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OPTIMAL ENERGY DECAY FOR A NONHOMOGENEOUS FLEXIBLE BEAM WITH A TIP MASS

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ABSTRACT. In this paper, a simple boundary feedback control moment is proposed to stabilize a nonhomogeneous flexible beam with a tip mass. By adopting the Riesz basis approach, it is shown that the close-loop system is a Riesz spectral system. Consequently, the exponential stability, spectrum-determined growth condition, and optimal decay rate are obtained.

1. Introduction

Consider a nonhomogeneous flexible beam clamped at one end and free at the other end where a tip mass is attached. The vibrations of the beam are described by the following system (see [3, 13]):

$$\begin{cases} \rho(x)y_{tt}(x,t) + \left(EI(x)y_{xx}\right)_{xx}(x,t) = 0, & 0 < x < \ell, \quad t > 0, \\ y(0,t) = y_x(0,t) = 0, & t > 0, \\ my_{tt}(\ell,t) - \left(EI(x)y_{xx}\right)_{x}(\ell,t) = 0, & t > 0, \\ EI(\ell)y_{xx}(\ell,t) = \Gamma(t), & t > 0, \end{cases} \tag{1.1}$$

where x denotes the position and t denotes the time, $y(x,t)$ represents the transversal displacement of the beam, ℓ is the length of the beam, and the coefficients $\rho(x)$ and $EI(x)$ are, respectively, the mass distribution and the flexural rigidity of the beam along the spatial variable x satisfying the conditions

$$0 < \rho_0 < \rho(x) \in C^4[0,\ell], \quad 0 < EI_0 < EI(x) \in C^4[0,\ell]. \tag{1.2}$$

Moreover, $m > 0$ is the mass of the rigid body attached to the beam and $\Gamma(t)$ is the bending *moment* control applied at the free end of the beam. The description of physical background of the model with constant coefficients can be found in [13]. Furthermore, the coefficients are supposed to

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be variable because it is common, in engineering, to adopt problems with nonhomogeneous materials such as smart materials [12].

The stabilization problem of system (1.1) has been the subject of many studies. Indeed, there are two categories of works: one where the coefficients ρ and EI are supposed to be constants (see [1, 3, 4, 6, 11] and the references therein), whereas in the another one, they are nonuniform [16] (see also [21] for other systems). More precisely, in the case of constant coefficients, it is shown in [3] that the system can be uniformly stabilized by means of a feedback control *force of high order*. In a very special case, where the constant coefficients and the feedback gains of the closed-loop system satisfy a certain condition, the spectrum-determined growth condition as well as the optimal decay rate are obtained. Next, it is shown in [1, 11] that the system can be uniformly stabilized by means of a *simple* boundary feedback control *moment* under a condition on the constant coefficients. Later, the results obtained in [3] were re-proved in an elegant way and without any condition on the physical constants by using a Riesz basis method [6]. Based on the results of [1], the authors showed in [4] that the optimal energy decay rate is given by the spectrum of the system under the same condition on the constant coefficients as in [1,11]. In turn, in the case of variable coefficients, a uniform stability result is derived in [16] under some conditions on the coefficients $\rho(x)$ and $EI(x)$. Note that similar systems have been studied in literature (see, e.g., [2, 8, 14, 15, 19] and the references therein).

The main contribution of this note is to show that the system can be uniformly stabilized by means of the following *simple* boundary feedback control *moment* applied at the free end of the beam:

$$\Gamma(t) = -\alpha y_{xt}(\ell, t), \tag{1.3}$$

where $\alpha > 0$ is a feedback gain that can be tuned in practice. This is important because exponential stability is a very desirable property for such elastic systems and the feedback scheme using bending moment only is simple and attractive. In addition, we are able to show that there exists a sequence of generalized eigenfunctions which forms a Riesz basis in the state space. As a consequence, the spectrum-determined growth condition as well as the optimal energy decay rate of the closed-loop system are obtained. Comparing our results with those of [1, 4, 11] (where the coefficients are constant) and [16] (where the coefficients are non-uniform), we can claim that

(1) for the case of constant coefficients:
   (a) the conditions on the coefficients, as $m > 1/3$, imposed in [1,4,11] are removed;
   (b) we extend the results obtained in [1, 4, 11] to variable coefficients;
   (c) we provide an alternative proof for the Riesz basis property and the optimal energy decay rate obtained in [4]. The advantage

of our approach, which is based on Guo's results [6] (see also [7]), lies in the fact that it is simpler in the sense that there is less computations. Moreover, we are able to deal with variable coefficients which is not the case in [4] where Langer results [10] and Shkalikov theory [20] are used;

(2) for the case of variable coefficients:
   (a) a simpler proof of [16, Theorem 3.2], which is related to asymptotic stability of the system, is given. In addition, the conditions on the variable coefficients, like $\rho + x\rho' + \frac{\ell}{m}\rho > 0$ and $3EI - x\rho' + \frac{\ell}{m}EI > 0$, assumed to be true in this theorem are removed;
   (b) most importantly, we are able, in contrast to [16], to show that there exists a sequence of generalized eigenfunctions, which forms a Riesz basis in the state space and prove the spectrum-determined growth condition as well as the optimal energy decay rate of the closed-loop system;
   (c) using Guo's method [6], we propose a simpler proof of the exponential stability result obtained in [16] where the frequency multiplier method is used, which is based on the well-known Huang's result [9].

The rest of the paper is organized as follows. In Sec. 2, the well-posedness and the basic properties of the closed-loop system (1.1)–(1.3) are established. Section 3 is devoted to the asymptotic analysis for the eigenpairs of the closed-loop system. Finally, in Sec. 4 we prove the Riesz basis property, spectrum-determined growth condition, and optimal decay rate.

2. Preliminaries

We consider system (1.1)-(1.3) on the following complex Hilbert space:

$$\mathcal{H} := H^2_E(0,\ell) \times L^2(0,1) \times \mathbb{C},\ H^2_E(0,\ell) := \Big\{ f \in H^2(0,\ell);\ f(0) = f'(0) = 0 \Big\}$$

equipped with the norm

$$\|\Phi\|^2 := \int_0^\ell \left[EI(x)|f''(x)|^2 + \rho(x)|g(x)|^2 \right] dx + m|\xi|^2$$

for any $\Phi := (f, g, \xi) \in \mathcal{H}$. Then, we define an operator as follows: for all $\Phi := (f, g, \xi) \in \mathcal{D}(\mathcal{A})$,

$$\mathcal{A}\Phi := \left(g,\ -\frac{1}{\rho(x)} \Big((EI(x)f''(x))'' \Big),\ \frac{1}{m} \left(EIf''\right)'(\ell) \right) \tag{2.1}$$

with

$$\mathcal{D}(\mathcal{A}) := \Big\{(f, g, \xi) \in (H^4 \cap H_E^2)(0, \ell) \times H_E^2(0, \ell) \times \mathbb{C}; \\ EI(\ell)f''(\ell) = -\alpha g'(\ell), \xi = g(\ell)\Big\}. \quad (2.2)$$

This allows one to write the closed-loop system (1.1)–(1.3) as an evolution equation

$$\frac{d}{dt}\Phi(t) = \mathcal{A}\Phi(t), \quad \Phi(0) = \Phi_0,$$

where $\Phi := (y(\cdot, t), y_t(\cdot, t), y_t(\ell, t))$ and Φ_0 is the initial condition.

