2D Conformal Field Theories

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Two-dimensional CFTs occupy a central role in modern theoretical physics, bridging concepts in string theory, statistical mechanics, condensed matter physics and quantum gravity. We review fundamental concepts and building blocks of 2D CFTs, focusing mainly on the underlying symmetry algebra, operator formalism and construction of the Hilbert space of the theory. This review aims to provide readers with a concise summary of 2D CFTs and their applications in the existing literature.

INTRODUCTION

Conformal Field Theories (CFTs) are a special class of quantum field theories characterized by their invariance under conformal transformations. Such transformations locally scale the metric as

$$g(x) \to \Omega^2(x) g(x). \tag{1}$$

Consequently, spacetime intervals get stretched and contracted, while angles are preserved. Thanks to their scaling symmetry, CFTs play a crucial role in theoretical physics, ranging from statistical physics to string theory, cosmology and gravity. The particular importance of CFTs in two dimensions comes from the underlying symmetry group being infinite dimensional. This enables us to seek for solutions to complex physical problems without the need for perturbative expansion. Moreover, the stringent constraints imposed by the infinite-dimensional symmetry group facilitate the prediction of the form of certain mathematical quantities, even without explicitly solving the governing equations.

Two-dimensional CFTs were first studied extensively in the context of critical phenomena, where they describe the scaling limit of systems undergoing second-order phase transitions. The seminal work of Belavin, Polyakov, and Zamolodchikov (BPZ)[1] introduced the conformal bootstrap framework for classifying 2D CFTs based on their central charge and primary operators, paving the way for exact solutions of many models, such as the Ising model and minimal models. During the same period, Polyakov made the connection between 2D CFT and string theory by considering a path integral formulation in which the string worldsheet is treated as a two-dimensional random surface[2]. The introduction of holography in the 1990s by Maldacena [3], particularly through the AdS/CFT correspondence, established profound connections between conformal field theories (CFTs) and higher-dimensional gravity theories. In particular, the exact solvability of correlation functions in 2D CFTs provides an opportunity to study black hole dynamics and computation of black hole entropy in 3D, thanks to the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. Furthermore, recent literature has demonstrated that 2D conformal field theories (CFTs) are valuable tools for providing deformations of gravitational theories, such as the $T\bar{T}$ and $J\bar{T}$ deformations [4, 5], as well as for calculating entanglement entropy [6].

This review provides a brief overview of key elements in 2D CFTs, with an emphasis on the mathematical base blocks, such as the fundamental fields of the theory, operator formalism, algebra of the symmetry group and construction of the Hilbert space.

FIELDS

In 2D CFTs, or CFTs in any dimension for that matter, we are mainly interested in how fundamental quantities respond to conformal transformations. In particular, we would like to analyze how the fields of our theory, spinless or with spin, behave under scaling and rotation. Hence, we define two useful quantities related to conformal scaling: holomorphic conformal dimension h and anti-holomorphic conformal dimension $\bar{h}$

$$h \equiv \frac{1}{2}(\Delta + s), \qquad \bar{h} = \frac{1}{2}(\Delta - s), \tag{2}$$

where Δ is the scaling dimension and s is the planar spin for a given field.

We then call a field primary if it transforms under all local conformal transformations $z \to \lambda(z,\bar{z})z$, $\bar{z} \to \bar{\lambda}(z,\bar{z})\bar{z}$ as

$$\phi'(\lambda z, \bar{\lambda}\bar{z}) = \left(\frac{\mathrm{d}(\lambda z)}{\mathrm{d}z}\right)^{-h} \left(\frac{\mathrm{d}(\bar{\lambda}\bar{z})}{\mathrm{d}\bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}), \tag{3}$$

where by convention we suppress the spin indices if any. If instead the field satisfies (3) under global conformal transformations, then the field is called quasi-primary. All the primary fields are eventually quasi-primary fields, while the reverse does not hold true in general. Usually, non-primary fields are called secondary.

