Name:	
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Midterm Examination - 3

Phys331: Electromagnetic Theory I

2025/12/25

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) all calculations are to be conducted in the notations and conventions of the formulae sheets provided during the exam unless explicitly stated otherwise in the question;

(2) I take *full responsibility* for any ambiguity in my selections in “multiple choice questions”;

(3) incorrect selections will receive $-1/7$ of the question's points;

(4) I am expected to provide *step-by-step explanation of how I solved the question* and am expected to do so *only within the answer boxes* provided with the questions: the explanation is supposed to be succinct, well-articulated, and correct both scientifically and mathematically;

(5) no partial credit is awarded for the explanations provided in the answer boxes;

(6) some questions of some students will be randomly selected for inspection: *a question (if selected for inspection) might be awarded negative points* if its explanation is incorrect or insufficient to get the correct answer, even if the correct option is selected;

(7) any page which does not contain *both my name and student id* may not be graded;

(8) any extra sheet that I may use are for my own calculations and will not be graded.

Signature: ______________________________

This exam has a total of 3 questions, some of which may be for bonus points. You can obtain a maximum grade of 105+0 from this examination.

Question:	1	2	3	Total
Points:	42	42	21	105

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Question: 1: Electrostatics in spherical coordinates .... *(42 points)*
(Midterm Two Question Reloaded)

In this question, you may use the Laplacian operator in the spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2 f}{\partial \phi^2} \tag{1}$$

You may also use the information that the Legendre polynomials $P_\ell(x)$ (also equivalent to the notation $P_\ell^0(x)$) satisfy for all non-negative integer ℓ the differential equation $\frac{d}{dx}\left((1-x^2)\frac{dP_\ell(x)}{dx}\right) + \ell(\ell+1)P_\ell(x) = 0.$

Consider a spherical insulator of negligible thickness and of radius R. Set up a spherical coordinate system (r, θ, ϕ) such that the origin coincides with the center of the sphere. In that coordinate system, let the electric potential on the surface of the sphere be given as $V_0(\theta)$.

(a) **(10½ points)** Assume that the sphere is in an otherwise empty space and that the electric potential satisfies the behavior $\lim_{r\to\infty} V(r, 0, 0) = 0$. If we are also given $V_0(\theta) = \cos(\theta)$ and $R = 3/2$, what is the x−component of the electric field at the point $(x, y, z) = (2, 1, 2)$?

☐ $-1/9$ ☐ $-1/3$ ☐ $-1/6$ ☐ $-1/12$ ☐ $1/12$ ☐ $1/6$ ☐ $1/3$ ■ $1/9$

(b) **(10½ points)** Assume that the sphere is in an otherwise empty space and that the electric potential satisfies the behavior $\lim_{r\to\infty} V(r, 0, 0) = 0$. If we are also given $V_0(\theta) = \cos(\theta)$ and $R = 3/2$, what is the x−component of the electric field at the point $(x, y, z) = (2, -1, 2)$?

☐ $-1/9$ ☐ $-1/3$ ☐ $-1/6$ ☐ $-1/12$ ☐ $1/12$ ☐ $1/6$ ☐ $1/3$ ■ $1/9$

(c) **(10½ points)** Assume that the original sphere is in an otherwise empty space, except for a second sphere which is conducting, grounded, of radius $2R$, and is centered at the origin. If we are also given $V_0(\theta) = \cos(\theta)$, find the potential $V(r, \theta = 0, \phi = 0)$ between the spheres: what is the coefficient for the *increasing piece* of the electric potential (i.e. find $\lim_{r\to\infty} \frac{d\left(r^\alpha V(r,0,0)\right)}{dr}$ for the unique value of α with which the limit yields a finite non-zero result)?

