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Midterm Examination

Phys210: Mathematical Methods in Physics II

2024/04/18

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) any of my work *outside the designated areas* in the "fill-in the blank questions" will not be graded;

(2) I take *full responsibility* for any ambiguity in my selection of the correct option in "multiple choice questions";

(3) any of my work *outside the answer boxes* in the "classical questions" will not be graded;

(4) any page which does not contain *both my name and student id* will not be graded;

(5) any extra sheet that I may use are for my own calculations and will *not* be graded.

Signature: ____________________

This exam has a total of 7 questions, some of which are for bonus points. You can obtain a maximum grade of 22+2 from this examination.

Question	Points	Score
1	6	
2	2	
3	3	
4	3	

Question	Points	Score
5	2	
6	6	
7	0	
Total:	22	

Name:	
Student ID:	

1 Notations & Conventions

- **The non-negative integer power** of an object A (denoted A^n) is defined recursively as follows:

$$A^0 = \mathbb{I}\,, \quad A^n = A \cdot A^{n-1} \;\; \forall n \geq 1 \tag{1}$$

where the operation $\cdot$ is matrix multiplication if A is a matrix, application of differentiation if A is a differential operator (such as $\frac{\mathrm{d}}{\mathrm{d}x}$), or ordinary multiplication if A is simply a scalar number. $\mathbb{I}$ is the identity object with respect to the operation —identity matrix for matrix multiplication, the number 1 for ordinary multiplication, and so on.

- **Exponentiation of an object** A (denoted e^A) is defined as

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{2}$$

where A^n is the $n-$th power of the object A. For instance, we can write down

$$e^{\frac{\mathrm{d}}{\mathrm{d}x}} = \cos\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) + i \sin\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{3}$$

in accordance with the Euler formula.

- **The Kronecker symbol** (also called Kronecker-delta) is defined as

$$\delta :: \{\mathbb{Z}, \mathbb{Z}\} \to \mathbb{Z} \tag{4a}$$

$$\delta = \{i, j\} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{4b}$$

- **The Dirac-delta generalized function** δ is (for all practical purposes of a Physicist) defined via the relation

$$\int_{\mathcal{A}} f(y)\delta(x-y)dy = \begin{cases} f(x) & \text{if } x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

A useful representation of Dirac-delta generalized function is

$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} \tag{6}$$

- **A particular permutation of n objects** is denoted as $(i_1 i_2 \ldots i_n)$ where $i_1 \neq i_2 \neq \ldots \neq i_n \in \{1, \ldots, n\}$. A permutation $(i_1 \ldots i_n)$ is said to be an even (odd) permutation of $(k_1 \ldots k_n)$ if the two are identical after the permutation of an even (odd) number of adjacent indices. For example, (2431) is an even permutation of (2143) and an odd permutation of (2134).

- **Levi-Civita symbol** is denoted as

$$\epsilon :: \{\mathbb{Z}^+, \ldots, \mathbb{Z}^+\} \to \mathbb{Z} \tag{7a}$$

$$\epsilon = \{a_1, \ldots, a_n\} \to \begin{cases} 1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an even permutation of } (12\ldots n) \\ -1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an odd permutation of } (12\ldots n) \\ 0 & \text{otherwise} \end{cases} \tag{7b}$$

- **The determinant function** (denoted det) is defined as

$$\det :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{8a}$$

$$\det = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \to \sum_{i_1, \ldots, i_n} \epsilon_{i_1 \ldots i_n} a_{1i_1} \ldots a_{ni_n} \tag{8b}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

- **The adjugate function** (denoted adj) is defined as

$$\mathrm{adj} :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{9a}$$

$$\mathrm{adj} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \to \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{pmatrix} \tag{9b}$$

$$\text{where } b_{i_n k_n} = \frac{1}{(n-1)!} \epsilon_{i_1 \ldots i_n} \epsilon_{k_1 \ldots k_n} a_{i_1 k_1} \ldots a_{i_{n-1} k_{n-1}} \tag{9c}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

Name:	
Student ID:	

- **Inverse of a matrix** A is to be denoted as A^{-1}: it satisfies the equations $A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{I}$ where $\mathbb{I}$ is the identity matrix. One can prove (which is beyond the scope of this course) that the inverse of a matrix A can be computed through its adjugate and its determinant:

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det A} \tag{10}$$

- **The trace function** (denoted tr) is defined as

$$\mathrm{tr} :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{11a}$$

$$\mathrm{tr} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_i a_{ii} \tag{11b}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

- **Wronskian matrix** of a set of functions $\{f_1(x), \dots, f_n(x)\}$ is defined as a square matrix where the first row is the set of the functions and the $i-$th row is $(i-1)-$th derivative of the functions for all $n \geq i \geq 2$.

- **A complex number** is (for all practical purposes of a Physicist) a pair of two real numbers, i.e. $(z \in \mathbb{C}) \leftrightarrow (x \in \mathbb{R}, y \in \mathbb{R})$ where one can construct z via $z = x + iy$ (i is called *the imaginary unit* with the property $i^2 = -1$); conversely, one can extract x and y via the functions Re and Im: $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

- **Complex conjugation** (denoted $*$) is a function defined to act on complex numbers as

$$* :: \mathbb{C} \to \mathbb{C} \tag{12a}$$

$$* = z \to (z^* = \mathrm{Re}(z) - i\,\mathrm{Im}(z)) \tag{12b}$$

- **Matrix transpose** (denoted T) is a function defined to act on matrices as

$$T :: \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{13a}$$

$$T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \tag{13b}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i, j$. Usually, we take $\mathcal{A} = \mathbb{C}$.

