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Midterm Examination - 2

Phys331: Electromagnetic Theory I

2025/12/4

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) all calculations are to be conducted in the notations and conventions of the formulae sheets provided during the exam unless explicitly stated otherwise in the question;

(2) I take *full responsibility* for any ambiguity in my selections in "multiple choice questions";

(3) incorrect selections will receive $-1/7$ of the question's points;

(4) I am expected to provide *step-by-step explanation of how I solved the question* and am expected to do so *only within the answer boxes* provided with the questions: the explanation is supposed to be succinct, well-articulated, and correct both scientifically and mathematically;

(5) no partial credit is awarded for the explanations provided in the answer boxes;

(6) some questions of some students will be randomly selected for inspection: *a question (if selected for inspection) might be awarded negative points* if its explanation is incorrect or insufficient to get the correct answer, even if the correct option is selected;

(7) any page which does not contain *both my name and student id* may not be graded;

(8) any extra sheet that I may use are for my own calculations and will not be graded.

Signature: ____________________________

This exam has a total of 3 questions, some of which may be for bonus points. You can obtain a maximum grade of 105+0 from this examination.

Question:	1	2	3	Total
Points:	21	21	63	105

Name:	
Student ID:	

Question: 1: Electric field in Cartesian coordinates . . . . *(21 points)*

In this question, you may use the following Taylor series expansion:

$$x \int_{-1/x}^{1/x} da \int_{-1/x}^{1/x} db \left(1 + a^2 + b^2\right)^{-1/2} = constant - 2\pi x + 2\sqrt{2}x^2 - \frac{5}{6\sqrt{2}}x^4 + \dots$$

A square panel of sidelength ℓ and of negligible thickness is of electric charge uniformly distributed with a constant surface charge density $\sigma > 0$. Consider this panel in an otherwise empty space and do so in a coordinate system chosen such that the geometric center of the panel is at the origin and that the panel lies on the $x - y$ plane with its two sides orthogonal to the $x-$axis. Below, we consider the electric field $\mathbf{E}(x, y, z)$ in this coordinate system.

(a) **(7 points)** For some constant $c < 0$, which one of the following statements is true?

- ☐ $\lim_{\kappa \to 0^+} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} \to -\infty$
- ☐ $\lim_{\kappa \to 0^-} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} \to -\infty$
- ☐ $\lim_{\kappa \to 0^+} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} \to \infty$
- ☐ $\lim_{\kappa \to 0^-} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} \to \infty$
- ☐ $\lim_{\kappa \to 0^+} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} = 0$
- ☐ $\lim_{\kappa \to 0^-} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} = 0$
- ☐ $\lim_{\kappa \to 0^+} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} = c$
- ■ $\lim_{\kappa \to 0^-} \mathbf{E}(0, 0, \kappa\ell) \cdot \hat{z} = c$

(b) **(7 points)** Assume that $|z| \ll |\ell|$. Which of the below would give the correct potential energy $V(0, 0, z)$ under this assumption and a suitable chosen reference point?

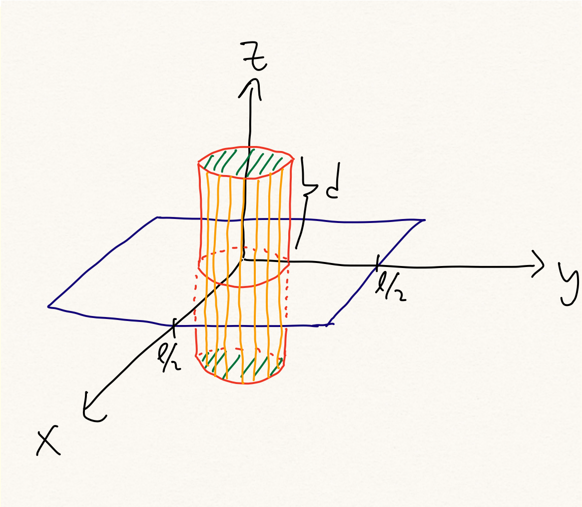
- ☐ $V(0,0,z) \approx \frac{\sigma}{2\epsilon_0}$
- ☐ $V(0,0,z) \approx -\frac{\sigma}{2\epsilon_0}$
- ☐ $V(0,0,z) \approx \frac{\sigma}{2\epsilon_0} + \frac{\sigma z}{2\epsilon_0}$
- ☐ $V(0,0,z) \approx \frac{\sigma}{2\epsilon_0} - \frac{\sigma z}{2\epsilon_0}$
- ☐ $V(0,0,z) \approx \frac{\sigma z}{2\epsilon_0}$
- ☐ $V(0,0,z) \approx -\frac{\sigma z}{2\epsilon_0}$
- ■ $V(0,0,z) \approx -\frac{\sigma}{2\epsilon_0} - \frac{\sigma |z|}{2\epsilon_0}$
- ☐ $V(0,0,z) \approx -\frac{\sigma}{2\epsilon_0} + \frac{\sigma |z|}{2\epsilon_0}$