We have the following lemma.

Lemma 2.1. *Let the operator $\mathcal{A}$ be defined by* (2.1) *and* (2.2)*. Then $\mathcal{A}$ is a densely defined, closed, dissipative operator in $\mathcal{H}$, and $\mathcal{A}^{-1}$ exists and is compact on $\mathcal{H}$. Moreover, $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{\mathcal{A}t}$ on $\mathcal{H}$ and the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists only of the isolated eigenvalues.*

Proof. Let $\Phi := (f, g, \xi) \in \mathcal{D}(\mathcal{A})$. Then we have

$$\langle \mathcal{A}\Phi,\ \Phi\rangle = \int_0^1 \left[EI(x)g''(x)\overline{f''(x)} - EI(x)f''(x)\overline{g''(x)}\right] dx - \alpha|g'(\ell)|^2$$

and hence $\operatorname{Re}\langle \mathcal{A}\Phi, \Phi\rangle = -\alpha|g'(\ell)|^2 \le 0$. Thus, $\mathcal{A}$ is dissipative in $\mathcal{H}$. Next, we show that $\mathcal{A}^{-1}$ exists. Let $\Psi = (\phi, \psi, \eta) \in \mathcal{H}$. We will find $\Phi = (f, g, \xi) \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}\Phi = \Psi$, which yields

$$g = \phi, \quad \xi = g(\ell) = \phi(\ell),$$
$$(EI(x)f''(x))'' = -\rho(x)\psi(x),$$
$$f(0) = f'(0) = 0, \quad EI(\ell)f''(\ell) = -\alpha g'(\ell) = -\alpha\phi'(\ell),$$
$$(EIf'')'(\ell) = m\eta.$$

A direct computation shows that the above solution is given by

$$g = \phi, \quad \xi = g(\ell) = \phi(\ell),$$
$$f(x) = \int_0^x \int_0^t dsdt \left[\frac{m\eta(s-\ell) - \alpha\phi'(\ell)}{EI(s)} + \frac{1}{EI(s)}\int_\ell^s \int_\xi^\ell \rho(\zeta)\psi(\zeta)d\zeta d\xi\right].$$

Thus, $\mathcal{A}^{-1}$ exists and is bounded on $\mathcal{H}$. Furthermore, the Sobolev embedding theorem implies that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$, and the Lumer–Phillips theorem [18] can be applied to conclude that $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{\mathcal{A}t}$ in $\mathcal{H}$. The lemma is proved. □

Now, we turn our attention to the asymptotic stability of the system.

Lemma 2.2. *Let $\mathcal{A}$ be the operator defined by* (2.1)–(2.2)*. Then* $\operatorname{Re}(\mathcal{A}) < 0$ *and hence system* (1.1)–(1.3) *is asymptotically stable.*

Proof. It suffices to show that $\{i\eta; \eta \in \mathbb{R}\} \subset \rho(\mathcal{A})$. Assume that this is false. This, together with Lemma 2.1, implies that there exists nonzero $\eta \in \mathbb{R}$ such that $i\eta \in \sigma_p(\mathcal{A})$, where $\sigma_p(\mathcal{A})$ is the point spectrum, i.e., there exists $\Phi = (f, g, \xi) \in \mathcal{D}(\mathcal{A})$ satisfying, without loss of generality, the conditions $\|\phi\|_{\mathcal{H}} = 1$ and

$$(i\eta - \mathcal{A})\Phi = 0, \tag{2.3}$$

i.e.,

$$\begin{gathered}
(EI(x)f''(x))'' - \eta^2\rho(x)f(x) = 0,\\
f(0) = f'(0) = 0,\\
EI(\ell)f''(\ell) = -i\alpha\eta f'(\ell), \quad (EIf'')'(\ell) = -m\eta^2 f(\ell),\\
g = i\eta f, \quad \xi = i\eta f(\ell).
\end{gathered} \tag{2.4}$$

Using (2.3), we obtain $\operatorname{Re}\langle\mathcal{A}\Phi, \Phi\rangle = -\alpha|g'(\ell)|^2 = 0$, which further implies by means of (2.4) that $f'(\ell) = 0$ and $f''(\ell) = 0$. System (2.4) yields

$$\begin{gathered}
(EI(x)f''(x))'' - \eta^2\rho(x)f(x) = 0,\\
f(0) = f'(0) = 0, \quad f'(\ell) = f''(\ell) = 0,\\
(EIf'')'(\ell) = -m\eta^2 f(\ell).
\end{gathered} \tag{2.5}$$

We claim that $f = 0$ is the unique solution of the above system, which obviously contradicts the fact that $\|\Phi\|_{\mathcal{H}} = 1$. First, assume that

$$f(\ell) < 0 \quad \text{(the positive case is similar)}, \tag{2.6}$$

which implies by the last boundary condition that

$$(EIf'')'(\ell) > 0. \tag{2.7}$$

Then, because of $f(0) = 0$ and $f(\ell) < 0$ as well as the continuity of $f(x)$ in $[0, \ell]$, there exits $y \in [0, \ell)$ such that $f(y) = 0$ and $f(x) < 0$ for any $x \in (y, \ell]$. Thus, we have the following results:

(i) in terms of (1.2) and (2.5), the function $(EI(x)f''(x))'$ is decreasing on the set $(y, \ell]$;

(ii) by (2.7) and (i), the function $(EI(x)f''(x))'$ is positive on the set $(y, \ell]$ and hence $EI(x)f''(x)$ is increasing on $(y, \ell]$;

(iii) due to $f''(\ell) = 0$ and (ii), the function $f''(x)$ is negative on (y, ℓ), i.e., $f'(x)$ is decreasing on (y, ℓ);

(iv) since $f'(\ell) = 0$, assertion (iii) implies that $f'(x)$ is positive on (y, ℓ) and, therefore, $f(x)$ is increasing on (y, ℓ);

(v) by (2.6), we finally have $f(y) < 0$.