Under infinitesimal conformal map of the form $z \to \lambda z \approx z + \epsilon(z)$, the transformation rule for a quasi-primary field can be found from (3) as

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\phi &\equiv \phi'(z,\bar{z}) - \phi(z,\bar{z}) \\ &= -(h\phi\partial\epsilon + \epsilon\partial\phi) - (\bar{h}\phi\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\phi). \end{aligned} \tag{4}$$

VIRASORO ALGEBRA

We can expand the holomorphic part of the energy-momentum tensor in its Laurent modes as

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \tag{5}$$

which can be inverted to yield

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z). \tag{6}$$

A similar procedure applies to the anti-holomorphic counterpart. Operators in (6), along with their antiholomorphic counterparts, generate the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{7}$$

The anomalous constant term c in front of the last term in equation (7) is referred to as the central charge in 2D CFTs. The central charge is a fundamental parameter that plays a crucial role in the classification and analysis of 2D CFTs.

Out of the generators in (6) L_0, $L_{\pm 1}$ constitute the $\mathfrak{sl}(2, \mathbb{C})$ sub-algebra and generates global conformal transformations on the Riemann sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$. In particular,

- L_{-1} generates translations ($z \mapsto z + b$),
- L_0 generates rotations and dilatations ($z \mapsto az$),
- L_1 generates special conformal transformations ($z \mapsto \dfrac{z}{cz+1}$).

CORRELATORS

In CFTs, and also QFTs for that matter, we are primarily interested in computing the correlation function of operators

$$\langle \mathcal{O}(x) \rangle = \frac{1}{\mathcal{Z}} \int [\mathcal{D}\phi] \, \phi(x) e^{-S[\phi]}, \tag{8}$$

where the integral is to be taken over all possible configurations of ϕ. In an ordinary QFT, one would need to evaluate the integral to calculate this quantity. However, in 2D CFTs we adopt a bootstrap approach to predict the form of (8) from the strong symmetry condition (3). For instance, translation invariance restricts the two-point function to be of the form

$$\langle \phi_1(z)\phi_2(\omega) \rangle = g(z - \omega), \tag{9}$$

with two quasi-primary fields $\phi_1(z)$ and $\phi_2(\omega)$. Demanding (9) to be invariant under scaling $z \to \lambda z$ yields yet another condition:

$$\lambda^{-(h_1+h_2)} g(z-\omega) = g(\lambda(z-\omega)), \tag{10}$$

which puts the function into the form

$$g(z-\omega) = \frac{C_{12}}{(z-\omega)^{h_1+h_2}}, \tag{11}$$

where C_{12} is a constant. Finally, the two-point function should also be left invariant under the conformal transformation $z \to -1/z$. Applying this transformation to (11), the two-point function is completely fixed up to a constant:

$$\langle \phi_1(z)\phi_2(\omega) \rangle = \begin{cases} \dfrac{C_{12}}{z_{12}^{2h} \bar{z}_{12}^{2\bar{h}}}, & \begin{array}{l} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{array}, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

where $z_{ij} \equiv |z_i - z_j|$, $\bar{z}_{ij} \equiv |\bar{z}_i - \bar{z}_j|$. A similar procedure may be applied to fix the form of the three-point function as

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) \rangle = \frac{C_{123}}{z_{12}^{h_{123}} z_{23}^{h_{231}} z_{13}^{h_{312}} \bar{z}_{12}^{\bar{h}_{123}} \bar{z}_{23}^{\bar{h}_{231}} \bar{z}_{13}^{\bar{h}_{312}}} \tag{13}$$

However, computation of 4-point function is more complex and requires the so called crossing symmetry, which is diagrammatically given as