- **Hermitian conjugation** (also called *conjugate transpose*, *adjoint*, or *dagger*) is a function to act on matrices of complex entries as

$$\dagger :: \mathfrak{M}_{n\times n}(\mathbb{C}) \to \mathfrak{M}_{n\times n}(\mathbb{C}) \tag{14a}$$

$$\dagger = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11}^* & a_{21}^* & \dots & a_{n1}^* \\ a_{12}^* & a_{22}^* & \dots & a_{n2}^* \\ \dots \\ a_{1n}^* & a_{2n}^* & \dots & a_{nn}^* \end{pmatrix} \tag{14b}$$

- **Characteristic polynomial of** any square matrix A is defined as

$$\det(A - \lambda_i \mathbb{I}) = 0 \tag{15}$$

- **Fourier transforms** are widely-used integral transformations (and are the simplest example of the harmonic analysis) which can be defined with any self-consistent convention. For this examination, please stick to the following conventions for Fourier transformation (and its different versions):

$$f :: \mathbb{R} \to \mathbb{C}\,, \qquad f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \hat{f}(k) \tag{16a}$$

$$\hat{f} :: \mathbb{R} \to \mathbb{C}\,, \qquad \hat{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \tag{16b}$$

$$f :: [a, a+T] \to \mathbb{C}\,, \quad f(x) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{T}x} \hat{f}(n) \tag{17a}$$

$$\hat{f} :: \mathbb{Z} \to \mathbb{C}\,, \qquad \hat{f}(n) = \int_{a}^{a+T} dx e^{-i\frac{2\pi n}{T}x} f(x) \tag{17b}$$

$$f :: \mathbb{Z} \to \mathbb{C}\,, \qquad f(n) = \frac{1}{T} \int_{a}^{a+T} dx e^{i\frac{2\pi n}{T}k} \hat{f}(k) \tag{18a}$$

$$\hat{f} :: [a, a+T] \to \mathbb{C}\,, \quad \hat{f}(k) = \sum_{n=-\infty}^{\infty} e^{-i\frac{2\pi n}{T}k} f(n) \tag{18b}$$

$$f :: \mathbb{Z}_N \to \mathbb{Z}_N\,, \qquad f(n) = \frac{1}{N} \sum_{m=0}^{N-1} e^{i\frac{2\pi nm}{N}} \hat{f}(m) \tag{19a}$$

$$\hat{f} :: \mathbb{Z}_N \to \mathbb{Z}_N\,, \qquad \hat{f}(m) = \sum_{n=0}^{N-1} e^{-i\frac{2\pi nm}{N}} f(n) \tag{19b}$$

Name:	
Student ID:	

where (16), (17), (18), and (19) are called *Fourier Transform*, *Fourier Series*, *Discrete-time Fourier Transform*, and *Discrete Fourier Series* respectively. We will stick to this naming in this examination, but please be reminded that different communities (engineering, math, physics, etc.) use different naming conventions in general.

- **"Even part of" and "odd part of"** (denoted E and O) are higher order functions defined as

$$E :: (\mathcal{A} \to \mathcal{A}) \to (\mathcal{A} \to \mathcal{A}) \tag{20a}$$

$$E = (x \to f(x)) \to \left(x \to f_E(x) = \frac{f(x) + f(-x)}{2} \right) \tag{20b}$$

$$O :: (\mathcal{A} \to \mathcal{A}) \to (\mathcal{A} \to \mathcal{A}) \tag{20c}$$

$$O = (x \to f(x)) \to \left(x \to f_E(x) = \frac{f(x) - f(-x)}{2} \right) \tag{20d}$$

with which any single-argument function satisfies $f = E \cdot f + O \cdot f$, or with a more common notation, $f(x) = f_E(x) + f_O(x)$. Here $\mathcal{A}$ is any field, but we usually take it to be $\mathbb{C}$.

- **Inner product between two functions** f and g shall be denoted in this exam as $\langle f, g \rangle_\omega^{\mathcal{A}}$:

$$\langle , \rangle_\omega^{\mathcal{A}} :: (\mathcal{A} \to \mathbb{C}, \mathcal{A} \to \mathbb{C}) \to \mathbb{C} \tag{21a}$$

$$\langle f, g \rangle_\omega^{\mathcal{A}} = \int_A \Big(f(x) \Big)^* g(x) \omega(x) dx \tag{21b}$$

for $\mathcal{A} \subseteq \mathbb{R}$.

- **Group** is defined as a pair (S, o) where S :: `Set` and where $o :: (S, S) \to S$ for which the following statements are true:
  1. $(\exists e \in S)(\forall s \in S)\ o(e, s) = o(s, e) = s$
  2. $(\forall s \in S)\ o(s, i(s)) = o(i(s), s) = e$
  3. $(\forall a, b, c \in S)\ o(a, o(b, c)) = o(o(a, b), c)$

for a unique function $i :: S \to S$.

