Multi-particle Representations of Poincare Group

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This review focuses on the extension of the definition of asymptotic multi-particle states of the S-matrix, beyond the tensor products of one-particle states. This extended definition results in a new quantum number which is called as pairwise helicity. I start with the derivation of the pairwise helicity by the method similar to the Wigner's construction of one-particle representations. Then, I review the work which shows that this pairwise helicity can also be derived from topology. Finally, I discuss the current literature, the importance of the subject and some open questions.

I. INTRODUCTION

In quantum field theory, a particle can be defined as a set of states that mix only among themselves under Poincare transformations. In that sense, representations of the Poincare group form the foundation of particle physics, especially in the construction of the S-matrix. Without interactions at asymptotic times, the S-matrix can be taken as the overlap between the free particles in the in- and out-states, and these states are usually taken as products of one-particle representations of the Poincare group. To classify the one-particle representations, Wigner introduced the method of induced representations. Using this method, one-particle states can be represented by their masses and little group representations.

The little group is the subgroup of Lorentz transformations that leaves the chosen reference momentum k^μ invariant. After the representation of the little group is constructed, the general states are obtained by applying a Lorentz boost to the reference momentum k^μ to get a general momentum p^μ. For instance, for massive particles the reference momentum can be defined by choosing the rest frame of the particle and then the little group becomes the $SU(2)$ double cover of $SO(3)$. On the other hand, there is no rest frame for the massles particles but one can choose the reference momentum on the z-direction and little group becomes $U(1)$, which corresponds to the rotation around z-axis.

In the construction of an S-matrix, in- and out- states are described by multiparticle states, and the representations of these multiparticle states are generally taken as tensor product of one-particle states. Although this assumption seems consistent for the states at asymptotic times, in 1972 Zwanziger showed that in the scattering of electric and magnetic charges an extra quantum number is necessary to fully describe the process [1]. The observation Zwanziger made was that in this process, besides the orbital and spin angular momentum of each particle, there is also a residual angular momentum in the electromagnetic field of the in- and out- states. The generator associated to this angular momentum is added to the generators of the Lorentz transformations and as a result, the scattering states do not transform as free particles. This modification has a simple group-theoretical description and this description uses the idea of little group for the pairs of particles.

In 2021, the result of Zwanziger about the multiparticle representations of the Poincare group is extended in a more systematic way [2]. In this work, the authors extend the definition of asymptotic multi-particle states of the S-matrix, beyond the tensor products of one-particle states. This extension is made by the generalization of Wigner construction to the pairs of particles by introducing the so-called pairwise little groups. These pairwise little groups are defined as the subgroups of Lorentz transformations which leave a pair of momenta invariant. For a pair of particles, one can go to the center of momentum frame and the corresponding pairwise little group becomes $U(1)$. The authors call the charge related to this $U(1)$ transformation as pairwise helicity.

In a recent work, it was shown that the pairwise helicity can be seen as a direct consequnce of topology [3]. The key insight in this work is that the wavefunctions in quantum mechanics are not functions on configuration space but sections of line bundles over configuration space. These line bundles can have non-trivial topology especially for the case of dyons and this non-trivial topology results in the pairwise helicity.

In this review, I summarize the results on this extended definition of multi-particle state representations. Firstly, I start with the work [2] and show how the pairwise helicity can be derived from the extended version of Wigner's construction. Then, in the next section, I review the work which shows how this result can also be derived as a consequence of topology [3]. Finally, I discuss the current literature and why this subject is important for the future works.

II. PAIRWISE LITTLE GROUP

The Hilbert space for a single-particle consists of the momentum eigenstates $|p;\sigma\rangle$ where p corresponds to momentum and σ corresponds to the other quantum numbers defining the state. The quantum number σ is re-

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lated to the little group and to understand its precise meaning one can start by choosing a reference momentum k. For massive particles, one can choose the rest frame by $k = (m, 0, 0, 0)$ and for massless particles can choose $k = (E, 0, 0, E)$. Then the corresponding little groups are are $SU(2)$ and $U(1)$ respectively. The quantum number σ stands for spin for massive particles and helicity for the massless particles. Hence, the little group transformation of a massive particle with given reference momentum k takes the form