(c) **(7 points)** In part (a), we considered the asymptotic value of the electric field; in this part, we will instead consider the asymptotic value of $\lim_{z \to 0^+} \frac{d^3\left(\hat{z} \cdot \mathbf{E}(0,0,z)\right)}{dz^3}$ in the region $z > 0$, and evaluate it for a panel with charge density $\sigma = (\pi\epsilon_0\ell^3)/20$. What is the result?

- ☐ $-\frac{1}{\sqrt{2}}$
- ☐ -1
- ☐ $-\sqrt{2}$
- ☐ -2
- ☐ $\frac{1}{\sqrt{2}}$
- ☐ 1
- ■ $\sqrt{2}$
- ☐ 2

Solution 1.1 - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

Name:	
Student ID:	



We can schematically show the situation in the image to the left: the panel is on the $x - y$ plane and oriented as instructed. In part (a), we consider electric field infinitesimally close to the plane, hence we can apply the infinite-plane approximation for which the electric field becomes purely in the $z-$direction. One can then considers the Gaussian surface in the schematic image and apply the divergence theorem:

$$\oint \mathbf{E} \cdot \mathbf{dS} = \frac{Q_{enclosed}}{\epsilon_0}$$

The sides of the surface does not contribute, hence the only contribution comes from the top and bottom:

$$\mathbf{E}(0,0,z) \xrightarrow{|z|\to 0} \frac{\sigma}{2\epsilon_0}\text{sgn}(z)\hat{z} \tag{1}$$

Clearly, as $\sigma, \epsilon_0, \ell > 0$, we see that $\lim_{\kappa\to 0^+} \mathbf{E}(0,0,\kappa\ell)\cdot\hat{z} = \frac{\sigma}{2\epsilon_0} > 0$ and $\lim_{\kappa\to 0^-} \mathbf{E}(0,0,\kappa\ell)\cdot\hat{z} = -\frac{\sigma}{2\epsilon_0} < 0$.

We can similarly answer part (b) as the $|z| \ll \ell$ assumption is same as the infinite-plane assumption at the leading order. With our E found above, we can solve the electric potential:

$$\nabla V(x,y,z) = -\mathbf{E}(x,y,z) \quad \Rightarrow \quad \frac{\partial V(x,y,z)}{\partial z} = -\frac{\sigma}{2\epsilon_0}\text{sgn}(z) \quad \Rightarrow V(x,y,z) = -\frac{\sigma|z|}{2\epsilon_0} + f(x,y) \tag{2}$$

for an undetermined function f. Then, we see that

$$V(0,0,z) = \textit{constant} - \frac{\sigma|z|}{2\epsilon_0} \tag{3}$$

For part (c), we need to consider the more general case, i.e. beyond the infinite-plane approximation. The contribution to the electric potential field at the point $(0,0,z)$ due to an infinitesimal rectangle of dimensions $dx \times dy$ at the point $(x,y,0)$ reads via Coulomb law as

$$dV(0,0,z) = \frac{1}{4\pi\epsilon_0}\frac{\sigma\, dx\, dy}{\sqrt{x^2+y^2+z^2}} \tag{4}$$

thus the full potential due to the square panel becomes

$$V(0,0,z) = \frac{\sigma}{4\pi\epsilon_0}\int_{-\ell/2}^{\ell/2}\int_{-\ell/2}^{\ell/2}\frac{dx\,dy}{\sqrt{x^2+y^2+z^2}} \tag{5}$$

Name:	
Student ID:	

