Multiparticle Representations of The Poincare Group

Rasim Yılmaz

November 27, 2024

Introduction

- Representations of Poincare group are important in the construction of S-matrix
- S-matrix is the overlap between quantum states representing free particles in the asymptotic times
- In-out states of S-matrix approach products of one particle representations of Poincare group
- Should we extend the definition of multiparticle representations beyond tensor products?

One-Particle Representations

- Method of induced representations (Wigner) to classify one-particle states by mass and little group
- Little group is subgroup of Lorentz transformations that leaves a particular reference momentum invariant
- For massive particles, choose the rest frame and the little group becomes SU(2) double cover of SO(3) rotations
- For massless particles no rest frame, choose reference momentum in z-direction and the little group is U(1)

- Hilbert space for single particle spanned by momentum eigenstates $|p;\sigma>$ which satisfy

$$P^{\mu}|p;\sigma>= p^{\mu}|p;\sigma> \quad (1)$$

- To define precise meaning of extra quantum numbers σ, choose a common reference momentum k for every particle

- For massive particles k=(m,0,0,0), little group is SU(2)
  For massless particles k=(E,0,0,E), little group is U(1)

- Label σ fix transformation of $|k,\sigma>$ under little group

- For massive particles, σ stands for both total spin s and z-component of spin s_z and for all $W \in SU(2)$

$$U(W)|k; s, s_z >= D^s_{s_{z'} s_z}(W)|k; s, s'_z > \quad (2)$$

- For massless particles σ stands for helicity h and little group transformation is just phase $e^{ih\phi}$, ϕ is U(1) rotation angle

- Define a Lorentz transform by $p = L_p k$. Then the corresponding quantum state becomes

$$|p; \sigma >= U(L_p)|k; \sigma > \quad (3)$$

- Its Lorentz transformation

$$U(\Lambda)|p; \sigma >= U(L_{\Lambda_p})U(W)|k; \sigma >= D_{\sigma' \sigma}(W)|\Lambda p; \sigma' > \quad (4)$$

Multiparticle Representation

- In standard construction of S-matrix, multiparticle scattering states assumed to be tensor products of one particle states
- For two-particle states one needs to specify two masses and two spins
- 1972: Zwanziger pointed out that for scattering of electric and magnetic charges, these quantum numbers not sufficient
- Additional quantum number needed to characterize the relative transform of two-particle state w.r.t. tensor product state

- Extend the definition of asymptotic multi-particle states of the S-matrix, beyond the tensor products of one-particle states.
- Identify new quantum numbers called pairwise helicities
- The states will reduce to
    - Tensor product states for vanishing pairwise helicity
    - Zwanziger's scalar dyon states for vanishing spin
- Key insight: For multiparticle states, in addition to little groups of each particle, one also needs pairwise litte groups

- Define generalized 2-particle state as

$$|p_1, p_2, (\tilde{p}_1, \tilde{p}_2); \sigma >= |p_1; \sigma_1 > \otimes |p_2; \sigma_2 > \otimes |(\tilde{p}_1, \tilde{p}_2); q_{12} > \quad (5)$$

- Similar to single particle case, define the reference momenta.
- For single particle momenta p_1 and p_2, we can choose k_1, k_2 defined as in the single particle case.
- To define reference momenta $(\tilde{k}_1, \tilde{k}_2)$ corresponding to the pair $(\tilde{p}_1, \tilde{p}_2)$, go to the pair's COM frame with the two particles are both moving along the z-axis.

$$\tilde{k}_1 = (\tilde{E}_1, 0, 0, \tilde{p}_c) \quad \tilde{k}_2 = (\tilde{E}_2, 0, 0, \tilde{p}_c) \quad (6)$$

- Corresponding Lorentz transformations

$$p_1 = L^1_{p_1} k_1, \quad p_2 = L^2_{p_2} k_2, \quad (\tilde{p}_1, \tilde{p}_2) = (\tilde{L}^{12}_{\tilde{p}_1,\tilde{p}_2} \tilde{k}_1, \tilde{L}^{12}_{\tilde{p}_1,\tilde{p}_2} \tilde{k}_2) \tag{7}$$

- Generic state defined by

$$\begin{aligned} |p_1, p_2, (\tilde{p}_1, \tilde{p}_2); \sigma >= (U(L^1_{p_1}) \ |k_1; \sigma_1 >)\otimes \\ (U(L^2_{p_2}) \ |k_2; \sigma_2 >) \otimes (U(\tilde{L}^{12}_{\tilde{p}_1\tilde{p}_2}) \ |(\tilde{k}_1, \tilde{k}_2); q_{12} >) \end{aligned} \tag{8}$$

- Now find representations of Lorentz transformations $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_{12}) \in P_1 \times P_2 \times \tilde{P}_{12}$ on this state

- The transformation of the generic state defined by

$$U(\Lambda)|p_1, p_2, (\tilde{p}_1, \tilde{p}_2); \sigma >=$$

$$(D_{\sigma_1' \sigma_1}(W_1)\ |\Lambda_1 p_1; \sigma_1' >) \otimes (D_{\sigma_2' \sigma_2}(W_2)\ |\Lambda_2 p_2; \sigma_2' >) \otimes \tag{9}$$

$$(U(\tilde{L}^{12}_{\tilde{\Lambda}_{12}\tilde{p}_1, \tilde{\Lambda}_{12}\tilde{p}_2})\ U(\tilde{W}_{12})\ |(\tilde{k}_1, \tilde{k}_2); q_{12} >)$$