☐ $\frac{8R^2}{7}$ ☐ $\frac{27}{7R}$ ☐ $\frac{R^2}{7}$ ☐ $\frac{1}{7R}$ ■ $-\frac{1}{7R}$ ☐ $-\frac{R^2}{7}$ ☐ $-\frac{8}{7R}$ ☐ $-\frac{8R^2}{7}$

(d) **(10½ points)** Assume that the original sphere is in an otherwise empty space, except for a second sphere which is conducting, grounded, of radius $2R$, and is centered at the origin. If we are also given $V_0(\theta) = \cos(\theta)$, find the potential $V(r, \theta = 0, \phi = 0)$ between the spheres: what is the coefficient for the *decreasing piece* of the electric potential (i.e. find $\lim_{r\to 0} \frac{d\left(r^\alpha V(r,0,0)\right)}{dr}$ for the unique value of α with which the limit yields a finite non-zero result)?

■ $\frac{8R^2}{7}$ ☐ $\frac{27}{7R}$ ☐ $\frac{R^2}{7}$ ☐ $\frac{1}{7R}$ ☐ $-\frac{1}{7R}$ ☐ $-\frac{R^2}{7}$ ☐ $-\frac{8}{7R}$ ☐ $-\frac{8R^2}{7}$

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Solution 1.1 - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

In part (a) & (b), we are trying to find electric field due to some electric potential. We can then achieve this by solving the Laplace equation and applying the sphere and the behavior at the infinity as appropriate boundary conditions. We are already provided with

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2 f}{\partial\phi^2} \tag{2}$$

and due to the azimuthal symmetry (i.e. absence of ϕ dependence on V_0), we can just take $V(r,\theta,\phi) = R(r)\Theta(\theta)$ and apply seperation of variables:

$$0 = \nabla^2 V(r,\theta,\phi) = \Theta\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + R\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) \tag{3}$$

hence

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) \tag{4}$$

Since the right hand side is independent of r and left hand side can only depend on r, both sides are equal constants. Let's call that constant λ for now: we then have two ordinary differential equations:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda R = 0\ , \quad \frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda\Theta\sin\theta = 0 \tag{5}$$

We can massage these equations: for the first one, we simply carry out the differentiation's:

$$r^2 R'' + 2rR' - \lambda R = 0 \tag{6}$$

For the second one, we observe that the given differential equation is actually equivalent to the one provided in the question for $\lambda = \ell(\ell+1)$; indeed, by changing $x = \cos(\theta)$, we see that the given differential equation

$$\frac{d}{dx}\left((1-x^2)\frac{df(x)}{dx}\right) + \ell(\ell+1)f(x) = 0 \tag{7}$$

becomes via the chain rule $\frac{1}{\sin\theta}\frac{d}{d\theta} = \frac{1}{\sin\theta}\frac{dx}{d\theta}\frac{d}{dx} = \frac{d}{dx}$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\right]\left((\sin\theta)^2\left[\frac{1}{\sin\theta}\frac{df(\cos^{-1}(\theta))}{d\theta}\right]\right) + \ell(\ell+1)f(\cos^{-1}(\theta) = 0 \tag{8}$$

which identifies with (5) for $\lambda = \ell(\ell+1)$. Therefore, we conclude that

$$V_\ell(r,\theta,\phi) = R(r)P_\ell^0(\cos(\theta)) \tag{9}$$

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for $R(r)$ satisfying $r^2R'' + 2rR' - \ell(\ell+1)R = 0$. Obviously, this is an example of Euler's differential equation (r^n accompanies $n-th$ derivative), hence its solutions are of the form r^α. By inserting this and solving for α, we find that

$$R_\ell(r) = a_\ell r^\ell + b_\ell r^{-1-\ell} \tag{10}$$

Since the Legendre differential equation restricts ℓ to integers, the most general solution is then the integer sum of V_ℓ, i.e.

