Phys209: Mathematical Methods in Physics I

Homework 11

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) are for your own practice!
- Unless stated otherwise, you are expected to show your derivation of the results.
- The homework is due December 22nd 2023, 23:59 TSI.

(1) Problem One *(3 points)*

In class, we have discussed the Fourier transform, which we have defined with the conventions

$$
\begin{aligned}
f &:: \mathbb{C} \to \mathbb{C} \\
f &= x \to \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \frac{dk}{2\pi} \\
\left(\hat{f} \equiv \text{F.T.}(f)\right) &:: \mathbb{C} \to \mathbb{C} \\
\hat{f} &= k \to \int_{-\infty}^{\infty} f(x) e^{-ikx} dx
\end{aligned}
$$

We'll work with this integral transformation in this homework.

(1.1) (1pt)

Define *the generalized function* (also called *distribution*)

$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx} \frac{dk}{2\pi}$$

which is commonly known as Dirac delta function. Note that we are following Physicists' notation here: as δ is actually not a function, a mathematician would not like the expression $\delta(x)$.

In terms of δ distribution, compute the functions $\text{F.T.}(f)$ for $f = \cos$ and $f = \sin$.

Hint: As $\sin$ *and* $\cos$ *are real functions along the real line (e.g.* $\cos(x) \in \mathbb{R}, \forall x \in \mathbb{R}$*), we expect their Fourier transforms to satisfy the following relation that we have shown in class:* $\text{F.T.}(f)(-k) = [\text{F.T.}(f)(k)]^*$.

(1.2) (1pt)

Insert the definition of $\hat{f}$ in terms of f into the expansion of f in terms of $\hat{f}$. By using the definition of delta distribution and its property

$f(x) = \int_{-\infty}^{\infty} \delta(x-y) f(y) dy$, show that the Fourier transform and its inverse are consistent!

(1.3) (1pt)

Consider a function $s :: \mathbb{R} \to \mathbb{R}$. This *signal* has a fourier representation $S \equiv \text{F.T.}(s)$. We define a new function m such that

$$\begin{aligned} m &:: \mathbb{R} \to \mathbb{R} \\ m &= t \to \cos(f\,t) s(t) \end{aligned}$$

where we will call f *the carrier signal*.

Find out the the function $M \equiv \text{F.T.}(m)$ in terms of the function S.

(1.4) Bonus question *(not graded)*

One can carry signals via electromagnetic waves: this is how your cellphone or your wifi works. To extract information from an electromagnetic wave, we need antennas, which should not be smaller in length than the frequencies of the signal. In the past, we transmitted low frequency signals which is why we needed long antennas; nowadays, we need shorter and shorter antennas; in fact, the newest technology is called 5G and it is in such a high frequency that we need only tiny antennas.

It is clearly advantageous to transmit high frequency signals, but how do we increase frequency of a signal without altering the content? You show in the previous part that multiplying a function with

$$\cos(f\,t)$$

shifts the fourier components without changing the form of the function: this is precisely what we need! This very simple approach is called *amplitude modulation*, and was widely used in radious in the past (am radios). More advanced techniques have developed since those times (such as *frequency modulation*, or fm radios), which are beyond the scope of this course.

(2) Problem Two *(3 points)*

We have also discussed the Fourier series in the class, which we define with the conventions

$$
\begin{aligned}
f &:: [0, T] \to \mathbb{C} \\
f &= x \to \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}(n) \exp\left(\frac{2\pi i x n}{T}\right) \\
\left(\hat{f} \equiv \text{F.T.}(f)\right) &:: \mathbb{Z} \to \mathbb{C} \\
\hat{f} &= n \to \int_0^T f(x) \exp\left(-\frac{2\pi i x n}{T}\right) dx
\end{aligned}
$$

Note that this transformation is still valid for a periodic function $f_P :: \mathbb{R} \to \mathbb{C}$ as any output of f_P can be identified with an output of f:

$$f_P(x) = f\left(x - T\left\lfloor \frac{x}{T} \right\rfloor\right)$$

where $x \to \lfloor x \rfloor$ is the *floor* function defined as

$$
\begin{aligned}
\text{floor} &:: \mathbb{R} \to \mathbb{Z} \\
\text{floor} &= x \to \left(\text{floor}(x) \equiv \lfloor x \rfloor = \max\{m \in \mathbb{Z} | m \leq x\}\right)
\end{aligned}
$$

Floor literally gives the largest integer less than or equal to the given input!

(2.1) (0.5pt)

Consider a function

$$
\begin{aligned}
f_P &:: \mathbb{R} \to \mathbb{R} \\
f_P &= x \to \cos(x) + \sin(2x)
\end{aligned}
$$

What is the lowest value of T such that you can find a function $f :: [0, T] \to \mathbb{R}$ with the property $f_P(x) = f\left(x - T\left\lfloor \frac{x}{T} \right\rfloor\right)$?

(2.2) (0.5pt)

For $T = 4\pi$, write down the values of $f_P(a)$ (for $a = 10, 11, 21, 22$) in terms of values of $f(x)$.

(2.3) (2pt)

Find out the Fourier series representation of the function g defined as

$$g :: [-\pi, \pi] \to \mathbb{R}$$
$$g = x \to x + x^2 + x^3$$

by computing the unknowns c_0, c_1, c_2 in the expansion

$$f(x) = c_0 + c_1 \cos(x) + c_2 \sin(x) + \text{ higher frequenct components}$$

(2.4) Bonus question *(not graded)*

Previous question can be solved with the Mathematica code

```
FourierSeries[x + x^2 + x^3, x, 1, FourierParameters -> {-1, 1}] //
    FullSimplify
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