Name:	
Student ID:	

Midterm Examination - 1

Phys210: Mathematical Methods in Physics II

2025/04/24

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) any of my work *outside the designated areas* in the "fill-in the blank questions" will not be graded;

(2) I take *full responsibility* for any ambiguity in my selections in "multiple choice questions";

(3) any of my work *outside the answer boxes* in the "classical questions" will not be graded;

(4) any page which does not contain *both my name and student id* will not be graded;

(5) any extra sheet that I may use are for my own calculations and will *not* be graded.

Signature: ______________________

This exam has a total of 3 questions, some of which may be for bonus points. You can obtain a maximum grade of 34+0 from this examination.

Question	Points	Score
1	6	
2	10½	

Question	Points	Score
3	17½	
Total:	34	

1 Notations & Conventions

This section contains various useful definitions to refer while solving the problems. Note that it might contain additional information not covered in class, so please do not panic: the questions do not necessarily refer to *everything* in this section.

- **The non-negative integer power** of an object A (denoted A^n) is defined recursively as
$$A^0 = \mathbb{I}\ , \quad A^n = A \cdot A^{n-1} \ \ \forall n \geq 1 \tag{1}$$
with respect to the operation $\cdot$ (such as matrix multiplication or differentiation) and its identity object $\mathbb{I}$.

- **Exponentiation of an object** A (denoted e^A) is
$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{2}$$
where A^n is the $n-$th power of the object A.

- **Logarithm of an object** A (denoted $\log A$) is defined as the inverse of the exponentiation. For objects for which the exponentiation is not a monomorphism (such as complex numbers), logarithm is a *relation* (also called multi-valued function). Conventionally, one imposes restrictions on the domain to ensure that logarithm acts as a function; for instance, for a complex number $z = re^{i\theta} \in \mathbb{C}$ with $(r, \theta) \in (\mathbb{R}^+, \mathbb{R})$, we can define $\log z = i\theta_p + \log r$ where $0 \leq \theta_p < 2\pi$ is called *the principal value of* θ.

- **The generalized power of an object** A (denoted A^α) is defined as
$$A^\alpha = e^{\alpha \log A} \tag{3}$$
If exponentiation is not a monomorphism when acting on the domain of A, A^α is not a function but a relation *unless* a principle domain is selected (similar to the logarithm).

- **Generalized exponentiation of an object** A (denoted α^A) is defined as
$$\alpha^A = e^{A \log \alpha} \tag{4}$$
Depending on the available values for $\log \alpha$, α^A may mean multiple different functions. However, each one is *still* a proper function, not a multi-valued function.

- **Trigonometric functions** cos, sin, tan, cot, csc , sec are defined in terms of the exponential via the equations
$$e^{\pm iA} = \cos(A) \pm i \sin(A)\ , \ \tan(A) = \frac{1}{\cot(A)} = \frac{\sin(A)}{\cos(A)} \tag{5}$$
$$\csc(A)\sin(A) = 1\ , \ \sec(A)\cos(A) = 1 \tag{6}$$

- **Hyperbolic functions** cosh, sinh, tanh, coth, csch, sech are defined in terms of the exponential via equations
$$e^{\pm A} = \cosh(A) \pm \sinh(A), \tanh(A) = \frac{1}{\coth(A)} = \frac{\sinh(A)}{\cosh(A)} \tag{7}$$
$$\operatorname{csch}(A)\sinh(A) = 1\ , \ \operatorname{sech}(A)\cosh(A) = 1 \tag{8}$$

- **Inverse Trigonometric/Hyperbolic functions** are denoted with an *arc* prefix in their naming, i.e. $\arcsin(x) \coloneqq \sin^{-1}(x)$. Like logarithm, these objects are *relations* (not functions) unless their domain is restricted.