$$U(W)\,|k;s,s_z\rangle = D^s_{s'_z s_z}(W)\,|k;s,s'_z\rangle\,, \tag{1}$$

where s is the total spin, s_z is its z-component and $D^s_{s'_z s_z}(W)$ is spin s-representation of $SU(2)$ group. To obtain a representation in an arbitrary reference frame, we can define a Lorentz transformation L_p by $p = L_p k$ and the state becomes $|p;\sigma\rangle \equiv U(L_p) \mid k;\sigma\rangle$. Then, the total Lorentz transformation for a state $|p;\sigma\rangle$ is given by

$$U(\Lambda)|p;\sigma\rangle = U\left(L_{\Lambda p}\right)U(W)|k;\sigma\rangle = D_{\sigma'\sigma}(W)\,|\Lambda p;\sigma'\rangle \tag{2}$$

where $D_{\sigma'\sigma}$ either stands for $D^s_{s'_z s_z}$ for massive particles or $e^{ih\phi}$ for massless particles. After obtaining this one-particle representations, it is possible to obtain multi-particle representations by tensor product of these states. However, as its was first shown by Zwanziger there some cases where this tensor product representation cannot fully describe the situation. To construct more general representations, one can extend this Wigner's induced representation method for the pairs of particles [2].

One can start the analysis by considering two-particle representation for simplicity. For a particle pair given by 1 and 2, consider the representations of the product group $P_1 \times P_2 \times \tilde{P}_{12}$. While P_1 and P_2 are the Poincare groups acting on particles with momenta p_1 and p_2, $\tilde{P}_{12}$ acts on the pair of momenta $(\tilde{p}_1, \tilde{p}_2)$. At this point $\tilde{p}_1$ and $\tilde{p}_2$ should be thought as distinct from p_1 and p_2 and $(\tilde{p}_1, \tilde{p}_2)$ should be thought as a simple tool instead of a two-particle state. Then the generalized two-particle state can be written as

$$|p_1, p_2, (\tilde{p}_1, \tilde{p}_2)\,;\sigma\rangle \equiv |p_1;\sigma_1\rangle \otimes |p_2;\sigma_2\rangle \otimes |(\tilde{p}_1, \tilde{p}_2)\,;q_{12}\rangle \tag{3}$$

where the last term corresponds to pairwise state with q_{12} defined as pairwise helicity. To understand the pairwise helicity, one can follow the procedure which has been used for the single-particle representation. The reference momenta for the individual particles can be chosen as similarly to the single particle case with $p_1 = k_1$ and $p_2 = k_2$. The reference momenta $(\tilde{k}_1, \tilde{k}_2)$ for the pair $(\tilde{p}_1, \tilde{p}_2)$ can be chosen by going to the center of momentum frame of the pair with two particles are both moving along the z-axis,

$$\tilde{k}_1 = \left(\tilde{E}_1, 0, 0, \tilde{p}_c\right), \quad \tilde{k}_2 = \left(\tilde{E}_2, 0, 0, -\tilde{p}_c\right), \tag{4}$$

where $\tilde{E}_{1,2} = \sqrt{m^2_{1,2} + \tilde{p}^2_c}$. To pass to a generic frame, the relevant Lorenzt transformations are given by

$$\begin{aligned} &p_1 = L^1_{p_1} k_1, \quad p_2 = L^2_{p_2} k_2 \\ &(\tilde{p}_1, \tilde{p}_2) = \left(\tilde{L}^{12}_{\tilde{p}_1,\tilde{p}_2}\tilde{k}_1, \tilde{L}^{12}_{\tilde{p}_1,\tilde{p}_2}\tilde{k}_2\right). \end{aligned} \tag{5}$$

Following Wigner's method, one can define the representation of the Lorentz transformations of the form $\Lambda \equiv (\Lambda_1, \Lambda_2, \tilde{\Lambda}_{12}) \in P_1 \times P_2 \times \tilde{P}_{12}$ on this state as

$$\begin{aligned} &U(\wedge)\,|p_1, p_2, (\tilde{p}_1, \tilde{p}_2)\,,\sigma\rangle = \\ &\left(D_{\sigma'_1\sigma_1}(W_1)\,|\Lambda_1 p_1;\sigma'_1\rangle\right) \otimes \left(D_{\sigma'_2\sigma_2}(W_2)\,|\Lambda_2 p_2;\sigma'_2\rangle\right) \otimes \\ &\left(U(\tilde{L}_{\tilde{\Lambda}_{12}\tilde{p}_1,\tilde{\Lambda}_{12}\tilde{p}_2})U(\tilde{W}_{12})|(\tilde{k}_1,\tilde{k}_2);q_{12}\rangle\right), \end{aligned} \tag{6}$$