Obviously, assertion (v) violates the fact that $f(y) = 0$. This implies that $f(\ell) = 0$, and system (2.5) becomes

$$\begin{gathered}
(EI(x)f''(x))'' - \eta^2\rho(x)f = 0,\\
f(0) = f'(0) = 0, \quad f'(\ell) = f''(\ell) = 0, \quad f'''(\ell) = f(\ell) = 0.
\end{gathered}$$

Using the idea of [5], it has been proved in [7] that the above system has only the trivial solution, i.e., $f \equiv 0$. Lemma 2.2 is proved. □

3. Spectral analysis

For convenience, we assume that $\ell = 1$ in the rest of this paper. Note that

$$\mathcal{A}\Phi = \lambda\Phi \tag{3.1}$$

yields

$$\begin{gathered}
(EIf'')''(x) + \lambda^2\rho(x)g(x) = 0, \quad x \in (0,1), \\
f(0) = f'(0) = 0, \\
m\lambda^2 f(1) - (EIf'')'(1) = 0, \quad EI(1)f''(1) = -\alpha\lambda f'(1), \\
g(x) = \lambda f(x), \quad \xi = g(1).
\end{gathered} \tag{3.2}$$

Writing (3.2) in the standard form of a linear differential operator with homogeneous boundary conditions, we obtain

$$\begin{gathered}
f^{(4)}(x) + \frac{2EI'(x)}{EI(x)}f'''(x) + \frac{EI''(x)}{EI(x)}f''(x) + \lambda^2\frac{\rho(x)}{EI(x)}f(x) = 0, \\
f(0) = f'(0) = 0, \\
\lambda^2 f(1) - a_1 f'''(1) - a_2 f''(1) = 0, \quad \lambda f'(1) + a_3 f''(1) = 0,
\end{gathered} \tag{3.3}$$

where

$$a_1 := \frac{EI(1)}{m}, \quad a_2 := \frac{EI'(1)}{m}, \quad a_3 := \frac{EI(1)}{\alpha}. \tag{3.4}$$

In order to simplify the computations, we introduce a spatial-scale transformation in x (see [7, 21]):

$$\phi(z) := f(x), \quad z := \frac{1}{h}\int_0^x \left(\frac{\rho(\zeta)}{EI(\zeta)}\right)^{1/4} d\zeta, \quad z \in (0,1), \tag{3.5}$$

where

$$h := \int_0^1 \left(\frac{\rho(\zeta)}{EI(\zeta)}\right)^{1/4} d\zeta.$$

Then Eq. (3.3) has the form

$$\begin{gathered}
\phi^{(4)}(z) + \widetilde{a}(z)\phi'''(z) + \widetilde{b}(z)\phi''(z) + \widetilde{c}(z)\phi'(z) + \lambda^2 h^4\phi(z) = 0, \\
\phi(0) = \phi'(0) = 0, \\
\lambda^2\phi(1) - a_4\phi'''(1) - a_5\phi''(1) - a_6\phi'(1) = 0, \\
\lambda\phi'(1) + a_7\phi''(1) + a_8\phi'(1) = 0,
\end{gathered} \tag{3.6}$$

where

$$\widetilde{a}(z) := 6\frac{z_{xx}}{z_x^2} + \frac{2EI'(x)}{z_x EI(x)},$$
$$\widetilde{b}(z) := \frac{3z_{xx}^2}{z_x^4} + \frac{(6z_{xx}EI'(x))}{(z_x^3 EI(x))} + \frac{(EI''(x))}{(z_x^2 EI(x))} + \frac{4z_{xxx}}{z_x^3},$$
$$\widetilde{c}(z) := \frac{z_{xxxx}}{z_x^4} + \frac{(2z_{xxx}EI'(x))}{(z_x^4 EI(x))} + \frac{(z_{xx}EI''(x))}{(z_x^4 EI(x))},$$

and

$$\begin{aligned} &a_4 := a_1 z_x^3(1), \quad a_5 := 3a_1 z_x(1) z_{xx}(1) + a_2 z_x^2(1), \\ &a_6 := a_1 z_{xxx}(1) + a_2 z_{xx}(1), \quad a_7 := a_3 z_x(1), \quad a_8 := \frac{z_{xx}(1)}{z_x(1)}. \end{aligned} \tag{3.7}$$

Equation (3.6) can be further simplified by applying another invertible transformation (see [17]):

$$\varphi(z) := \exp\left(\frac{1}{4}\int_0^z \widetilde{a}(\zeta)d\zeta\right)\phi(z), \quad z \in (0,1), \tag{3.8}$$

which allows one to cancel the term $\widetilde{a}(z)\phi'''(z)$ in (3.6). Hence we arrive at an equivalent eigenvalue problem

$$\begin{aligned} &\varphi^{(4)}(z) + b(z)\varphi''(z) + c(z)\varphi'(z) + d(z)\varphi(z) + \lambda^2 h^4 \varphi(z) = 0, \\ &\varphi(0) = \varphi'(0) = 0, \\ &\lambda^2\varphi(1) - a_4\varphi'''(1) + \Psi_1\big(\varphi(1), \varphi'(1), \varphi''(1)\big) = 0, \\ &\lambda\left(\varphi'(1) - \frac{1}{4}\widetilde{a}(1)\varphi(1)\right) + a_7\varphi''(1) + \Psi_2\big(\varphi(1), \varphi'(1)\big) = 0, \end{aligned} \tag{3.9}$$

where $\Psi_1(x_1, x_2, x_3)$ and $\Psi_2(x_1, x_2)$ are linear combinations of x_1, x_2, x_3 and x_1, x_2, respectively,

$$b(z) := -\frac{3}{2}\widetilde{a}'(z) - \frac{3}{8}\widetilde{a}^2(z) + \widetilde{a}(z),$$

and $c(z)$ and $d(z)$ are smooth functions of $\widetilde{a}(z)$, $\widetilde{b}(z)$, and $\widetilde{c}(z)$,

$$c(z) := c(\widetilde{a}(z), \widetilde{b}(z), \widetilde{c}(z)), \quad d(z) := d(\widetilde{a}(z), \widetilde{b}(z), \widetilde{c}(z)).$$

To estimate asymptotically the solutions to the eigenvalue problem (3.9), we proceed as in [17]. First, due to Lemma 2.2 and the fact that eigenvalues of $\mathcal{A}$ are symmetric with respect to the real axis, we only need to consider those $\lambda \in \sigma(\mathcal{A})$ that satisfy $\pi/2 \leq \arg(\lambda) \leq \pi$, which we assume in the sequel. Next, we set $\lambda := \gamma^2$ and hence

$$\frac{\pi}{2} \leq \arg(\lambda) \leq \pi \quad \Leftrightarrow \quad \gamma \in S := \left\{\gamma \in \mathbb{C};\ \frac{\pi}{4} \leq \arg\gamma \leq \frac{\pi}{2}\right\}. \tag{3.10}$$