Let us introduce new variables $a = x/z$ and $b = y/z$. The integration then becomes

$$V(0,0,z) = \frac{\sigma z}{4\pi\epsilon_0}\int_{-\frac{\ell}{2z}}^{\frac{\ell}{2z}}\int_{-\frac{\ell}{2z}}^{\frac{\ell}{2z}} \frac{da\,db}{\sqrt{1+a^2+b^2}} \tag{6}$$

where we assume that $z > 0$: this is fine as we are interested in positive z values as told in the question. We can now set $z = \kappa\ell/2$ and then use the given Taylor series expansion to get

$$V(0,0,\kappa\ell/2) = \frac{\sigma\kappa\ell}{8\pi\epsilon_0}\int_{-\frac{1}{\kappa}}^{\frac{1}{\kappa}}\int_{-\frac{1}{\kappa}}^{\frac{1}{\kappa}} \frac{da\,db}{\sqrt{1+a^2+b^2}} = \frac{\sigma\ell}{8\pi\epsilon_0}\left[constant - 2\pi\kappa + 2\sqrt{2}\kappa^2 - \frac{5}{6\sqrt{2}}\kappa^4 + \mathcal{O}(\kappa^5)\right] \tag{7}$$

where $\mathcal{O}(\kappa^5)$ denotes terms that depend on κ with fifth or higher orders. By switching back to the parameter z via $\kappa = 2z/\ell$, we get

$$V(0,0,z) = constant - \frac{\sigma}{2\epsilon_0}z + \frac{\sqrt{2}\sigma}{\pi\ell\epsilon_0}z^2 - \frac{5}{3\sqrt{2}}\frac{\sigma}{\pi\epsilon_0\ell^3}z^4 + \mathcal{O}(z^5) \tag{8}$$

hence

$$\hat{z}\cdot\mathbf{E}(0,0,z) = -\hat{z}\cdot\nabla V(0,0,z) = -\frac{dV(0,0,z)}{dz} = \frac{\sigma}{\pi\epsilon_0}\left[\frac{\pi}{2} - 2\sqrt{2}\left(\frac{z}{\ell}\right) + \frac{10\sqrt{2}}{3}\left(\frac{z}{\ell}\right)^3 + \mathcal{O}(z^4)\right] \tag{9}$$

We see that, as expected, we get back the infinite-plane result when $z/\ell \ll 1$, i.e. $\frac{\sigma}{2\epsilon_0}$, which we derived using the simpler Gaussian surface method. The current derivation allows us to consider term-by-term corrections to that simple idealization once we take into account the finite nature of the size of the plane!

Finding the answer for part (c) is now straightforward:

$$\frac{d^3\left(\hat{z}\cdot\mathbf{E}(0,0,z)\right)}{dz^3} = \frac{20\sqrt{2}\sigma}{\pi\epsilon_0\ell^3} + \mathcal{O}(z) \tag{10}$$

hence

$$\lim_{z\to 0}\frac{d^3\left(\hat{z}\cdot\mathbf{E}(0,0,z)\right)}{dz^3} = \frac{20\sqrt{2}\sigma}{\pi\epsilon_0\ell^3} = \sqrt{2} \tag{11}$$

as we are told to take $\sigma = \dfrac{\pi\epsilon_0\ell^3}{20}$.

Question: 2: Insulators vs Conductors . . . . *(21 points)*

Please take $\log(2) \simeq 0.7$ *and* $\pi\epsilon_0 \simeq 3.6^{-1}\times 10^{-10}$ *in this question*

Consider two electrically charged objects of negligible thickness in an otherwise empty space. Their shape is geometrically described by the following two surfaces:

- a hyperboloid segment described by $x^2 + y^2 - z^2 = a^2$ and $b < z < b + h$, with total charge Q

Name:	
Student ID:	

- a cone segment described by $x^2 + y^2 - z^2 = 0$ and $b < z < b + h$, with total charge $-Q$

for the parameters $a, b, h, Q \in \mathbb{R}^+$. Proceed with the following parts under the assumption that the electric field $\mathbf{E}(x, y, z)$ can be approximated to satisfy $\hat{z} \cdot \mathbf{E}(x, y, z) = 0$.

(a) **(7 points)** Assume that material of these objects is a perfect insulator and the charge is distributed uniformly. For $a = 8$, $b = 1/2$, $h = 5$, $Q = 12$, what is $|\mathbf{E}(x, y, z)|$ at $(x, y, z) = (7.2, 0, 6)$?

☐ 0 ☐ 2×10^9 ☐ 3×10^9 ☐ 5×10^9 ☐ 6×10^9 ☐ 10^{10} ■ 1.2×10^{10} ☐ 1.5×10^{10}

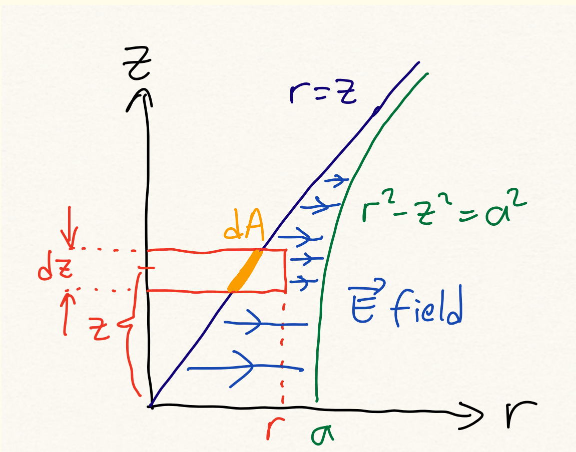
(b) **(7 points)** Assume that material of these objects is a perfect insulator and the charge is distributed uniformly. For $a = 8$, $b = 1/2$, $h = 5$, $Q = 12$, what is $|\mathbf{E}(x, y, z)|$ at $(x, y, z) = (0, 7.2, 6)$?