where W_i correspond to single particle little group transformations and $\tilde{W}_{12}$) corresponds to a liitle group transformation which preserve both $\tilde{k}_1$ and $\tilde{k}_2$. Remembering that the reference momentum for the pair is chosen on the z-axis, the pairwise little group is $U(1)$ which corresponds to the rotation around z-axis. Hence, the pairwise little group is always $U(1)$ whether the particles are massive or not. Defining a rotation angle by $R_z\left(\tilde{\phi}_{12}\right) \equiv \tilde{W}_{12}$, transformation of a two-particle state is given by

$$\begin{aligned} &U(\Lambda)\,|p_1, p_2, (\tilde{p}_1, \tilde{p}_2)\,;\sigma\rangle = e^{iq_{12}\tilde{\phi}_{12}}. \\ &D_{\sigma'_1\sigma_1}(W_1)\,D_{\sigma'_2\sigma_2}(W_2)\left|\Lambda_1 p_1, \Lambda_2 p_2, (\tilde{\Lambda}_{12}\tilde{p}_1, \tilde{\Lambda}_{12}\tilde{p}_2);\sigma\right\rangle. \end{aligned} \tag{7}$$

Note that at this point there are still three separate copies of Poincare group and all the momenta $p_1, p_2, \tilde{p}_1, \tilde{p}_2$ are independent. To find the true representation, one can perform a projection onto the physical states, where $\tilde{p}_1 = p_1$ and $\tilde{p}_2 = p_2$. The transformation of a physical two-particle state is then given by [2]

$$\begin{aligned} &U(\Lambda)\,|p_1, p_2;\sigma_1, \sigma_2;q_{12}\rangle = \\ &e^{iq_{12}\tilde{\phi}_{12}} D_{\sigma'_1\sigma_1}(W_1)\,D_{\sigma'_2\sigma_2}(W_2)\,|\Lambda p_1, \Lambda p_2;\sigma'_1, \sigma'_2;q_{12}\rangle\,. \end{aligned} \tag{8}$$

This is the extended form of two-particle state representation of the Poincare group which reduces to the usual tensor product states when the pairwise helicity is zero, and reduces to the Zwanziger's two scalar dyon states when the spins are zero and $q_{12} = 1$. To generalize this extension to the n-particles, one can start with the tensor product of $2^n - 1$ copies of the Poincare transformations

$$\begin{aligned} &P_1 \times \ldots \times P_n \times P_{12} \times \ldots \times P_{n-1,n} \times \\ &P_{123} \times \ldots \times P_{n-2,n-1,n} \times \ldots \times P_{123\ldots n}, \end{aligned} \tag{9}$$

where $P_{i_1\ldots i_k}$ each represents an independent k-tuple of momenta. In 4D, all k-tuple little groups for $k > 2$ is trivial since there is no Lorentz transformation that leaves three general momenta invariant. Hence the product group can be represented by states which have only single particle momenta and pairwise momenta as

$$|p_1, \ldots, p_n;(\tilde{p}_1, \tilde{p}_2)\,, \ldots, (\tilde{p}_{n-2}, \tilde{p}_n)\,, (\tilde{p}_{n-1}, \tilde{p}_n)\,;\sigma\rangle\,. \tag{10}$$

Projecting onto the physical states, the general transformation rule takes the form

$$U(\Lambda)\left|p_1,\ldots,p_n;\sigma_1,\ldots,\sigma_n;q_{12},\ldots,q_{n-1,n}\right\rangle=\prod_{i>j}e^{iq_{ij}\phi_{ij}}\prod_i D_{\sigma_i'\sigma_i}\left(W_i\right)\left|\Lambda p_1,\ldots,\Lambda p_n,;\sigma_1',\ldots,\sigma_n';q_{12},\ldots q_{n-1,n}\right\rangle. \tag{11}$$

Note that the multi-particle states defined by (11) involve an extra angular momentum associated with their "pairwise" part. This can be seen by using the angular momentum operator on the pairwise part as

$$J_z\left|\tilde{k}_i,\tilde{k}_j;q_{ij}\right\rangle=q_{ij}\left|\tilde{k}_i,\tilde{k}_j;q_{ij}\right\rangle. \tag{12}$$

The extra angular momentum is associated with particle pairs but independent of the distance between them. This extra asymptotic contribution to the angular momentum is the result observed by Zwanziger in 1972 for the scattering of electric and magnetic charges [1]. So, this can be seen by calculating the overall angular momentum stored in the electromagnetic field in the presence of particles with electric (magnetic) charges [4]. The relativistic S-matrix for electric-magnetic scattering can be constructed by using (10) and identifying pairwise helicity as the half-integer $q_{ij}=e_ig_j-e_jg_i$.