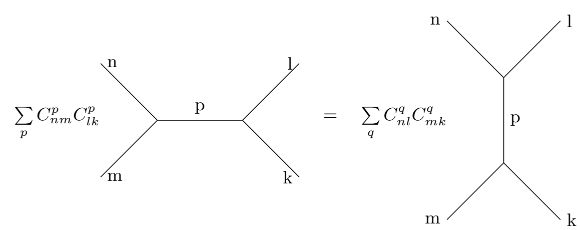

FIG. 1. Crossing symmetry

RADIAL QUANTIZATION

In this section, we adopt the most commonly used formalism and work on a cylinder of fixed radius with coordinate $\xi = t_{\mathrm{E}} + ix$ (t_{E} corresponding to Euclidean time, which can be obtained through application of a Wick's rotation to Minkowskian time). Time coordinate ranges from $-\infty$ to ∞ and the space coordinate is compactified by identifying $x = 0$ and $x = l$. Then, we map the cylinder onto a complex plane by the conformal transformation

$$\omega \mapsto z = e^{2\pi\xi/l}. \tag{14}$$

Through this mapping, the distant past corresponds to the origin $z = 0$, while the distant future maps to infinity $z = \infty$. Constant time slices of the cylinder are represented as circles of constant radii on the complex plane. In other words, the complex plane is foliated radially (see fig. (2)).

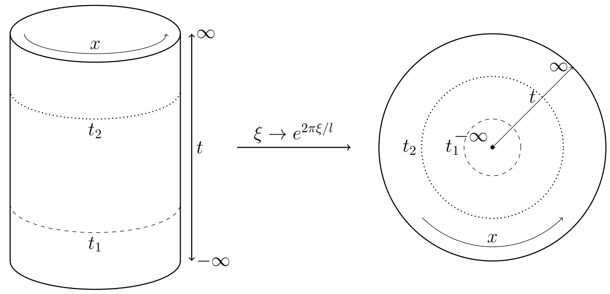

FIG. 2. Mapping from cylinder to complex plane

The unitary time and space translation operators then become radial translation and angular translation operators:

$$H = L_0 + \bar{L}_0, \qquad P = i(L_0 - \bar{L}_0) \tag{15}$$

Time ordering operation in usual a usual QFT is replaced with radial ordering, in which operators are ordered depending on their distance to the origin of the complex plane:

$$\mathcal{T}\{\phi(t_1,x_1)\varphi(t_2,x_2)\} = \begin{cases} \phi(t_1,x_1)\varphi(t_2,x_2), & t_1 > t_2 \\ \varphi(t_2,x_2)\phi(t_1,x_1), & t_1 < t_2 \end{cases}$$
$$\downarrow$$
$$\mathcal{R}\{\phi'(z_1)\varphi'(z_2)\} = \begin{cases} \phi'(z_1)\varphi'(z_2), & |z_1| > |z_2| \\ \varphi'(z_2)\phi'(z_1), & |z_1| < |z_2| \end{cases}. \tag{16}$$

After choosing the time coordinate to the radial distance, OPE's can be related to commutation relations, and vice-versa:

$$[\mathcal{O}_1, \mathcal{O}_2] = \oint_0 \mathrm{d}\omega \oint_\omega \mathrm{d}z f_1(z) f_2(\omega), \tag{17}$$

where $\mathcal{O}_{1(2)}$ is the contour integral of $f_{1(2)}$ at fixed radius $|\omega| + \epsilon$ around $z = 0$.

CONSTRUCTION OF THE HILBERT SPACE

To construct the Hilbert spaces of our theory, we first define the vacuum state $|0\rangle$, which corresponds to no operator inserted. Since the vacuum state is equivalent to "empty" spacetime, it is by construction invariant under global conformal transformations. In other words, the ground state should be annihilated by L_0 and $L_{\pm 1}$

$$L_n |0\rangle = 0, \quad \bar{L}_n |0\rangle = 0 \qquad (|n| \le 1). \tag{18}$$

Asymptotic states can be generated by inserting field operators at the origin. Consequently, initial states, represented as kets in the Hilbert space, are defined on a circle of constant radius and can be obtained as

$$|\phi_{\mathrm{in}}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z}) |0\rangle. \tag{19}$$

Equation (19) establishes a one-to-one correspondence between states and operators: operators inserted at the origin can be scaled out to states on circles of constant radii through the use of conformal invariance. This is the so-called state-operator correspondence.

