Phys209: Mathematical Methods in Physics I

Homework 7

Soner Albayrak[†]

[†]*Middle East Technical University, Ankara 06800, Turkey*

Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) are for your own practice!
- Unless stated otherwise, you are expected to show your derivation of the results.
- The homework is due November 24th 2023, 23:59 TSI.

(1) Levi-Civita Symbol *(2 points)*

We have introduced in class the Levi-Civita symbol ϵ:

$$\epsilon :: \{\mathbb{Z}^+, \ldots, \mathbb{Z}^+\} \to \mathbb{Z}$$
$$\epsilon = \{i_1, \ldots, i_n\} \to \begin{cases} 1 & (i_1 i_2 \ldots i_n) \text{ is an even permutation of } (12\ldots n) \\ -1 & (i_1 i_2 \ldots i_n) \text{ is an odd permutation of } (12\ldots n) \\ 0 & i \neq j \end{cases}$$

where the arguments are conventionelly written as subscripts, e.g. $\epsilon_{231} = 1$ and so on. In this question, we will see an alternative way to define this symbol. For this, we first need to define *Kronecker delta symbol*.

Kronecker delta symbol is defined as follows:

$$\delta :: \{\mathbb{Z}^+, \mathbb{Z}^+\} \to \mathbb{Z}$$
$$\delta = \{i, j\} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

which is also conventionally written with arguments as subscripts, e.g. $\delta_{12} = 0$. We can use this function to define Levi-Civita symbol; for instance, we can find out by inspection that

$$\epsilon_{ij} = \delta_{i1}\delta_{j2} - \delta_{j1}\delta_{i2}$$
$$\epsilon_{ijk} = \delta_{i1}\delta_{j2}\delta_{k3} - \delta_{i1}\delta_{k2}\delta_{j3} + \delta_{k1}\delta_{i2}\delta_{j3} - \delta_{k1}\delta_{j2}\delta_{i3} + \delta_{j1}\delta_{k2}\delta_{i3} - \delta_{j1}\delta_{i2}\delta_{k3}$$

(1.1) (0.5pt)

Write down ϵ_{2ij} in terms of δ symbols.

(1.2) (0.5pt)

Write down $\sum_{i=1}^{3} \epsilon_{ij}\epsilon_{ik}$ in terms of δ symbols.

(1.3) (0.5pt)

Consider the relation $\epsilon_{i_1 i_2 i_3 \dots i_{98} i_{99} i_{100}} = \alpha \epsilon_{i_1 i_{99} i_3 \dots i_{98} i_2 i_{100}}$ where i_2 and i_{99} are interchanged in the second ϵ symbol. What is α?

(1.4) (0.5pt)

Write down $\sum_{\ell=1}^{4} \epsilon_{ijk\ell} \delta_{\ell 4}$ in terms of ϵ symbols.

(2) Determinant *(2 points)*

We have seen in class that det is a function defined as

$$\det :: \mathfrak{M}_{n,n}(\mathbb{C}) \to \mathbb{C}$$
$$\det = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1 i_2 \dots i_n} a_{1 i_1} a_{2 i_2} \dots a_{n i_n}$$

where $\mathfrak{M}_{n,n}(\mathbb{C})$ denotes the set of $n \times n$ matrices with complex entries.

Consider two 2×2 matrices A and B. Let $C = A \cdot B$ where $\cdot$ denotes the matrix multiplication. One can show that

$$\det C = \alpha \ (\det A)(\det B)$$

for a constant α. Find α (show your derivation)!

(3) Wronskian *(2 points)*

We have defined in class that the Wronskian determinant W for a set of solutions $f_i(x)$ is given by

$$W = \det \begin{pmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \dots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{pmatrix}_{x=a}$$

where a is usually determined by boundary conditions. Let us take $a = 1$ in this question.

Compute the Wronskian of the three solutions you have found in homework 6, i.e.

$$f_1(x) = \frac{1}{x}\,, \quad f_2(x) = \frac{e^{-x/2}}{x}\,, \quad f_3(x) = x - 4\,,$$