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Final Examination

Phys209: Mathematical Methods in Physics I

2024/01/10

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) any of my work *outside the designated areas* in the "fill-in the blank questions" will not be graded;

(2) I take *full responsibility* for any ambiguity in my selection of the correct option in "multiple choice questions";

(3) any of my work *outside the answer boxes* in the "classical questions" will not be graded;

(4) any page which does not contain *both my name and student id* will not be graded;

(5) any extra sheet that I may use are for my own calculations and will *not* be graded.

Signature: ____________________

This exam has a total of 8 questions, some of which are for bonus points. You can obtain a maximum grade of 20+4 from this examination.

Question	Points	Score
1	6	
2	2	
3	4	
4	2	
5	1	

Question	Points	Score
6	1	
7	4	
8	0	
Total:	20	

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1 Notations & Conventions

- The non-negative integer power of an object A (denoted A^n) is defined recursively as follows:

$$A^0 = \mathbb{I}\,, \quad A^n = A \cdot A^{n-1}\ \forall n \geq 1 \tag{1}$$

  where the operation $\cdot$ is matrix multiplication if A is a matrix, application of differentiation if A is a differential operator (such as $\frac{\mathrm{d}}{\mathrm{d}x}$), or ordinary multiplication if A is simply a scalar number. $\mathbb{I}$ is the identity object with respect to the operation —identity matrix for matrix multiplication, the number 1 for ordinary multiplication, and so on.

- Exponentiation of an object A (denoted e^A) is defined as

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{2}$$

  where A^n is the $n-$th power of the object A. For instance, we can write down

$$e^{\frac{\mathrm{d}}{\mathrm{d}x}} = \cos\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) + i \sin\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{3}$$

  in accordance with the Euler formula.

- The Kronecker symbol (also called Kronecker-delta) is defined as

$$\delta :: \{\mathbb{Z}^+, \mathbb{Z}^+\} \to \mathbb{Z} \tag{4a}$$

$$\delta = \{i, j\} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{4b}$$

- The Dirac-delta generalized function δ is (for all practical purposes of a Physicist) defined via the relation

$$\int_{\mathcal{A}} f(y)\delta(x-y)dy = \begin{cases} f(x) & \text{if } x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

  A useful representation of Dirac-delta generalized function is

$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} \tag{6}$$

- We denote a particular permutation of n objects as $(i_1 i_2 \ldots i_n)$ where $i_1 \neq i_2 \neq \ldots \neq i_n \in \{1, \ldots, n\}$. A permutation $(i_1 \ldots i_n)$ is said to be an even (odd) permutation of $(k_1 \ldots k_n)$ if the two are identical after the permutation of an even (odd) number of adjacent indices. For example, (2431) is an even permutation of (2143) and an odd permutation of (2134).

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- We define Levi-Civita symbol as

$$\epsilon :: \{\mathbb{Z}^+, \dots, \mathbb{Z}^+\} \to \mathbb{Z} \tag{7a}$$

$$\epsilon = \{a_1, \dots, a_n\} \to \begin{cases} 1 & \text{if } (a_1 a_2 \dots a_n) \text{ is an even permutation of } (12\dots n) \\ -1 & \text{if } (a_1 a_2 \dots a_n) \text{ is an odd permutation of } (12\dots n) \\ 0 & \text{otherwise} \end{cases} \tag{7b}$$

- We define the determinant function (denoted det) as

$$\det :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{8a}$$

$$\det = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_{i_1,\dots,i_n} \epsilon_{i_1\dots i_n} a_{1i_1} \dots a_{ni_n} \tag{8b}$$

  where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

- We defined the adjugate function (denoted adj) as

$$\text{adj} :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{9a}$$

$$\text{adj} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \quad \text{where} \tag{9b}$$

$$b_{i_n k_n} = \frac{1}{(n-1)!} \epsilon_{i_1\dots i_n} \epsilon_{k_1\dots k_n} a_{i_1 k_1} \dots a_{i_{n-1} k_{n-1}} \tag{9c}$$