III. PAIRWISE HELICITY FROM TOPOLOGY

One can show that the pairwise helicity obtained in the previous section can be also seen as a direct consequence of the topology [3]. The key insight is that the wavefunctions in quantum mechanics are not functions on configuration space but sections of line bundles over configuration space. For particles moving in $\mathbb{R}^3$, these line bundles can have non-trivial topology, especially for the case of dyons. To derive the pairwise helicity from topology, one can start with the dynamics of n non-relativistic particles, each moving in $\mathbb{R}^3$. At this point, it is crucial to impose the condition that the particles cannot sit top of each other. This means that the configuration space is not $\mathbb{R}^3$ but takes the form

$$\mathcal{M}_n=\left\{\mathrm{x}_i:\mathrm{x}_i\neq\mathrm{x}_j\ \text{for}\ i\neq j\right\}. \tag{13}$$

The particles interact with some fixed, background $U(1)$ connection over the configuration space $\mathcal{M}_n$. So, the target space of our theory is a $U(1)$ principal bundle

$$U(1)\hookrightarrow P\xrightarrow{\pi}\mathcal{M}_n \tag{14}$$

where these bundles can be topologically nontrivial and classified by second cohomology group as

$$H^2\left(\mathcal{M}_n,\mathbb{Z}\right)=\mathbb{Z}^{n(n-1)/2}. \tag{15}$$

Note that to classify these bundles one needs $\binom{n}{2}=\frac{1}{2}n(n-1)$ integers, which gives the number of pairs of particles. After constructing these bundles explicitly one can see that this classification of bundles is related to the pairs of particles and hence to the pairwise helicity.

To understand the physics, one can start with a classical action describing the motion of n particles. Introducing a $U(1)$ gauge field A_a and defining a global connection as

$$\omega=d\sigma+A, \tag{16}$$

the classical action takes the form [5]

$$S[\bar{\gamma}]=\int_\gamma dt\frac{1}{2}g_{ab}(x)\dot{x}^a\dot{x}^b+k\int_{\bar{\gamma}}\omega, \tag{17}$$

where $\bar{\gamma}$ is a curve in P and γ is its projection onto the base space $\mathcal{M}_n$. Then, we want to understand how the symmetries of our theory act, with a particular interest on rotations. The base configuration space $\mathcal{M}_n$ is symmetric under $SO(3)$, generated by vector fields

$$\xi^\alpha=-i\sum_i\epsilon^{\alpha\beta\gamma}\mathbf{x}_i^\beta\frac{\partial}{\partial\mathbf{x}_i^\gamma}. \tag{18}$$

The important point is that this symmetry transformation should be applied to the full target space, which means one should move in the fibre as rotating in the base. Then the first task is to lift the SO(3) action to an action on the full bundle P, which generically results in an action of $SU(2)$ double cover of $SO(3)$. Such a lift exists and unique up to conjugation by a gauge transformation [6]. It is also necessary that the $SU(2)$ action preserves our chosen connection; which can be ensured if and only if the field strength $F=dA$ is globally defined on the configuration space, is rotationally symmetric

$$\mathcal{L}_{\xi^\alpha}F=0. \tag{19}$$

In this case the unique lift of $SU(2)$ action to the full bundle is generated by the modified vector fields

$$\tilde{\xi}^\alpha=\xi_H^\alpha-\frac{i}{2}\epsilon^{\alpha\beta\gamma}F\left(\xi^\beta,\xi^\gamma\right)\eta, \tag{20}$$

where $\eta=\frac{\partial}{\partial\sigma}$ is the vector field moving along the fibre and ξ_H^α is the horizontal lift of the original generators which locally takes the form