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(a_\ell r^\ell + b_\ell r^{-1-\ell}\right) P_\ell^0(\cos\theta) \tag{11}$$

The boundary condition $\lim_{r\to\infty} V(r,0,0) = 0$ ensures that $a_\ell = 0$ for all ℓ, hence

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{b_\ell}{r^{\ell+1}} P_\ell^0(\cos\theta) \tag{12}$$

The other boundary condition, i.e. $V(R,\theta,\phi) = V_0(\theta) = \cos\theta$ means

$$\sum_{\ell=0}^{\infty} \frac{b_\ell}{R^{\ell+1}} P_\ell^0(\cos\theta) = \cos\theta \tag{13}$$

Since $\cos\theta = P_1^0(\cos\theta)$ (you may know this by memory or you can derive it using equation (cf) in the cheatsheet), we see that choosing $b_\ell = R^2\delta_{1,\ell}$ satisfy the boundary condition, hence

$$V(r,\theta) = \frac{R^2}{r^2}\cos\theta \quad \Rightarrow \quad V(x,y,z) = \frac{R^2 z}{(x^2+y^2+z^2)^{3/2}} \tag{14}$$

We can now compute the $x-$component of the $\mathbf{E}-$field easily:

$$\hat{x}\cdot\mathbf{E}(x,y,z) = -\hat{x}\cdot\nabla\cdot V(x,y,z) = -\frac{\partial V(x,y,z)}{\partial x} = \frac{3R^2xz}{(x^2+y^2+z^2)^{5/2}} \tag{15}$$

which is valid outside the sphere, i.e. if $r > R$. This condition is satisfied for $R = 3/2$, hence we have

$$\hat{x}\cdot\mathbf{E}(x=2, y=\pm1, z=2) = \frac{1}{9} \tag{16}$$

Let us move on to part (c) & (d): our general solution to the differential equation in (11) is still valid, i.e.

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(a_\ell r^\ell + b_\ell r^{-1-\ell}\right) P_\ell^0(\cos\theta) \tag{17}$$

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however, we no longer have the boundary condition at the infinity; instead, we need to impose $V(2R,\theta)=0$ *as the conducting sphere is grounded. Therefore, we have the following conditions:*

$$V(R,\theta)=\sum_{\ell=0}^{\infty}\left(a_\ell R^\ell+b_\ell R^{-1-\ell}\right)P_\ell^0(\cos\theta)=P_1^0(\cos\theta) \tag{18a}$$

$$V(2R,\theta)=\sum_{\ell=0}^{\infty}\left(a_\ell(2R)^\ell+b_\ell(2R)^{-1-\ell}\right)P_\ell^0(\cos\theta)=0 \tag{18b}$$

Since Legendre polynomials are orthogonal, the first equation is possible only if $a_\ell=b_\ell=0$ *for all* $\ell\neq 1$, *hence the equations become*

$$V(R,\theta)=\left(a_1R+b_1R^{-2}\right)P_1^0(\cos\theta)=P_1^0(\cos\theta) \tag{19a}$$

$$V(2R,\theta)=\left(a_1(2R)+b_1(2R)^{-2}\right)P_1^0(\cos\theta)=0 \tag{19b}$$

hence

$$\begin{pmatrix} R & \frac{1}{R^2} \\ 2R & \frac{1}{4R^2}\end{pmatrix}\begin{pmatrix} a_1 \\ b_1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix} a_1 \\ b_1\end{pmatrix}=\begin{pmatrix} R & \frac{1}{R^2} \\ 2R & \frac{1}{4R^2}\end{pmatrix}^{-1}\begin{pmatrix}1\\0\end{pmatrix} \tag{20}$$

Equations (x) of the cheatsheet can be used to invert the matrix: we then get

$$a_1=-\frac{1}{7R}\,,\qquad b_1=\frac{8R^2}{7} \tag{21}$$

hence (17) gives the final result:

$$V(r,\theta)=\left(\frac{8R^2}{7r^2}-\frac{r}{7R}\right)\cos\theta \tag{22}$$

We can now immediately compute

$$\frac{d}{dr}\left(r^\alpha V(r,0)\right)=\frac{8R^2}{7}(\alpha-2)r^{\alpha-3}-\frac{1}{7R}(\alpha+1)r^\alpha \tag{23}$$