- **Ring** is defined as a triplet $(S, +, \cdot)$ where S :: `Set`, $+, \cdot :: (S, S) \to S$ for which the following statements are true:
  1. $(S, +)$ :: `Commutative Group`
  2. $(\forall a, b, c \in S)\ a \cdot (b + c) = a \cdot b + a \cdot c$
  3. $(\forall a, b, c \in S)\ (b + c) \cdot a = b \cdot a + c \cdot a$

- **Skew field** is defined as a triplet $(S, +, \cdot)$ where S :: `Set`, $+, \cdot :: (S, S) \to S$ for which the following statements are true:
  1. $(S, +, \cdot)$ :: `Ring`
  2. $(S \backslash \{0\}, \cdot)$ :: `Group`

where 0 denotes the identity element with respect to $+$.

- **Field** is defined as a triplet $(S, +, \cdot)$ where S :: `Set`, $+, \cdot :: (S, S) \to S$ for which the following statements are true:
  1. $(S, +, \cdot)$ :: `Ring`
  2. $(S \backslash \{0\}, \cdot)$ :: `Commutative Group`

where 0 denotes the identity element with respect to $+$.

- **Linear space** (also called *vector space*) over a field $F = (S, +, \cdot)$ shall be denoted as $V(F)$ and is defined as a triplet $(V, \oplus, \odot)$ $\big(V$ :: `Set`, $\oplus :: (V, V) \to V$, and $\odot :: (S, V) \to V\big)$ for which the following statements are true:
  1. $(V, \oplus)$ :: `Commutative Group`
  2. $(\forall v \in V)\ 1 \odot v = v \quad$ (1 is the identity element of $\cdot$)
  3. $(\forall v \in V)(\forall s \in S)\ s \odot v \in V$
  4. $(\forall v \in V)(\forall a, b \in S)\ (a \cdot b) \odot v = a \odot (b \odot v)$
  5. $(\forall v \in V)(\forall a, b \in S)\ (a + b) \odot v = (a \odot v) \oplus (b \odot v)$
  6. $(\forall v, w \in V)(\forall s \in S)\ s \odot (v \oplus w) = (s \odot v) \oplus (s \odot w)$

The elements of the set S (V) are called *scalars* (*vectors*).

- **Linear algebra** (also called *vector algebra*) over a field $F = (S, +, \cdot)$ shall be denoted as $L(F)$ and is defined as a quadruple $(V, \oplus, \odot, \otimes)$ $\big(V$ :: `Set`, $\oplus, \otimes :: (V, V) \to V$, and $\odot :: (S, V) \to V\big)$ for which the following statements are true:
  1. $(V, \oplus, \odot)$ :: `Linear Space`
  2. $(\forall x, y, z \in V)\ x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
  3. $(\forall x, y, z \in V)\ (x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$
  4. $(\forall x, y \in V)(\forall a, b \in S)\ (a \odot x) \otimes (b \odot y) = (a \cdot b) \odot (x \otimes y)$

- **Lie algebra** is a linear algebra $(V, \oplus, \odot, \otimes)$ with the additional condition that $(\forall x, y \in V)\ x \otimes y = -y \otimes x$.

- **Commutator** is a higher order function which takes two functions $f, g :: \mathcal{A} \to \mathcal{A}$ for any type $\mathcal{A}$, and gives a new function $[f, g] :: \mathcal{A} \to \mathcal{A}$ by cascading their action. It is defined on an object $x \in \mathcal{A}$ as $[f, g](x) = f(g(x)) - g(f(x))$.

- **Basis** B of a vector space V is $(B \supset V)$ :: `Set` for which following statements are true:

Name:	
Student ID:	

1. $(\forall k \in \{1, 2, \dim B\})(\forall e_1, \dots, e_k \in B)(\forall c_1, \dots, c_k \in S)[c_1 = \dots = c_k = 0] \vee [c_1 e_1 + \dots + c_k e_k \neq 0]$
2. $(\forall v \in V)(\exists! a_1, \dots, a_{\dim B} \in S)$ $$v = a_1 e_1 + \dots + a_{\dim B} e_{\dim B}$$