$$\xi_H=\xi^a\partial_a-\left(\xi^aA_a\right)\eta. \tag{21}$$

This horizontal lift is the naive way to lift a vector field from base space to the full bundle and gives rise to the covariant derivative when the theory is quantized. The second term, which can be written in coordinates as $F\left(\xi^{\beta}, \xi^{\gamma}\right) = F_{ab}\xi^{\beta a}\xi^{\gamma b}$, is more interesting and it results in the additional phase in the multi-particle states. This additional phase can be shown to be equal to the pairwise helicity. The action of (20) on the principal bundle $P_n[w]$ for n particle case induces a representation on the Hilbert space $\mathcal{H}_n[w]$ for n particles. This representation is generated by the operators $\mathbf{J} = (J_1, J_2, J_3)$, which can be locally written as

$$\mathbf{J} = -i\sum_i \mathbf{x}_i \times \left(\frac{\partial}{\partial \mathbf{x}_i} - ik\mathbf{A}_i\right) - \frac{k}{2}\sum_{i<j} w_{ij}\frac{\mathbf{x}_{ij}}{|\mathbf{x}_{ij}|}. \quad (22)$$

To make contact with the pairwise helicity, one can define the momentum eigenstates and show that n-particle states take a phase factor related to the second term in (22). This additional phase factor is pairwise helicity.

IV. DISCUSSION AND CONCLUSION

While one-particle representations have been an important subject for many years, surprisingly little attention has been paid to multi-particle representations [2]. In the construction of S-matrix, the asymptotic multi-particle states are usually taken as simple tensor product states of one-particle states. In 1972, Zwanziger showed that this tensor product representation can be insufficient in some scattering cases, for example for the scattering of electric and magnetic charges [1]. This important result gained some attention again in the last couple of years. In [2], the authors extended this result in a more systematic way by using a method similar to the one introduced by Wigner for the one-particle case. Recently, it was shown that the same result can also be derived as a consequence of topology.

As it was discussed in [2], the extended representation of multi-particle states do not form a Fock space, which is by definition is a product space of one-particle states. This can be an important result about how we construct and understand the quantum field theories. In [2], the relevance of this result to the infrared divergence of QED is also discussed. To avoid the QED infrared problem, the traditional approach to describe an S-matrix of charged particles sourcing a classical field is to dress the particles with coherent photon states, as it was done in the Faddeev-Kulish formalism [7]. The construction of [2] for the multi-particle states mirrors the Faddeev-Kulish dressing.

Another important implication of [2] is about the violation of crossing symmetry for electric-magnetic processes. In nonrelativistic quantum mechanics, the angular momentum of the electromagnetic field flips its sign between the in and out state. In the language of pairwise helicity this means that the pairwise helicities of the in- and out- states should be differ by a minus sign. According to authors of [2], this can be the origin of the violation of crossing symmetry.

The work [3] takes the subject from a very different point of view. In [3], it is shown that the pairwise helicity can be understood as a direct consequence of topology. The key point for this result is the realization that the wavefunctions in quantum mechanics are not functions but section of line bundles. The nontrivial topology of the line bundles results in an extra phase factor in the transformation of multi-particle states, and this extra phase factor corresponds to the pairwise helicity. The first challenge which is faced with for this formalism is the construction of $U(1)$ principle bundles over the configuration space for n particles. Although this construction is straightforward for two-particle case, for $n > 2$ the situation is less tractable. The authors of [3] uses pull-back bundles to solve this problem. An open question for this topology point of view arises for the case of higher rank bundles. When we consider the higher rank bundles, which correspond to the particles carrying their own internal degrees of freedom, classifying the bundles is significantly more involved and is an open question [3]. Even if we can construct and classify such bundles, there may be many lifts of the symmetries of the base space. Solving these problems can be an important step to understand the full classification of multi-particle Hilbert spaces.

In these three main works [1–3] we reviewed, the multi-particle representations are mostly considered for the scattering of dyons. However, in the recent years, there are some works which implies that this result can be extended to the case of electromagnetism without magnetic charges [8, 9] and gravity [10]. Another important work deals with the pairwise boost quantum number [11], which can provide evidence for a universal breakdown of the description of multi-particle sates in terms of Fock space due to infrared back-reaction. The generalization of pairwise helicity idea to the higher dimensions is considered in [12].

The S-matrix formalism is one of the key points of quantum field theories and understanding the true representations of the multi-particle states in the construction of S-matrix can have important implications. The discussions in the recent literature imply that an immediate consequence can be understanding the true nature of infrared divergences [11]. This extended representation is also shown to be useful for monopole scattering cases (see for example [13–15]). There are still some open questions about these multi-particle representations, their implications for topology and their importance for the recent works. Solving these problems can help us to understand the S-matrix formalism and the infrared divergences further.

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