In the limit $r\to\infty$, *this expression is divergent if* $\alpha>0$ *and is zero if* $\alpha<0$, *hence* $\alpha=0$ *is the only possibility for a non-zero non-divergent result, with*

$$\lim_{r\to\infty}\frac{d}{dr}\left(V(r,0)\right)=-\frac{1}{7R} \tag{24}$$

In comparison, in the limit $r\to 0$, *the expression is divergent if* $\alpha<3$ *and is zero if* $\alpha>3$, *hence* $\alpha=3$ *is the only possibility for a non-zero non-divergent result, with*

$$\lim_{r\to 0}\frac{d}{dr}\left(r^3V(r,0)\right)=\frac{8R^2}{7} \tag{25}$$

Question: 2: Electric displacement field .... *(42 points)*

(Modified version of Example 7 of §4.2 of Griffiths, which we partially solved in class)

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In this question, you may use the Laplacian operator in the spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2 f}{\partial \phi^2} \tag{26}$$

You may also use the information that the Legendre polynomials $P_\ell(x)$ (also equivalent to the notation $P_\ell^0(x)$) satisfy for all non-negative integer ℓ the differential equation $\dfrac{d}{dx}\left((1-x^2)\dfrac{dP_\ell(x)}{dx}\right) + \ell(\ell+1)P_\ell(x) = 0$. Please take $\pi\epsilon_0 \simeq 3.6^{-1}\times 10^{-10}$ and $\pi\epsilon \simeq 1.8^{-1}\times 10^{-10}$ (a good approximation for polytetrafluoroethylene)

Consider an empty space filled with a uniform electric field of magnitude E_0, whose source will not be relevant for our purposes. Assume that an electrically uncharged polytetrafluoroethylene (colloquially known as *teflon*) droplet, of perfect spherical shape, is put into this otherwise empty space. Set up a coordinate system such that the geometric center of the droplet coincides with the origin. Furthermore, in the parts below, take A and B to be two points that are collinear with the origin such that B is in the middle. Additionally, the length of the line segments, $|AB|$ and $|BO|$, satisfy the relation $|AB| = |BO| = 2R$ where R is the radius of the droplet.

(a) **(10½ points)** Given that E_0 is sufficiently small such that the linear optics is a valid approximation (i.e. $\mathbf{D} = \epsilon\mathbf{E}$), what would be the magnitude of the electric field at the geometric center of the droplet?

■ $\frac{3E_0}{4}$ ☐ $\frac{E_0}{4}$ ☐ $\frac{4E_0}{5}$ ☐ $\frac{E_0}{5}$ ☐ $\frac{5E_0}{6}$ ☐ $\frac{E_0}{6}$ ☐ $\frac{6E_0}{7}$ ☐ $\frac{E_0}{7}$

(b) **(10½ points)** Given that E_0 is sufficiently small such that the linear optics is a valid approximation (i.e. $\mathbf{D} = \epsilon\mathbf{E}$), what would be the ratio V_A/V_B where V_A & V_B are electric potentials evaluated at the points A & B?

☐ $\frac{2^7-2^0}{2^6-2^2}$ ☐ $\frac{2^8-2^0}{2^6-2^2}$ ☐ $\frac{2^9-2^0}{2^6-2^2}$ ☐ $\frac{2^{10}-2^0}{2^6-2^2}$ ☐ $\frac{2^7-2^0}{2^7-2^2}$ ■ $\frac{2^8-2^0}{2^7-2^2}$ ☐ $\frac{2^9-2^0}{2^7-2^2}$ ☐ $\frac{2^{10}-2^0}{2^7-2^2}$

(c) **(10½ points)** Given that E_0 is sufficiently small such that the linear optics is a valid approximation with the caveat that the polytetrafluoroethylene droplet has a constant uniform permanent polarization (i.e. polarization in the absence of an external electric field) in the direction of the external electric field, and with the magnitude same as $\epsilon_0 E_0$, what would be the ratio V_A/V_B where V_A & V_B are electric potentials evaluated at the points A & B?

