Name:	
Student ID:	

Midterm Examination - 1

Phys209: Mathematical Methods in Physics I

2024/11/27

Please carefully read below before proceeding!

I acknowledge by taking this examination that I am aware of all academic honesty conducts that govern this course and how they also apply for this examination. I therefore accept that I will not engage in any form of academic dishonesty including but not limited to cheating or plagiarism. I waive any right to a future claim as to have not been informed in these matters because I have read the syllabus along with the academic integrity information presented therein.

I also understand and agree with the following conditions:

(1) any of my work *outside the designated areas* in the "fill-in the blank questions" will not be graded;

(2) I take *full responsibility* for any ambiguity in my selection of the correct option in "multiple choice questions";

(3) any of my work *outside the answer boxes* in the "classical questions" will not be graded;

(4) any page which does not contain *both my name and student id* will not be graded;

(5) any extra sheet that I may use are for my own calculations and will *not* be graded.

Signature: ____________________________

This exam has a total of 4 questions, some of which are for bonus points. You can obtain a maximum grade of 34+0 from this examination.

Question	Points	Score
1	14	
2	8	
3	8	

Question	Points	Score
4	4	
Total:	34	

1 Notations & Conventions

This section contains various useful definitions to refer while solving the problems. Note that it might contain additional information not covered in class, so please do not panick: the questions do not necessarily refer to *everything* in this section.

- **The non-negative integer power** of an object A (denoted A^n) is defined recursively as

$$A^0 = \mathbb{I}\ , \quad A^n = A \cdot A^{n-1} \ \ \forall n \geq 1 \tag{1}$$

with respect to the operation $\cdot$ (such as matrix multiplication or differentiation) and its identity object $\mathbb{I}$.

- **Exponentiation of an object** A (denoted e^A) is

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{2}$$

where A^n is the $n-$th power of the object A.

- **Logarithm of an object** A (denoted $\log A$) is defined as the inverse of the exponentiation. For objects for which the exponentiation is not a monomorphism (such as complex numbers), logarithm is a *relation* (also called multi-valued function). Conventionally, one imposes restrictions on the domain to ensure that logarithm acts as a function; for instance, for a complex number $z = re^{i\theta} \in \mathbb{C}$ with $(r, \theta) \in (\mathbb{R}^+, \mathbb{R})$, we can define $\log z = i\theta_p + \log r$ where $0 \leq \theta_p < 2\pi$ is called *the principal value of* θ.

- **The generalized power of an object** A (denoted A^α) is defined as

$$A^\alpha = e^{\alpha \log A} \tag{3}$$

If exponentiation is not a monomorphism when acting on the domain of A, A^α is not a function but a relation *unless* a principle domain is selected (similar to the logarithm).

- **Generalized exponentiation of an object** A (denoted α^A) is defined as

$$\alpha^A = e^{A \log \alpha} \tag{4}$$

Depending on the available values for $\log \alpha$, α^A may mean multiple different functions. However, each one is *still* a proper function, not a multi-valued function.

- **Trigonometric functions** $\cos$, $\sin$, $\tan$, $\cot$, $\csc$, $\sec$ are defined in terms of the exponential via the equations

$$e^{\pm iA} = \cos(A) \pm i \sin(A)\ , \ \tan(A) = \frac{1}{\cot(A)} = \frac{\sin(A)}{\cos(A)} \tag{5}$$

$$\csc(A)\sin(A) = 1\ , \ \sec(A)\cos(A) = 1 \tag{6}$$

- **Hyperbolic functions** $\cosh$, $\sinh$, $\tanh$, $\coth$, csch, sech are defined in terms of the exponential via equations

$$e^{\pm A} = \cosh(A) \pm \sinh(A), \tanh(A) = \frac{1}{\coth(A)} = \frac{\sinh(A)}{\cosh(A)} \tag{7}$$

$$\mathrm{csch}(A)\sinh(A) = 1\ , \ \mathrm{sech}(A)\cosh(A) = 1 \tag{8}$$

- **Inverse Trigonometric/Hyperbolic functions** are denoted with an *arc* prefix in their naming, i.e. $\arcsin(x) \coloneqq \sin^{-1}(x)$. Like logarithm, these objects are *relations* (not functions) unless their domain is restricted.

