# Belinfante Symmetrization and Improved Symmetric Energy-Momentum Tensors

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We give a review on the Belinfante symmetrization of the relativistic stress-energy tensor. In discussion, we mentioned further developments on tensor, necessary conditions for extended symmetries of space-time such as scale and conformal transformations.

## I. INTRODUCTION

The Canonical (relativistic) Energy-Momentum Stress Tensor is an essential ingredient of physical part of Einstein's field equations in general relativity. Its symmetric form is the source of gravitational field in Lorentz local frame. However, the stress-energy tensor might not be symmetric. Generically, symmetric tensor is produced by exploiting Hillbert's method through the derivation of Lagrangian with respect to metric. Since metric is a symmetric, consequent object is also symmetric. That is the well-known and popular method, but it does not mean that is only and unique way to design the machine. Matrix form of so-called tensor and its components are as in the following form.

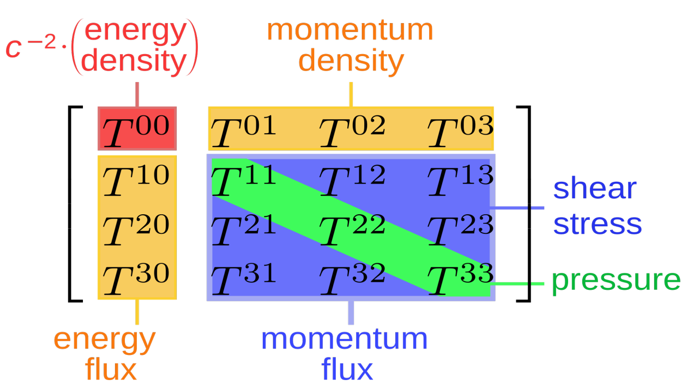


FIG. 1. Components of The Stress-Energy Tensor [1]

Constructing stress-energy tensor is the first deal, but the second deal is to answer the question is it symmetric or not. While Lagrangian functioning with vector field, spin contribution spoils symmetry. In 1939, Belinfante[2] added an anti-symmetric tensor to remove spin contribution. This medication was proposed and shown success in (3+1) space-time dimensions. Symmetric Energy-Momentum tensor is the source of weakly coupled gravitational field locally, however if one needs to make theory scale or conformal invariant, one need to satisfy additional conditions. Nakayama gives an extensive treatment on subject, and Mahapatra summarizes well.[3],[4]

## II. THEORY

One can write angular momentum tensor in terms of energy-momentum on scalar fields as

$$J_{ijk} = x_i T_{jk} - x_j T_{ik} \tag{1}$$

This may be regarded as orbital angular momentum. With conservation of tensor implied by Noether's theorem.

$$\partial_j T^{ij} = 0 \tag{2}$$

$$\partial_k J^{ijk} = T^{ji} - T^{ij} = 0 \tag{3}$$

It means for scalar fields, relativistic energy-momentum tensor is symmetric. For vector fields, angular momentum tensor gets an extra term called spin. If one sees antisymmetric part of tensor as orbital angular momentum, sees total angular momentum with contribution from vector, tensor or spinor fields.

$$\begin{aligned} \partial_k J^{ijk} &= \partial_k (L^{ijk} + S^{ijk}) \\ &= \partial_k (x^i T^{jk} - x^j T^{ik} + S^{ijk}) \\ &= T^{ji} - T^{ij} + \partial_k S^{ijk} = 0 \end{aligned} \tag{4}$$

As is seen, To satisfy the angular momentum, the antisymmetric part of $T^{ij}$ equals spin tensor. That is expressed in terms of derivative of Lagrangian with respect to first order of field multiplied by transformed field by Lorentz group. Lorentz group transforms field, by rotations and Lorentz boosts.

$$S_{\alpha\beta}{}^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} (M_{\alpha\beta})_{AB} \phi_B \tag{5}$$

$M_{\alpha\beta}$ matrices are six generators of the Lorentz group. One can see [5] for calculations of these matrices. To fix the asymmetry, Belinfante added a tensor anti-symmetric in its last two indices.

$$B_{ijk} = \frac{1}{2}[S_{jki} + S_{ikj} - S_{ijk}] \tag{6}$$

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Symmetric tensor comes as,

$$\Theta^{ij} = T^{ij} + \partial_k B^{ijk} \tag{7}$$

One acts partial on second index, can see Belinfante tensor be zero due to anti-symmetry, and new tensor satisfies conservation. Either the energy-momentum tensor is symmetric or not, it must satisfy four momentum.

$$\int_M \partial_\nu \Theta^{\mu\nu} d^4x = \int_M \partial_\nu T^{\mu\nu} d^4x + \int_M \partial_\nu \partial_\rho B^{\mu\nu\rho} d^4x \tag{8}$$

Anti-symmetry of Belinfante tensor in last two indices, improvement term vanishes, and by using Stoke's theorem, rest of the terms give us four-momentum.

$$\int_{\partial M} \Theta^{\mu 0} dV = \int_{\partial M} T^{\mu 0} dV = P^\mu \tag{9}$$

One can look at [6] for explicit explanations of taking integral to get four-momentum or one can say that momentum and energy densities. To prove our new improved tensor is symmetric, one can have a look at angular momentum conservation written in terms of new tensor[7].

$$\begin{aligned} \partial_k J^{ijk} &= \partial_k (x^i \Theta^{jk} - x^j \Theta^{ik}) \\ &= \Theta^{ji} - \Theta^{ij} \\ &= T^{jk} + \partial_m B^{jkm} - (T^{kj} + \partial_m B^{kjm}) \\ &= T^{jk} - T^{kj} + \partial_m (B^{jkm} - B^{kjm}) \\ &= T^{jk} - T^{kj} - \partial_m S^{jkm} = 0 \end{aligned} \tag{10}$$

As is seen in eq.(4), last line equals zero. When seeing the second line i.e. anti-symmetric part, also equals zero, meaning our new tensor is symmetric, satisfying our expectations.

## III. DISCUSSION

Belinfante constructed symmetric energy-momentum tensor by adding term corresponding spin tensor contribution that stems from space-time symmetries. This symmetrization is the first stone on the path to extended symmetries albeit that tensor is not unique object. Having symmetric tensor is not a condition to satisfy scale invariance. One needs to have trace of stress tensor divergence of virial current[8] to have scale invariant tensor. Although, scale invariance is implied by conformal invariance[9], just for some theories converse is true. Therefore, conformal invariance possess traceless condition besides symmetric form of tensor. One should also look at pivotal work of Callan et. al. [10] on improved tensor with cut-off independent elements for renormalizable field theories.


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