■ $\frac{2^7-2^0}{2^6-2^2}$ ☐ $\frac{2^8-2^0}{2^6-2^2}$ ☐ $\frac{2^9-2^0}{2^6-2^2}$ ☐ $\frac{2^{10}-2^0}{2^6-2^2}$ ☐ $\frac{2^7-2^0}{2^7-2^2}$ ☐ $\frac{2^8-2^0}{2^7-2^2}$ ☐ $\frac{2^9-2^0}{2^7-2^2}$ ☐ $\frac{2^{10}-2^0}{2^7-2^2}$

(d) **(10½ points)** If your theoretician friend tells you that the electric susceptibility of polytetrafluoroethylene is not entirely isotropic and that you (an experimentalist) can verify this by measuring V_0 (potential at the center of the droplet) and V_∞ (potential sufficiently far away from the droplet), which measurement below would confirm your theoretician friend's claim?

☐ $\nabla^2 V_0 = 0$ ☐ $\nabla^2 V_0 \neq 0$ ☐ $(\nabla V_0)\times(\nabla V_\infty) = 0$ ■ $(\nabla V_0)\times(\nabla V_\infty) \neq 0$

☐ $(\nabla V_0)\cdot(\nabla V_\infty) = 0$ ☐ $(\nabla V_0)\cdot(\nabla V_\infty) \neq 0$ ☐ $\nabla\cdot(V_0\nabla V_\infty) = 0$ ☐ $\nabla\cdot(V_\infty\nabla V_0) = 0$

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Solution 2.1 - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

Let us first set our coordinate system such that the electric field is along the $z-$coordinate, i.e. $\mathbf{E} = E_0\hat{z}$, and choose the radius of the water droplet to be R. Considering the spherical nature of the water and the overall azimuthal symmetry, let us proceed by solving the Laplace equation in the spherical coordinates. We already derived this above, see (11), hence

$$V(r,\theta) = \sum_{\ell=0}^{\infty} P_\ell^0(\cos\theta) \times \begin{cases} a_\ell r^\ell + b_\ell r^{-1-\ell} & r > R \\ c_\ell r^\ell + d_\ell r^{-1-\ell} & r < R \end{cases} \tag{27}$$

where we are to fix the undetermined coefficients a_ℓ, b_ℓ, c_ℓ, and d_ℓ through the boundary conditions. Indeed,

$$\lim_{r\to\infty} \mathbf{E}(r,\theta) = E_0\hat{z} \tag{28}$$

means $\lim_{r\to\infty} V(r,\theta) = -E_0 z$ hence

$$V(r,\theta) = \begin{cases} -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} P_\ell^0(\cos\theta) b_\ell r^{-1-\ell} & r > R \\ \sum_{\ell=0}^{\infty} P_\ell^0(\cos\theta) c_\ell r^\ell & r < R \end{cases} \tag{29}$$

where we also used the fact that $\lim_{r\to 0} V(r,\theta) < \infty$ (potential being non-singular at the origin) forced $d_\ell = 0$.

The next boundary condition is

$$\lim_{r\to R^+} V(r,\theta) = \lim_{r\to R^-} V(r,\theta) \tag{30}$$

This leads to

$$\left[\frac{b_0}{R} - c_0\right] + \left[-E_0 R + \frac{b_1}{R^2} - c_1 R\right] P_1(\cos\theta) + \sum_{\ell=1}^{\infty}\left[\frac{b_\ell}{R^{\ell+1}} - c_\ell R^\ell\right] P_\ell(\cos\theta) = 0 \tag{31}$$

which can be utilized to fix c_ℓ in terms of b_ℓ through the orthogonality of the Legendre polynomials:

$$V(r,\theta) = \begin{cases} -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} \frac{b_\ell}{r^{\ell+1}} P_\ell^0(\cos\theta) & r > R \\ -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} \frac{b_\ell r^\ell}{R^{2\ell+1}} P_\ell^0(\cos\theta) & r < R \end{cases} \tag{32}$$

The final boundary condition can be derived via the Gauss's law of the electric displacement $\mathbf{D}$, i.e.

$$\lim_{r\to R^+} [\hat{r}\cdot\mathbf{D}] - \lim_{r\to R^-} [\hat{r}\cdot\mathbf{D}] = \sigma_f \tag{33}$$

where σ_f is the surface free charge density which is zero.

