Name:	
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Midterm Examination - 2

Phys209: Mathematical Methods in Physics I

2024/12/25

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) any of my work *outside the designated areas* in the "fill-in the blank questions" will not be graded;

(2) I take *full responsibility* for any ambiguity in my selection of the correct option in "multiple choice questions";

(3) any of my work *outside the answer boxes* in the "classical questions" will not be graded;

(4) any page which does not contain *both my name and student id* will not be graded;

(5) any extra sheet that I may use are for my own calculations and will *not* be graded.

Signature: ____________________

This exam has a total of 3 questions, some of which may be for bonus points. You can obtain a maximum grade of 34+0 from this examination.

Question	Points	Score
1	14	
2	8	

Question	Points	Score
3	12	
Total:	34	

1 Notations & Conventions

This section contains various useful definitions to refer while solving the problems. Note that it might contain additional information not covered in class, so please do not panick: the questions do not necessarily refer to *everything* in this section.

- **The non-negative integer power** of an object A (denoted A^n) is defined recursively as

$$A^0 = \mathbb{I}\ , \quad A^n = A \cdot A^{n-1} \ \ \forall n \geq 1 \tag{1}$$

with respect to the operation $\cdot$ (such as matrix multiplication or differentiation) and its identity object $\mathbb{I}$.

- **Exponentiation of an object** A (denoted e^A) is

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{2}$$

where A^n is the $n-$th power of the object A.

- **Logarithm of an object** A (denoted $\log A$) is defined as the inverse of the exponentiation. For objects for which the exponentiation is not a monomorphism (such as complex numbers), logarithm is a *relation* (also called multi-valued function). Conventionally, one imposes restrictions on the domain to ensure that logarithm acts as a function; for instance, for a complex number $z = re^{i\theta} \in \mathbb{C}$ with $(r, \theta) \in (\mathbb{R}^+, \mathbb{R})$, we can define $\log z = i\theta_p + \log r$ where $0 \leq \theta_p < 2\pi$ is called *the principal value of* θ.

- **The generalized power of an object** A (denoted A^α) is defined as

$$A^\alpha = e^{\alpha \log A} \tag{3}$$

If exponentiation is not a monomorphism when acting on the domain of A, A^α is not a function but a relation *unless* a principle domain is selected (similar to the logarithm).

- **Generalized exponentiation of an object** A (denoted α^A) is defined as

$$\alpha^A = e^{A \log \alpha} \tag{4}$$

Depending on the available values for $\log \alpha$, α^A may mean multiple different functions. However, each one is *still* a proper function, not a multi-valued function.

- **Trigonometric functions** cos, sin, tan, cot, csc , sec are defined in terms of the exponential via the equations

$$e^{\pm iA} = \cos(A) \pm i \sin(A)\ ,\ \tan(A) = \frac{1}{\cot(A)} = \frac{\sin(A)}{\cos(A)} \tag{5}$$

$$\csc(A)\sin(A) = 1\ ,\ \sec(A)\cos(A) = 1 \tag{6}$$

- **Hyperbolic functions** cosh, sinh, tanh, coth, csch, sech are defined in terms of the exponential via equations

$$e^{\pm A} = \cosh(A) \pm \sinh(A), \tanh(A) = \frac{1}{\coth(A)} = \frac{\sinh(A)}{\cosh(A)} \tag{7}$$

$$\operatorname{csch}(A)\sinh(A) = 1\ ,\ \operatorname{sech}(A)\cosh(A) = 1 \tag{8}$$

- **Inverse Trigonometric/Hyperbolic functions** are denoted with an *arc* prefix in their naming, i.e. $\arcsin(x) := \sin^{-1}(x)$. Like logarithm, these objects are *relations* (not functions) unless their domain is restricted.