- **The Kronecker symbol** (Kronecker-delta) is defined

$$\delta : \{\mathbb{Z}, \mathbb{Z}\} \to \mathbb{Z} \tag{9}$$

$$\delta = \{i, j\} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{10}$$

- **The Dirac-delta generalized function** δ is (for all practical purposes of a Physicist) defined via the relation

$$\int_{\mathcal{A}} f(y)\delta(x-y)dy = \begin{cases} f(x) & \text{if } x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

A useful representation of this generalized function is

$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi} \tag{12}$$

- **Heaviside generalized function** θ is (for all practical purposes of a Physicist) defined via the relations

$$\int_a^b \theta(x) f(x) dx = \begin{cases} \displaystyle\int_a^b f(x)dx & \text{if } a \geq 0 \\ \displaystyle\int_0^b f(x)dx & \text{if } a < 0 \end{cases} \tag{13}$$

This definition implies that $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$; however, it *does not fix* $f(0)$. We choose *the convention* $f(0) = 1/2$; this ensures

$$\mathrm{sgn}(x) = 2\theta(x) - 1 = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases} \tag{14}$$

- **A particular permutation of n objects** is denoted as $(i_1 i_2 \ldots i_n)$ where $i_1 \neq i_2 \neq \ldots \neq i_n \in \{1, \ldots, n\}$. A permutation $(i_1 \ldots i_n)$ is said to be an even (odd) permutation of $(k_1 \ldots k_n)$ if the two are identical after the permutation of an even (odd) number of adjacent indices. For example, (2431) is an even permutation of (2143) and an odd permutation of (2134).

- **Levi-Civita symbol** ϵ is defined as

$$\epsilon : \{\mathbb{Z}^+, \ldots, \mathbb{Z}^+\} \to \mathbb{Z} \tag{15}$$

$$\epsilon = \{a_1, \ldots, a_n\} \to \begin{cases} 1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an even permutation of } (12\ldots n) \\ -1 & \text{if } (a_1 a_2 \ldots a_n) \text{ is an odd permutation of } (12\ldots n) \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

- **The determinant function** (denoted det) is defined

$$\det : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{17}$$

$$\det = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_{i_1,\dots,i_n} \epsilon_{i_1\dots i_n} a_{1i_1}\dots a_{ni_n} \tag{18}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i,j$.

- **The adjugate function** (denoted adj) is defined as

$$\mathrm{adj} : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{19}$$

$$\mathrm{adj} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \tag{20}$$

for

$$b_{i_nk_n} = \sum_{\substack{i_1,\dots,i_{n-1}\\ k_1,\dots,k_{n-1}}} \frac{\epsilon_{i_1\dots i_n}\epsilon_{k_1\dots k_n} a_{i_1k_1}\dots a_{i_{n-1}k_{n-1}}}{(n-1)!} \tag{21}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i,j$.

- **Inverse of an object** A is denoted as A^{-1} and is defined with respect to an operation "$\cdot$" and its identity element $\mathbb{I}$ via the the equations $A\cdot A^{-1} = A^{-1}\cdot A = \mathbb{I}$. If "$\cdot$" is matrix multiplication, then

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det A} \tag{22}$$

- **The trace function** (denoted tr) is defined as

$$\mathrm{tr} : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathcal{A} \tag{23}$$

$$\mathrm{tr} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \sum_i a_{ii} \tag{24}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i,j$.

- **Wronskian matrix** of a set of functions $\{f_1(x),\dots,f_n(x)\}$ is defined as a square matrix where the first row is the set of the functions and the $i-$th row is $(i-1)-$th derivative of the functions for all $n \geq i \geq 2$.

- **A complex number** z is (for all practical purposes of a Physicist) a pair of two real numbers (x,y) where one can construct z via $z = x+iy$ (i is called *the imaginary unit* with the property $i^2=-1$); conversely, one can extract x and y via $x=\mathrm{Re}(z)$, $y=\mathrm{Im}(z)$.