- **The Kronecker symbol** (Kronecker-delta) is defined
$$\delta : \{\mathbb{Z}, \mathbb{Z}\} \to \mathbb{Z} \tag{9}$$
$$\delta = \{i, j\} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{10}$$

- **The Dirac-delta generalized function** δ is (for all practical purposes of a Physicist) defined via the relation
$$\int_{\mathcal{A}} f(y)\delta(x-y)dy = \begin{cases} f(x) & \text{if } x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$
A useful representation of this generalized function is
$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} \tag{12}$$

- **Heaviside generalized function** θ is (for all practical purposes of a Physicist) defined via the relations
$$\int_a^b \theta(x) f(x) dx = \begin{cases} \displaystyle\int_a^b f(x)dx & \text{if } a \geq 0 \\ \displaystyle\int_0^b f(x)dx & \text{if } a < 0 \end{cases} \tag{13}$$
This definition implies that $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$; however, it *does not fix* $f(0)$. We choose *the convention* $f(0) = 1/2$; this ensures
$$\operatorname{sgn}(x) = 2\theta(x) - 1 = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases} \tag{14}$$

- **A particular permutation of n objects** is denoted as $(i_1 i_2 \ldots i_n)$ where $i_1 \neq i_2 \neq \ldots \neq i_n \in \{1, \ldots, n\}$. A permutation $(i_1 \ldots i_n)$ is said to be an even (odd) permutation of $(k_1 \ldots k_n)$ if the two are identical after the permutation of an even (odd) number of adjacent indices. For example, (2431) is an even permutation of (2143) and an odd permutation of (2134).

- **Levi-Civita symbol** ϵ is defined as
$$\epsilon : \{\mathbb{Z}^+, \ldots, \mathbb{Z}^+\} \to \mathbb{Z} \tag{15}$$
$$\epsilon = \{a_1, \ldots, a_n\} \to \begin{cases} 1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an even permutation of } (12\ldots n) \\ -1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an odd permutation of } (12\ldots n) \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

- **A complex number** z is (for all practical purposes of a Physicist) a pair of two real numbers (x, y) where one can construct z via $z = x + iy$ (i is called *the imaginary unit* with the property $i^2 = -1$); conversely, one can extract x and y via $x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z)$.

- **Complex conjugation** (denoted $*$) is a function defined to act on complex numbers as

$$* : \mathbb{C} \to \mathbb{C} \tag{17}$$

$$* = z \to (z^* = \mathrm{Re}(z) - i\,\mathrm{Im}(z)) \tag{18}$$

• **Hermitian conjugation** (also called *conjugate transpose*, *adjoint*, or *dagger*) is a function defined as

$$\dagger : \mathfrak{M}_{n\times n}(\mathbb{C}) \to \mathfrak{M}_{n\times n}(\mathbb{C}) \tag{19}$$

$$\dagger = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11}^* & a_{21}^* & \dots & a_{n1}^* \\ a_{12}^* & a_{22}^* & \dots & a_{n2}^* \\ \dots & & & \\ a_{1n}^* & a_{2n}^* & \dots & a_{nn}^* \end{pmatrix} \tag{20}$$

• **Commutator** is a higher order function which takes two functions $f, g : \mathcal{A} \to \mathcal{A}$ for any type $\mathcal{A}$, and gives a new function $[f, g] : \mathcal{A} \to \mathcal{A}$ by cascading their action. It is defined on an object $x \in \mathcal{A}$ as $[f, g](x) = f(g(x)) - g(f(x))$.

• **Polar coordinates in $\mathbb{R}^d$** $(r, \theta_1, \dots, \theta_{d-1})$ are defined in terms of the Cartesian coordinates $(x_1, \dots, x_d)$ as

$$x_1 = r\cos(\theta_1)\ , \quad x_d = x_{d-1}\tan(\theta_{d-1}) \tag{21}$$

$$x_i = x_{i-1}\tan(\theta_{i-1})\cos(\theta_i) \quad \text{for } 1 < i < d \tag{22}$$

In two-dimensions, this reduces to the familiar polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$; in 3 ($> 3$) dimensions, it is also called *(hyper)spherical coordinates.*

• **Cylindrical coordinates in $\mathbb{R}^d$** $(\rho, \theta_1, \dots, \theta_{n-1}, x_n, x_{n+1}, \dots, x_d)$ is a coordinate system such that a subset $\mathbb{R}^n$ of the total space $\mathbb{R}^d$ (for $n < d$) is converted into the polar coordinates. For instance, if we convert $\mathbb{R}^2$ of $\mathbb{R}^3$ into polar coordinates, we obtain the familiar *3d cylindrical coordinates*, i.e. $(x, y, z) = (\rho\cos\theta, \rho\sin\theta, z)$.