- **The Kronecker symbol** (Kronecker-delta) is defined

$$\delta : \{\mathbb{Z}, \mathbb{Z}\} \to \mathbb{Z} \tag{9}$$

$$\delta = \{i, j\} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{10}$$

- **The Dirac-delta generalized function** δ is (for all practical purposes of a Physicist) defined via the relation

$$\int_{\mathcal{A}} f(y)\delta(x-y)dy = \begin{cases} f(x) & \text{if } x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

A useful representation of this generalized function is

$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} \tag{12}$$

- **Heaviside generalized function** θ is (for all practical purposes of a Physicist) defined via the relations

$$\int_a^b \theta(x) f(x) dx = \begin{cases} \displaystyle\int_a^b f(x)dx & \text{if } a \geq 0 \\ \displaystyle\int_0^b f(x)dx & \text{if } a < 0 \end{cases} \tag{13}$$

This definition implies that $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$; however, it *does not fix* $f(0)$. We choose *the convention* $f(0) = 1/2$; this ensures

$$\operatorname{sgn}(x) = 2\theta(x) - 1 = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases} \tag{14}$$

- **A particular permutation of n objects** is denoted as $(i_1 i_2 \dots i_n)$ where $i_1 \neq i_2 \neq \dots \neq i_n \in \{1, \dots, n\}$. A permutation $(i_1 \dots i_n)$ is said to be an even (odd) permutation of $(k_1 \dots k_n)$ if the two are identical after the permutation of an even (odd) number of adjacent indices. For example, (2431) is an even permutation of (2143) and an odd permutation of (2134).

- **Levi-Civita symbol** ϵ is defined as

$$\epsilon : \{\mathbb{Z}^+, \dots, \mathbb{Z}^+\} \to \mathbb{Z} \tag{15}$$

$$\epsilon = \{a_1, \dots, a_n\} \to \begin{cases} 1 & \text{if } (a_1 a_2 \dots a_n) \text{ is an even permutation of } (12\dots n) \\ -1 & \text{if } (a_1 a_2 \dots a_n) \text{ is an odd permutation of } (12\dots n) \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

- **The determinant function** (denoted det) is defined

$$\det : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{17}$$

$$\det = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_{i_1,\dots,i_n} \epsilon_{i_1\dots i_n} a_{1i_1}\dots a_{ni_n} \tag{18}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

- **The adjugate function** (denoted adj) is defined as

$$\mathrm{adj} : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{19}$$

$$\mathrm{adj} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \tag{20}$$

for

$$b_{k_n i_n} = \sum_{\substack{i_1,\dots,i_{n-1} \\ k_1,\dots,k_{n-1}}} \frac{\epsilon_{i_1\dots i_n}\epsilon_{k_1\dots k_n} a_{i_1k_1}\dots a_{i_{n-1}k_{n-1}}}{(n-1)!} \tag{21}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

- **Inverse of an object** A is denoted as A^{-1} and is defined with respect to an operation "$\cdot$" and its identity element $\mathbb{I}$ via the the equations $A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{I}$. If "$\cdot$" is matrix multiplication, then

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det A} \tag{22}$$

- **The trace function** (denoted tr) is defined as

$$\mathrm{tr} : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{23}$$

$$\mathrm{tr} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_i a_{ii} \tag{24}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

- **Wronskian matrix** of a set of functions $\{f_1(x), \dots, f_n(x)\}$ is defined as a square matrix where the first row is the set of the functions and the $i-$th row is $(i-1)-$th derivative of the functions for all $n \geq i \geq 2$.

- **A complex number** z is (for all practical purposes of a Physicist) a pair of two real numbers (x, y) where one can construct z via $z = x + iy$ (i is called *the imaginary unit* with the property $i^2 = -1$); conversely, one can extract x and y via $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

- **Complex conjugation** (denoted $*$) is a function defined to act on complex numbers as

$$* : \mathbb{C} \to \mathbb{C} \tag{25}$$

$$* = z \to (z^* = \mathrm{Re}(z) - i\,\mathrm{Im}(z)) \tag{26}$$

- **Matrix transpose** (denoted T) is a function defined

$$T : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{27}$$

$$T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \tag{28}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$.

- **Hermitian conjugation** (also called *conjugate transpose*, *adjoint*, or *dagger*) is a function defined as

$$\dagger : \mathfrak{M}_{n\times n}(\mathbb{C}) \to \mathfrak{M}_{n\times n}(\mathbb{C}) \tag{29}$$

$$\dagger = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a^*_{11} & a^*_{21} & \dots & a^*_{n1} \\ a^*_{12} & a^*_{22} & \dots & a^*_{n2} \\ \dots \\ a^*_{1n} & a^*_{2n} & \dots & a^*_{nn} \end{pmatrix} \tag{30}$$