For part (a) & (b), we can proceed further using the linear optics approximation, i.e. the condition in (33) becomes

$$\epsilon_0 \lim_{r\to R^+} [\hat{r}\cdot \mathbf{E}] = \epsilon \lim_{r\to R^-} [\hat{r}\cdot \mathbf{E}] \quad \Rightarrow \quad \epsilon_0 \lim_{r\to R^+} \frac{\partial V}{\partial r} = \epsilon \lim_{r\to R^-} \frac{\partial V}{\partial r} \tag{34}$$

with which (32) becomes

$$-\epsilon_0 E_0 \cos\theta - \sum_{\ell=0}^{\infty} \frac{\epsilon_0 b_\ell(\ell+1)}{R^{\ell+2}} P_\ell^0(\cos\theta) = -\epsilon E_0 \cos\theta + \sum_{\ell=0}^{\infty} \frac{\epsilon \ell b_\ell}{R^{\ell+2}} P_\ell^0(\cos\theta) \tag{35}$$

indicating

$$\frac{\epsilon_0}{R^2} b_0 + \left[(\epsilon_0 - \epsilon)E_0 + \frac{2\epsilon_0 + \epsilon}{R^3} b_1\right]\cos\theta + \sum_{\ell=2}^{\infty} \frac{\epsilon\ell + \epsilon_0(\ell+1)}{R^{\ell+2}} b_\ell P_\ell^0(\cos\theta) = 0 \tag{36}$$

Orthogonality of P_ℓ means

$$\frac{\epsilon_0}{R^2} b_0 = 0\ , \quad (\epsilon_0 - \epsilon)E_0 + \frac{2\epsilon_0+\epsilon}{R^3} b_1 = 0\ , \quad \frac{\epsilon\ell + \epsilon_0(\ell+1)}{R^{\ell+2}} b_\ell = 0 \text{ for } \ell \geq 2 \tag{37}$$

with which the fact that $\ell, \epsilon_0, \epsilon, R \in \mathbb{R}^+$ leads to

$$b_\ell = \delta_{\ell,1} \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} R^3 E_0 \tag{38}$$

hence (32) becomes

$$V(r,\theta) = \begin{cases} -E_0 r\cos\theta \left[1 - \dfrac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\left(\dfrac{R}{r}\right)^3\right] & r > R \\ -E_0 r \cos\theta \left[1 - \dfrac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right] & r < R \end{cases} \tag{39}$$

We are told that $\pi\epsilon_0 \simeq 3.6^{-1}\times 10^{-10}$ and $\pi\epsilon \simeq 1.8^{-1}\times 10^{-10}$, hence $\epsilon = 2\epsilon_0$, leading to

$$V(r,\theta) = \begin{cases} -E_0 r \cos\theta \left[1 - \dfrac{1}{4}\left(\dfrac{R}{r}\right)^3\right] & r > R \\ -\dfrac{3}{4} E_0 r\cos\theta & r < R \end{cases} \tag{40}$$

For part (a), the electric field at the origin is simply

$$\lim_{x,y,z\to 0} \mathbf{E}(x,y,z) = -\lim_{x,y,z\to 0} \nabla V(x,y,z) = -\lim_{x,y,z\to 0} \nabla\left(-\frac{3}{4}E_0 z\right) = \frac{3E_0}{4}\hat{z} \tag{41}$$

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For part (b), we proceed as follows: since the points A, B, and O are collinear, and since we have the azimuthal symmetry, we can choose the x coordinate such that the line AB lies in the $x-z$ plane, meaning we can choose the points A and B as

$$\begin{aligned} A: &\quad (r,\theta,\phi) = (4R,\alpha,0) \\ B: &\quad (r,\theta,\phi) = (2R,\alpha,0) \end{aligned} \tag{42}$$

for some undetermined angle α where the radial distances are fixed via the given information $|AB| = |BO| = 2R$. Note that we have also ensured that B is between A and the origin.