Now, let us choose ω_i $(i=1,2,3,4)$ as follows:

$$\omega_1 := e^{(3\pi/4)i}, \quad \omega_2 := e^{(\pi/4)i}, \quad \omega_3 := -\omega_2, \quad \omega_4 := -\omega_1. \tag{3.11}$$

Consequently, we have for $\gamma \in S$:

$$\begin{gathered}
\operatorname{Re}(\gamma\omega_1) = -|\gamma| \sin\left(\arg\gamma + \frac{\pi}{4}\right) \le -\frac{\sqrt{2}}{2}|\gamma| < 0, \\
\operatorname{Re}(\gamma\omega_2) = |\gamma| \cos\left(\arg\gamma + \frac{\pi}{4}\right) \le 0, \\
\omega_1^2 = -i, \quad \omega_1^3 = -i\omega_1, \quad \omega_2^2 = i, \quad \omega_2^3 = i\omega_2, \quad \omega_1\omega_2 = -1, \\
\omega_1\omega_2^{-1} = i, \quad \omega_1 + \omega_2 = \sqrt{2}i, \quad \omega_1 - \omega_2 = -\sqrt{2}.
\end{gathered} \tag{3.12}$$

In order to analyze the asymptotic distribution of eigenpairs for (3.9), we need the following result [17] (see also [21]).

Lemma 3.1. *For $\gamma \in S$ with sufficiently large $|\gamma|$, the equation*

$$\varphi^{(4)}(z) + b(z)\varphi''(z) + c(z)\varphi'(z) + d(z)\varphi(z) + \gamma^4 h^4 \varphi(z) = 0 = 0, \quad x \in (0,1),$$

has four linearly independent asymptotic fundamental solutions:

$$\varphi_i(z,\gamma) := e^{\gamma\omega_i z}\left(1 + \frac{\varphi_{i1}(z)}{\gamma} + \mathcal{O}(\gamma^{-2})\right), \quad i = 1,2,3,4,$$

and hence their derivatives for $i = 1,2,3,4$ and $j = 1,2,3$ are given by

$$\frac{d^j}{dz^j}\varphi_i(z,\gamma) = (\gamma\omega_i)^j e^{\gamma\omega_i z}\left(1 + \frac{\varphi_{i1}(z)}{\gamma} + \mathcal{O}(\gamma^{-2})\right),$$

where

$$\varphi_{i1}(z) = -\frac{1}{4\omega_i}\int_0^z b(\zeta)d\zeta.$$

Hence, for $i = 1,2,3,4$,

$$\varphi_{i1}(0) = 0, \quad \varphi_{i1}(1) = -\frac{1}{4\omega_i}\int_0^1 b(\zeta)d\zeta := \frac{1}{\omega_i}\mu, \quad \mu := -\frac{1}{4}\int_0^1 b(\zeta)d\zeta.$$

For simplicity, we introduce the following notation: $[a]_i := a + \mathcal{O}(\gamma^{-i})$ for $i = 1,2$. From Lemma 3.1, one can write the asymptotic solution of (3.9) as follows:

$$\varphi(x) = \sum_{i=1}^{4} e_i \varphi_i(x,\gamma), \quad x \in (0,1), \quad \gamma \in S,$$

where e_i is chosen so that f satisfies the boundary conditions, i.e., $\Delta(\gamma)(e_1,e_2,e_3,e_4)^\top = 0$, where

$$\Delta(\gamma) := [\Delta_1(\gamma),\ \Delta_2(\gamma)],$$

$$\Delta_1(\gamma) := \begin{bmatrix} [1]_2 & [1]_2 \\ \gamma\omega_1[1]_2 & \gamma\omega_2[1]_2 \\ \gamma^4 e^{\gamma\omega_1}\left[1+\dfrac{\mu+a_4}{\omega_1\gamma}\right]_2 & \gamma^4 e^{\gamma\omega_2}\left[1+\dfrac{\mu+a_4}{\omega_2\gamma}\right]_2 \\ \gamma^3 e^{\gamma\omega_1}\left[\omega_1+\dfrac{a_9+\omega_1^2 a_7}{\gamma}\right]_2 & \gamma^3 e^{\gamma\omega_2}\left[\omega_2+\dfrac{a_9+\omega_2^2 a_7}{\gamma}\right]_2 \end{bmatrix},$$

$$\Delta_2(\gamma) = \begin{bmatrix} [1]_2 & [1]_2 \\ \gamma\omega_3[1]_2 & \gamma\omega_4[1]_2 \\ \gamma^4 e^{\gamma\omega_3}\left[1+\dfrac{\mu+a_4}{\omega_3\gamma}\right]_2 & \gamma^4 e^{\gamma\omega_4}\left[1+\dfrac{\mu+a_4}{\omega_4\gamma}\right]_2 \\ \gamma^3 e^{\gamma\omega_3}\left[\omega_3+\dfrac{a_9+\omega_3^2 a_7}{\gamma}\right]_2 & \gamma^3 e^{\gamma\omega_4}\left[\omega_4+\dfrac{a_9+\omega_4^2 a_7}{\gamma}\right]_2 \end{bmatrix}.$$

Here

$$a_9 := \mu - \frac{1}{4}\widetilde{a}(1).$$

By virtue of (3.11) and (3.12), we have

$$\begin{aligned}
&-\omega_2\gamma^{-8}e^{\gamma(\omega_1+\omega_2)}\det(\Delta(\gamma)) \\
&= \begin{vmatrix} 1 & 1 & e^{\gamma\omega_2} & 0 \\ 1 & -i & ie^{\gamma\omega_2} & 0 \\ 0 & e^{\gamma\omega_2}\left(1+\dfrac{\mu+a_4}{\omega_2\gamma}\right) & 1-\dfrac{\mu+a_4}{\omega_2\gamma} & 1-\dfrac{\mu+a_4}{\omega_1\gamma} \\ 0 & e^{\gamma\omega_2}\left(\omega_2+\dfrac{a_9+ia_7}{\gamma}\right) & -\omega_2+\dfrac{a_9+ia_7}{\gamma} & -\omega_1+\dfrac{a_9-ia_7}{\gamma} \end{vmatrix} + O(\gamma^{-2}) \\
&= \begin{vmatrix} 1 & 1 \\ 1 & -i \end{vmatrix} \begin{vmatrix} 1-\dfrac{\mu+a_4}{\omega_2\gamma} & 1-\dfrac{\mu+a_4}{\omega_1\gamma} \\ -\omega_2+\dfrac{a_9+ia_7}{\gamma} & -\omega_1+\dfrac{a_9-ia_7}{\gamma} \end{vmatrix} \\
&\quad - e^{2\gamma\omega_2}\begin{vmatrix} 1 & 1 \\ 1 & i \end{vmatrix} \begin{vmatrix} 1+\dfrac{\mu+a_4}{\omega_2\gamma} & 1-\dfrac{\mu+a_4}{\omega_1\gamma} \\ \omega_2+\dfrac{a_9+ia_7}{\gamma} & -\omega_1+\dfrac{a_9-ia_7}{\gamma} \end{vmatrix} + O(\gamma^{-2}) \\
&= (i+1)\left(\omega_1-\omega_2-\frac{a_9-ia_7}{\gamma}+\frac{a_9+ia_7}{\gamma}-\omega_1\frac{\mu+a_4}{\omega_2\gamma}+\omega_2\frac{\mu+a_4}{\omega_1\gamma}\right) \\
&\quad + e^{2\gamma\omega_2}(1-i)\biggl(-\omega_1-\omega_2+\frac{a_9-ia_7}{\gamma}-\frac{a_9+ia_7}{\gamma}-\omega_1\frac{\mu+a_4}{\omega_2\gamma} \\
&\quad +\omega_2\frac{\mu+a_4}{\omega_1\gamma}\biggr)+O(\gamma^{-2}) = -(i+1)\left(\sqrt{2}-\frac{i2(a_7-a_{10})}{\gamma}\right) \\
&\quad - e^{2\gamma\omega_2}(1-i)\left(i\sqrt{2}+\frac{i2(a_7+a_{10})}{\gamma}\right)+O(\gamma^{-2})
\end{aligned}$$