- **Characteristic polynomial** of any square matrix A:

$$\det(A - \lambda_i \mathbb{I}) = 0 \tag{31}$$

- **Laplace transform** is an integral transform (denote $\mathcal{L}$) which converts a function $f : \mathbb{R} \to \mathbb{R}$ into another function $\hat{f} = \mathcal{L}(f)$ such that

$$\hat{f} : \mathbb{C} \to \mathbb{C} , \qquad \hat{f}(s) = \int_0^\infty f(x) e^{-xs} dx \tag{32}$$

For *meromorphic* $\hat{f}$ (i.e. $\frac{\text{polynomial}}{\text{polynomial}}$), the inverse is computed by rewriting $\hat{f}(s)$ as a sum $\sum_i a_i(s + r_i)^{-n_i-1}$ which is clearly (for some $c_{k,\ell}$) the Laplace transform of $f(x) = \sum_i e^{-r_i x} (c_{i,1} + c_{i,2}x + \dots c_{i,n_i}x^{n_i})$. Formally,

$$f : \mathbb{R} \to \mathbb{R} , \qquad f(x) = \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{f}(s) e^{xs} \frac{ds}{2\pi i} \tag{33}$$

where the *contour integral* in the complex plane is chosen appropriately based on the convergence.

- **Convolution** of two functions f and g (denote $f * g$) is the operation that becomes multiplication in the Laplace domain, i.e. $\mathcal{L}(f * g) \equiv \mathcal{L}(f)\mathcal{L}(g)$; equivalently,

$$(f * g)(x) = \int_0^x f(y) g(x - y) dy \tag{34}$$

- **Commutator** is a higher order function which takes two functions $f, g : \mathcal{A} \to \mathcal{A}$ for any type $\mathcal{A}$, and gives a new function $[f, g] : \mathcal{A} \to \mathcal{A}$ by cascading their action. It is defined on an object $x \in \mathcal{A}$ as $[f, g](x) = f(g(x)) - g(f(x))$.

- **Polar coordinates in $\mathbb{R}^d$** $(r, \theta_1, \dots, \theta_{d-1})$ are defined in terms of the Cartesian coordinates $(x_1, \dots, x_d)$ as

$$x_1 = r\cos(\theta_1) , \quad x_d = x_{d-1}\tan(\theta_{d-1}) \tag{35}$$

$$x_i = x_{i-1}\tan(\theta_{i-1})\cos(\theta_i) \quad \text{for } 1 < i < d \tag{36}$$

In two-dimensions, this reduces to the familiar polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$; in 3 ($> 3$) dimensions, it is also called *(hyper)spherical coordinates.*

- **Cylindrical coordinates in $\mathbb{R}^d$** $(\rho, \theta_1, \dots, \theta_{n-1}, x_n, x_{n+1}, \dots, x_d)$ is a coordinate system such that a subset $\mathbb{R}^n$ of the total space $\mathbb{R}^d$ (for $n < d$) is converted into the polar coordinates. For instance, if we convert $\mathbb{R}^2$ of $\mathbb{R}^3$ into polar coordinates, we obtain the familiar *3d cylindrical coordinates*, i.e. $(x, y, z) = (\rho\cos\theta, \rho\sin\theta, z)$.

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2 Fill-in the blanks

Each correct answer is worth 1.4 point.

Question: 1 . . . . *(14 points)*

In the first midterm, we discussed classification of differential equations and various techniques to solve them, such as reduction of order. Our discussion was mainly focused on solving *linear, ordinary, and* ______***homogeneous***______ differential equations, that is $\left(\sum_{i=0}^{n}\alpha_i(x)\frac{d^i}{dx^i}\right)f(x)=g(x)$ with $g(x)=0$. Let us now discuss what happens when $g(x)\neq 0$.

Unlike the homogeneous solutions that come with undetermined coefficients which can only be fixed via boundary/initial conditions, the ______***particular***______ solution (solution due to nonzero $g(x)$) is unique; since verification of a solution candidate is usually way faster than solving the differential equation itself, guessing a solution (and then verifying it) is the most effective way. However, most likely it is impossible to guess the precise form of the solution; one then simply guess the *functional form* of the solution (which is called making an ansatz) and then fix the parameters in the ansatz by ensuring the ansatz does solve the differential equation: this approach is called *method of* ***undetermined coefficients***. As the last resort, we can always stick to the algorthmic technique, *method of* ***variation of parameters***, if we do not have any idea regarding the form of the solution.