We can now find the required ratio as

$$\frac{V_A}{V_B} = \frac{V(4R,\alpha,0)}{V(2R,\alpha,0)} = \frac{-E_0(4R)\cos\alpha\left[1-\frac{1}{4}\left(\frac{R}{4R}\right)^3\right]}{-E_0(2R)\cos\alpha\left[1-\frac{1}{4}\left(\frac{R}{2R}\right)^3\right]} = \frac{2^8-2^0}{2^7-2^2} \tag{43}$$

For part (c), given information translates into

$$\mathbf{D} = \begin{cases} \epsilon_0 E_0 \hat{z} + \epsilon \mathbf{E} & r<R \\ \epsilon_0 \mathbf{E} & r>R \end{cases} \tag{44}$$

in our coordinate system. The fourth boundary condition in (33) then leads to

$$-\epsilon_0 \lim_{r\to R^+} \frac{\partial V}{\partial r} = \epsilon_0 E_0 \cos\theta - \epsilon \lim_{r\to R^-} \frac{\partial V}{\partial r} \tag{45}$$

with which (32) becomes

$$\epsilon_0 E_0\cos\theta + \sum_{\ell=0}^{\infty} \frac{\epsilon_0 b_\ell(\ell+1)}{R^{\ell+2}} P_\ell^0(\cos\theta) = (\epsilon+\epsilon_0)E_0\cos\theta - \sum_{\ell=0}^{\infty}\frac{\epsilon\ell b_\ell}{R^{\ell+2}} P_\ell^0(\cos\theta) \tag{46}$$

indicating

$$\frac{\epsilon_0}{R^2} b_0 + \left[-\epsilon E_0 + \frac{2\epsilon_0+\epsilon}{R^3} b_1\right]\cos\theta + \sum_{\ell=2}^{\infty} \frac{\epsilon\ell+\epsilon_0(\ell+1)}{R^{\ell+2}} b_\ell P_\ell^0(\cos\theta) = 0 \tag{47}$$

Orthogonality of P_ℓ means

$$\frac{\epsilon_0}{R^2} b_0 = 0\,, \quad -\epsilon E_0 + \frac{2\epsilon_0+\epsilon}{R^3} b_1 = 0\,, \quad \frac{\epsilon\ell+\epsilon_0(\ell+1)}{R^{\ell+2}} b_\ell = 0 \text{ for } \ell\geq 2 \tag{48}$$

with which the fact that $\ell,\epsilon_0,\epsilon,R\in\mathbb{R}^+$ leads to

$$b_\ell = \delta_{\ell,1}\frac{\epsilon}{\epsilon+2\epsilon_0}R^3E_0 \tag{49}$$

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hence (32) becomes

$$V(r,\theta) = \begin{cases} -E_0 r\cos\theta\left[1 - \dfrac{\epsilon}{\epsilon+2\epsilon_0}\left(\dfrac{R}{r}\right)^3\right] & r > R \\ -E_0 r\cos\theta\left[1 - \dfrac{\epsilon}{\epsilon+2\epsilon_0}\right] & r < R \end{cases} \tag{50}$$

We are told that $\pi\epsilon_0 \simeq 3.6^{-1}\times 10^{-10}$ *and* $\pi\epsilon \simeq 1.8^{-1}\times 10^{-10}$, *hence* $\epsilon = 2\epsilon_0$, *leading to*

$$V(r,\theta) = \begin{cases} -E_0 r\cos\theta\left[1 - \dfrac{1}{2}\left(\dfrac{R}{r}\right)^3\right] & r > R \\ -\dfrac{1}{2}E_0 r\cos\theta & r < R \end{cases} \tag{51}$$

As before, we have $\dfrac{V_A}{V_B} = \dfrac{V(4R,\alpha,0)}{V(2R,\alpha,0)}$, *hence*

$$\frac{V_A}{V_B} = \frac{V(4R,\alpha,0)}{V(2R,\alpha,0)} = \frac{-E_0(4R)\cos\alpha\left[1-\frac{1}{2}\left(\frac{R}{4R}\right)^3\right]}{-E_0(2R)\cos\alpha\left[1-\frac{1}{2}\left(\frac{R}{2R}\right)^3\right]} = \frac{2^7-2^0}{2^6-2^2} \tag{52}$$