$$= -\sqrt{2}(i+1)\left(1 - \frac{i\sqrt{2}(a_7 - a_{10})}{\gamma}\right)$$
$$- e^{2\gamma\omega_2}(1+i)\sqrt{2}\left(1 + \frac{\sqrt{2}(a_7 + a_{10})}{\gamma}\right) + O(\gamma^{-2}),$$

where $a_{10} := \mu + a_4$. Now we are ready to state the following result.

Theorem 3.1. *Let* $\lambda := \gamma^2$, *where* $\gamma \in S$. *The characteristic determinant* $\det(\Delta(\gamma))$ *of the eigenvalue problem* (3.9) *has the following asymptotic expression in the sector* S:

$$\frac{\omega_2\gamma^{-8}}{\sqrt{2}(i+1)} e^{\gamma(\omega_1+\omega_2)} \det(\Delta(\gamma))$$
$$= 1 - \frac{i\sqrt{2}(a_7 - a_{10})}{\gamma} + e^{2\gamma\omega_2}\left(1 + \frac{\sqrt{2}(a_7 + a_{10})}{\gamma}\right) + O(\gamma^{-2}),$$

where a_7 *is given in* (3.7) *and* (3.4) *and can be simplified here by*

$$a_7 = a_3 z_x(1) = \frac{EI(1)}{\alpha} z_x(1). \tag{3.13}$$

Moreover, let $\sigma(\mathcal{A}) = \{\lambda_n, \overline{\lambda}_n : n \in \mathbb{N}\}$ *be the eigenvalues of* $\mathcal{A}$. *Then, for* $k = n - 1/2$ *and* $\gamma_n \in S$, *the following asymptotic expansion holds*:

$$\begin{aligned} \gamma_n &= \frac{1}{\omega_2} k\pi i - \frac{\sqrt{2}}{2} \frac{a_7(1+i) + a_{10}(1-i)}{k\pi i} + O(n^{-2}), \\ \lambda_n &= -2a_7 + (k\pi)^2 i + a_{10} i + \mathcal{O}(n^{-1}), \end{aligned} \tag{3.14}$$

for sufficiently large positive integers n. *Moreover, by* (3.13), *we obtain that*

$$\lambda_n, \overline{\lambda}_n \to -2a_7 = -2\frac{EI(1)}{\alpha} z_x(1) < 0 \quad \text{as } n \to \infty. \tag{3.15}$$

Proof. Note that $\lambda = \gamma^2 \in \sigma(\mathcal{A})$, where $\gamma \in S$ (see (3.10)) if and only if

$$1 - \frac{i\sqrt{2}(a_7 - a_{10})}{\gamma} + e^{2\gamma\omega_2}\left(1 + \frac{\sqrt{2}(a_7 + a_{10})}{\gamma}\right) + O(\gamma^{-2}) = 0,$$

which is equivalent to

$$e^{-\gamma\omega_2} - \frac{i\sqrt{2}(a_7 - a_{10})}{\gamma} e^{-\gamma\omega_2} + e^{\gamma\omega_2} + \frac{\sqrt{2}(a_7 + a_{10})}{\gamma} e^{\gamma\omega_2} + O(\gamma^{-2}) = 0, \tag{3.16}$$

which further can be rewritten as

$$e^{\gamma\omega_2} + e^{-\gamma\omega_2} + O(\gamma^{-1}) = 0. \tag{3.17}$$

Obviously, the equation

$$e^{\gamma\omega_2} + e^{-\gamma\omega_2} = 0$$

has solutions

$$\tilde{\gamma}_n = \frac{1}{\omega_2} k\pi i, \quad n \in \mathbb{N}, \quad k = n - \frac{1}{2}.$$

Applying Rouché's theorem to (3.17) we obtain

$$\gamma_n = \tilde{\gamma}_n + \alpha_n = \frac{1}{\omega_2} k\pi i + \alpha_n, \quad \alpha_n = \mathcal{O}(n^{-1}), \quad n = N, N+1, \ldots, \tag{3.18}$$

where N is a large positive integer. Substituting γ_n into (3.16) and using the fact that $\exp(\tilde{\gamma}_n \omega_2) = -\exp(-\tilde{\gamma}_n \omega_2)$, we obtain

$$e^{\alpha_n \omega_2} - e^{-\alpha_n \omega_2} + \sqrt{2}(a_7 + a_{10})\gamma_n^{-1} e^{\alpha_n \omega_2} + i\sqrt{2}(a_7 - a_{10})\gamma_n^{-1} e^{-\alpha_n \omega_2} + O(\gamma_n^{-2}) = 0.$$

On the other hand, expanding the exponential function according to its Taylor series, we obtain

$$\alpha_n = -\frac{\sqrt{2}}{2} \frac{a_7(1+i) + a_{10}(1-i)}{k\pi i} + O(n^{-2}).$$

Substituting this estimate in (3.18), we have

$$\gamma_n = \frac{1}{\omega_2} k\pi i - \frac{\sqrt{2}}{2} \frac{a_7(1+i) + a_{10}(1-i)}{k\pi i} + O(n^{-2}), \quad n = N, N+1, \ldots.$$

Finally, recall that $\lambda_n = \gamma_n^2$, $\omega_2 = \exp(i\pi/4)$, and $\omega_2^2 = i$, and hence the last estimate yields

$$\lambda_n = -2a_7 + (k\pi)^2 i + a_{10} i + \mathcal{O}(n^{-1}), \quad n = N, N+1, \ldots,$$

where N is sufficiently large. The theorem is proved. □

We have also the following theorem.