☐ 0 ☐ 2×10^9 ☐ 3×10^9 ☐ 5×10^9 ☐ 6×10^9 ☐ 10^{10} ■ 1.2×10^{10} ☐ 1.5×10^{10}

(c) **(7 points)** Assume that the material of these objects is a perfect conductor and the charge distributes itself through the surface accordingly. What would be the quantity $\left(a \dfrac{d \log(\sigma(z))}{dz} \right) \Big|_{z=a}$ where $\sigma(z)$ denotes the surface charge density of the cone as a function of the position z?

☐ $7/10$ ☐ $4/7$ ■ $3/7$ ☐ 0 ☐ $-3/7$ ☐ $-4/7$ ☐ $-7/10$ ☐ $-10/7$

Solution 2.1 - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.



Any given information in the question remains invariant under a rotation in the $x - y$ plane, hence we can leverage this by working with cylindrical coordinates, i.e.

$$x = r\cos(\theta) \quad , \quad y = r\sin(\theta) \tag{12}$$

for $r \in \mathbb{R}^+$ and $\theta \in [0, 2\pi)$. Then, our surfaces are described as

$$r = z \quad , \quad r = \sqrt{z^2 + a^2} \tag{13}$$

as we are interested in $z > 0$ region since the relevant segment is defined there. We can see a schematic drawing of the cross-section of these surfaces in the left.

In reality, we have no right to expect to have an $\mathbf{E}-$field that is orthogonal to the $z-$axis; however, we are given this piece of information since we are told that $\hat{z} \cdot \mathbf{E}(x, y, z) = 0$: this is only possible if $\mathbf{E}(x, y, z)$ does not have any z component! Therefore we can find the $\mathbf{E}-$field between the surfaces by considering a Gaussian surface of cylindrical geometry (the schematic red drawing in the image) of infinitesimal thickness dz and of radius r: the only flux that passes this closed volume is through its infinitesimal vertical

Name:	
Student ID:	

side, hence

$$\oint \mathbf{E}\cdot d\mathbf{S} = \frac{Q_{enclosed}}{\epsilon_0} \quad \Rightarrow \quad |\mathbf{E}(r,z)|(2\pi r dz) = \frac{|\sigma(z)|dA}{\epsilon_0} \tag{14}$$

where dA is the infinitesimal area segment in this infinitesimal Gaussian surface, drawn in orange in the schematic above. It is a cylindrical shell of length $2\pi z$ and of width $\sqrt{2}dz$, hence $dA = 2\sqrt{2}\pi z dz$, leading to

$$|\mathbf{E}(r,z)| = \frac{\sqrt{2}|\sigma(z)|}{\epsilon_0}\frac{z}{r} \tag{15}$$

One might get confused regarding the appearance of $\sqrt{2}$: it is because of the fact that we are considering an area element on the surface $r = z$. In general situations, one needs to consider the tangent plane to the surface; in our case, our surface is already planar so we do not need to do any complicated computations: by simple geometry, we see that the "length" of the orange line is $\sqrt{2}$ times dz.

Another point of possible confusion might be our choice of $\sigma(z)$: in principle, the surface charge density depends on two variables that parametrize our surface; in the case of the cone, these can be chosen as z and θ (as $r = z$). However, the polar symmetry ensures that σ does not depend on θ hence the most general surface charge density for our setup is simply $\sigma(z)$.

