Moduli Space of Superconformal Field Theories

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We will investigate the moduli space of conformal field theories that arise from string theory compactifications, specifically for string theory compactified on specific compact, complex manifolds with $SU(n)$ holonomies, namely Calabi-Yau manifolds. The moduli space of such theories are determined by the shape and the size of the associated Calabi-Yau manifolds.

I. INTRODUCTION

It has been almost 40 years since anomaly cancellation of the string theory was done [1], which was a major step for the validity of consistency of superstring theories in 10 dimensions. With this, the inclusion of the known superstring theories in 10 dimensions increased with the $E_8 \times E_8$ gauge group. This increasing number of seemingly different theories also questioned the validity of the theory. At this point, an important realization has been made, that these seemingly different theories are actually connected by various dualities, for instance, Type IIA Superstring is T-dual to the Type IIB superstring theory in $d = 10$ [2], or Type I superstring theory is S-dual to the Heterotic string theory in $d = 10$ dimensions [3]. This idea of duality led to important results such as AdS/CFT [4] and some other dualities that arise from string theory compactifications, such as Mirror Symmetry [5].

The idea of duality is extensively used in the search for principles of possible quantum gravity theories [6],[7]. Theories that arise from string compactifications in the low-energy scale constitute the consistent theories that can arise from quantum gravity, which gives the Landscape of theories [8].

The dimensionality of spacetime in string theory is fixed by the CFT arguments, namely that it should be free of ghosts, in other words, it should be anomaly-free CFT. For bosonic string theory, this condition is satisfied for $d = 26$ and for superstring theory, this is satisfied for $d = 10$. Our concern will be of superstring theories, for which, a single partner of bosonic and fermionic fields adds up to a 3/2 central charge. In the end, we need a theory with a central charge of 15.

Lets us think that our universe has the form of $M_4 \times M$, where M_4 denotes the homogenous and isotropic 4 dimensional flat space and M is the space in which we compactify our theory. By construction, we can say that we have fundamentally two theories, a theory in M_4 with a CFT that has a central charge $c = 6$, and a six-dimensional theory in M with a CFT that has a central charge $c = 9$. For perturbative stability, space-time supersymmetry is required for the compactification, and for analytic control, we will also impose $\mathcal{N} = 2$ supersymmetry on theory in M.

In order to incorporate the supersymmetric structure in M, one needs to consider the holonomy of the supersymmetric transformation, which requires M to be a complex manifold with a holonomy group $SU(n)$. As we compactify ten-dimensional theory in this space, it is required that this space be compact. These conditions, including that this complex manifold should also be Kähler, imply that this complex manifold should be a Calabi-Yau manifold [9],[10]. The topological characteristics of Calabi-Yau manifolds are described by Hodge-diamond, which is the dimension of the cohomology class. Different Hodge-diamond structures will give different Calabi-Yau manifolds. A certain symmetry of this Hodge diamond gives a theory, which is dual to the original theory compactified on the original Calabi-Yau manifold. This is what is called Mirror Symmetry.

This big Landscape of solutions is greatly reduced by the ideas of dualities, and the parameterization of this landscape is done via moduli fields, which corresponds to different theories in the context of string theory, as there is no free parameter and all parameters are vacuum expectation values of some fields. In this review, we will try to understand what operators make us move in this moduli space and how do the dualities of the theories effect them.

II. SUPERCONFORMAL FIELD THEORY

We will start with the discussion of $\mathcal{N} = 0$ conformal field theory, and we will do the discussion from the algebra point of view. One can show that in $d = 2$, there is a one-to-one correspondance between the conformal algebra and the operator product expansions (OPE). Conformal algebra can be derived from the following OPE for the energy-momentum tensor:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + ..., \quad (1)$$

where c stands for the central charge, ... stands for the analytical terms, which will be of no importance to us. One can expand the energy-momentum tensor in terms of Laurent modes, $T(z) = \sum_n L_n z^{-n-2}$, then the OPE above implies the following algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}. \quad (2)$$

This algebra forms the complete algebra for the $\mathcal{N}=0$. In order to expand this result and find the $\mathcal{N}=1$ algebra, we have to introduce the superpartner for the energy-momentum tensor $T(z)$. We will call this generator $G(z)$, which will have conformal weight $3/2$, which can be seen from the OPE with energy-momentum tensor. The OPE's that include this new field is given by:

$$\begin{aligned} T(z)G(w) =& \frac{3/2}{(z-w)^2}G(w) + \frac{\partial_w G(w)}{(z-w)} + ..., \\ G(z)G(w) =& \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{(z-w)} + ... \end{aligned} \quad (3)$$

These OPE's, including the OPE of the energy-momentum tensor with itself(1), form the complete set of OPE's for the $\mathcal{N}=1$ SCFT, from which one can Laurent expand the corresponding fields and find the associated algebra between the operators.

We will need one more step to find $\mathcal{N}=2$ SCFT [11], which is to introduce one more superpartner to the theory. The other superpartner is $G^2(z)$ and $J(z)$ fields, where $G^2(z)$ is another copy of $G(z)$, and $J(z)$ is a bosonic $U(1)$ current field.