Theorem 3.2. *Let $\lambda_n := \gamma_n^2$, where $\gamma_n \in S$ is given by (3.14). Then the corresponding eigenfunctions $\{\Phi_n := (f_n, \lambda_n f_n, \xi_n),\ \overline{\Phi}_n := (\overline{f}_n, \overline{\lambda}_n \overline{f}_n, \overline{\xi}_n)\}$ have the following asymptotics*:

$$\begin{cases} \lambda_n f_n(x) = \exp\left(-\dfrac{1}{4}\displaystyle\int_0^z a(\zeta)d\zeta\right)\Big((1-i)e^{\gamma_n\omega_2}e^{\gamma_n\omega_1 z} - e^{\gamma_n\omega_2(1+z)} \\ \qquad - (1+i)e^{\gamma_n\omega_1(1-z)} + ie^{\gamma_n\omega_2(1-z)} + \mathcal{O}(n^{-1})\Big), \\ f_n''(x) = z_x^2(1)\exp\left(-\dfrac{1}{4}\displaystyle\int_0^z a(\zeta)d\zeta\right)\Big((1+i)e^{\gamma_n\omega_1(1-z)} \\ \qquad - (1-i)e^{\gamma_n\omega_2}e^{\gamma_n\omega_1 z} - e^{\gamma_n\omega_2(1+z)} + ie^{\gamma_n\omega_2(1-z)} + \mathcal{O}(n^{-1})\Big), \\ \xi_n = \mathcal{O}(n^{-1}) \end{cases} \tag{3.19}$$

for sufficiently large positive integers n. Furthermore, $(f_n, \lambda_n f_n, \xi_n)$ are approximately normalized in $\mathcal{H}$ in the sense that there exist positive constants β_1 and β_2 independent of n, such that for all sufficiently large n,

$$\beta_1 \le \|f_n''\|_{L^2(0,1)}, \quad \|\lambda_n f_n\|_{L^2(0,1)}, \quad |\xi_n| \le \beta_2. \tag{3.20}$$

Proof. As was already mentioned, $\lambda \in \sigma(\mathcal{A})$ if and only if (3.1) holds for a nonzero $\Phi := (f, g, \xi)$, i.e., f, g, and ξ satisfy the characteristic equation (3.2). We only need to prove the first two equalities of (3.19) since if they are valid, then it follows from (3.20) that

$$\begin{aligned}\xi_n = \lambda_n f_n(1) &= \exp\left(-\frac{1}{4}\int_0^1 a(\zeta)d\zeta\right) \\ &\quad\times\Big((1-i)e^{\gamma_n\omega_2}e^{\gamma_n\omega_1} - e^{2\gamma_n\omega_2} - (1+i) + i + \mathcal{O}(n^{-1})\Big) \\ &= \exp\left(-\frac{1}{4}\int_0^1 a(\zeta)d\zeta\right)\Big((1-i)e^{\gamma_n\omega_2}e^{\gamma_n\omega_1} - e^{2\gamma_n\omega_2} - 1 + \mathcal{O}(n^{-1})\Big),\end{aligned}$$

and, therefore, by using (3.12) and (3.17), we obtain $\xi_n = \mathcal{O}(n^{-1})$. Now we find f. Alternatively, by (3.3), (3.5), (3.8), and (3.9), we have

$$f(x) = \phi(z) = \exp\left(-\frac{1}{4}\int_0^z a(\zeta)d\zeta\right)\varphi(z) \tag{3.21}$$

and hence it suffices to find $\varphi \neq 0$ satisfying (3.9). In view of (3.9), (3.11), and Lemma 3.1 as well as simple facts of linear algebra, the eigenfunction φ corresponding to the eigenvalue $\lambda = \gamma^2$ with $\gamma \in S$ is given by

$$\begin{aligned}&\omega_2 e^{\gamma(\omega_1+\omega_2)}\varphi(z) \\ &\quad= \omega_2 e^{\gamma(\omega_1+\omega_2)}\begin{vmatrix}[1]_1 & [1]_1 & [1]_1 & [1]_1 \\ \gamma\omega_1[1]_1 & \gamma\omega_2[1]_1 & \gamma\omega_3[1]_1 & \gamma\omega_4[1]_1 \\ \gamma^4 e^{\gamma\omega_1}[1]_1 & \gamma^4 e^{\gamma\omega_2}[1]_1 & \gamma^4 e^{\gamma\omega_3}[1]_1 & \gamma^4 e^{\gamma\omega_4}[1]_1 \\ e^{\gamma\omega_1 z}[1]_1 & e^{\gamma\omega_2 z}[1]_1 & e^{\gamma\omega_3 z}[1]_1 & e^{\gamma\omega_4 z}[1]_1\end{vmatrix} \\ &\quad= -\gamma^5\begin{vmatrix}1 & 1 & e^{\gamma\omega_2} & 0 \\ 1 & -i & ie^{\gamma\omega_2} & 0 \\ 0 & e^{\gamma\omega_2} & 1 & 1 \\ e^{\gamma\omega_1 z} & e^{\gamma\omega_2 z} & e^{\gamma\omega_2(1-z)} & e^{\gamma\omega_1(1-z)}\end{vmatrix} + \mathcal{O}(\gamma^{-1}) = -\gamma^5 \\ &\times\left\{e^{\gamma\omega_1(1-z)}\begin{vmatrix}1 & 1 & e^{\gamma\omega_2} \\ 1 & -i & ie^{\gamma\omega_2} \\ 0 & e^{\gamma\omega_2} & 1\end{vmatrix} - \begin{vmatrix}1 & 1 & e^{\gamma\omega_2} \\ 1 & -i & ie^{\gamma\omega_2} \\ e^{\gamma\omega_1 z} & e^{\gamma\omega_2 z} & e^{\gamma\omega_2(1-z)}\end{vmatrix}\right\} + \mathcal{O}(\gamma^{-1}) \\ &= -\gamma^5\Big\{\big((1-i)e^{2\gamma\omega_2} - (1+i)\big)e^{\gamma\omega_1(1-z)} + (i-1)e^{\gamma\omega_2(1+z)} - 2ie^{\gamma\omega_2}e^{\gamma\omega_1 z}\end{aligned}$$

$$+ (1+i)e^{\gamma\omega_2(1-z)}\Big\} + \mathcal{O}(\gamma^{-1}) = -\gamma^5(1-i)\Big\{ \left(e^{2\gamma\omega_2} - i\right) e^{\gamma\omega_1(1-z)} - e^{\gamma\omega_2(1+z)} + (1-i)e^{\gamma\omega_2}e^{\gamma\omega_1 z} + ie^{\gamma\omega_2(1-z)}\Big\} + \mathcal{O}(\gamma^{-1}).$$