Let us switch gears and talk about matrices: any *diagonalizable* matrix A can be written as $A=U.D.U^{-1}$ for a diagonal matrix D: this relation is called ______***similarity***______ *transformation*. As we explicitly discussed in class, the procedure to find D and U is as follows: **(1)** solve ______***characteristic***______ *equation*, i.e. $\det(A-\lambda\mathbb{I})=0$, which is a polynomial in λ; **(2)** find the roots of this polynomial, which are called *the* ______***eigenvalues***______ *of the matrix* A; **(3)** for each such λ, find the column matrix u which satisfies $(A-\lambda\mathbb{I}).u=0$: they are called *the* ______***eigenvectors***______ *of the matrix* A; **(4)** combine the column matrices to obtain the square matrix U and construct D by putting λ's on the diagonal.

The sudden discussion of matrices in the middle of a differential equations course was not unjustified: we did have seen in class explicitly that any *linear, ordinary, and finite-order differential equation* can be converted to a system of *linear, ordinary, and* ______***first***______ *order* differential equations; in other words, understanding differential equations of the form $\frac{d}{dt}V(t)=A(t).V(t)$ for an $n\times n$ matrix $A(t)$ has far-reaching consequences than it would suggest at the face value. We have explicitly solved the case when $A(t)$ is a constant matrix; however, things get complicated if it is not a constant matrix: we now have to solve *Volterra integral equation*, which has a complicated answer. Even though the solution is actually quite simple when $A(t)$ is a 1×1 matrix, i.e. $V(t)=c\exp\left(\int_1^t A(\tau)d\tau\right)$ for an undetermined coefficient c, the answer gets complicated once it is no longer a 1×1 matrix. The reason behind this is that, unlike 1×1 matrices, two generic $n\times n$ matrices do not necessarily ______***commute***______, i.e. $A.B=B.A$ is not necessarily true!

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3 Choose the correct option

You do not need to show your derivation in this part.
Incorrect answer for a question of X point is worth $-X/4$ points: this ensures that the randomly given answer has an expectation value of 0 point.

Question: 2 .... *(8 points)*

In their paper titled *Neural Ordinary Differential Equations*, Chen and their collaborators introduce "deep neural network models for artificial intelligence" that are modelled via a first order ordinary differential equation. As can be seen in their paper https://arxiv.org/pdf/1806.07366 (published December 2019), the general form of the differential equation they consider is

$$\frac{dh(t)}{dt} = f(h(t), t, \theta) \tag{37}$$

for an external parameter θ, a given function f, and the unknown column matrix $h(t)$.

Since non-linear equations are incredibly hard to solve by hand (and we have only covered linear differential equations up to this point anyway), let us consider a subclass of this equation

$$\frac{dh(t)}{dt} = A.h(t) + \theta \tag{38}$$

for

$$h(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & \omega \\ 0 & -1 \end{pmatrix}, \quad \theta = \begin{pmatrix} \omega \\ -1 \end{pmatrix} \tag{39}$$

for some constant ω. We will now analyze this equation and solve it for generic ω.

(a) **(3 points)** With the matrix A being time independent, we can immediately propose that the homogeneous solution is $h(t) = \exp(At).C$ for a column matrix C of undetermined entries. Which one of the below is $\exp(At)$? *Hint: use the definition of the exponential function and the definitions for the hyperbolic functions.*

■ $\begin{pmatrix} e^t & \omega\sinh(t) \\ 0 & e^{-t} \end{pmatrix}$ □ $\begin{pmatrix} e^t & \omega\sinh(t) \\ 0 & e^{t} \end{pmatrix}$ □ $\begin{pmatrix} e^{-t} & \omega\sinh(t) \\ 0 & e^{-t} \end{pmatrix}$ □ $\begin{pmatrix} e^{-t} & \omega\sinh(t) \\ 0 & e^{t} \end{pmatrix}$ □ None

Solution 2.1 *Observe that the matrix A squares to the identity matrix, $A^2 \equiv A \cdot A = \mathbb{I}$. This means that any even integer power of A is identity matrix and any odd integer power of A is itself. We can then use the definition of the exponential:*

$$\exp(At) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \mathbb{I}\left(\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\right) + A\left(\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\right) \tag{40}$$

hence

$$\exp(At) = \begin{pmatrix} \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\right) & \omega \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \\ 0 & \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\right) \end{pmatrix} \tag{41}$$