We can write the total set of OPE by diagonalizing the action of $J(z)$ to $G^1(z)$ and $G^2(z)$, by redefining fields as $G^{\pm}(z) = \frac{1}{\sqrt{2}}(G^1(z) \pm iG^2(z))$. The complete set of OPE's for $\mathcal{N}=2$ SCFT will be:

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + ..., \\ T(z)G^{\pm}(w) &= \frac{3/2}{(z-w)^2}G^{\pm}(w) + \frac{\partial_w G^{\pm}(w)}{(z-w)} + ..., \\ T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)} + ..., \\ G^+(z)G^-(w) &= \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} \\ &\quad + \frac{2T(w) + \partial_w J(w)}{(z-w)} + ..., \\ J(z)G^{\pm}(w) &= \pm\frac{G^{\pm}(w)}{(z-w)}, \\ J(z)J(w) &= \frac{c/3}{(z-w)^2}. \end{aligned}$$

One can study the representation theory of this $\mathcal{N}=2$ SCFT [12], and its relation to the string theory compactifications [13].

One can consider the following four cases of $\mathcal{N}=2$ SCFT's. The first one would be the free field theory, in which we have a complex scalar field and associated with it, we have two fermionic fields as supersymmetric partners. Due to this, we have $c=3$ for this theory. This is important, as the M_4 theory can be constructed by taking two copies of this theory.

The second example would be a non-linear sigma model (NLSM). In this theory, fields live in the moduli space, which can be curved. So the metric in the kinetic terms depends on the metric of the moduli space, and the Riemann tensor can be included in the action.

The third example is the Landau-Ginzburg model. While studying the representation theory of $\mathcal{N}=2$ SCFT's, one can find chiral representations of this theory. The basic ingredient of the Landau-Ginzburg theory as an SCFT is born out of these chiral representations, which can be easily seen from the superspace formalism.

The fourth example would be the minimal models. Necessary conditions for the unitary highest weight representations of the Virasoro algebra highly constrain the values of the central charge. This also constrains the conformal weights of the primary fields. There are interesting relations between the chiral rings of $\mathcal{N}=2$ SCFT minimal models and Landau-Ginzburg models, [14].

III. MODULI SPACE OF $\mathcal{N}=2$ SCFT

We can deform theories by including additional terms to the Lagrangian. Not all terms will affect the low-energy behavior of the theory [15]. Such operators are distinguished by dimensional analysis. In $d=2$ conformal field theories, the relevant dimension is the conformal dimension, which corresponds to the conformal weight of the operators. These are given by the coefficients of the $\frac{1}{(z-w)^2}$ term in the OPE of the relevant primary field with the energy momentum tensor. Careful that due to the central charge term in (1), we know that the energy momentum tensor is not a primary field in $d=2$ CFT.

For a primary field, we have the conformal dimensions $(h, \bar{h})$, which will determine whether the field has an effect when the Renormalization group (RG) acts on the theory. There are three cases that should be considered:

- Irrelevant: If $h+\bar{h} > 2$, then these operators will have no effect on the infrared theory, as RG action will drive its coefficient to 0.

- Relevant: If $h+\bar{h} < 2$, the operator is called relevant and they can significantly affect the theory in the IR region. In some cases, it can even make the theory trivial after RG action.

- Marginal: If $h+\bar{h} = 2$, the operator is called marginal operator. For spinless operators, the left and right conformal weights will be equal $h=\bar{h}=1$. These operators will be of great importance as they give us different conformal field theories with the same central charge. In a sense, these operators make us move in the moduli space.

We also have operators that are called truly marginal operators, if the corresponding operator still remain as a marginal operator after the RG action.

A. Boson on a Circle

We can see such a scenario when we study the free boson on a circle. The action for the free boson in $d = 2$ compactified on a circle is given by:

$$S_{R_0} = \int d^2z \partial X \cdot \bar{\partial} X, \tag{4}$$

with the periodic identification of the field, $X \sim X + 2\pi R_0$ where R_0 is the radius of compactification. One can see that the field X has conformal dimension $(h, \bar{h}) = (0, 0)$. Consider the operator $\mathcal{O} = \partial X \cdot \bar{\partial} X$ which has conformal dimension $(h, \bar{h}) = (1, 1)$. We can deform the theory by adding this term to the action:

$$S_R = S_{R_0} + \epsilon \int d^2z \mathcal{O}, \tag{5}$$

which is just $S_R = (1 + \epsilon) S_{R_0}$ by a redefenition of the field $X \to \tilde{X} = \sqrt{1+\epsilon} X$ the action becomes $S_R = \int d^2z \partial \tilde{X} \cdot \bar{\partial} \tilde{X}$, where we have the new periodicity condition as $\tilde{X} \sim \tilde{X} + 2\pi R_0(\sqrt{1+\epsilon})$. It can be seen that the operator $\mathcal{O}$ changed the radius of compactification of the $d = 2$ CFT. This tells us that the change in the CFT can be characterized by the change in the radius of compactification. The more generalized version of these kinds of relations can be found in the literature, [16]. As we know from the string compactification, if this free boson is associated with a string, then we have T-duality, [17], meaning that two theories are equivalent for $R \sim 1/R$. For two different radius of compactifications related to this, we have dual theories. Overall, the moduli space of a free boson on a circle can be given by real values $R \in (1, \infty)$ where R is the radius of compactification, that can be changed by the marginal operator $\mathcal{O}$.