With the magnitude of electric field between the surfaces computed, we can immediately write down the $\mathbf{E}-$*field as follows:*

$$\mathbf{E}(r,z) = \frac{\sqrt{2}\sigma(z)}{\epsilon_0}\frac{z}{r}\hat{r} \tag{16}$$

Note that we take $\mathbf{E}-$*field to be in $\hat{r}$ direction: this is the only direction that **(1)** has polar symmetry in the $x-y$ plane and **(2)** also orthogonal to $z-$direction. Also, we choose $\hat{r}$ but not $-\hat{r}$ as we want the electric field away from the cone for positive σ.*

It is important to emphasize that this is actually not *the full electric field, it has to have $\hat{z}$ component! (otherwise we can not actually satisfy $\nabla\times\mathbf{E} = 0$!) However, we are simply ignoring that component in our computations as requested in the question!*

Let's now focus on part (a): as the charge density is uniform, we can take $\sigma(r,z) = \dfrac{-Q}{\text{area of cone segment}}$. We already see geometrically that the area element is simply $\sqrt{2}dxdy$; however, we can also derive this explicitly using our calculus knowledge; indeed, we know that the area element for a surface described by $z = f(x,y)$ is simply $dA = \sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + 1}\,dxdy$, which becomes $\sqrt{2}dxdy$ for $z = r$ as expected. Thus, in cylindrical coordinates

$$\text{area of cone segment} = \int_0^{2\pi}\int_b^{b+h}\sqrt{2}r\,dr\,d\theta = \sqrt{2}\pi\left[(b+h)^2 - b^2\right] = \sqrt{2}\pi h(2b+h) \tag{17}$$

Name:	
Student ID:	

hence

$$\sigma(r,z) = -\frac{Q}{\sqrt{2}\pi h(2b+h)} \tag{18}$$

meaning

$$\mathbf{E}(r,z) = -\frac{Q}{\epsilon_0 \pi h(2b+h)}\frac{z}{r}\hat{r} \tag{19}$$

We can now immediately answer part (a): observe that $a = 8$ with $z = 6$ implies that any point with $6 < r < 10$ is between the surfaces, hence at $r = 7.2$, we have

$$|\mathbf{E}(r=7.2, z=6)| = \left|\frac{12}{3.6^{-1}\times 10^{-10}\times 5\times 6}\frac{6}{7.2}\right| = 1.2\times 10^{10} \tag{20}$$

where we are also given $a = 8$, $b = 1/2$, $h = 5$, $Q = 12$ and $\pi\epsilon_0 \simeq 3.6^{-1}\times 10^{-10}$.

Let us move on to part (b). We actually do not need to do much here as this question is identical to part (a) due to the cylindrical symmetry, i.e. the relevant parameter is r which is same in both parts. Hence the result is same as that of part (a).

In part (c), we are told to assume that the material is a perfect conductor: this means that all points on the cone has a unique electric potential (say V_c) and all points on the hyperboloid has a unique electric potential (say V_h): note that this follows from the fact that there can not be electric field inside these conductors hence the electric potential is constant throughout these surfaces. As electric field is a conservative vector field in electrostatics, we then immediately have

$$V_h - V_c = -\int_{\text{any point on the cone}}^{\text{any point on the hyperboloid}} \mathbf{E}\cdot\mathbf{dl} \tag{21}$$

Since the electric field is radial in the given approximation, we can choose $\mathbf{dl} = \hat{r}\,dr$, hence if we choose to integrate over a radial line at some z, (16) becomes

$$\begin{aligned} V_h - V_c &= -\int_{r=z}^{r=\sqrt{a^2+z^2}} \frac{\sqrt{2}\sigma(z)}{\epsilon_0}\frac{z}{r}dr = -\frac{\sqrt{2}\sigma(z)z}{\epsilon_0}\log(r)\Big|_{r=z}^{r=\sqrt{a^2+z^2}} \\ &= -\frac{\sqrt{2}}{\epsilon_0}\left[\sigma(z)z\log\left(\frac{\sqrt{a^2+z^2}}{z}\right)\right] \\ &= -\frac{1}{\sqrt{2}\epsilon_0}\left[\sigma(z)z\log\left(1+\frac{a^2}{z^2}\right)\right] \end{aligned} \tag{22}$$

Since the left hand side is a constant, the right hand side has to be independent of the variable z, which means

$$\sigma(z) = \frac{\alpha}{z\log\left(1+\frac{a^2}{z^2}\right)} \tag{23}$$

Name:	
Student ID:	

for some undetermined coefficient α. We can actually find this constant as well by integrating $\sigma(z)$ over the whole cone and equating it to $-Q$; however, we will not need it as it disappears under the derivative of the log. Indeed, observe that

$$\log(\sigma(z)) = \log(\alpha) - \log(z) - \log\left(\log\left(1 + \frac{a^2}{z^2}\right)\right) \tag{24}$$

hence

$$\begin{aligned} \frac{d\log(\sigma(z))}{dz} &= -\frac{1}{z} - \frac{1}{\log\left(1+\frac{a^2}{z^2}\right)} \frac{\frac{d}{dz}\left(1+\frac{a^2}{z^2}\right)}{1+\frac{a^2}{z^2}} \\ &= \frac{1}{z}\left(-1 + \frac{2a^2}{z^2+a^2}\frac{1}{\log\left(1+\frac{a^2}{z^2}\right)}\right) \end{aligned} \tag{25}$$

leading to

$$\left(a\frac{d\log(\sigma(z))}{dz}\right)\Bigg|_{z=a} = \frac{1-\log(2)}{\log(2)} \simeq \frac{3}{7} \tag{26}$$

as we are given $\log(2) \simeq 0.7$.