  where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

- We denote the inverse of a matrix A as A^{-1}: it satisfies the equations $A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{I}$ where $\mathbb{I}$ is the identity matrix. One can prove (which is beyond the scope of this course) that the inverse of a matrix A can be computed through its adjugate and its determinant:

$$A^{-1} = \frac{\text{adj}(A)}{\det A} \tag{10}$$

- We define the trace function (denoted tr) as

$$\text{tr} :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{11a}$$

$$\text{tr} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_i a_{ii} \tag{11b}$$

  where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

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- We define *Wronskian matrix* of a set of functions $\{f_1(x), \ldots, f_n(x)\}$ as a square matrix where the first row is the set of the functions and the $i-$th row is $(i-1)-$th derivative of the functions for all $n \geq i \geq 2$.

- A complex number $\mathbb{C}$ is a pair of two real numbers, i.e. $\mathbb{C} \simeq (\mathbb{R}, \mathbb{R})$ where the first real number is called its *real part* and second one is called its *imaginary* part. $\mathbb{C} \simeq (\mathbb{R}, \mathbb{R})$ indicates $(z \in \mathbb{C}) \leftrightarrow (x \in \mathbb{R}, y \in \mathbb{R})$ where one can construct z via $z = x + iy$ (i is called *the imaginary unit* with the property $i^2 = -1$); conversely, one can extract x and y via the functions Re and Im: $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

- The function *complex conjugation* (denoted $*$) is defined to act on complex numbers as

$$* :: \mathbb{C} \to \mathbb{C} \tag{12a}$$
$$* = z \to z^* = \mathrm{Re}(z) - i\,\mathrm{Im}(z) \tag{12b}$$

- The function *matrix transpose* (denoted T) is defined as

$$T :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{13a}$$
$$T = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \ldots & & & \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{pmatrix} \tag{13b}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

- The function *hermitian conjugation* (also called *conjugate transpose*, *adjoint*, or *dagger*) is defined over matrices of complex entries as

$$\dagger :: \mathfrak{M}_{n\times n}(\mathbb{C}) \to \mathfrak{M}_{n\times n}(\mathbb{C}) \tag{14a}$$
$$\dagger = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11}^* & a_{21}^* & \ldots & a_{n1}^* \\ a_{12}^* & a_{22}^* & \ldots & a_{n2}^* \\ \ldots & & & \\ a_{1n}^* & a_{2n}^* & \ldots & a_{nn}^* \end{pmatrix} \tag{14b}$$

- For any square matrix A, the *characteristic polynomial of* A is defined as

$$\det\left(A - \lambda_i \mathbb{I}\right) = 0 \tag{15}$$

- Fourier transforms are widely-used integral transformations (and are the simplest example of the harmonic analysis) which can be defined with any self-consistent convention. For this examination, please stick to the following conventions for Fourier transformation (and its different versions):

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$$f :: \mathbb{R} \to \mathbb{C} \tag{16a}$$
$$\hat{f} :: \mathbb{R} \to \mathbb{C} \tag{16b}$$
$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \hat{f}(k) \tag{16c}$$
$$\hat{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \tag{16d}$$

$$f :: \mathbb{Z} \to \mathbb{C} \tag{18a}$$
$$\hat{f} :: [a, a+T] \to \mathbb{C} \tag{18b}$$
$$f(n) = \frac{1}{T} \int_{a}^{a+T} dx e^{i\frac{2\pi n}{T}k} \hat{f}(k) \tag{18c}$$
$$\hat{f}(k) = \sum_{n=-\infty}^{\infty} e^{-i\frac{2\pi n}{T}k} f(n) \tag{18d}$$

$$f :: [a, a+T] \to \mathbb{C} \tag{17a}$$
$$\hat{f} :: \mathbb{Z} \to \mathbb{C} \tag{17b}$$
$$f(x) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{T}x} \hat{f}(n) \tag{17c}$$
$$\hat{f}(n) = \int_{a}^{a+T} dx e^{-i\frac{2\pi n}{T}x} f(x) \tag{17d}$$