- **Complex conjugation** (denoted $*$) is a function defined to act on complex numbers as

$$* : \mathbb{C} \to \mathbb{C} \tag{25}$$

$$* = z \to (z^* = \mathrm{Re}(z) - i\,\mathrm{Im}(z)) \tag{26}$$

- **Matrix transpose** (denoted T) is a function defined

$$T : \mathfrak{M}_{n\times n}(\mathcal{A}) \to \mathfrak{M}_{n\times n}(\mathcal{A}) \tag{27}$$

$$T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \tag{28}$$

where $\mathcal{A}$ is any field such that $a_{ij} \in \mathcal{A}$, $\forall i,j$.

- **Hermitian conjugation** (also called *conjugate transpose*, *adjoint*, or *dagger*) is a function defined as

$$\dagger : \mathfrak{M}_{n\times n}(\mathbb{C}) \to \mathfrak{M}_{n\times n}(\mathbb{C}) \tag{29}$$

$$\dagger = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} a^*_{11} & a^*_{21} & \dots & a^*_{n1} \\ a^*_{12} & a^*_{22} & \dots & a^*_{n2} \\ \dots \\ a^*_{1n} & a^*_{2n} & \dots & a^*_{nn} \end{pmatrix} \tag{30}$$

- **Characteristic polynomial** of any square matrix A:

$$\det(A - \lambda_i\mathbb{I}) = 0 \tag{31}$$

- **Laplace transform** is an integral transform (denote $\mathcal{L}$) which converts a function $f : \mathbb{R} \to \mathbb{R}$ into another function $\hat{f} = \mathcal{L}(f)$ such that

$$\hat{f} : \mathbb{C} \to \mathbb{C}\ , \qquad \hat{f}(s) = \int_0^\infty f(x)e^{-xs}dx \tag{32}$$

For *meromorphic* $\hat{f}$ (i.e. $\frac{\text{polynomial}}{\text{polynomial}}$), the inverse is computed by rewriting $\hat{f}(s)$ as a sum $\sum_i a_i(s+r_i)^{-n_i-1}$ which is clearly (for some $c_{k,\ell}$) the Laplace transform of $f(x) = \sum_i e^{-r_ix}\left(c_{i,1} + c_{i,2}x + \dots c_{i,n_i}x^{n_i}\right)$. Formally,

$$f : \mathbb{R} \to \mathbb{R}\ , \qquad f(x) = \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{f}(s)e^{xs}\frac{ds}{2\pi i} \tag{33}$$

where the *contour integral* in the complex plane is chosen appropriately based on the convergence.

- **Convolution** of two functions f and g (denote $f*g$) is the operation that becomes multiplication in the Laplace domain, i.e. $\mathcal{L}(f*g) \equiv \mathcal{L}(f)\mathcal{L}(g)$; equivalently,

$$(f*g)(x) = \int_0^x f(y)g(x-y)dy \tag{34}$$

- **Commutator** is a higher order function which takes two functions $f,g : \mathcal{A} \to \mathcal{A}$ for any type $\mathcal{A}$, and gives a new function $[f,g] : \mathcal{A} \to \mathcal{A}$ by cascading their action. It is defined on an object $x \in \mathcal{A}$ as $[f,g](x) = f(g(x)) - g(f(x))$.

- **Polar coordinates in $\mathbb{R}^d$** $(r,\theta_1,\dots,\theta_{d-1})$ are defined in terms of the Cartesian coordinates $(x_1,\dots,x_d)$ as

$$x_1 = r\cos(\theta_1)\ , \quad x_d = x_{d-1}\tan(\theta_{d-1}) \tag{35}$$

$$x_i = x_{i-1}\tan(\theta_{i-1})\cos(\theta_i) \quad \text{for } 1 < i < d \tag{36}$$

In two-dimensions, this reduces to the familiar polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$; in 3 ($>3$) dimensions, it is also called *(hyper)spherical coordinates*.

- **Cylindrical coordinates in $\mathbb{R}^d$** $(\rho,\theta_1,\dots,\theta_{n-1},x_n,x_{n+1},\dots,x_d)$ is a coordinate system such that a subset $\mathbb{R}^n$ of the total space $\mathbb{R}^d$ (for $n<d$) is converted into the polar coordinates. For instance, if we convert $\mathbb{R}^2$ of $\mathbb{R}^3$ into polar coordinates, we obtain the familiar *3d cylindrical coordinates*, i.e. $(x,y,z) = (\rho\cos\theta, \rho\sin\theta, z)$.