It follows from (3.17) that $e^{2\gamma\omega_2} = -1+\mathcal{O}(\gamma^{-1})$ and hence the last estimate yields

$$\frac{\omega_2 e^{\gamma(\omega_1+\omega_2)}}{(i-1)\gamma^5}\varphi(z) = (1-i)e^{\gamma\omega_2}e^{\gamma\omega_1 z} - e^{\gamma\omega_2(1+z)} - (1+i)e^{\gamma\omega_1(1-z)} + ie^{\gamma\omega_2(1-z)} + \mathcal{O}(\gamma^{-1}). \quad (3.22)$$

Similarly,

$$\omega_2 e^{\gamma(\omega_1+\omega_2)}\varphi''(z) = \omega_2 e^{\gamma(\omega_1+\omega_2)} \times \begin{vmatrix} [1]_1 & [1]_1 & [1]_1 & [1]_1 \\ \gamma\omega_1[1]_1 & \gamma\omega_2[1]_1 & \gamma\omega_3[1]_1 & \gamma\omega_4[1]_1 \\ \gamma^4 e^{\gamma\omega_1}[1]_1 & \gamma^4 e^{\gamma\omega_2}[1]_1 & \gamma^4 e^{\gamma\omega_3}[1]_1 & \gamma^4 e^{\gamma\omega_4}[1]_1 \\ (\gamma\omega_1)^2 e^{\gamma\omega_1 z}[1]_1 & (\gamma\omega_2)^2 e^{\gamma\omega_2 z}[1]_1 & (\gamma\omega_3)^2 e^{\gamma\omega_3 z}[1]_1 & (\gamma\omega_4)^2 e^{\gamma\omega_4 z}[1]_1 \end{vmatrix}$$

$$= -i\gamma^7 \begin{vmatrix} 1 & 1 & e^{\gamma\omega_2} & 0 \\ 1 & -i & ie^{\gamma\omega_2} & 0 \\ 0 & e^{\gamma\omega_2} & 1 & 1 \\ -e^{\gamma\omega_1 z} & e^{\gamma\omega_2 z} & e^{\gamma\omega_2(1-z)} & -e^{\gamma\omega_1(1-z)} \end{vmatrix} + \mathcal{O}(\gamma^{-1}) = -i\gamma^7$$

$$\times \left\{ -e^{\gamma\omega_1(1-z)} \begin{vmatrix} 1 & 1 & e^{\gamma\omega_2} \\ 1 & -i & ie^{\gamma\omega_2} \\ 0 & e^{\gamma\omega_2} & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & e^{\gamma\omega_2} \\ 1 & -i & ie^{\gamma\omega_2} \\ -e^{\gamma\omega_1 z} & e^{\gamma\omega_2 z} & e^{\gamma\omega_2(1-z)} \end{vmatrix} \right\} + \mathcal{O}(\gamma^{-1})$$

$$= -\gamma^7\Big\{ \left((i-1)e^{2\gamma\omega_2} + (1+i)\right) e^{\gamma\omega_1(1-z)} + (i-1)e^{\gamma\omega_2(1+z)} + 2ie^{\gamma\omega_2}e^{\gamma\omega_1 z} + (1+i)e^{\gamma\omega_2(1-z)}\Big\} + \mathcal{O}(\gamma^{-1})$$

$$= -\gamma^7(1-i)\Big\{ \left(i - e^{2\gamma\omega_2}\right) e^{\gamma\omega_1(1-z)} - e^{\gamma\omega_2(1+z)} - (1-i)e^{\gamma\omega_2}e^{\gamma\omega_1 z} + ie^{\gamma\omega_2(1-z)}\Big\} + \mathcal{O}(\gamma^{-1})$$

and, therefore,

$$\frac{\omega_2 e^{\gamma(\omega_1+\omega_2)}}{(i-1)\gamma^7}\varphi''(z) = (1+i)e^{\gamma\omega_1(1-z)} - (1-i)e^{\gamma\omega_2}e^{\gamma\omega_1 z} - e^{\gamma\omega_2(1+z)} + ie^{\gamma\omega_2(1-z)} + \mathcal{O}(\gamma^{-1}). \quad (3.23)$$

Expression (3.19) can then be deduced from (3.21), (3.22), and (3.23) after setting

$$\varphi_n(z) = -\frac{1+i}{2}\gamma_n^{-7}\omega_2 e^{\gamma_n(\omega_1+\omega_2)}\varphi(z),\ \ f_n(x) = \exp\left(-\frac{1}{4}\int_0^z a(\zeta)d\zeta\right)\varphi_n(z).$$

Finally, to prove (3.20), we note from (3.11) and (3.14) that

$$\gamma_n\omega_1 = -\left(n-\frac{1}{2}\right)\pi + \mathcal{O}(n^{-1}), \quad \gamma_n\omega_2 = \left(n-\frac{1}{2}\right)\pi i + \mathcal{O}(n^{-1}).$$

Therefore,

$$\|e^{\gamma_n\omega_1 z}\|^2_{L^2(0,1)} = \mathcal{O}(n^{-1}), \quad \|e^{\gamma_n\omega_1(1-z)}\|^2_{L^2(0,1)} = \mathcal{O}(n^{-1}),$$
$$\|e^{\gamma_n\omega_2(1-z)}\|^2_{L^2(0,1)} = 1+\mathcal{O}(n^{-1}), \quad \|e^{\gamma_n\omega_2(1+z)}\|^2_{L^2(0,1)} = 1+\mathcal{O}(n^{-1}).$$

This, together with (3.19), yields (3.20). □

4. Riesz basis property

Let us recall that for a closed operator $\mathbf{A}$ in a Hilbert space $\mathbf{H}$, a nonzero element $\Phi \in \mathbf{H}$ is called a generalized eigenvector of $\mathbf{A}$ corresponding to an eigenvalue λ of $\mathbf{A}$ if there exists a nonnegative integer n such that $(\lambda I - \mathbf{A})^n\Phi \neq 0$ and $(\lambda I - \mathbf{A})^{n+1}\Phi = 0$. A sequence $\{\Phi_n\}_{n=1}^\infty$ in $\mathbf{H}$ is called a Riesz basis for $\mathbf{H}$ if there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ in $\mathbf{H}$ and a linear bounded invertible operator T such that

$$T\Phi_n = e_n, \quad n = 1, 2, \ldots.$$

Let $\{\lambda_n\}_{n=1}^\infty = \sigma(\mathbf{A})$ be the spectrum of $\mathbf{A}$. Assume that each λ_n has a finite algebraic multiplicity m_n. Let $\{\Psi_{n_i}\}_1^{m_n}$ be the set of generalized eigenvectors of $\mathbf{A}$ corresponding to λ_n. Then if $\{\Psi_{n_i} \mid 1 \le i \le m_n,\ n = 1, 2, \ldots\}$ form a Riesz basis for $\mathbf{H}$, then the C_0-semigroup generated by $\mathbf{A}$ can be represented as