• **(Anti)holomorphic function** of a complex variable is a function f for which the derivative with respect to z ($\bar{z}$) is uniquely defined, i.e.

$$\frac{df(x,y)}{dz} := \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x + i\Delta y}$$
$$\left(\frac{df(x,y)}{d\bar{z}} := \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x - i\Delta y}\right) \tag{23}$$

is well-defined and independent of the order of limits (this condition leads to *Cauchy-Riemann equations*). As any antiholomorphic function can be written as *complex conjugate of a holomorphic function*, one usually focuses on the analysis of holomorphic functions alone.

• **An analytic function** is a function expandable as a convergent power series. Cauchy's integral formula ensures that *a complex analytic function* (with a series expansion in z) is equivalent to a *holomorphic function.*

• **Cauchy's integral formula** for a function f complex-analytic in the region $D \subset \mathbb{C}$ can be written as

$$f(z) = \oint_{\partial D} \frac{f(\omega)}{\omega - z}\frac{d\omega}{2\pi i} \tag{24}$$

• **Laurent series** of a function complex analytic for $R_1 < |z - a| < R_2$ is the convergent series expansion $f(z) = \sum\limits_{n=-\infty}^{\infty} c_n(z-a)^n$.

• **A pole of a complex analytic function** f is a point $a \in \mathbb{C}$ such that $f(a)$ is singular and S is a non-empty set for $S = \{m \in \mathbb{Z} \mid (z-a)^m f(z) \text{ is analytic at } a\}$. $\min(S)$ is called the order of the pole.

• **A zero of a function** f is the value a such that $f(a) = 0$. $\min\{m \in \mathbb{Z} \mid \lim\limits_{z\to a}(z-a)^{-m}f(z) \neq 0\}$ is called the order of the zero.

• **A meromorphic function** f in a domain D is a holomorphic function in D except a set of points at which f has a pole. For example, $z \to \dfrac{1}{\sin(z)}$, $z \to \dfrac{e^z}{z}$ are meromorphic functions in $D = \mathbb{C}$. If we also include infinity ($D = \mathbb{C} \cup \{\infty\}$), they are no longer meromorphic as they are singular at infinity but that singularity is not a pole. In fact, *the only meromorphic functions in $D = \mathbb{C} \cup \{\infty\}$* are rational functions, e.g. $z \to \dfrac{(z-1)(z+i)}{(2z+\pi)(z-i+1)}$.

• **Residue** of a complex function at an isolated singularity a is defined as

$$\mathrm{Res} : \big(\mathbb{C} \to \mathbb{C}\ ,\ \mathbb{C}\big) \to \mathbb{C} \tag{25}$$

$$\mathrm{Res}(f, a) = \frac{1}{2\pi i}\oint_{C_a} f(z)dz \tag{26}$$

for an infinitesimal closed contour C_a centered at a.

• **Cauchy's principal value** (denoted p.v.) is for our purposes defined via the relation

$$\text{p.v.}\int_a^c f(x)dx = \lim_{\epsilon\to 0}\left[\int_a^{b-\epsilon} f(x)dx + \int_{b+\epsilon}^c f(x)dx\right] \tag{27}$$

for $a < b < c$, where $f(x)$ is assumed to be analytic in $[a, c]\backslash\{b\}$. If $f(x)$ is analytic at b, the principle value gives the same result with the ordinary integral; on the other hand, if $f(x)$ is not analytic at b, the principle value assigns a well-defined value to the integral which would be otherwise ill-defined as a function.

