9893 Time Series — Assignment 1

2. let $dt = 1$, then

$$dX_t = X_t - X_{t-1} \qquad dt = 1$$

$$dW_t = W_t - W_{t-1} = \varepsilon_t / \sigma$$

so we get

$$X_t - X_{t-1} = \lambda(\mu - X_t) + \gamma\, \varepsilon_t / \sigma$$

$$(1+\lambda) X_t = X_{t-1} + \lambda\mu + \gamma \varepsilon_t / \sigma$$

$$X_t = \frac{\lambda\mu}{1+\lambda} + \frac{X_{t-1}}{1+\lambda} + \frac{\gamma \varepsilon_t}{\sigma(1+\lambda)}$$

$$\Rightarrow \alpha = \frac{\lambda\mu}{1+\lambda} \qquad \beta = \frac{1}{1+\lambda} \qquad \cancel{\frac{\gamma}{\sigma(1+\lambda)} = \sigma}$$

$$\cancel{\text{i.e. } \lambda =} \qquad \sigma = \frac{\gamma}{1+\lambda}$$

3. $$\tau_k = \beta_1 \tau_{k-1} + \beta_2 \tau_{k-2} \qquad k > 0$$

$$\tau_0 = \beta_1 \tau_1 + \beta_2 \tau_2 + \sigma^2 \quad ①$$

set $k = 1$ or 2 and if ov stationary, $\tau_1 = \tau_{-1}$, $\tau_2 = \tau_{-2}$

$$\tau_1 = \beta_1 \tau_0 + \beta_2 \tau_1 \Rightarrow \tau_1 = \frac{\beta_1 \tau_0}{1-\beta_2} \quad \text{if } \beta_2 \neq 1$$

$$\tau_2 = \beta_1 \tau_1 + \beta_2 \tau_0 \Rightarrow$$

$$\tau_2 = \frac{\beta_1^2 \tau_0}{1-\beta_2} + \beta_2 \tau_0$$

put in ①

$$\tau_0 = \frac{\beta_1^2 \tau_0}{1-\beta_2} + \beta_2^2 \tau_0 + \frac{\beta_2 \beta_1^2 \tau_0}{1-\beta_2} + \sigma^2$$

$$\cancel{\frac{\tau_0 - \tau_0 \beta_2}{1-\beta_2} = \frac{\beta_1^2 \tau_0 + \beta_2^2 \tau_0 - \beta_2^3 \tau_0 + \beta_2 \beta_1^2 \tau_0}{1-\beta_2} + \sigma^2}$$

$$\tau_0 = \frac{\sigma^2 (1-\beta_2)}{(1+\beta_2)\left((1-\beta_2)^2 - \beta_1^2\right)}$$

3、

$\phi(z) = 1 - \beta_1 z - \beta_2 z^2$, L be the lag operator

$\phi(L) X_t = \alpha + \varepsilon_t$, 1. $\phi(L)\alpha = \phi(1)\alpha$

2. $\phi(L)^{-1}\alpha = \phi(1)^{-1}\alpha$

$\Rightarrow \quad X_t = \frac{\alpha}{\phi(1)} + \phi(L)^{-1}\varepsilon_t$

Assume $\phi(z)$ has two roots z_1 and z_2

$\cancel{\phi(z) = (1-z_1)(1-z_2)}$

$z_1 + z_2 = \frac{-\beta_1}{\beta_2}$ $\qquad z_1 z_2 = -\frac{1}{\beta_2}$

$-\beta_1 = (z_1 + z_2)\left(-\frac{1}{z_1 z_2}\right)$ $\qquad \beta_2 = -\frac{1}{z_1 z_2}$

$\beta_1 = \frac{z_1 + z_2}{z_1 z_2}$

$$\phi(z) = \frac{(z - z_1)(z - z_2)}{z_1 z_2}$$

$$\phi(z)^{-1} = \frac{1}{\left(1 - \frac{z}{z_1}\right)\left(1 - \frac{z}{z_2}\right)} = \frac{1}{\left(1 - \frac{z}{z_2}\right)}\left(\sum_{i=0}^{\infty} \frac{z^i}{z_1^i}\right)$$

$$= \left(\sum_{j=0}^{\infty} \frac{z^j}{z_2^j}\right)\left(\sum_{i=0}^{\infty} \frac{z^i}{z_1^i}\right)$$

$$= \sum_{k=0}^{\infty} \gamma_k z^k$$ γ_k are some constants

if $|z_1^{-1}| < 1$ and $|z_2^{-1}| < 1$

then $\phi(L)^{-1} = \sum_{k=0}^{\infty} \gamma_k L^k$, $X_t = \frac{\alpha}{\phi(1)} + \sum_{k=0}^{\infty} \gamma_k \varepsilon_{t-k}$

$E(X_t) = \frac{\alpha}{\phi(1)}$ $\qquad Cov(X_t, X_{t-m}) = \sum_{k=0}^{\infty} \gamma_k \gamma_{k+m}$,

m is any constant

~~sta~~ so if $|z_1^{-1}| < 1$ and $|z_2^{-1}| < 1$, AR(2) is covariance stationary

z_1 and z_2 are roots of $1 - \beta_1 z - \beta_2 z^2 = 0$