- **Normed vector space** over a field F is a vector space $V(F)$ over which a function $\mathsf{norm} :: V \to \mathbb{R}$ exists with the notation $\mathsf{norm} = x \to \|x\|$, for which following statements are true:
  1. $(\forall v \in V)[\|v\| \neq 0] \vee [v = 0]$
  2. $(\forall v \in V)(\forall s \in F)\|s \odot v\| = |s|.\|v\|$
  3. $(\forall v, w \in V)\|v \oplus w\| \leq \|v\| + \|w\|$
- **Inner product vector space** over a field F is a vector space $V(F)$ over which a function $\langle\rangle :: (V, V) \to \mathbb{C}$ exists for which following statements are true:
  1. $(\forall v, w \in V)\ \langle v, w\rangle = \langle w, v\rangle^*$
  2. $(\forall u, v, w \in V)(\forall a, b \in F)$ $$\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$$
  3. $(\forall v \in V\backslash\{0\})\ \langle v, v\rangle > 0$
  4. $\langle 0, 0\rangle = 0$
- **Dual of a vector space** $V(F)$ is a vector space denoted as $V^*(F)$ whose elements are linear functions from the vector space $V(F)$ to the underlying field F.
- **Type (r, s) tensor on a vector space V** is an element of vector space $\underbrace{V \otimes V \otimes \cdots V}_{r} \otimes \underbrace{V^* \otimes V^* \cdots \otimes V^*}_{s}$ where $\otimes$ is an associative bilinear map.
- **Tensor algebra $T(V)$** over a vector space V is the direct sum of all possible (r, s) tensor spaces, with the $\otimes$ being the natural product between different tensors.
- **Multivector** (also called $k-$vector) is an element of the vector space whose elements are constructed via the associative antisymmetric *wedge product* $\wedge$ of the underlying vectors; for instance, $u \wedge v \wedge w$ is a 3-vector if u, v, w are vectors.
- **Exterior algebra $\Lambda(V)$** over a vector space V is the direct sum of all possible multivectors, with the wedge product $\wedge$ being the natural product between different multivectors.
- **Covariant & Contravariant indices** in our conventions refer to *downstairs* and *upstairs* indices of a tensor's components, hence are multiplied with basis vectors of V^* and V to yield the full tensor, e.g. $T = T^{ij}_{\ \ k}\ e_i \textcircled{?} e_j \textcircled{?} e^k$ with $T^{ij}_{\ \ k}$ having one covariant and two contravariant indices where $\textcircled{?}$ is the associative binary operation appropriate to the algebra considered $(\otimes, \wedge, \dots)$.
- **Contraction** is the action of applying a dual vector $(V \to S)$ to a vector (V), hence reducing a $(r, s)-$tensor to a $(r-1, s-1)-$tensor. In an orthonormal basis with $e^i(e_k) = \delta^i_k$ (such as Cartesian coordinates), this amounts to summing over a covariant and a contravariant indices.
- **Manifold** is (for our purposes) any space that resembles $\mathbb{R}^d$ near its every point, for instance the sphere S^2.
- **(Co)tangent space** to a manifold M at a point x is $\mathbb{R}^d$ centered at x and is denoted as $T_x M$ ($T^*_x M$). The (co)tangent space is inhabited by the (co)vectors at $x \in M$, with the basis vectors usually chosen as $\frac{\partial}{\partial x^i}$ (dx_i).
- **(Co)tangent bundle** is the *disjoint union* of all (co)tangent spaces of a manifold M, and is denoted as TM (T^*M).
- **Musical isomorphism** between a tangent and cotangent bundle is initiated with two functions: $\flat :: TM \to T^*M$ and $\sharp :: T^*M \to TM$, hence for instance $(x^i e_i)^\flat = (x_i e^i)$, and $(x_i e^i)^\sharp = (x^i e_i)$.
- **Field in Physicist terminology** broadly refers to any map from a manifold M to *something* $(\mathbb{R}, TM, \dots)$. The field is *named* appropriately depending on the output: scalar field $(M \to \mathbb{R})$, vector field $(M \to TM)$, tensor field $(M \to (TM \otimes TM \otimes T^*M \otimes \cdots))$, and so on.
- **Differential forms** (or forms for short) are functions that takes a point x from a Manifold M and yields a multi(co)vector from the exterior algebra of the (co)tangent space of M at x, e.g. $\omega = (x, y) \to dx + y dy$.
- **Hodge dual of a multivector or a form** α is denoted as $\star\alpha$, and their components in $\mathbb{R}^d$ are related to one another for $\alpha = \alpha_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k}$ and $\star\alpha = (\star\alpha)_{i_{k+1} \dots i_d} e^{i_{k+1}} \wedge \dots \wedge e^{i_d}$ as $$(\star\alpha)_{i_{k+1} \dots i_d} = \frac{1}{(d-k)!} \alpha_{i_1 \dots i_k} \epsilon^{i_1 \dots i_k \ell_{k+1} \dots \ell_d} \delta_{i_{k+1}\ell_{k+1}} \cdots \delta_{i_d \ell_d}$$
- **Exterior derivative** takes a $p-$form ω to $p+1$ form $d\omega$; with the basis vectors $\{dx^i\}$, it reads as $$\omega = \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \tag{22a}$$ $$d\omega = \frac{\partial \omega_{i_1 \dots i_p}}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \tag{22b}$$
- **Gradient** (denoted grad) is a function `Scalar Field` $\to$ `Vector Field`, defined as $\mathsf{grad} = f \to (df)^\sharp$. $\boldsymbol{\nabla} f$ is also used as a notation for $\mathsf{grad}(f)$. In Cartesian coordinates, $$\begin{aligned} \mathsf{grad} = &\big((x_1, \dots, x_d) \to f(x_1, \dots, x_d)\big) \\ &\to \left((x_1, \dots, x_d) \to \frac{\partial f(x_1, \dots, x_d)}{\partial x_i} \hat{x_i}\right) \end{aligned} \tag{23}$$

Name:	
Student ID:	