$$f :: \mathbb{Z}_N \to \mathbb{Z}_N \tag{19a}$$
$$\hat{f} :: \mathbb{Z}_N \to \mathbb{Z}_N \tag{19b}$$
$$f(n) = \frac{1}{N} \sum_{m=0}^{N-1} e^{i\frac{2\pi nm}{N}} \hat{f}(m) \tag{19c}$$
$$\hat{f}(m) = \sum_{n=0}^{N-1} e^{-i\frac{2\pi nm}{N}} f(n) \tag{19d}$$

where (16), (17), (18), and (19) are called *Fourier Transform*, *Fourier Series*, *Discrete-time Fourier Transform*, and *Discrete Fourier Series* respectively. We will stick to this naming in this examination, but please be reminded that different communities (engineering, math, physics, etc.) use different naming conventions in general.

- The higher order functions *"even part of"* and *"odd part of"* (denoted E and O) are defined as

$$E :: (\mathcal{A} \to \mathcal{A}) \to (\mathcal{A} \to \mathcal{A}) \tag{20a}$$
$$E = (x \to f(x)) \to \left(x \to f_E(x) = \frac{f(x) + f(-x)}{2}\right) \tag{20b}$$
$$O :: (\mathcal{A} \to \mathcal{A}) \to (\mathcal{A} \to \mathcal{A}) \tag{20c}$$
$$O = (x \to f(x)) \to \left(x \to f_E(x) = \frac{f(x) - f(-x)}{2}\right) \tag{20d}$$

with which any single-argument function satisfies $f = E \cdot f + O \cdot f$, or with a more common notation, $f(x) = f_E(x) + f_O(x)$. Here $\mathcal{A}$ is any field, but we usually take it to be $\mathbb{C}$.

- We shall denote the inner product between two functions f and g as $\langle f, g \rangle_\omega^{\mathcal{A}}$ and define it via

$$\langle , \rangle_\omega^{\mathcal{A}} :: (\mathcal{A} \to \mathbb{C}, \mathcal{A} \to \mathbb{C}) \to \mathbb{C} \tag{21a}$$
$$\langle f, g \rangle_\omega^{\mathcal{A}} = \int_{A} \left(f(x)\right)^* g(x) \omega(x) dx \tag{21b}$$

for $\mathcal{A} \subseteq \mathbb{R}$.

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2 Fill-in the blanks

Each correct answer is worth 0.4 *point.*

Question: 1 . . . . *(6 points)*

The differential equations with constant coefficients form the simplest class of ordinary differential equations as they are explicitly solvable at any order! In contrast, it is not always possible to solve other classes of differential equations! Nevertheless, we can always convert a general linear ordinary differential equation to *a system of first order differential equations* which always have ___***formal***___ *solutions*, solutions that can not necessarily be written down in terms of elementary functions (for instance, $\int e^{x^4} dx$). Yet these solutions are still useful as we can analyze their limiting behavior or compute them numerically.

If we forego our overarching aim of computing general linear ordinary equations, we can actually do much more by focusing our attention to special classes of equations. For instance, the differential equations of the form $\mathcal{D}_1 \cdot \mathcal{D}_2 \cdot \dots \cdot \mathcal{D}_n \cdot f(x) = 0$ are fully solvable if the first order differential operators $\mathcal{D}_i$ ___***commute***___, i.e. $[\mathcal{D}_i, \mathcal{D}_j] = 0$. Even if they do not satisfy this condition, we can still make progress as $\mathcal{D}_n \cdot f(x) = 0$ is one of the solutions, and we can make use of this information with the method ___***reduction of order***___ which will convert our differential equation to a simpler one, i.e. $\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \cdots\right) \cdot g(x) = 0$. Another example for a special class of differential equations that are generically solvable is that which can be converted to a differential equation with constant coefficients via a ___***reparametrization***___; for instance, Euler equations are solvable because of this!