• **Conformal transformation** (for our purposes) is any mapping $x \to x'$ of the coordinates for which the angles between (co)tangent vectors do not change, e.g. $\dfrac{\langle dx, dy\rangle}{\sqrt{|dx||dy|}} = \dfrac{\langle dx', dy'\rangle}{\sqrt{|dx'||dy'|}}$; for instance, translation $x' = x + a$, rotation $x' = e^{i\theta}x$ or scaling $x' = \lambda x$ are so.

Name:	
Student ID:	

2 Fill-in the blanks

Each unambigiously correct answer is worth 6 point respectively.

Question: 1 .... *(6 points)*

Sofya Vasilyevna Kovalevskaya was a 19th century Russian mathematican who prominently contributed to the field of the analysis and partial differential equations. She was actively involved in the civil rights movement in Russia, advocated feminist movement, and participated in progressive political actions during her time.

Amalie Emmy Noether was a 19th/20th century German mathematican who made important contributions to abstract algebra and mathematical physics. She hold strong social-democratic political beliefs and was outspoken in civil rights, which eventually led to her exile after nazis came into power in 1930s.

Alexandre Grothendieck and *Laurent-Moïse Schwartz* were two 20th/21th century French mathematicans, and they made substantial contributions to algebraic geometry and the theory of distributions. They were both well-known intellectuals, opposing several anti-democratics phenomena of their time like the totalitarianism of the Soviet Union and the military interventions of the United States (such as the Vietnam war).

Although history is filled with many such brave souls opposing injustice and oppression of their time, I wanted to commemorate these mathematicians on behalf of all! For that, please write the name of any one of these four intellectuals in the following blank to get your six points:

Sofya Vasilyevna Kovalevskaya

3 Choose the correct option

You do not need to show your derivation in this part.
Incorrect answer for a question of X point is worth $-X/7$ points: this ensures that the randomly given answer has an expectation value of 0 point.

Question: 2 .... *($10\frac{1}{2}$ points)*

In the first month of the class (2025/02/17 - 2025/03/14), we discussed *Cauchy Riemann equations* and how we check those equations to verify if a given function is analytic. In this question, we will cover this.

For a function f where $f(x,y) = u(x,y) + iv(x,y)$ for $u(x,y), v(x,y) \in \mathbb{R}$, the Cauchy Riemann equations read as

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{28}$$

Name:	
Student ID:	

If these equations are satisfied at some (x, y), the function $f(x, y)$ is said to be complex analytic at the point $z = x + iy$.

In the parts below, choose the correct $g(x, y)$ such that $f(x, y)$ is an analytic function (each correct answer is worth 2.1 points).

(a) $$f(x, y) = 2 + ig(x, y)$$

☐ $x^{-2}y^{-2}$ ☐ $x^{-1}y^{-2}$ ☐ $x^{-1}y^{-1}$ ☐ $x^{-1}y^{0}$
■ $x^{0}y^{0}$ ☐ $x^{1}y^{0}$ ☐ $x^{1}y^{1}$ ☐ $x^{1}y^{2}$

(b) $$f(x, y) = (1 + i)x + ig(x, y)$$

☐ $-x - y - 1$ ☐ $-x - y + 1$ ■ $-x + y - 1$ ☐ $x - y - 1$
☐ $-x^2 - y^2 - 1$ ☐ $-x^2 - y^2 + 1$ ☐ $-x^2 + y^2 - 1$ ☐ $x^2 - y^2 - 1$

(c) $$f(x, y) = g(x, y) + ix^2 - iy^2$$

☐ $1 + xy$ ☐ $1 - xy$ ☐ $2(1 + xy)$ ■ $2(1 - xy)$
☐ $x + y$ ☐ $x - y$ ☐ $-x + y$ ☐ $-x - y$