• **Divergence** (denoted div) is a function `Vector Field` $\to$ `Scalar Field`, defined as $\mathsf{grad} = v \to \left(\star\, d \star v^\flat\right)$. $\boldsymbol{\nabla}\cdot v$ is also used as a notation for $\mathsf{div}(v)$. In Cartesian coordinates,

$$\begin{aligned}\mathsf{div} =& \left((x_1,\ldots,x_d) \to v^i(x_1,\ldots,x_d)\hat{x}_i\right)\\ &\to \left((x_1,\ldots,x_d) \to \frac{\partial v^i(x_1,\ldots,x_d)}{\partial x^i}\right)\end{aligned} \tag{24}$$

• **Curl** (denoted curl) is a function `Vector Field` $\to$ $(d-2)-$`Vector Field`, defined as $\mathsf{curl} = v \to \left(\star\, dv^\flat\right)^\sharp$. In $d=3$, $\boldsymbol{\nabla}\times v$ is also used as a notation for $\mathsf{curl}(v)$; in Cartesian coordinates,

$$\begin{aligned}\boldsymbol{\nabla}\times v = &\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y}\\ &+ \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}\end{aligned} \tag{25}$$

• **Laplacian** (for our purposes) is a function `Tensor Field` $\to$ `Tensor Field`, denoted as Δ, and is defined as follows for practical purposes:

$$\begin{aligned}R &:: TM\otimes\cdots\otimes TM\otimes T^*M\otimes\cdots\otimes T^*M\\ R &= R^{i_1\ldots i_r}{}_{k_1\ldots k_s}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{k_1}\otimes\cdots\otimes dx^{k_s}\\ \Delta R &:: TM\otimes\cdots\otimes TM\otimes T^*M\otimes\cdots\otimes T^*M\\ \Delta R &= \frac{\partial^2 R^{i_1\ldots i_r}{}_{k_1\ldots k_s}}{\partial x^m\partial x_m}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{k_1}\otimes\cdots\otimes dx^{k_s}\end{aligned} \tag{26}$$

• **Helmholtz decomposition of a $3d$ vector field** E is a way of rewriting it in terms of its *scalar potential* Φ (related to the divergence of the vector field) and its *vector potential* V (related to the curl of the vector field): $E = \text{constant } -\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\times V$ where

$$\begin{aligned}\Phi(r) =& \frac{1}{4\pi}\int\limits_{\text{manifold}}\frac{\boldsymbol{\nabla}'\cdot E(r')}{|r-r'|}dV'\\ &-\frac{1}{4\pi}\oint\limits_{\text{boundary}}\frac{\hat{n}'\cdot E(r')}{|r-r'|}dS'\end{aligned} \tag{27a}$$

$$\begin{aligned}V(r) =& \frac{1}{4\pi}\int\limits_{\text{manifold}}\frac{\boldsymbol{\nabla}'\times E(r')}{|r-r'|}dV'\\ &-\frac{1}{4\pi}\oint\limits_{\text{boundary}}\frac{\hat{n}'\times E(r')}{|r-r'|}dS'\end{aligned} \tag{27b}$$

• **Arc-length** is the length of a curve (denoted by s), which satisfies $s = \int\limits_{t_0}^{t}\left|\frac{d\mathbf{x}(t)}{dt'}\right|dt'$. In this equation, $\mathbf{x}(t)$ is the position of a point on the curve, t is the parametrization parameter, and t_0 is the value of t at the starting point of the curve. The arc-length itself can be used to parametrize the curve.

• **Tangent vector to a curve** in the arc-length parametrization is the function $\mathbf{t}(s) = \frac{d\mathbf{x}(s)}{ds}$. It has unit norm, and can be likened to the ratio velocity per speed.

• **Curvature of a curve** κ is a function of the arc-length whose value is $\kappa(s) = \left|\frac{d\mathbf{t}(s)}{ds}\right|$.

• **Principle normal of a curve** $\mathbf{n}$ is a function of the arc-length whose value is $\mathbf{n}(s) = \frac{1}{\kappa(s)}\frac{d\mathbf{t}(s)}{ds}$. It has unit norm, and can be likened to the acceleration unit vector.

• **Binormal vector of a curve** $\mathbf{b}$ is a function of the arc-length whose value is $\mathbf{b}(s) = \mathbf{t}(s)\times\mathbf{n}(s)$ ($|\mathbf{b}(s)| = 1$).

• **Torsion of a curve** τ is a function of the arc-length whose value is $\tau(s) = -\mathbf{n}(s)\cdot\frac{d\mathbf{b}(s)}{ds}$.

• **The Frenet-Serret equations** is a closed system of equations which completely determine the properties of a curve as a function of the curvature and torsion functions. They read as

$$\begin{aligned}\frac{d\mathbf{t}(s)}{ds} = \kappa(s)\mathbf{n}(s)\ , \quad \frac{d\mathbf{b}(s)}{ds} = -\tau(s)\mathbf{n}(s)\ ,\\ \frac{d\mathbf{n}(s)}{ds} = \tau(s)\mathbf{b}(s) - \kappa(s)\mathbf{t}(s)\end{aligned} \tag{28}$$

• **Generalized Stokes theorem** equates the integration of a $p-$form ω over the boundary of a manifold ∂M to the integration of the exterior derivative of the $p-$form $d\omega$ over the manifold M: $\int_{\partial M}\omega = \int_M d\omega$.