Even though we may have to put a lot of effort into solving general differential equations, the complexity of computations actually scales with the order of the differential equation and most of the differential equations we need to solve in Physics are of smaller orders. Furthermore, in some cases, the order may even be secretly lower! Indeed, a diffential equation $\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} + \dots\right) f(x) = 0$ may secretly be equivalent to $\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \dots\right) g(x)\right] = 0$: such equations are called ___***exact***___. In other cases, the equation may be secretly of a lower order for a whole another reason; for instance, the differential equation $f^n(x) + \cdots = 0$ can be converted to $g^{n-1}(x) + \cdots = 0$ for $g(x) = f'(x)$ if ___***$f(x)$ is missing***___ in the original equation.

Despite the existence of special classes of differential equations, sometimes it is unavoidable to try to solve a hard differential equation. We have seen that we can always find a *local* solution by expanding our function around an ordinary point of the differential equation, i.e. by taking $f(x) = \sum_{n=0}^{\infty} a_n x^n$. However, this expansion does not work if we are around ___***a singular***___ point; nevertheless, if this point is regular, we can use the ___***Frobenius***___ method, i.e. take $f(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n$.

Solving the homogeneous part of a linear ordinary differential equation (equivalently computing the ___***kernel***___ of the differential operator) is sufficient to get the full result: indeed, the particular solution can be computed once we have the full homogeneous solution! We have seen that this is possible through the method of ***variation of parameters***: this relates the particular solution to the inverse of the Wronskian matrix!

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Finding the solutions of a system may not be the only aim of a Physicist: indeed, sometimes it is more important to understand the *normal mods* of a system, i.e. the parameters of a system that behave independently. To find those, one analyzes the *eigensystem* of the system, which is the computation of ___*eigenvalues*___ of the matrix A (which describes this system) through the characteristic polynomial, and then calculation of ___*eigenvectors*___ through which one can construct the U matrix for which we have the equality $A = U \cdot D \cdot U^{-1}$. The diagonal matrix D here is called the ___*spectrum*___.

Although we have learned of quite advanced techniques to analyze differential equations, they are only applicable to certain classes of differential equations. For instance, a differential equation such as $\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) f(x,t) = 0$ is not really amenable to many of these techniques as it is a ___*partial*___ differential equation; likewise, it is also not easy to analyze $f''(x) + \cos(f(x)) = 0$ as it is a ___*nonlinear*___ differential equation.

3 Choose the correct option

Incorrect answers are worth -0.25 points: this ensures that the randomly given answer has an expectation value of 0 point.

Question: 2 . . . . *(2 points)*

Compute the *adjugate* of the matrix A, i.e. $\operatorname{adj}(A)$

(a) **(1 point)** for $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$:

■ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ □ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ □ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ □ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ □ None

(b) **(1 point)** for $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$:

□ $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$ □ $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ □ $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$ □ $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ ■ **None**

Question: 3 . . . . *(4 points)*

Compute the Fourier coefficient $\hat{f}(1)$

(a) **(1 point)** for $f :: [-5\pi, 5\pi] \to \mathbb{C}$ and $f(x) = \dfrac{\delta(x+2)}{(x-3)(x+4)}$:

□ $\dfrac{e^{-2i}}{-20\pi}$ □ $\dfrac{e^{2i}}{-8\pi}$ □ $\dfrac{e^{2i}}{-10}$ □ $\dfrac{e^{-2i}}{-4}$ ■ **None**

(b) **(1 point)** for $f :: \mathbb{R} \to \mathbb{C}$ and $f(x) = \dfrac{\delta(x+2)}{(x-3)(x+4)}$:

□ $\dfrac{e^{-2i}}{-20\pi}$ □ $\dfrac{e^{2i}}{-8\pi}$ ■ $\dfrac{e^{2i}}{-10}$ □ $\dfrac{e^{-2i}}{-4}$ □ None

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(c) **(1 point)** for $f :: \mathbb{R} \to \mathbb{C}$ and $f(x) = \cos(4x)\cos(2x)$:

☐ π ☐ $\frac{\pi}{2}$ ■ 0 ☐ $\frac{\pi}{-2}$ ☐ None

(d) **(1 point)** for $f :: \{0,1,2,3,4,5\} \to \mathbb{C}$ and $f(n) = \delta_{n0} - 2\delta_{n1} + 3\delta_{n2} - 2\delta_{n4} + 3\delta_{n5}$:

☐ i ■ 1 ☐ 0 ☐ -1 ☐ None

Question: 4 . . . . *(2 points)*

Compute the Wronskian matrix evaluated at $x = \frac{\pi}{2}$ for the differential operator $\mathcal{D}$ with the

(a) **(1 point)** $\ker(\mathcal{D}) = \{x \to 0, x \to \sin(x), x \to x\sin(x), x \to x^2\sin(x)\}$:

■ $\begin{pmatrix} 1 & \frac{\pi}{2} & \frac{\pi^2}{4} \\ 0 & 1 & \pi \\ -1 & -\frac{\pi}{2} & 2-\frac{\pi^2}{4} \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & \frac{\pi}{2} & \frac{\pi^2}{4} \\ 0 & 1 & \pi \\ -1 & \frac{\pi}{2} & 2-\frac{\pi^2}{4} \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & \frac{\pi}{2} & \frac{\pi^2}{4} \\ 0 & -1 & \pi \\ -1 & -\frac{\pi}{2} & 2+\frac{\pi^2}{4} \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & \frac{\pi}{2} & \frac{\pi^2}{4} \\ 0 & -1 & \pi \\ -1 & \frac{\pi}{2} & 2+\frac{\pi^2}{4} \end{pmatrix}$ ☐ None

(b) **(1 point)** $\ker(\mathcal{D}) = \left\{x \to 0, x \to e^{\cos(x)}, x \to e^{\sin(x)}, x \to \cot(x)\right\}$:

☐ $\begin{pmatrix} 1 & e & 0 \\ -1 & 0 & 1 \\ 1 & -e & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & e & 0 \\ 1 & 0 & 1 \\ 1 & e & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & e & 0 \\ 1 & 0 & -1 \\ 1 & e & 0 \end{pmatrix}$ ■ $\begin{pmatrix} 1 & e & 0 \\ -1 & 0 & -1 \\ 1 & -e & 0 \end{pmatrix}$ ☐ None

Question: 5 . . . . *(1 points)*

Compute the inner product

(a) **(½ point)** $\langle \sin, \cos \rangle_{\sin}^{[0,\pi/2]}$:

☐ $\frac{1}{5}$ ☐ $\frac{1}{4}$ ■ $\frac{1}{3}$ ☐ $\frac{1}{2}$ ☐ None

(b) **(½ point)** $\langle \sin, \cos \rangle_{\mathcal{I}}^{[0,\pi/2]}$ for $\mathcal{I} : x \to 1$:

☐ $\frac{1}{5}$ ☐ $\frac{1}{4}$ ☐ $\frac{1}{3}$ ■ $\frac{1}{2}$ ☐ None

Question: 6 . . . . *(1 points)*

Which one of the below is the set of the eigenvalues of the matrix $A = \begin{pmatrix} 24 & 5 & -4 \\ -16 & 0 & 16 \\ 4 & 5 & 16 \end{pmatrix}$?

☐ $\{20, 10, 8\}$ ☐ $\{20, 10, 6\}$ ☐ $\{20, 8, 6\}$ ☐ $\{5, 8, 6\}$ ■ **None**

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4 Classical questions

Question: 7 .... *(4 points)*

Leyla ile Mecnun (*majnūn laylā* in Arabic, and *laylâ-o-majnun* in Persian) is an old story based on the love life of a 7th century poet —you may liken it to the famous play of Shakespeare *Romeo & Juliet*. Our main protagonists are two lovers: Leyla, and Mecnun. We will consider a made-up version of their story to see if we can model their love life.

Assume that these lovers would like to marry each other, but living in 7th century, things are more complicated compared to the modern times: several obstacles lie before them (such as their parents) and their marriage is only probabilistic, not deterministic!