Question: 3: Electrostatics in spherical coordinates . . . . *(63 points)*

In this question, you may use the Laplacian operator in the spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2 f}{\partial\phi^2} \tag{27}$$

You may also use the information that the Legendre polynomials $P_\ell(x)$ (also equivalent to the notation $P_\ell^0(x)$) satisfy for all non-negative integer ℓ the differential equation $\frac{d}{dx}\left((1-x^2)\frac{dP_\ell(x)}{dx}\right) + \ell(\ell+1)P_\ell(x) = 0.$

In this question, we refer $c_1(r)$ in $f(r,\theta) = \sum_{\ell=0}^{\infty} c_\ell(r)P_\ell(\cos\theta)$ as dipole term in multipole expansion *of $f(r,\theta)$.*

Consider a spherical insulator of negligible thickness and of radius R. Set up a spherical coordinate system (r,θ,ϕ) such that the origin coincides with the center of the sphere. In that coordinate system, let the electric potential on the surface of the sphere be given as $V_0(\theta)$.

(a) **(10½ points)** Assume that the sphere is in an otherwise empty space and that the electric potential satisfies the behavior $\lim_{r\to\infty} V(r,0,0) = 0$. For $V_0(\pi) \in \mathbb{R}^+$, which one of the below is correct for $\lim_{r\to\infty}\frac{V(r,\pi,0)}{V_0(\pi)}$?

☐ $-\pi$ ☐ $-\pi/2$ ☐ $-2/\pi$ ☐ $-1/\pi$ ■ 0 ☐ $1/\pi$ ☐ $2/\pi$ ☐ $\pi/2$

(b) **(10½ points)** Assume that the sphere is in an otherwise empty space and that the electric potential satisfies the behavior $\lim_{r\to\infty} V(r,0,0) = 0$. If we are also given $V_0(\theta) = \cos(\theta)$ and $R = 4/3$, what is the $x-$component of the electric field at the point $(x,y,z) = (1,\sqrt{2},1)$?

☐ $-1/12$ ☐ $-1/6$ ☐ $-1/4$ ☐ $-1/3$ ☐ $1/3$ ☐ $1/4$ ■ $1/6$ ☐ $1/12$

Name:	
Student ID:	

(c) **(10½ points)** Assume that the sphere is in an otherwise empty space and that the electric potential satisfies the behavior $\lim_{r\to\infty} V(r,0,0) = 0$. If we are also given $V_0(\theta) = \cos(\theta)$ and $R = 4/3$, what is the $x-$component of the electric field at the point $(x,y,z) = (-1, \sqrt{2}, 1)$?

☐ $-1/12$ ■ $-1/6$ ☐ $-1/4$ ☐ $-1/3$ ☐ $1/3$ ☐ $1/4$ ☐ $1/6$ ☐ $1/12$

(d) **(10½ points)** Assume that the original sphere is in an otherwise empty space, except for a second sphere which is conducting, grounded, of radius $3R$, and is centered at the origin. If we are also given $V_0(\theta) = \cos(\theta)$, find the potential $V(r, \theta = 0, \phi = 0)$ between the spheres: what is the coefficient for the *increasing piece* of the electric potential (i.e. find $\lim_{r\to\infty} \frac{d\,(r^\alpha V(r,0,0))}{dr}$ for the unique value of α with which the limit yields a finite non-zero result)?

☐ $-\frac{27R^2}{26}$ ☐ $-\frac{27}{26R}$ ☐ $-\frac{R^2}{26}$ ■ $-\frac{1}{26R}$ ☐ $\frac{1}{26R}$ ☐ $\frac{R^2}{26}$ ☐ $\frac{27}{26R}$ ☐ $\frac{27R^2}{26}$

(e) **(10½ points)** Assume that the original sphere is in an otherwise empty space, except for a second sphere which is conducting, grounded, of radius $3R$, and is centered at the origin. If we are also given $V_0(\theta) = \cos(\theta)$, find the potential $V(r, \theta = 0, \phi = 0)$ between the spheres: what is the coefficient for the *decreasing piece* of the electric potential (i.e. find $\lim_{r\to 0} \frac{d\,(r^\alpha V(r,0,0))}{dr}$ for the unique value of α with which the limit yields a finite non-zero result)?