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which simplifies as

$$\exp(At) = \begin{pmatrix} e^t & \omega \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \\ 0 & e^{-t} \end{pmatrix} \tag{42}$$

How do we find that sum? Well, there is a hint that we should use the definition of hyperbolic functions! We are given that

$$e^{\pm x} = \cosh(x) \pm \sinh(x) \tag{43}$$

hence

$$\cosh(x) + \sinh(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,, \quad \cosh(x) - \sinh(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \tag{44}$$

By summing or subtracting, we see that

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\,, \quad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \tag{45}$$

with which we get the final answer.

(b) **(3 points)** Since θ is time-independent, we can immediately conclude that the particular solution for $h(t)$ (call $p(t)$) is constant and satisfies $p(t) = -A^{-1}.\theta$. Which one below is the particular solution?

☐ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ☐ $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ■ $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ ☐ None

Solution 2.2 *We need to compute the inverse of the matrix A. The relevant formula is already provided:*

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det A} \tag{46}$$

hence with the definitions of adjugate and determinant provided, the components of the inverse matrix reads as

$$(A^{-1})_{k_2 i_2} = \frac{\sum\limits_{i_1,k_1} \epsilon_{i_1 i_2}\epsilon_{k_1 k_2} A_{i_1 k_1}}{\sum\limits_{\ell_1,\ell_2} \epsilon_{\ell_1 \ell_2} A_{1\ell_1} A_{2\ell_2}} \tag{47}$$

hence

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{21}A_{12}} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} = \begin{pmatrix} 1 & \omega \\ 0 & -1 \end{pmatrix} \tag{48}$$

Of course, in the previous part, we already showed that $A^2 = \mathbb{I}$ so it was obvious from the start that $A^{-1} = A$. Anyway, we can then compute $p(t) = -A^{-1}.\theta$ as

$$p(t) = -\begin{pmatrix} 1 & \omega \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \omega \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{49}$$

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(c) **(2 points)** If you are additionally given the information that $y(0) = 0$, what is the approximate value for $y(-1/100)$?
Hint: you may take $e^{1/100} \simeq 1.01$ *and* $e^{-1/100} \simeq 0.99$

☐ -0.02 ☐ -0.01 ■ 0.01 ☐ 0.02 ☐ None

Solution 2.3 *We were told in part a that the homogeneous soultion is* $\exp(At).C$ *for a column matrix* C *of undetermined entries, and we just computed the particular solution in part b. Therefore, the full solution is*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t & \omega \sinh(t) \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{50}$$

which means

$$y(t) = e^{-t} - 1 \tag{51}$$

where we also fixed $c_2 = 1$ *by using the given information* $y(0) = 0$*. Then*

$$y(-1/100) = e^{1/100} - 1 \simeq 1.01 - 1 = 0.01 \tag{52}$$

Question: 3 . . . . *(12 points)*

A black hole is a complicated object which requires knowledge far beyond what is covered in this course. Nevertheless, let us consider the following case study.

Assuming bunch of technical details (a non-rotating spherically symmetric black hole that can be modelled as an asymptotically AdS spacetime), consider how light would be affected near a black hole (under the assumptions that light passes sufficiently away from the black hole and that there is nothing else nearby to the light in this universe). The brightness of the light can be shown to be related to the *amplitude* $A(r)$ where r is the radial distance between the light and the black hole. This amplitude satisfies *Bessel's differential equation* in a judiciously chosen coordinate system:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left(1 - \frac{1}{4r^2}\right)\right) A(r) = g(r) \tag{53}$$

(a) **(2 points)** The homogeneous solutions to this differential equation (denote them as $A_\pm(r)$) are $A_\pm(r) = \dfrac{\exp(\pm ar)}{\sqrt{r}}$. Which one of the below expressions is the correct one?
Hint: we define i*, the imaginary unit, as the solution to the equation* $i^2 + 1 = 0$.