• **Integral theorems** are special cases of the generalized Stokes theorem. For a volume $\mathbf{V}\in\mathbb{R}^3$, a surface $\mathbf{S}\in\mathbb{R}^3$, a curve $\gamma\in\mathbb{R}^3$, and a region $\mathbf{D}\in\mathbb{R}^2$ (and for the notation ∂A being boundary of A), we have

$$\begin{aligned}&\int\limits_{\mathbf{V}}\boldsymbol{\nabla}\cdot F dV = \oint\limits_{\partial\mathbf{V}}F\cdot dS\ , \quad \int\limits_{\mathbf{S}}\boldsymbol{\nabla}\times F\cdot dS = \oint\limits_{\partial\mathbf{S}}F\cdot d\Gamma\\ &\int\limits_{\gamma}\boldsymbol{\nabla}f\cdot dr = f\Big|_{\text{initial}}^{\text{final}}\ , \quad \int\limits_{\mathbf{D}}\left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y}\right)dxdy\\ &\qquad = \oint\limits_{\partial\mathbf{D}}\left(L(x,y)dx + M(x,y)dy\right)\end{aligned} \tag{29}$$

Name:	
Student ID:	

2 Fill-in the blanks

Each correct answer is worth 0.5 *point.*

Question: 1 . . . . *(6 points)*

Vectors (and their generalization tensors) are extremely important in science as they enable *observer-covariant* expression of fundamental laws, however the proper way to understand them requires us to analyze the abstract nature of linear spaces. Indeed, we started this semester by discussion the chain topics of *groups*, *rings*, *fields*, and then **_linear/vector spaces_** of which vectors are defined as elements. Let us review these concepts.

The group is a pair $(S, f) :: (\texttt{Set}, (S, S) \to S)$ where f is a **_binary_** operation on S, hence converting a pair of S elements into an element of S. Of course, not all (S, f) pairs are groups, we need three properties to be satisfied: **(1)** there is an identity element, **(2)** all elements have inverses, and **(3)** f is an **_associative_** function: $(f(f(a, b), c) = f(a, f(b, c)))$.

A particularly important subset of groups is those for which $(\forall a, b \in S)\ f(a, b) = f(b, a)$: these are called **_commutative_** groups. If we add a *second* binary operation g to such a group (S, f), then the triplet (S, f, g) becomes a *ring* if g **_distributes_** over f. Denote 0 the **_identity element_** of the group (S, f); if $(S\backslash\{0\}, g)$ is a group, then ring (S, f, g) actually becomes **_skew field_** (e.g. quaternions). From here, one moves on to *fields* and *linear spaces* to define vectors properly.

The path described above to define vectors is rather different than the *freshman approach* of stating that "vectors are quantities with magnitude and direction". Indeed, we have seen that not all vectors have magnitudes to begin with; this actually requires the existence of a function $f :: \texttt{Vector} \to \mathbb{R}$ with certain properties (such as triangle inequality). The vector spaces for which this function exists are called **_normed_** vector spaces, which is the overwhelming majority of the vector spaces considered in Physics. In fact, the most of the vector spaces we deal with are of a more restricted kind, called **_inner product spaces_**, in which a function $g :: (\texttt{Vector}, \texttt{Vector}) \to \mathbb{C}$ exist with certain properties (such as conjugate symmetry): $\sqrt{g(v, v)}$ can then be interpreted as the magnitude of any given vector v. Bereft of such a function in the most general linear spaces, we can only obtain a scalar out of a vector by using the **_dual vector_** (also called *covector*) which is a linear function on the space of vectors.

Vector laws include the existence of a *scaling* operation of a vector (e.g. $(3, \hat{\imath} - 2\hat{\jmath}) \to (3\hat{\imath} - 6\hat{\jmath})$), whereas **_multiplication_** of vectors is not necessarily defined. We call a linear space *a linear algebra* if such an operation is defined; depending on the properties of this operation, the algebra gets a different adjective. For instance, we call it **_an exterior algebra_** if the operation is an associative antisymmetric bilinear mapping.

3 Choose the correct option

You do not need to show your derivation in this part.
Incorrect answer for a question of X point is worth $-X/4$ points: this ensures that the randomly given answer has an expectation value of 0 point.

Name:	
Student ID:	

Question: 2 . . . . *(2 points)*

Consider a *family of groups* $\mathcal{A}_k = \left(\{e^{ikx} \mid x \in \mathbb{R}\}, (x, y) \to x.y\right)$ where . is the arithmetic multiplication and k is a label that parametrizes different groups.

(a) **(1 point)** What is the identity element of the group $\mathcal{A}_1$?

☐ 0 ■ 1 ☐ e^{ix} ☐ e^{ik} ☐ None

(b) **(1 point)** Consider the function $f = k \to e^{i2k}$. For any given k, let $g(k)$ be the inverse of $f(k)$ in the group $\mathcal{A}_k$. If we integrate $g(k)$ over the real line against $h(x, k)$, it yields 0 unless $x = 3$ (i.e. the result is proportional to a Dirac-delta function). What is $h(x, k)$?