Finally, let us answer part (d): as external electric field is $-\nabla V_\infty$ *and an isomorphic material produces a polarization in the same direction with the external electric field, the net electric field inside an isomorphic material,* $-\nabla V_0$, *would be directionally proportional to* $-\nabla V_\infty$, *leading to* $(\nabla V_0)\times(\nabla V_\infty) = 0$. *Thus, if there is a measurement contradicting this, i.e.* $(\nabla V_0)\times(\nabla V_\infty) \neq 0$, *we are then led to conclude that the electric susceptibility of the material is not entirely isotropic.*

Question: 3: Nature of magnetic field . . . . *(21 points)*

Assume that a big press release tomorrow announces the discovery of the magnetic monopoles and we begin to use the equations

$$\nabla\cdot\mathbf{B} = \mu_0\rho_B \quad , \quad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \tag{53}$$

for magnetostatics. Here, $\mathbf{J}$ and μ_0 are the current density and the vacuum permeability as usual, and ρ_B is the newly discovered magnetic monopole: the press release basically tells that ρ_B can be nonzero. Answer the questions below for this fictitious scenario.

(a) **(10½ points)** What would be the SI unit for ρ_B?

■ $A^1\ m^{-2}\ s^0$ ☐ $A^1\ m^{-1}\ s^0$ ☐ $A^1\ m^0\ s^0$ ☐ $A^1\ m^1\ s^0$

☐ $A^1\ m^{-2}\ s^1$ ☐ $A^1\ m^{-1}\ s^1$ ☐ $A^1\ m^0\ s^1$ ☐ $A^1\ m^1\ s^1$

(b) **(10½ points)** For an experimental setup with no current flow, which equation below would magnetic field be measured to satisfy?

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☐ $\mathbf{B}(\mathbf{r}) = \frac{1}{4\pi\mu_0}\int \frac{\rho_B(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3\mathbf{r}'$

■ $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\rho_B(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3\mathbf{r}'$

☐ $\mathbf{B}(\mathbf{r}) = \frac{1}{4\pi\mu_0}\int \frac{\rho_B(\mathbf{r})(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3\mathbf{r}'$

☐ $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\rho_B(\mathbf{r})(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3\mathbf{r}'$

☐ $\mathbf{B}(\mathbf{r}) = \frac{1}{4\pi\mu_0}\int \frac{\rho_B(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2} d^3\mathbf{r}'$

☐ $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\rho_B(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2} d^3\mathbf{r}'$

☐ $\mathbf{B}(\mathbf{r}) = \frac{1}{4\pi\mu_0}\int \frac{\rho_B(\mathbf{r})(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2} d^3\mathbf{r}'$

☐ $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\rho_B(\mathbf{r})(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2} d^3\mathbf{r}'$

Solution 3.1 - Below is the full derivation of the result; in the exam, you are only supposed to describe the procedure without any actual derivation in the answer box, and do so in a succinct, coherent and well-articulated manner.

For part (a), we can immediately see that ρ_B has to have the same unit with $\mathbf{J}$ *as the left hand side of the equations in (53) have the same units. This means $[\rho_B]$ is $\frac{\textit{electric current}}{\textit{area}}$ which in SI units is $\frac{\mathrm{A}}{\mathrm{m}^2}$.*

For part (b), we are essentially looking for the solution to the equations

$$\nabla \cdot \mathbf{B} = \mu_0 \rho_B \quad , \quad \nabla \times \mathbf{B} = 0 \tag{54}$$

which is (53) with $\mathbf{J} = 0$ as we are interested in an experimental setup with no current flow. We can compare these equations with the equations of electrostatic, i.e

$$\nabla \cdot \mathbf{E} = \epsilon_0^{-1} \rho \quad , \quad \nabla \times \mathbf{E} = 0 \tag{55}$$

which we know to have the solution as the Coulomb's law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3\mathbf{r}' \tag{56}$$

By comparison then, we can find $\mathbf{B}$ *from the Coulumb's law with the replacement $(\mathbf{E}, \rho, \epsilon_0) \Rightarrow (\mathbf{B}, \rho_B, \mu_0^{-1})$:*

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\rho_B(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3\mathbf{r}' \tag{57}$$

« « « Congratulations, you have made it to the end! » » »