■ **a=i** ☐ a=2i ☐ a=3i ☐ a=4i ☐ None

Solution 3.1 *Let us simply insert the ansatz into the differential equation to find a: with the*

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results of the basic calculus

$$\begin{aligned}\frac{d}{dr}\frac{\exp(\pm ar)}{\sqrt{r}} &= \frac{\exp(\pm ar)(-1\pm 2ar)}{2r^{3/2}}\\ \frac{d^2}{dr^2}\frac{\exp(\pm ar)}{\sqrt{r}} &= \frac{\exp(\pm ar)(3\pm 4ar(-1\pm ar))}{4r^{5/2}}\end{aligned} \tag{54}$$

we see that

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\left(1-\frac{1}{4r^2}\right)\right)\frac{\exp(\pm ar)}{\sqrt{r}} = \frac{(1+a^2)\exp(\pm ar)}{\sqrt{r}} \tag{55}$$

This becomes zero if $a^2+1=0$. But we are told in the hint that i called the imaginary unit satisfies by definition the equation $i^2+1=0$, hence we conclude $a=i$.

(b) **(3 points)** Let us make the ansatz for the full solution as $A(r) = A_1(r)c_1(r) + A_2(r)c_2(r)$ for two arbitrary functions $c_{1,2}(r)$ where we choose $A_1(r) = \dfrac{\cos(r)}{\sqrt{r}}$ and $A_2(r) = \dfrac{\sin(r)}{\sqrt{r}}$ as linear combinations of $A_\pm(r)$. Inserting this in the differential equation, we get

$$\cos(r)(c_1''(r)+\ell c_2'(r)) + \sin(r)(c_2''(r)-\ell c_1'(r)) = r^k g(r) \tag{56}$$

What is $k+\ell$?

☐ $-\frac{1}{2}$ ☐ $\frac{1}{2}$ ☐ $\frac{3}{2}$ ■ $\frac{5}{2}$ ☐ None

Solution 3.2 *By chain rule, we see that*

$$\begin{aligned}\frac{dA(r)}{dr} &= c_1(r)A_1'(r)+c_2(r)A_2'(r)+A_1(r)c_1'(r)+A_2(r)c_2'(r)\\ \frac{d^2A(r)}{dr^2} &= 2A_1'(r)c_1'(r)+2A_2'(r)c_2'(r)+c_1(r)A_1''(r)+c_2(r)A_2''(r)+A_1(r)c_1''(r)+A_2(r)c_2''(r)\end{aligned} \tag{57}$$

Inserting these into the differential equation and using the fact that

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\left(1-\frac{1}{4r^2}\right)\right)A_{1,2}(r) = 0 \tag{58}$$

which is true as $A_{1,2}(r)$ are linear combinations of $A_\pm(r)$, we obtain the result

$$\frac{\sin(r)\,(c_2''(r)-2c_1'(r))+\cos(r)\,(2c_2'(r)+c_1''(r))}{\sqrt{r}} = g(r) \tag{59}$$

Comparing with the form given in the question, we conclude that $\ell = 2$ and $k = 1/2$, hence $\ell + k = 5/2$.

(c) **(4 points)** By choosing $c_{1,2}'(r)$ to be linearly dependent with a judiciously chosen propor-

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tionality, we can bring this equation to the form:

$$\begin{pmatrix} c_1'(r) \\ c_2'(r) \end{pmatrix} = \begin{pmatrix} \dfrac{\cos(r)}{\sqrt{r}} & \dfrac{\sin(r)}{\sqrt{r}} \\ -\dfrac{2r\sin(r)+\cos(r)}{2r^{3/2}} & -\dfrac{\sin(r)-2r\cos(r)}{2r^{3/2}} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ g(r) \end{pmatrix} \tag{60}$$

What is $c_2'(r)$?

☐ $\dfrac{\cos(r)}{\sqrt{r}}g(r)$ ■ $\sqrt{r}\cos(r)g(r)$ ☐ $\dfrac{\sin(r)}{\sqrt{r}}g(r)$ ☐ $\sqrt{r}\sin(r)g(r)$ ☐ None