■ $e^{ik(x-1)}$ ☐ $e^{ik(x+1)}$ ☐ $e^{-ik(x-1)}$ ☐ $e^{-ik(x+1)}$ ☐ None

Question: 3 . . . . *(3 points)*

Consider a vector field $v :: T\mathbb{R}^2$ orthogonal to the direction $\mathbf{i}+\mathbf{j}$. If we also know that the vector field satisfies the differential equation

(a) **(1½ points)** $\boldsymbol{\nabla} \cdot v(x, y) = n \cdot v(x, y)$ for the vector $n = \mathbf{i} - \mathbf{j}$, which of the following can be $\mathbf{i} \cdot v(x, y)$?

☐ $\dfrac{e^{-2y}}{\cos(x+y)+1}$ ☐ $e^{4x+2y}(x+y)^2$ ☐ e^{x-y} ☐ e^{2x} ■ **All**

(b) **(1½ points)** $\Delta v(x, y) = -v(x, y)$, which of the following can be $\mathbf{i} \cdot v(x, y)$?

☐ e^{x-iy} ☐ e^{x} ☐ e^{x+iy} ■ $e^{x+i\sqrt{2}y}$ ☐ All

Question: 4 . . . . *(3 points)*

A vector field $\mathbf{E} :: T\mathbb{R}^3$ is defined as follows

$$\mathbf{E} = (x, y, z) \to \left(\cos(x+z)\frac{\partial}{\partial x} + \frac{y^2}{1+az}\frac{\partial}{\partial y} + \left(\cos(x)\cos(z) - b\sin(x)\sin(z)\right)\frac{\partial}{\partial z}\right) \tag{30}$$

(a) **(1½ points)** For which parameters (a, b) can this vector field be described without a vector potential in its Helmholtz decomposition?

☐ $(0, -1)$ ☐ $(-1, 0)$ ■ $(0, 1)$ ☐ $(1, 0)$ ☐ None

(b) **(1½ points)** What is $\boldsymbol{\nabla} \cdot \mathbf{E}$ evaluated at a point on the line $(x, y, z) = (t, 0, t)$?

■ $-2\sin(2t)$ ☐ $2\sin(2t)$ ☐ $-2\cos(2t)$ ☐ $2\cos(2t)$ ☐ None

Question: 5 . . . . *(2 points)*

Consider a vector field $E :: T\mathbb{R}^4$ defined as $E(x, y, z, t) = x\dfrac{\partial}{\partial y}$. Which of the following *bivector field* is the curl of this vector field in this four dimensional Euclidean space?

☐ $\dfrac{\partial}{\partial x} \wedge \dfrac{\partial}{\partial y}$ ☐ $-\dfrac{\partial}{\partial x} \wedge \dfrac{\partial}{\partial y}$ ■ $\dfrac{\partial}{\partial z} \wedge \dfrac{\partial}{\partial t}$ ☐ $-\dfrac{\partial}{\partial z} \wedge \dfrac{\partial}{\partial t}$ ☐ None

Name:	
Student ID:	

4 Classical questions

Question: 6 .... *(6 points)*

Mahmûd bin Hüseyîn bin Muhammed el-Kâşgarî (known as *Kaşgarlı Mahmut* among the Turkish speaking population) was an important 11^{th} century lexicographer who specialized in Turkic languages. His legacy, *Dîvânu Lugâti't-Türk* (compendium of the languages of the Turks), was written in 1074 and is widely accepted as one of the oldest Turkish lexicons; in it, we learn about the famous Turkish mythological hero, *Alp Er Tunga*. In this question, we will consider the epic of Alp Er Tunga and see how that epic can be endowed with a Mathematical group structure.

Alp Er Tunga, believed to have lived in the 7^{th} century BC, was the ruler of Saka, a group of normadic people in the central Asia with disputed Turkish/Persian origin. After several legendary battles between the Iranians and Saka, Alp Er Tunga dies for whom the following timeless requiem is written:

Alp Er Tunga öldi mü?
Isiz ajun kaldı mu?
Ödlek öçin aldı mu?
Emdi yürek yırtılur.
Ödlek yırag közetti,
Ogrı tuzak uzattı,
Begler begin azıttı,
Kaçan kalı kurtulur.
Ulşıp eren börleyü,
Yırtıp yaka urlayu,
Sıkrıp üni yurlayu,
Sıgtap közi örtülür.

(Note: You are not required to understand the semantics of this requiem for this question.)

Let us try to see what kind of a mathematical structure we would obtain if we were to shuffle the letters in this requiem. For this, we *define* the type `String`, which denotes any ordered collection of characters (letters, space, punctuation, etc.). As is the tradition in computer science, we will denote strings between double quotation; for instance, we can immediately write "iz ajun kal":: `String`, "lşı":: `String`, "":: `String`. The last one simply states that an expression with zero character can still be seen as a string.