$$e^{\mathbf{A}t}x = \sum_{n=1}^\infty e^{\lambda_n t}\sum_{j=1}^{m_n} a_{nj} f_{nj}(t)\Psi_{nj} \quad \forall x = \sum_{n=1}^\infty\sum_{j=1}^{m_n} a_{nj}\Psi_{nj} \in \mathbf{H},$$

where $f_{nj}(t)$ are the polynomials in t of the order not greater than m_n. In particular, if $m_n = 1$ for all sufficiently large n, then the spectrum determined growth condition holds, i.e., $\omega(\mathbf{A}) = s(\mathbf{A})$, where $\omega(\mathbf{A})$ is the growth bound of $e^{\mathbf{A}t}$ and $s(\mathbf{A})$ is the spectral bound of $\mathbf{A}$ (see [6]).

The following result [6] provides a useful way to verify the Riesz basis property for the generalized eigenvectors of a linear operator with compact resolvent in a Hilbert space.

Theorem 4.1 (see [6])**.** *Let $\mathbf{A}$ be a densely defined discrete operator (i.e., there exists the resolvent $\lambda \in \rho(\mathbf{A})$ of $\mathbf{A}$, such that $(\lambda I - \mathbf{A})^{-1}$ is compact on $\mathbf{H}$) in a Hilbert space $\mathbf{H}$. Let $\{\Phi_n\}_1^\infty$ be a Riesz basis for $\mathbf{H}$. If there exist an integer $N \ge 0$ and a sequence of generalized eigenvectors $\{\Psi_n\}_{N+1}^\infty$ of $\mathbf{A}$ such that*

$$\sum_{N+1}^\infty \|\Phi_n - \Psi_n\|^2 < \infty,$$

then

(1) *there exist an integer* $M > N$ *and generalized eigenvectors* $\{\Psi_{n0}\}_1^M$ *of* $\mathbf{A}$ *such that* $\{\Psi_{n0}\}_1^M \cup \{\Psi_n\}_{M+1}^\infty$ *form a Riesz basis for* $\mathbf{H}$;

(2) *if* $\{\Psi_{n0}\}_1^M \cup \{\Psi_n\}_{M+1}^\infty$ *are the generalized eigenvectors corresponding to eigenvalues* $\{\sigma_n\}_1^\infty$ *of* $\mathbf{A}$, *then* $\sigma(\mathbf{A}) = \{\sigma_n\}_1^\infty$, *where* σ_n *is accounted according to its algebraic multiplicity*;

(3) *if there exists an integer* $M_0 > 0$ *such that* $\sigma_n \neq \sigma_m$ *for all* $m, n > M_0$, *then there exists an integer* $N_0 > M_0$ *such that all* σ_n *are algebraically simple for all* $n > N_0$.

Our main result is as follows.

Theorem 4.2. *There exists a sequence of generalized eigenfunctions of the operator* $\mathcal{A}$ *defined by* (2.1) *and* (2.2), *which forms a Riesz basis for* $\mathcal{H}$. *Moreover, all eigenvalues with sufficiently large modulus are algebraically simple. Consequently, the spectrum-determined growth condition* $\omega(\mathcal{A}) = s(\mathcal{A})$ *holds for the* C_0*-semigroup* $e^{\mathcal{A}t}$ *generated by* $\mathcal{A}$. *Hence system* (1.1)–(1.3) *is exponentially stable.*

Proof. It suffices to show that the pair

$$\left\{\Phi_n := (f_n, \lambda_n f_n, \xi_n),\ \overline{\Phi}_n := (\overline{f}_n, \overline{\lambda}_n \overline{f}_n, \overline{\xi}_n)\right\}$$

obtained in Theorem 3.2 satisfies the hypotheses of Theorem 4.1 with respect to a suitably chosen reference Riesz basis of $\mathcal{H}$. To do this, let $\alpha = 0$. Then we define another operator $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0)\ (\subset \mathcal{H}) \to \mathcal{H}$ as follows:

$$\mathcal{D}(\mathcal{A}_0) := \Big\{(f, g, \xi) \in (H^4 \cap H_E^2)(0,1) \times H_E^2(0,1) \times \mathbb{C}^2; \\ EI(1)f''(1) = 0, \xi = g(1)\Big\},$$

and for all $\Phi := (f, g, \xi) \in \mathcal{D}(\mathcal{A}_0)$,

$$\mathcal{A}_0\Phi := \left(g,\ -\frac{1}{\rho(x)}\left((EI(x)f''(x))''\right),\ \frac{1}{m}\left(EIf''\right)'(1)\right).$$

It is easy to verify that $\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$ with compact resolvents and hence the generalized eigenfunctions

$$\left\{\Phi_{n0} := (f_{n0}, \lambda_{n0} f_{n0}, \xi_{n0}),\ \overline{\Phi}_{n0} := (\overline{f}_{n0}, \overline{\lambda}_{n0} \overline{f}_{n0}, \overline{\xi}_{n0})\right\}$$

of $\mathcal{A}_0$ form a Riesz basis for $\mathcal{H}$. Moreover, using the arguments of the previous subsection, one can verify that λ_{n0} and Φ_{n0} have the same asymptotics (3.14) and (3.19) with $\alpha \equiv 0$, respectively. This, together with Theorems 3.1 and 3.2, leads us to find $N > 0$ such that

$$\sum_{n\geq N}^\infty \|\Phi_n - \Phi_{n0}\|^2 = \sum_{n\geq N}^\infty \mathcal{O}(n^{-2}) < \infty.$$

The same is true for their conjugates. Hence all hypotheses of Theorem 4.1 are satisfied and the generalized eigenfunctions of $\mathcal{A}$ form a Riesz basis in $\mathcal{H}$. In addition, since for a skew-adjoint operator, the geometric multiplicity and algebraic multiplicity of each eigenvalue are the same, we see that all eigenvalues of $\mathcal{A}_0$ with sufficiently large modulus are algebraically simple. Since $\{\Phi_n, \overline{\Phi}_n\}_{n\in\mathbb{N}}$ forms a Riesz basis for $\mathcal{H}$, we also have that all eigenvalues of $\mathcal{A}$ with sufficiently large modulus are algebraically simple.

The spectrum-determined growth condition is an immediate and general consequence of the simplicity of the eigenvalues with large modulus and the Riesz basis property of system (1.1)–(1.3).

Finally, the exponential stability can be obtained by the spectrum-determined growth condition, Lemma 2.2, and the asymptotics of the eigenvalues of (3.14). □

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