(d) $$f(x, y) = 1 + 2x + g(x, y) + 2yi(1 + x)$$

■ $x^2 - y^2$ ☐ $x - y^2$ ☐ $x^2 - y$ ☐ $x - y$
☐ $x^2 + y^2$ ☐ $x + y^2$ ☐ $x^2 + y$ ☐ $x + y$

(e) $$f(x, y) = e^{-y}\cos(x)(1 - i) + g(x, y)(1 + i)$$

☐ $\cos(x)$ ☐ $\sin(x)$ ☐ $\cos(y)$ ☐ $\sin(y)$
☐ $e^{-y}\cos(x)$ ■ $e^{-y}\sin(x)$ ☐ $e^{-x}\cos(y)$ ☐ $e^{-x}\sin(y)$

Name:	
Student ID:	

Solution 2.1 *Let's solve the part* **(e)***: the solution for the other parts are similar (even simpler). We are given*

$$f(x,y) = e^{-y}\cos(x)(1-i) + g(x,y)(1+i) \tag{29}$$

which means

$$u(x,y) = e^{-y}\cos(x) + g(x,y)\ , \quad v(x,y) = -e^{-y}\cos(x) + g(x,y) \tag{30}$$

for which the Cauchy Riemann equations read

$$\begin{aligned} -e^{-y}\sin(x) + \frac{\partial g(x,y)}{\partial x} =& e^{-y}\cos(x) + \frac{\partial g(x,y)}{\partial y} \\ -e^{-y}\cos(x) + \frac{\partial g(x,y)}{\partial y} =& - e^{-y}\sin(x) - \frac{\partial g(x,y)}{\partial x} \end{aligned} \tag{31}$$

which can be reorganized as

$$\begin{aligned} \frac{\partial g(x,y)}{\partial x} - \frac{\partial g(x,y)}{\partial y} =& e^{-y}\cos(x) + e^{-y}\sin(x) \\ \frac{\partial g(x,y)}{\partial y} + \frac{\partial g(x,y)}{\partial x} =& - e^{-y}\sin(x) + e^{-y}\cos(x) \end{aligned} \tag{32}$$

By summing and subtracting, we obtain

$$\begin{aligned} \frac{\partial g(x,y)}{\partial x} =& e^{-y}\cos(x) \\ \frac{\partial g(x,y)}{\partial y} =& - e^{-y}\sin(x) \end{aligned} \tag{33}$$

The solution to the first equation is trivial, i.e.

$$g(x,y) = e^{-y}\sin(x) + h(y) \tag{34}$$

for the undetermined function h. If we now insert this into the second equation, we obtain

$$-e^{-y}\sin(x) + h'(y) = -e^{-y}\sin(x) \tag{35}$$

which leads to $h(y) =$constant. Therefore, we conclude that

$$g(x,y) = e^{-y}\sin(x) + c \tag{36}$$

for the undetermined coefficient c. Among the options, the only choice that fits this $g(x,y)$ is the one for $c = 0$.

Name:	
Student ID:	

Question: 3 .... *(17½ points)*

In the first month of the class (2025/02/17 - 2025/03/14), we also discussed *Laurent series expansions* and how *meromorphic functions* can easily be converted to Laurent series by simple algebra and by the judicious use of geometric series. In this question, we will go over that.

Consider the function

$$f(z) = \frac{1}{(z-1)(z-2)(z-3)} \tag{37}$$

which can be expanded as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad \text{for } \beta < |z| < \gamma \tag{38}$$

In the parts below, choose the correct c_n for the given quadruplet (a, β, γ, n) (each correct answer is worth 3.5 points).