☐ $-\frac{27R^2}{26}$ ☐ $-\frac{27}{26R}$ ☐ $-\frac{R^2}{26}$ ☐ $-\frac{1}{26R}$ ☐ $\frac{1}{26R}$ ☐ $\frac{R^2}{26}$ ☐ $\frac{27}{26R}$ ■ $\frac{27R^2}{26}$

(f) **(10½ points)** Assume that the original sphere is in an otherwise empty space, except for a second sphere which is conducting, grounded, of radius $3R$, and is centered at the origin. If we are also given $V_0(\theta) = \cos(\theta)$ and $R = 1$, what would be the dipole term in the multipole expansion of the electric potential V at $(x,y,z) = (5,0,0)$?

■ 0 ☐ $1/5$ ☐ $2/5$ ☐ $3/5$ ☐ $4/5$ ☐ 1 ☐ $6/5$ ☐ $7/5$

Solution 3.1 - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

In part (a), we are told that $V_0(\pi)$ is a finite nonzero number via the expression $V_0(\pi) \in \mathbb{R}^+$: since there is no other charge than the sphere, the electric potential has to *decrease to zero at infinity (independent of the angle θ), hence the ratio is simply zero.*

In part (b) & (c), we are trying to find electric field due to some electric potential. We can then achieve this by solving the Laplace equation and applying the sphere and the behavior at the infinity as appropriate

Name:	
Student ID:	

boundary conditions. We are already provided with

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2 f}{\partial\phi^2} \tag{28}$$

and due to the azimuthal symmetry (i.e. absence of ϕ dependence on V_0), we can just take $V(r,\theta,\phi) = R(r)\Theta(\theta)$ and apply seperation of variables:

$$0 = \nabla^2 V(r,\theta,\phi) = \Theta\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + R\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) \tag{29}$$

hence

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) \tag{30}$$

Since the right hand side is independent of r and left hand side can only depend on r, both sides are equal constants. Let's call that constant λ for now: we then have two ordinary differential equations:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda R = 0\ , \qquad \frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda\Theta\sin\theta = 0 \tag{31}$$

We can massage these equations: for the first one, we simply carry out the differentiation's:

$$r^2 R'' + 2rR' - \lambda R = 0 \tag{32}$$

For the second one, we observe that the given differential equation is actually equivalent to the one provided in the question for $\lambda = \ell(\ell+1)$; indeed, by changing $x = \cos(\theta)$, we see that the given differential equation

$$\frac{d}{dx}\left((1-x^2)\frac{df(x)}{dx}\right) + \ell(\ell+1)f(x) = 0 \tag{33}$$

becomes via the chain rule $\frac{1}{\sin\theta}\frac{d}{d\theta} = \frac{1}{\sin\theta}\frac{dx}{d\theta}\frac{d}{dx} = \frac{d}{dx}$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\right]\left((\sin\theta)^2\left[\frac{1}{\sin\theta}\frac{df(\cos^{-1}(\theta)}{d\theta}\right]\right) + \ell(\ell+1)f(\cos^{-1}(\theta) = 0 \tag{34}$$

which identifies with (31) for $\lambda = \ell(\ell+1)$. Therefore, we conclude that

$$V_\ell(r,\theta,\phi) = R(r)P_\ell^0(\cos(\theta)) \tag{35}$$

for $R(r)$ satisfying $r^2R'' + 2rR' - \ell(\ell+1)R = 0$. Obviously, this is an example of Euler's differential equation (r^n accompanies $n - th$ derivative), hence its solutions are of the form r^α. By inserting this and solving for α, we find that

$$R_\ell(r) = a_\ell r^\ell + b_\ell r^{-1-\ell} \tag{36}$$

Name:	
Student ID:	

Since the Legendre differential equation restricts ℓ to integers, the most general solution is then the integer sum of V_ℓ, i.e.