Keeping in mind the fact that the Euclidean time $t_{\mathrm{E}} = it_{\mathrm{M}}$ changes sign under Hermitian conjugation, i.e. $z \to 1/\bar{z}$ after radial quantization, we define the asymptotic out state:

$$\begin{aligned} \langle \phi_{\mathrm{out}}| &\equiv |\phi_{\mathrm{in}}\rangle^\dagger \\ &= \lim_{z,\bar{z}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0| \phi(1/\bar{z}, 1/\bar{z}). \end{aligned} \tag{20}$$

Using (19) with a primary field insertion, we obtain the primary states

$$|h, \bar{h}\rangle \equiv \lim_{z,\bar{z}\to 0} \phi(z,\bar{z}) |0\rangle. \tag{21}$$

Primary states are the fundamental building blocks of the Hilbert space, in that they are eigenstates of the L_0 operator:

$$L_0 |h, \bar{h}\rangle = h |h, \bar{h}\rangle. \tag{22}$$

The operators L_n $(n > 0)$ act as lowering operators and annihilate the primary states:

$$L_n |h, \bar{h}\rangle = 0, \tag{23}$$

while for $-n < 0$, the generators act as raising operators and increase the conformal dimension:

$$[L_0, L_{-n}] = nL_{-n}. \tag{24}$$

Equation (24) suggests that states with higher conformal weight, called descendant states, can be obtained by successive application of L_{-n}'s on $|h, \bar{h}\rangle$:

$$L_{-\mu_1-\mu_2-\cdots-\mu_k} |h, \bar{h}\rangle = L_{-\mu_1} L_{-\mu_2} \dots L_{-\mu_k} |h, \bar{h}\rangle. \tag{25}$$

The quantity $\mu \equiv \mu_1 + \mu_2 + \cdots + \mu_k$ is called the level of the descendant. The subset of the Hilbert space spanned by the primary state and its descendants forms a representation of the Virasoro algebra, called a Verma module. The states of a Verma module are given in table (I) for various descendant levels.

DISCUSSION & CONCLUSION

In this review, we have explored the fundamental aspects of two-dimensional conformal field theories (2D CFTs), with a particular focus on the definition of fields, the forms of correlation functions, the underlying symmetry structure, the concept of radial quantization, and the construction of the Hilbert space.

Two-dimensional CFTs have a wide area of applications in physics ranging from statistical mechanics to

μ	States
0	$\lvert h, \bar{h}\rangle$
1	$L_{-1}\lvert h, \bar{h}\rangle$
2	$L_{-1}^2\lvert h, \bar{h}\rangle$, $L_{-2}\lvert h, \bar{h}\rangle$
3	$L_{-1}^3\lvert h, \bar{h}\rangle$, $L_{-2}L_{-1}\lvert h, \bar{h}\rangle$, $L_{-3}\lvert h, \bar{h}\rangle$
$\vdots$	$\vdots$

TABLE I. States of a Verma Module

quantum gravity theories. In particular, they are useful in the analysis of systems at critical states, which exhibit scale invariance. Having an infinite-dimensional symmetry group, greatly limits the allowed fields and parameters, resulting in a more compact theory. Furthermore, the form of correlation functions can be predicted without expanding the fields in a perturbative sum, just by respecting the symmetries of the theory.

Despite its extensive range of applications and significant advancements in research, however, 2D CFT remains unable to give a complete classification of theories. In addition, a closed-form expression of Virasoro conformal blocks, which are fixed by the symmetry conditions, is missing.

In conclusion, while 2D CFTs have provided an invaluable tool for understanding various aspects of theoretical and mathematical physics, the field remains ripe for further exploration. The unanswered questions and the potential for new breakthroughs ensure that 2D CFTs will continue to play a central role in future research, inspiring new ideas and directions in the study of condensed matter physics, string theory, and beyond.

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