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2 Fill-in the blanks

Each correct answer is worth 1.4 point.

Question: 1 . . . . *(14 points)*

Differential equations are hard to solve! This has been the main theme of our classes so far, and it has only gotten worse as we have progressed in the class. Indeed, even though we only have focused on ***linear*** (hence can not contain terms like $f(x)^2$) and ***ordinary*** (hence does not have pieces like $\frac{\partial^2 f(x,y)}{\partial x \partial y}$) differential equations, they proved to be remarkably hard to solve. Of course, there are several methods to simplify the differential equation; for instance, if we already know one of the solutions, we can use the method called ***reduction of order*** to convert, say, a third order differential equation to a second order one; however, even such methods are quite lengthy hence can quickly become tedious.

Let us rewind back to the beginning of the semester: we started analyzing differential equations *with constant coefficients*. What was remarkable about such differential equations is that we can always solve them in an algorithmic way: indeed, we use *characteristic equation* to find ***homogeneous*** solutions, and then take the convolution of nonhomogeneous piece of the differential equation with the ***impulse response*** to obtain the particular solution.

Simplicity of differential equations with constant coefficients compell us to try converting other types of differential equations to this class if possible. Indeed, we have learned how to check if a reparemetrization of the differential equation can serve this purpose. An example where this approach is fruitful is the case of *Euler equations*, i.e. $x^2 f''(x) + axf'(x) + bf(x) = 0$; as a matter of fact, we have shown in class that this equation can be rewritten as $f''(y) + cf'(y) + df(y) = 0$ where $y(x)$ is given by the equation $y(x) = \log(x)$.

Sometimes we do not deal with a differential equation with constant coefficients, and no reparametrization is available. Nevertheless, we can still make progress by reducing the order of the differential equation via the insertion $g(x) = f'(x)$ if $f(x)$ is missing in the differential equation. And sometimes, we can reduce the order of the differential equation if we can write it in the form $\frac{d}{dx}\left[a_n(x)f^{(n)}(x) + \cdots + a_1(x)f'(x) + a_0(x)f(x)\right] = 0$: such differential equations are called ***exact*** differential equations.

Even with all these different techniques, some differential equations are simply impossible to solve directly; for them, we resort to *series solutions*. Choosing $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is sufficient to solve the differential equation if the point a is ***an ordinary / a regular*** point; in comparison, we need to use *Frobenius method* (i.e. choose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^{n+r}$) if the point a is ***a regular singular*** point. The last option is that $x = a$ is actually corresponding to an *essential singularity*, hence we can not use such a series solution around $x = a$ to find $f(x)$.

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3 Choose the correct option

You do not need to show your derivation in this part.
Incorrect answer for a question of X point is worth $-X/4$ points: this ensures that the randomly given answer has an expectation value of 0 point.

Question: 2 .... *(8 points)*

Consider the following fictitious story: you decided to analyze the orbit of Ganymede, the largest moon of the Jupiter, and you are only interested in a solution that is valid for a very short duration of time. This means that you can work with a simplistic (and importantly linearized) model instead. You ask your astophysicist friend for help, and she tells you that if we take r as the radial distance between Ganymede and Jupiter, then its time dependence $r(t)$ satisfies

$$f'(t) = a(f(t) - g(t)) \tag{37}$$

for some experimentally determined constant a. Here, $g(t)$ denotes the collective effect of all the other moons and celestial objects on Ganymede.

(a) **(1 point)** Assume that $g(t)$ does not have any time dependence, i.e. $g(t) = b$ for a constant b. Since we know that Ganymede is bounded in its orbit (i.e. the distance between this moon and Jupiter can not go to infinity as time increases), what can we say about the constant a? *Hint: take your differential equation to be valid only for $t > 0$!*

☐ $a < -|b|$ ■ $a < 0$ ☐ $a > 0$ ☐ $a > |b|$ ☐ None

(b) **(3 points)** Now assume that the change in the effect of the other objects on Ganymede depends on the distance between Ganymede and Jupiter, i.e.

$$g'(t) = \beta f(t) \tag{38}$$

For constants a and β, $f(t)$ now satisfies a linear ordinary homogeneous differential equation with constant coefficients: if its characteristic equation has roots at $r = -\frac{a^2}{1-2a}$ and $r = \frac{a-a^2}{1-2a}$, what is the value of β?