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(a_\ell r^\ell + b_\ell r^{-1-\ell}\right) P_\ell^0(\cos\theta) \tag{37}$$

The boundary condition $\lim_{r\to\infty} V(r,0,0) = 0$ ensures that $a_\ell = 0$ for all ℓ, hence

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{b_\ell}{r^{\ell+1}} P_\ell^0(\cos\theta) \tag{38}$$

The other boundary condition, i.e. $V(R,\theta,\phi) = V_0(\theta) = \cos\theta$ means

$$\sum_{\ell=0}^{\infty} \frac{b_\ell}{R^{\ell+1}} P_\ell^0(\cos\theta) = \cos\theta \tag{39}$$

Since $\cos\theta = P_1^0(\cos\theta)$ (you may know this by memory or you can derive it using equation (cf) in the cheatsheet), we see that choosing $b_\ell = R^2\delta_{1,\ell}$ satisfy the boundary condition, hence

$$V(r,\theta) = \frac{R^2}{r^2}\cos\theta \quad\Rightarrow\quad V(x,y,z) = \frac{R^2 z}{(x^2+y^2+z^2)^{3/2}} \tag{40}$$

We can now compute the $x-$component of the $\mathbf{E}-$field easily:

$$\hat{x}\cdot\mathbf{E}(x,y,z) = -\hat{x}\cdot\nabla\cdot V(x,y,z) = -\frac{\partial V(x,y,z)}{\partial x} = \frac{3R^2xz}{(x^2+y^2+z^2)^{5/2}} \tag{41}$$

which is valid outside the sphere, i.e. if $r > R$. This condition is satisfied for $R = 4/3$, hence we have

$$\hat{x}\cdot\mathbf{E}(x=\pm1, y=\sqrt{2}, z=1) = \pm\frac{1}{6} \tag{42}$$

Let us move on to part (d) & (e): our general solution to the differential equation in (37) is still valid, i.e.

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(a_\ell r^\ell + b_\ell r^{-1-\ell}\right) P_\ell^0(\cos\theta) \tag{43}$$

however, we no longer have the boundary condition at the infinity; instead, we need to impose $V(3R,\theta) = 0$ as the conducting sphere is grounded. Therefore, we have the following conditions:

$$V(R,\theta) = \sum_{\ell=0}^{\infty} \left(a_\ell R^\ell + b_\ell R^{-1-\ell}\right) P_\ell^0(\cos\theta) = P_1^0(\cos\theta) \tag{44a}$$

$$V(3R,\theta) = \sum_{\ell=0}^{\infty} \left(a_\ell (3R)^\ell + b_\ell (3R)^{-1-\ell}\right) P_\ell^0(\cos\theta) = 0 \tag{44b}$$

Name:	
Student ID:	

Since Legendre polynomials are orthogonal, the first equation is possible only if $a_\ell = b_\ell = 0$ for all $\ell \neq 1$, hence the equations become

$$V(R,\theta) = \left(a_1 R + b_1 R^{-2}\right) P_1^0(\cos\theta) = P_1^0(\cos\theta) \tag{45a}$$
$$V(3R,\theta) = \left(a_1 (3R) + b_1 (3R)^{-2}\right) P_1^0(\cos\theta) = 0 \tag{45b}$$

hence

$$\begin{pmatrix} R & \frac{1}{R^2} \\ 3R & \frac{1}{9R^2} \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} R & \frac{1}{R^2} \\ 3R & \frac{1}{9R^2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{46}$$

Equations (x) of the cheatsheet can be used to invert the matrix: we then get

$$a_1 = -\frac{1}{26R}\,, \quad b_1 = \frac{27R^2}{26} \tag{47}$$

hence (43) gives the final result:

$$V(r,\theta) = \left(\frac{27R^2}{26r^2} - \frac{r}{26R}\right)\cos\theta \tag{48}$$

We can now immediately compute

$$\frac{d}{dr}\left(r^\alpha V(r,0)\right) = \frac{27R^2}{26}(\alpha-2)r^{\alpha-3} - \frac{1}{26R}(\alpha+1)r^\alpha \tag{49}$$

In the limit $r \to \infty$, this expression is divergent if $\alpha > 0$ and is zero if $\alpha < 0$, hence $\alpha = 0$ is the only possibility for a non-zero non-divergent result, with

$$\lim_{r\to\infty} \frac{d}{dr}\left(V(r,0)\right) = -\frac{1}{26R} \tag{50}$$

In comparison, in the limit $r \to 0$, the expression is divergent if $\alpha < 3$ and is zero if $\alpha > 3$, hence $\alpha = 3$ is the only possibility for a non-zero non-divergent result, with

$$\lim_{r\to 0} \frac{d}{dr}\left(r^3 V(r,0)\right) = \frac{27R^2}{26} \tag{51}$$

Let's finally answer part (f) which is quiet trivial: $(x,y,z) = (0,0,5)$ is outside the grounded conducting sphere: potential is simply zero everywhere there, so is its all moments (including the dipole moment).

« « « Congratulations, you have made it to the end! » » »