We are interested in the excitement of Leyla and Mecnun, which we can model as two functions of the perceived probability that they believe they will be able to marry (call p for brevity). The change in the excitement of Mecnun with respect to a change in p is relatively simple: it is simply linearly proportional to the excitement of Leyla (which makes sense!). The proportionality function is exponentiation of minus the function $j(p)$ –*the jealousy function*. Again, this is consistent with our everyday observations: increased jealousies would diminish the coupling between partners.

Being a member of the female gender (hence endowed with a very complex hormonal structure due to the evolutionary history), Leyla's excitement is far more complicated to model! The change in the excitement of Leyla with respect to a change in p is linearly proportional to "her own excitement", "the excitement of Mecnun", and "the change in the excitement of Mecnun with rescpect to a change in p". The first proportionality function can be taken as $j'(p)$: change in the jealousies relates Leyla's excitement and the change in Leyla's excitement — not that surprising. We will choose our second proportionality function such that it will blow up if **(1)** the probability becomes 0, **(2)** the probability becomes 1, and **(3)** the jealousy becomes infinity. Such a proportionality function ensures that our model breaks down in these extreme cases: a reasonable proportionality function that satisfies all these requirements is $p^{-1}(1-p)^{-1}\exp(j(p))$. For simplicity, we will also take the third proportionality function of this form up to an unknown coefficient (denote c).

With all of these information, what would be the differential equation that describes Mecnun's excitement as a function of the probability that they believe they can marry (denote as $M(p)$)? More explicitly: write down the linear ordinary *homogeneous* differential equation for $M(p)$!

Solution: The given information is equivalent to these two equations:

$$M'(p) = e^{-j(p)}L(p) \tag{22a}$$

$$L'(p) = j'(p)L(p) + \frac{e^{j(p)}}{p(1-p)}M(p) + \frac{ce^{j(p)}}{p(1-p)}M'(p) \tag{22b}$$

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for some unknown constant c. If we take the derivative of the first equation and insert the second one, we get the result:

$$\boxed{p(1-p)M''(p) - cM'(p) - M(p) = 0} \tag{23}$$

Bonus Question: 8 .... *(4 points)*

The differential equation you have written above is a special case of *hypergeometric differential equation.* People started analyzing such cases in 17$^{\text{th}}$ century (unlikely to study love lives), and many math giants including Euler, Gauss, and Riemann spent considerable time advancing this branch of math.

You may easily find tables of formulas, integral identities, series expansions, and even books dedicated to the study of hypergeometric functions (i.e. solutions to hypergeometric differential equation), so you may solve the differential equation of the previous question by pen and paper. Nevertheless, in 21$^{\text{th}}$ century, it makes little sense to do this labor so we'll consider outsourcing it to the computers: *write down the Mathematica code to solve the differential equation you have found in the previous question, with the condition $M(1/2) = 1/2$.*

Note: I would also accept and give you the full credit if you do not write down the code but solve it yourself (if you happen to know hypergeometric functions), but be warned: the answer is

$$\frac{\left(2^{c+\sqrt[3]{-1}}p^{c+1}\,{}_2F_1\left(\ldots;p\right) - 2^{c+1}c_1\,{}_2F_1\left(\ldots;-1\right)p^{c+1}\,{}_2F_1\left(\ldots;p\right) + 2^{\sqrt[3]{-1}}c_1\,{}_2F_1\left(\ldots;\frac{1}{2}\right)\,{}_2F_1\left(\ldots;p\right)\right)}{2^{\sqrt[3]{-1}}\,{}_2F_1\left(c-\sqrt[3]{-1}+1, c+(-1)^{2/3}+1; c+2; \frac{1}{2}\right)} \tag{24}$$

with the actual arguments instead of "..." in the numerator, and I find it unlikely for you to compute this within the examination time (and I will not grant any partial credit).

Solution: The Mathematica code to solve this problem is simply this:

```
DSolve[{p (1 - p) M''[p] - c M'[p] - M[p] == 0, M[1/2] == 1/2}, M[p], p]
```

« « « Congratulations, you have made it to the end! » » »