Consider the set of *functions* whose both domain and codomain are `String`; among these, choose the subset (denote $\mathcal{S}$) of functions which *prepend* its input with a string from the requiem above. For instance, the functions $f = x \to \big(\text{"Kaçan kalı"} +\!\!+\, x\big)$ and $g = x \to \big(\text{"örtü"} +\!\!+\, x\big)$ are elements of $\mathcal{S}$, whereas $h = x \to \big(\text{"Youtube shorts suck"} +\!\!+\, x\big)$ is not. Here, the binary operation $+\!\!+$ is called *string concatenation* in computer science, and is a function from a pair of `String`'s to a `String`, i.e. "Phys" $+\!\!+$ "210" = "Phys210" and "hard " $+\!\!+$ "exam" = "hard exam".

Name:	
Student ID:	

Define the associative operation *function composition*, denoted by $\cdot$ as is the custom both in math and computer science. For this question, we will define it as follows:

$$\cdot :: (\texttt{String} \to \texttt{String}\ ,\ \texttt{String} \to \texttt{String}) \to (\texttt{String} \to \texttt{String}) \tag{31a}$$

$$\cdot = \big(x \to f(x)\ ,\ x \to g(x)\big) \to \big(x \to f(g(x))\big) \tag{31b}$$

We can now *endow* the set $\mathcal{S}$ with this operation; however, the pair $(\mathcal{S}, \cdot)$ is actually not a group.

(a) Explain why $(\mathcal{S}, \cdot)$ can not be a group!

Solution: $(\mathcal{S}, \cdot)$ can not be a group because elements do not have inverses. For instance, there is no element $f^{-1} \in \mathcal{S}$ such that $f^{-1}(f(\text{"some text"})) = \text{"some text"}$ as we need a function $f^{-1}(\text{"Kaçan kalısome text"}) = \text{"some text"}$ in $\mathcal{S}$, but all elements in $\mathcal{S}$ simply prepend something.

A necessary step in turning $(\mathcal{S}, \cdot)$ into a group is to restrict the elements of $\mathcal{S}$ and the binary operation $\cdot$ to a new type that we will call `Restricted String`. `Restricted String`'s are just ordinary `String`'s with the additional condition that *same characters at the beginning and end of the string are dropped.* For instance, the function $f = x \to \big(\text{"Kaçan kalı"} +\!\!+\ x\big)$ introduced above would yield

$$\begin{aligned}
f(\text{" from Star TreK"}) &= \text{"Kaçan kalı from Star TreK"} && \textit{(when acting on } \texttt{String}\textit{)}\\
f(\text{" from Star TreK"}) &= \text{"açan kalı from Star Tre"} && \textit{(when acting on } \texttt{Restricted String}\textit{)}\\
f(\text{" is not a cloaK"}) &= \text{"Kaçan kalı is not a cloaK"} && \textit{(when acting on } \texttt{String}\textit{)}\\
f(\text{" is not a cloaK"}) &= \text{"çan kalı is not a clo"} && \textit{(when acting on } \texttt{Restricted String}\textit{)}
\end{aligned} \tag{32}$$

(b) Is $(\mathcal{S}, \cdot)$ a group if we work with `Restricted String` instead of `String`? If yes, argue how it satisfies all group axioms. If no, find out the set $\mathcal{S}'$ such that $(\mathcal{S} \cup \mathcal{S}', \cdot)$ is a group if we work with `Restricted String`, and argue how group axioms are now satisfied!

Solution: With `Restricted String`, we could hope to avoid this problem as same characters are automatically removed; however, it is still not sufficient as the example above is still valid: we still do not have an inverse for f as the only possible candidate is $f^{-1} = x \to \big(\text{"txet emosılak naçaKsome text"} +\!\!+\ x\big)$ which ensures $f^{-1}(f(\text{"some text"})) = \text{"some text"}$. But this function is simply not an element of $\mathcal{S}$ (no such text in the requiem of Alp Er Tunga), hence $(\mathcal{S}, \cdot)$ can not be a group even if we restrict to `Restricted String`.

Name:	
Student ID:	

What we actually need is to enlarge the set $\mathcal{S}$ with another set $\mathcal{S}'$ defined as follows:

$\mathcal{S}$: set of functions which *prepend* its input with a string from the requiem
$\mathcal{S}'$: set of functions which *append* its input with an inverted string from the requiem

For instance, the function ϕ defined below is an element of $\mathcal{S}'$ and is actually inverse of the function f

$$\phi = x \to \left(x \; ++ \; \text{"ılak naçaK"}\right) \tag{34}$$

when we are working with `Restricted String`, i.e. $\phi(f(x)) = f(\phi(x))$ for any x.

With this definition, we can see that $(\mathcal{S} \cup \mathcal{S}', \cdot)$ is a group:

1. There is an identity element e with respect to the group operation: $e = x \to \left(x \mathbin{++} \text{""}\right)$.
2. Every element has an inverse with respect to the group operation (by our construction of $\mathcal{S}'$).
3. The group operation (function composition) is associative as explicitly stated in the question (as it is already given, you need not to prove this).

Bonus Question: 7 . . . . *(2 points)*
Which `Mathematica` code below would correctly define a *function composition* such that it takes the functions f and g and the value x as inputs and then yields $f(g(x))$ as the output?

- ☐ `Composition[f,g][x]`
- ☐ `f@*g@x`
- ☐ `RightComposition[g,f][x]`
- ☐ `g/*f@x`
- ■ All

« « « Congratulations, you have made it to the end! » » »