(a) $(a, \beta, \gamma, n) = (0, 0, 1, -1)$

■ 0 ☐ 1 ☐ 1/6 ☐ 2 ☐ 1/4 ☐ 4 ☐ 1/2 ☐ 6

(b) $(a, \beta, \gamma, n) = (0, 0, 1, 0)$

☐ 1/6 ☐ −1/2 ☐ 1/4 ☐ −1/3 ☐ 1/3 ☐ −1/4 ☐ 1/2 ■ −1/6

(c) $(a, \beta, \gamma, n) = (1, 1, 2, -1)$

☐ 1/6 ☐ −1/2 ☐ 1/4 ☐ −1/3 ☐ 1/3 ☐ −1/4 ■ 1/2 ☐ −1/6

(d) $(a, \beta, \gamma, n) = (2, 2, 3, 0)$

☐ 3/4 ☐ −3/4 ☐ 11/36 ☐ −11/36 ☐ 1 ☐ −1 ■ 0 ☐ 2025

(e) $(a, \beta, \gamma, n) = (2, 2, 3, 1)$

☐ 3/4 ☐ −3/4 ☐ 11/36 ☐ −11/36 ☐ 1 ■ −1 ☐ 0 ☐ 2025

Name:	
Student ID:	

Solution 3.1 *Let's solve the part* **(e)**: *the solution for the other parts are similar (and are actually easier as n is lower). We are given*

$$f(z) = \frac{1}{(z-1)(z-2)(z-3)} = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad \text{for } \beta < |z| < \gamma \tag{39}$$

and we are asked c_n for $(a, \beta, \gamma, n) = (2, 2, 3, 1)$. Therefore, we need to find c_1 such that

$$\frac{1}{(z-1)(z-2)(z-3)} = \cdots + c_1(z-2) + \cdots \qquad \text{for} \quad 2 < |z| < 3 \tag{40}$$

There are various ways to solve this; for instance, we could use Cauchy's integral formula to get the answer! However, note that $f(z)$ is a meromorphic function, i.e. it does not have any essential singularities! Therefore, its Laurent series expansion is bounded from below; for the case of an expansion around $z = 2$, we simply have

$$\frac{1}{(z-1)(z-2)(z-3)} = \frac{c_{-1}}{z-2} + c_0 + c_1(z-2) + \cdots \qquad \text{for} \quad 2 < |z| < 3 \tag{41}$$

as it is a simple pole! This means

$$\frac{1}{(z-1)(z-3)} = c_{-1} + c_0(z-2) + c_1(z-2)^2 + \cdots \tag{42}$$

If we now differentiate twice, first two terms on the right hand side will die! We then take $z \to 2$, killing the higher terms in the dots. That means

$$2c_1 = \lim_{z\to 2} \frac{d^2}{dz^2} \frac{1}{(z-1)(z-3)} \tag{43}$$

which we can solve as

$$4c_1 = \lim_{z\to 2} \frac{d^2}{dz^2} \left(\frac{1}{z-3} - \frac{1}{z-1} \right) = \lim_{z\to 2} \left(\frac{2}{(z-3)^3} - \frac{2}{(z-1)^3} \right) = -4 \tag{44}$$

hence $c_1 = -1$.

Alternatively, *we can use geometric series as hinted in the question. For $2 < |z| < 3$, we have*

$$\frac{1}{z-1} = \frac{1}{1+(z-2)} = \sum_{n=0}^{\infty} (-1)^n (z-2)^n \tag{45}$$

and

$$\frac{1}{z-3} = -\frac{1}{1-(z-2)} = -\sum_{n=0}^{\infty} (z-2)^n \tag{46}$$

Name:	
Student ID:	

meaning

$$
\begin{aligned}
\frac{1}{(z-1)(z-2)(z-3)} &= \left[\sum_{n=0}^{\infty}(-1)^n(z-2)^n\right](z-2)^{-1}\left[-\sum_{n=0}^{\infty}(z-2)^n\right] \\
&= \left[1-(z-2)+(z-2)^2+\dots\right]\left[-1-(z-2)-(z-2)^2+\dots\right](z-2)^{-1} \\
&= \left[-1-(z-2)^2+\dots\right](z-2)^{-1} \\
&= \frac{-1}{z-2}+0.(z-2)^0-(z-2)^1+\dots
\end{aligned} \tag{47}
$$

hence we conclude $c_{-1} = c_1 = -1$ *and* $c_0 = 0$.

« « « Congratulations, you have made it to the end! » » »