☐ $\beta = a - 1$ ☐ $\beta = \frac{a-1}{1-2a}$ ☐ $\beta = \frac{a-1}{(1-2a)^2}$ ■ $\beta = \frac{(a-1)a^2}{(1-2a)^2}$ ☐ None

(c) **(4 points)** Now assume that the change in the effect of the other objects on Ganymede not only depends on the distance between Ganymede and Jupiter, but also on how that distance changes with time! If we are then given that

$$g'(t) = f'(t) - \frac{t}{a+at}f'(t) - \frac{1}{a(1+t)^n}f(t) \tag{39}$$

what should be the value of n such that the differential equation for $f(t)$ actually becomes an *exact* differential equation?

☐ $n = 0$ ☐ $n = 1$ ■ $n = 2$ ☐ $n = 3$ ☐ None

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Question: 3 .... *(8 points)*

Consider the differential equation $f''(x) - 2\omega \tan(\omega x) f'(x) - \omega^2 f(x) = 1$ for some parameter $\omega \in \mathbb{R}$. We know that the most general solution to this differential equation takes the form $f(x) = c_1 f_1(x) + c_2 f_2(x) + p(x)$ for undetermined coefficients c_i, where $f_i(x)$ are the homogeneous solutions and $p(x)$ is the particular solution.

(a) **(1 point)** Assuming $\omega \neq 0$, what is $\omega^2 p(2)$?
Hint: The techniques to compute the particular solution for an arbitrary linear ordinary differential equation (such as method of variation of parameters) is beyond the scope of this exam; however, you do not need any of them for this question. In fact, you can simply guess the particular solution to be constant, which will uniquely fix it!

☐ -4 ■ -1 ☐ 1 ☐ 4 ☐ None

(b) **(1 point)** Assuming $\omega = 0$, what is $p(2)$?
Hint: This is now a differential equation with constant coefficients, for which you have learned how to find particular solution, although guessing it would be probably faster!

☐ -4 ☐ -1 ☐ 1 ☐ 4 ■ **None**

(c) **(4 points)** Let's say that $\omega = 1$ and $f_1(x) = \frac{1}{\cos(x)}$. Which one of the below can be $f_2(x)$ then? *Hint: use reduction of order*

☐ $f_2(x) = \frac{-2x^{-2}}{\cos(x)}$ ☐ $f_2(x) = \frac{-x^{-1}}{\cos(x)}$ ■ $f_2(x) = \frac{x}{\cos(x)}$ ☐ $f_2(x) = \frac{2x^2}{\cos(x)}$ ☐ None

(d) **(2 points)** If you are given the additional information $\{\omega \neq 0\ ,\ f(0) = \omega^{-2}\ ,\ f'(0) = 0\}$, what is $f\left(\frac{\pi}{\omega}\right)$?

■ $f\left(\frac{\pi}{\omega}\right) = \frac{-3}{\omega^2}$ ☐ $f\left(\frac{\pi}{\omega}\right) = \frac{-1}{\omega^2}$ ☐ $f\left(\frac{\pi}{\omega}\right) = \frac{1}{\omega^2}$ ☐ $f\left(\frac{\pi}{\omega}\right) = \frac{3}{\omega^2}$ ☐ None

Question: 4 .... *(4 points)*

Consider the differential equation $f''(x) + 2f'(x) + f(x) = e^{-x}$. What is the approximate value of the particular solution at $x = 9$?
Hint-1: This is a differential equation with constant coefficients!
Hint-2: You can take $e^9 \sim 90^2$.

☐ $f(9) \sim \frac{1}{200000}$ ■ $f(9) \sim \frac{1}{200}$ ☐ $f(9) \sim 200$ ☐ $f(9) \sim 200000$

☐ $f(9)$ should actually be a complex number!

« « « Congratulations, you have made it to the end! » » »