solution: Let us choose the following Fourier transform conventions:

$$\begin{aligned} f(x,y) &= \int_{-\infty}^{\infty} g(x,k) e^{iky} \frac{dk}{2\pi} \\ g(x,k) &= \int_{-\infty}^{\infty} f(x,y) e^{-iky} dy \end{aligned} \tag{2}$$

The differential equation then becomes:

$$\int_{-\infty}^{\infty} \left((ik)^3 \frac{\partial}{\partial x} - x^2 (ik) \frac{\partial}{\partial x} - 2x(ik) \right) g(x,k) e^{iky} \frac{dk}{2\pi} = 0 \tag{3}$$

hence

$$\left((k^2 + x^2) \frac{\partial}{\partial x} + 2x \right) g(x,k) = 0 \tag{4}$$

which can be rewritten as

$$d \log(g(x,k)) = -\frac{2x dx}{x^2 + k^2} \tag{5}$$

while taking k constant, thus

$$\log(g(x,k)) = \log\left(\frac{c(k)}{x^2 + k^2}\right) \tag{6}$$

for the integration constant $c(k)$ (note that it is constant with respect to x). Therefore, we arrive at

$$f(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c(k) e^{iky}}{x^2 + k^2} dk \tag{7}$$

Here, c is any function as long as the Fourier transform is defined. Since we are asked to find *a* solution, let us stick to the simplest option and take $c(k) = 1$.

Name:	
Student ID:	

For $y > 0$, the integrand dies exponentially fast in an integration arc at the upper half plane (UHP) of complex k plane, therefore we can immediately say

$$\int_{-\infty}^{\infty} \frac{e^{iky}}{x^2+k^2} dk = 2\pi i \sum_{\text{residues in UHP}} \frac{e^{iky}}{x^2+k^2} \qquad \text{for } y > 0 \tag{8}$$

As $\dfrac{e^{iky}}{x^2+k^2} = \dfrac{e^{iky}}{(k+ix)(k-ix)} = \dfrac{e^{iky}}{2ix}\dfrac{1}{k-ix} - \dfrac{e^{iky}}{2ix}\dfrac{1}{k+ix}$, we immediately find

$$\int_{-\infty}^{\infty} \frac{e^{iky}}{x^2+k^2} dk = (2\pi i) \times \begin{cases} \left.\dfrac{e^{iky}}{2ix}\right|_{k=ix} & x > 0 \\ -\left.\dfrac{e^{iky}}{2ix}\right|_{k=-ix} & x < 0 \end{cases} \qquad \text{for } y > 0 \tag{9}$$

hence

$$\int_{-\infty}^{\infty} \frac{e^{iky}}{x^2+k^2} dk = \pi \frac{e^{-|x|y}}{|x|} \qquad \text{for } y > 0 \tag{10}$$

In comparison, if $y < 0$, we close the contour in the lower half plane (LHP), therefore

$$\int_{-\infty}^{\infty} \frac{e^{iky}}{x^2+k^2} dk = -2\pi i \sum_{\text{residues in LHP}} \frac{e^{iky}}{x^2+k^2} \qquad \text{for } y < 0 \tag{11}$$

where minus sign comes because now the integration is clockwise. We then immediately write

$$\int_{-\infty}^{\infty} \frac{e^{iky}}{x^2+k^2} dk = -(2\pi i) \times \begin{cases} -\left.\dfrac{e^{iky}}{2ix}\right|_{k=-ix} & x > 0 \\ \left.\dfrac{e^{iky}}{2ix}\right|_{k=ix} & x < 0 \end{cases} \qquad \text{for } y > 0 \tag{12}$$

hence

$$\int_{-\infty}^{\infty} \frac{e^{iky}}{x^2+k^2} dk = \pi \frac{e^{|x|y}}{|x|} \qquad \text{for } y < 0 \tag{13}$$

Combining equations (7,10,13), we end up with

$$\boxed{f(x,y) = a\frac{e^{-|xy|}}{|x|} \qquad \text{for an arbitrary constant } a} \tag{14}$$

Question: 2 . . . . *(4 points)*

Discuss the other solutions besides the solution you have found above.

Name:	
Student ID:	

Solution: We arrived at an exponentially decreasing solution because we assumed the Fourier transform exists and this leaded to the absolute values in the answer. However, the differential operator

$$\mathcal{D}(x,y) = \frac{\partial^4}{\partial x \partial y^3} - x^2 \frac{\partial^2}{\partial x \partial y} - 2x \frac{\partial}{\partial y} \tag{15}$$

is actually separately invariant under $x \leftrightarrow -x$ and $y \leftrightarrow -y$ upto an overall sign, i.e.

$$\mathcal{D}(-x,y) = \mathcal{D}(x,-y) = -\mathcal{D}(x,y) \tag{16}$$

Therefore, if we find a solution $s(x,y)$ to $\mathcal{D}(x,y)s(x,y) = 0$, we immediately see that $s(-x,y)$ and $s(x,-y)$ are also solutions; therefore, we can immediately write down

$$f(x,y) = c_1 \frac{e^{-xy}}{x} + c_2 \frac{e^{xy}}{x} \tag{17}$$

where only a single combination vanishes at infinity (hence satisfies the condition for the existence of Fourier transform).

Since the differential equation is of degree four, we could expect other solutions to exist as well. By inspection, it is clear that any $y-$independent function is a solution too:

$$f(x,y) = c_0(x) + c_1 \frac{e^{-xy}}{x} + c_2 \frac{e^{xy}}{x} \tag{18}$$

which we could not derive as its $y-$Fourier transform is not functionally defined. We could also observe that the differential operator is scale invariant ($x \to \lambda x$) if the action of the first term vanishes, hence we could try solutions of the forms $x^n y$ and $x^m y^2$. Indeed, this leads to the form

$$f(x,y) = c_0(x) + c_1 \frac{e^{-xy}}{x} + c_2 \frac{e^{xy}}{x} + \frac{c_3 y + c_4 y^2}{x^2} \tag{19}$$

where again the last piece is not derivable as its Fourier transform does not exist.

Note that there might be other solutions as well, but a discussion such as this one (an even shorter version) is sufficient for the full grade in this question.

Bonus Question: 3 . . . . *(2 points)*

What Mathematica code would compute the gradient of the scalar function $f :: \mathbb{R}^2 \to \mathbb{R}$ for $f = (x,y) \to x^2 + y^2$?

Solution:

```
Grad[x^2 + y^2, {x, y}]
```