B. $\mathcal{N} = 2$ SCFT

Considering the Ward identity for the superconformal field theories [18] [19], one can find the following two truly marginal operators:

- Let $\phi \in (c, c)$ ring with $h = \bar{h} = 1/2$, and the charge with respect to the $J(z)$ field, we have $Q = \bar{Q} = 1$. Now define another field $\hat{\phi}$ as follows:

$$\hat{\phi}(w, \bar{w}) = \oint dz G^-(z) \phi(w, \bar{w}). \tag{6}$$

  With this, we have a field $\hat{\phi}$ with conformal weight $h = 1, \bar{h} = 1/2$, and $Q = 0, \bar{Q} = 1$. Then, define the field $\Phi_{(1,1)}$ by:

$$\Phi_{(1,1)}(w, \bar{w}) = \oint d\bar{z} \bar{G}^-(\bar{z}) \hat{\phi}(w, \bar{w}), \tag{7}$$

  where $\Phi_{(1,1)}$ has the conformal weight $h = \bar{h} = 1$ and $Q = \bar{Q} = 0$.

- Let $\phi \in (a, c)$ ring with $h = \bar{h} = 1/2$, and $Q = -\bar{Q} = 1$. From the idea that is used above, we will define a field $\hat{\phi}$:

$$\hat{\phi}(w, \bar{w}) = \oint d\bar{z} \bar{G}^-(\bar{z}) \phi(w, \bar{w}), \tag{8}$$

  and from this, define the corresponding operator $\Phi_{(-1,1)}$ as:

$$\Phi_{(-1,1)} = \oint dz G^+(z) \hat{\phi}(w, \bar{w}). \tag{9}$$

  One can check that the related field $\Phi_{(-1,1)}$ has the conformal weight $h = \bar{h} = 1$ and with charges $Q = \bar{Q} = 0$.

For the associated $\mathcal{N} = 2$ SCFT, these are the truly marginal operators that will make us move in the moduli space of the consistent theories. It is shown that, at least locally, the CFT Zamolodchikov metric is block diagonal in this moduli space between the marginal operators $\Phi_{(1,1)}$ and $\Phi_{(-1,1)}$, hence we can think of the metric as the product of these two deformations [20][21][22]. These operators will have geometrical meanings when they are added to the action. This comes from the association between the (c, c) and (a, c) fields with the $(2, 1)$ and $(1, 1)$ harmonic forms. This relation establishes the geometrical meaning. The $(1, 1)$ form has the structure of Kähler potential's derivative, which gives the metric for the associated complex manifold. As the metric changes, this operator changes the "size" of the associated Calabi-Yau manifold. Similarly, $\Phi_{1,1}$ gives the pure index type of metric perturbations, which only changes the "shape" of the complex manifold.

Elaborating more, one can think of the complex manifold as T^2. One can parameterize this torus with the moduli τ, which governs the shape and the volume of the associated torus. This torus, even though it is so real, is an example of a one dimensional Calabi-Yau manifold. By increasing $\tau \to \lambda\tau$ where λ is just a constant, we get a different lattice, in which we identify the opposite sides to get a big torus. Also, we can change the τ such that, at the end, we get a torus with the same size, but the shape is different. This can be done by a transformation $\tau \to \tau + 1$. In this sense, a Kähler transformation changes the volume of the torus, whereas a complex structure change the overall shape of the torus.

Thus, we have found that the moduli space of $\mathcal{N} = 2$ can be characterized by the volume and shape of the associated Calabi-Yau manifold, and the abovementioned operators change the size and shape of the Calabi-Yau manifold.

IV. CONCLUSION

We have analyzed the conformal field theories, and how different CFT's are related with each other. There are a few key points in analyzing such moduli spaces. First is that, the moduli space can be constructed by finding marginal operators, that changes the associated CFT to a nearby CFT without changing the central charge, and there is the idea of duality which gives an equivalence of points in the moduli space. In this scenario, we see that the moduli space of $\mathcal{N} = 0$ CFT of a free boson on a circle in $d = 2$ is characterized by the radius of compactification, which has also the T-duality. The associated marginal operator is seen to be changing the theory, meaning it changes the size of compactification.

For the case of $\mathcal{N} = 2$ SCFT on a Calabi-Yau manifold, we have seen that the moduli space is actually formed by product of two marginal operators. These operators have geometric meanings, one is changing the size and the other is changing the shape of the Calabi-Yau manifold. The example of T^2 is discussed. As usually happens in string theory, the duality for this kind of moduli space is in literature, which is called the Mirror Symmetry, arise from the topological nature of the Calabi-Yau manifolds and characteristics of string theory compactifications on them.

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