- Pairwise little group is U(1) rotation irrespective of whether the particles are massive or massless

- Defining rotation angle by $R_z(\tilde{\phi}_{12}) = \tilde{W}_{12}$

$$U(\Lambda)|p_1, p_2, (\tilde{p}_1, \tilde{p}_2); \sigma >= e^{iq_{12}\tilde{\phi}_{12}}$$
$$(D_{\sigma_1'\sigma_1}(W_1) D_{\sigma_2'\sigma_2}(W_2)\ |\Lambda_1 p_1, \Lambda_2 p_2, (\tilde{\Lambda}_{12}\tilde{p}_1, \tilde{\Lambda}_{12}\tilde{p}_2); \sigma >) \tag{10}$$

- Projecting onto physical states by $p_1 = \tilde{p}_1$, $p_2 = \tilde{p}_2$, transformation of physical 2-particle state takes the form

$$U(\Lambda)|p_1, p_2, ; \sigma_1, \sigma_2, q_{12} >=$$
$$e^{iq_{12}\tilde{\phi}_{12}} D_{\sigma_1'\sigma_1}(W_1) D_{\sigma_2'\sigma_2}(W_2)\ |\Lambda_1 p_1, \Lambda_2 p_2, ; \sigma_1', \sigma_2', q_{12} >) \tag{11}$$

- This 2-particle rep. reduces to tensor product states for $q_{12} = 0$ and for $q_{12} = 1$, $j_1 = j_2 = 0$ to Zwanziger's 2-scalar dyon states
- Generalization to n particles is straightforward and the general state can be written as

$$|p_1, \cdots, p_n; (\tilde{p}_1, \tilde{p}_2), \cdots, (\tilde{p}_{n-1}, \tilde{p}_n); \sigma > \tag{12}$$

- Projecting onto the physical states

$$\begin{aligned} &U(\Lambda)\ |p_1, \cdots, p_n; \sigma_1, \cdots, \sigma_n; q_{12}, \cdots, q_{n-1,n} > \\ &\prod_{i>j} e^{iq_{ij}\phi_{ij}} \prod_i D_{\sigma_i'\sigma_i}(W_i) |\Lambda p_1, \cdots, \Lambda p_n; \sigma_1', \cdots, \sigma_n'; q_{12}, \cdots, q_{n-1,n} > \end{aligned} \tag{13}$$

Pairwise Helicity From Topology

- The additional phase is a direct consequence of topology.
- The wavefunctions that describe dyons are not functions, but sections of a line bundle that may be topologically non-trivial.
- Consider dynamics of n particles on the config. space

$$\mathcal{M}_n = \{\mathbf{x}_i : \mathbf{x}_i \neq \mathbf{x}_j \text{ for } i \neq j\} \tag{14}$$

- Classification of U(1) principal bundles over $\mathcal{M}_n$ are given by

$$H^2(\mathcal{M}_n, \mathbb{Z}) = \mathbb{Z}^{n(n-1)/2} \tag{15}$$

- $\binom{n}{2} = \frac{1}{2}n(n-1)$ integers specify the bundles and these integers are associated with particle pairs

- To understand the physical meaning of these integers, consider a classical action on the target space

$$U(1) \hookrightarrow P \xrightarrow{\pi} \mathcal{M}_n \tag{16}$$

- The base space $\mathcal{M}_n$ admits $SO(3)$ symmetry, generated by

$$\xi^\alpha = -i \sum_i \epsilon^{\alpha\beta\gamma} \mathbf{x}_i^\beta \frac{\partial}{\partial \mathbf{x}_i^\gamma} \tag{17}$$

- The lift of the $SU(2)$ action to the full bundle is unique and is generated by

$$\tilde{\xi}^\alpha = \xi_H^\alpha - \frac{i}{2} \epsilon^{\alpha\beta\gamma} F\left(\xi^\beta, \xi^\gamma\right) \eta \tag{18}$$

- When we quantize the theory, the second term will result in the additional phase

Summary

- Usual description of multi-particle Fock spaces as the tensor product of single particle Hilbert spaces needs refinement
- This refinement comes in the form of pairwise helicity
- This pairwise helicity can be derived by extending Wigner's result using the little group of particle pairs
- A clean derivation comes from topology by noting wavefunctions in config. space are sections of line bundles
- Construct the principal bundle over $\mathcal{M}_n$, lift the symmetries of base space to the full bundle and quantize the theory

Future Directions

- Classifying higher rank bundles over $\mathcal{M}_n$, corresponding to particles carrying their own internal degrees of freedom
- Even with such a bundle in hand, there may be many lifts of the symmetries of $\mathcal{M}_n$
- Resolving these can help to derive a full classification of multi-particle Hilbert spaces
- The gravity analog of this situation
- Better understanding of the crossing symmetry violation in electric-magnetic processes

REFERENCES

1. Completing the Multi-particle Representations of the Poincare Group (Csaba Csaki et al.)
2. Scattering amplitudes for monopoles: pairwise little group and pairwise helicity (Csaba Csaki et al.)
3. Angular Distributions and a Selection Rule in Charge-Pole Reactions (Daniel Zwanziger)
4. On the Hilbert Space of Dyons (R. Mouland and D. Tong)