Scale Invariance vs. Conformal Invariance

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This review thoroughly investigates the relationship between scale invariance and conformal invariance. The main concept is whether scale invariance implies conformal invariance and what are the conditions for this implication? In two dimensions the proof is completed under some conditions that we discuss. However, the generalization to higher dimensions is not straightforward. To this end, we present many methods and works both written with the aim of disproof this implication or reverse. To see the importance of this problem, we also discuss several applications. Finally, we discuss recent works and possible future directions.

Keywords: scale invariance, conformal invariance, renormalization group flow

I. INTRODUCTION AND OUTLINE

Mathematically, scale invariance and conformal invariance are two very different symmetries. Conformal invariance is more restricted than scale invariance, and it implies scale invariance. The converse is not true, in general. However, in most physical models, when we only start with scale invariance, we also get the conformal invariance for granted.

In this paper, we are interested in understanding the underlying reasons why scale invariance implies conformal invariance, the counterexamples, and the importance of studying this problem. To this end, we first review the different approaches to define scale and conformal invariance. With these setups, we state the problem. Then, we review the works done in two-dimensional and higher-dimensional theories. Later, we discuss some of the applications of this problem in different fields such as statistical mechanics, string theory and quantum chromodynamics. Finally, we investigate the recent developments in the field and possible future directions.

II. SCALE INVARIANCE VS. CONFORMAL INVARIANCE

To state the problem firmly, we must first be sure what we mean by scale invariance and conformal invariance. This can be done in multiple ways. Conventionally, we start by considering a quantum field theory (QFT) in d dimensions with the symmetry group

$$\mathrm{ISO}(d-1,1) \rtimes \mathbb{R}^+ \tag{1}$$

which is the Poincaré group with scale invariance. Here $\mathbb{R}^+$ is generated by dilatation operators $\hat{D}$. An operator product expansion (OPE) example is

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{c_{12}}{(x_1-x_2)^{\Delta_1+\Delta_2}} \tag{2}$$

One important aspect of this OPE is that the convergence of it is not proven. Additionally, the 2-point functions of higher spin operators are not uniquely fixed. Also, bootstrap methods are not applicable. Under this symmetry group there exists a local and conserved scale (dilatation) current [1]

$$S^\mu(x) = x^\nu T_\nu{}^\mu + K^\mu(x) \tag{3}$$

Here $T_{\mu\nu}$ is a local, symmetric and conserved energy-momentum (EM) tensor, and $K^\mu(x)$ is some local operator, called the virial current. Using the conservation of $S^\mu(x)$, we can get the following condition for the trace of the EM tensor

$$T_\mu{}^\mu = -\partial_\mu K^\mu \tag{4}$$

On the other hand, if we have a conformal symmetry the symmetry group is enhanced to the conformal group

$$\text{Conf(d-1,1)} \simeq \text{SO(d,2)} \tag{5}$$

The OPE would be of the following form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{c_{12}\delta_{\Delta_1\Delta_2}}{(x_1-x_2)^{2\Delta_1}} \tag{6}$$

Convergence of this OPE is proved [2], and 2,3 - point functions are uniquely fixed. Also, we can apply bootstrap methods in theories with conformal symmetry group.

The corresponding conformal current is of the form

$$j^\mu(x) = v^\nu(x) T_\nu{}^\mu(x) + \partial\cdot v(x) K^\mu(x) + \partial_\nu \partial\cdot v(x) L^{\nu\mu}(x) \tag{7}$$

which implies a traceless EM tensor $T_\mu{}^\mu = 0$. Therefore, the aim is to search for a virial current whose divergence is zero.

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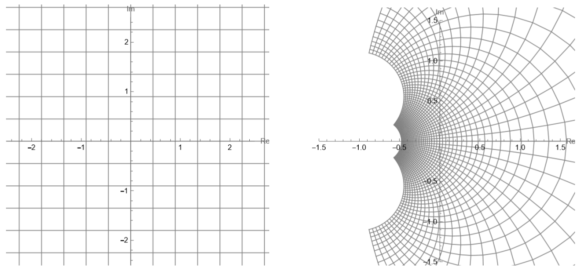

(a) Scale Transformations

(b) Conformal Transformations

Figure 1: Illustration of (a) Scale Transformations and (b) Conformal Transformations.

In this perspective, these two invariances appear to be highly different. If we apply scale and conformal transformations to a grid, we get as shown in Figures 1a and 1b, respectively.

There is another very useful perspective. In 1986, Zamolodchikov [3] showed the existence of a function $c(g)$ which monotonically decreases along the renormalization group (RG) flow. As a corollary to this theorem, the cyclic or chaotic behavior in the RG flow is not possible. These behaviors in the RG flow are closely related to scale invariant but non-conformal invariant theories [4].

Figure 2: Possible RG Flow Examples

The second figure shows a cyclic behavior and is related to scale invariant but non-conformal invariant field theories.

III. WORKS IN 2D AND HIGHER DIMENSIONS

Using the ideas of $c-$theorem, Polchinski [1] non-perturbatively showed that under the conditions (1) unitarity, (2) causality, (3) discrete spectrum in scaling dimension scale invariance implies conformal invariance. If we try to loose some of the conditions, counterexamples arise. For example, the elasticity theory in 2D [5] is scale invariant but not conformal invariant. However, it violates the unitarity condition. Another example is the 2D non-linear σ model proposed by Hull and Townsend [6], which violates the existence of finite correlators of the EM tensor.