Solution 3.3 *We need to compute the inverse of a matrix M. The relevant formula is already provided:*

$$M^{-1} = \frac{\mathrm{adj}(M)}{\det M} \tag{61}$$

hence with the definitions of adjugate and determinant provided, the components of the inverse matrix reads as

$$(M^{-1})_{k_2 i_2} = \frac{\sum_{i_1,k_1} \epsilon_{i_1 i_2}\epsilon_{k_1 k_2} M_{i_1 k_1}}{\sum_{\ell_1,\ell_2} \epsilon_{\ell_1 \ell_2} M_{1\ell_1} M_{2\ell_2}} \tag{62}$$

hence

$$M^{-1} = \frac{1}{M_{11}M_{22} - M_{21}M_{12}} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} = \frac{1}{1/r} \begin{pmatrix} -\dfrac{\sin(r)-2r\cos(r)}{2r^{3/2}} & -\dfrac{\sin(r)}{\sqrt{r}} \\ \dfrac{2r\sin(r)+\cos(r)}{2r^{3/2}} & \dfrac{\cos(r)}{\sqrt{r}} \end{pmatrix} \tag{63}$$

for $M = \begin{pmatrix} \dfrac{\cos(r)}{\sqrt{r}} & \dfrac{\sin(r)}{\sqrt{r}} \\ -\dfrac{2r\sin(r)+\cos(r)}{2r^{3/2}} & -\dfrac{\sin(r)-2r\cos(r)}{2r^{3/2}} \end{pmatrix}$. *We then finally see that*

$$\begin{pmatrix} c_1'(r) \\ c_2'(r) \end{pmatrix} = r \begin{pmatrix} -\dfrac{\sin(r)-2r\cos(r)}{2r^{3/2}} & -\dfrac{\sin(r)}{\sqrt{r}} \\ \dfrac{2r\sin(r)+\cos(r)}{2r^{3/2}} & \dfrac{\cos(r)}{\sqrt{r}} \end{pmatrix} \begin{pmatrix} 0 \\ g(r) \end{pmatrix} \tag{64}$$

leading to the result $c_2'(r) = \sqrt{r}\cos(r)g(r)$.

(d) **(3 points)** Given $g(r) = r^{-1/2}$ and $A\left(\frac{\pi}{2}\right) = A(\pi) = 0$, what is $A(2\pi)$?

☐ $\sqrt{\dfrac{3}{\pi}}$ ☐ $\sqrt{\dfrac{\pi}{3}}$ ☐ $\dfrac{3}{\pi}$ ☐ $\dfrac{\pi}{3}$ ■ **None**

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Solution 3.4 *From the previous part, we see that*

$$c_1'(r) = -\sqrt{r}\sin(r)g(r)\ , \quad c_2'(r) = -\sqrt{r}\cos(r)g(r) \tag{65}$$

which for $g(r) = r^{-1/2}$ *gives*

$$c_1(r) = \gamma_1 + \cos(r)\ , \quad c_2(r) = \gamma_2 - \sin(r) \tag{66}$$

upon integration. Here $\gamma_{1,2}$ *are integration constants to be fixed by the boundary conditions. Since in part b we made the ansatz for the full solution*

$$A(r) = \frac{\cos(r)}{\sqrt{r}}c_1(r) + \frac{\sin(r)}{\sqrt{r}}c_2(r) \tag{67}$$

we see that it becomes

$$A(r) = \gamma_1\frac{\cos(r)}{\sqrt{r}} + \gamma_2\frac{\sin(r)}{\sqrt{r}} + \frac{\cos(2r)}{\sqrt{r}} \tag{68}$$

where the firts two solutions are the homogeneous solutions and the last one is the particular solution —we wrote the particular solution in a simpler form using the identity $\cos(r)^2 - \sin(r)^2 = \cos(2r)$ *but it is not required to solve the question (one can actually derive this using the definitions provided in the exam). We are provided the information* $A(\pi/2) = A(\pi) = 0$ *which means*

$$\begin{aligned} A(\pi/2) =& \frac{\gamma_2 - 1}{\sqrt{\pi/2}} = 0 \\ A(\pi) =& \frac{-\gamma_1 + 1}{\sqrt{\pi}} = 0 \end{aligned} \tag{69}$$

which fixes $A(r)$ *completely:*

$$A(r) = \frac{\cos(r)}{\sqrt{r}} + \frac{\sin(r)}{\sqrt{r}} + \frac{\cos(2r)}{\sqrt{r}} \tag{70}$$

We then see that

$$A(2\pi) = \sqrt{\frac{2}{\pi}} \tag{71}$$

« « « Congratulations, you have made it to the end! » » »