The generalization of Polchinski's proof to higher dimensions is not straightforward. There is a simple counterexample proposed in [7], the free $U(1)$ gauge theory

$$S = \int d^d x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{8}$$

Since it is mass-free, it is scale invariant. We can work out to find the symmetric EM tensor of this theory and take the trace, which is

$$T_\mu{}^\mu = (d-4) F_{\mu\nu} F^{\mu\nu} = \frac{d-4}{2} \partial^\mu (F_{\mu\nu} A^\nu) \tag{9}$$

We conclude that if $d \neq 4$, there is no conformal invariance.

There is a particular importance in 4D, and a lot of work is done in that direction. In 4D, it has shown that in perturbation theory at one-loop order about a conformal fixed point, the only possible asymptotics is conformally invariant [8]. Later, this result has shown to be true at two-loop order. Also, in [9] it is proven that Lorentz invariance and unitarity are sufficient conditions for conformal invariance implied by scale invariance.

IV. APPLICATIONS

We start with the simplest example, the critical phenomena of boiling water. It is known that the second-order phase transition of the water shows universality. That is the critical exponents $\alpha \simeq 0.325$ and $\beta \simeq 0.11$ in

$$\delta_\rho(T) \sim (T - T_c)^\beta \quad \text{and} \quad C \sim (T - T_c)^\alpha \tag{10}$$

are universal. They are also seen in the 3D Ising model after appropriate substitutions. It is also observed that these thermodynamic quantities show a scaling behavior (sometimes called the "scaling hypothesis"). Using this scale invariance, we find relations between the critical exponents. However, to determine the critical exponents we also need conformal invariance. Therefore, we want to understand the necessary conditions for conformal invariance in critical phenomena better. Then, we may apply the ideas of 2D conformal field theories to higher dimensional theories to determine the critical exponents of higher dimensional critical phenomena [4].

Another application can be seen in quantum chromodynamics (QCD). The UV limit of QCD is asymptotic to the trivial Gaussian scale invariant fixed point [10], [11]. One common way of realizing the scale-invariant non-trivial fixed point in model-building strategies is to use the "anomalous dimensions" of different operators, deviating from naive dimensional analysis based on engineering calculations. Conformal invariance can put more constraints on the anomalous dimensions. However, if we have scale invariance without conformal invariance we lose predictive powers with relaxed constraints. It is arguable which one is better.

V. CURRENT WORKS AND FUTURE DIRECTIONS

The propagation of a string in a d dimensional background spacetime with a manifold M can be formulated in terms of a 2D non-linear σ model with M being its target space.

$$S = \frac{1}{4\pi\alpha'} \int_{\mathbb{R}^2} d^2x G_{IJ} \partial_\alpha X^I \partial^\alpha X^J \tag{11}$$

The consistency conditions of string theory in this setting require the σ model to be conformally invariant [6]. Witten and Papadopoulos [12], [13] recently started to work on scale invariance vs. conformal invariance from a different point of view. The results of Perelman on Poincaré conjecture are interpreted as a geometric proof of $c-$theorem in σ model perturbation theory. The dynamics of Ricci soliton and compactness of the target space have shown to be sufficient for scale invariance implying conformal invariance.

Another perspective is studied by Nakayama [14]. Using the AdS/CFT correspondence and holographic arguments they identify the $c-$function with the entropy of a gravitational system, which is called the holographic $c-$theorem. The metric on the $d+1$ dimensional AdS spacetime is given by

$$ds^2 = g_{MN} dx^M dx^N = L^2 \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2} \tag{12}$$

which is scale invariant. It is also invariant under the full AdS isometry by accident. Therefore, the holographic dual theory must be conformally invariant. There is no matter sector in this discussion. Some progress has been made in understanding the mechanism of breaking conformal symmetry by adding matter and analyzing the consequences of it in the context of scale invariance vs. conformal invariance [4].

A recent work [15] in critical random systems exhibiting Parisi-Sourlas supertranslation symmetry, nine non-trivial scale-invariant RG fixed points are identified, only one of which is conformal. It is believed that without a "sophisticated" mechanism, scale invariance without conformal invariance cannot occur.

VI. CONCLUSIONS

This review has discussed the subtle interplay between scale invariance and conformal invariance, pointing out the conditions under which one type of symmetry requires the other. In two dimensions, the relation is very precise when certain conditions are satisfied; in higher dimensions, it is more delicate and not so generally applicable. Counterexamples like the free $U(1)$ gauge theory and elasticity theory show the subtlety of the problem. The importance of establishing conditions under which scale invariance implies conformal invariance is underscored by numerous applications in such diverse fields as critical phenomena, quantum chromodynamics, and string theory. Recent progress, such as in the domains of holography and critical random systems, brings new insights and tools to tackle long-standing questions in this field.

While much has been achieved, many open questions remain, especially for higher-dimensional theories and the role of intricate mechanisms that may provide scale invariance without conformal invariance. Future work in this field will likely uncover more profound insights into the symmetry structures of nature, deepening our understanding of fundamental physics and its applications.

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