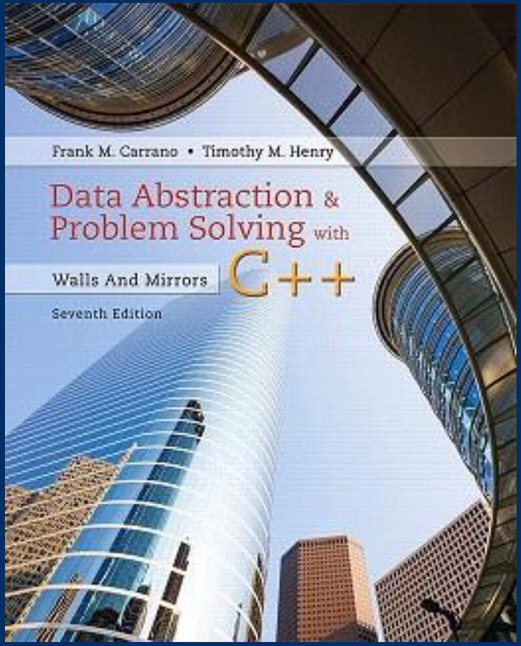
Chapter 5
Recursion as a Problem-Solving Technique

CS 302 - Data Structures

M. Abdullah Canbaz



Modified from authors' slides

Reminders

- Assignment 2 is available
  - Due Feb 14th at 2pm

- TA
  - Shehryar Khattak,
    Email: *shehryar [at] nevada {dot} unr {dot} edu,*
    Office Hours: Friday, 11:00 am - 1:00 pm at ARF 116

- Quiz 2 is available
  - Today between 4pm to 11:59pm

Recall

- Recursion
- Array-based implementation
- Linked List-based implementation

Content

- Defining Languages
  - Grammar Basics

- Algebraic Expressions
  - Prefix,postfix,infix evaluations
- Back Tracking
  - Airline Route
  - Eight Queens Problem

Defining Languages

- A language is
  - A set of strings of symbols
  - From a finite alphabet.
- Consider the C++ language

$$C{+}{+}Programs = \{\text{string } s : s \text{ is a syntactically correct C++ program}\}$$

- The set of algebraic expressions forms a language

$$AlgebraicExpressions = \{\text{string } s : s \text{ is an algebraic expression}\}$$

- A grammar states the rules of a language.

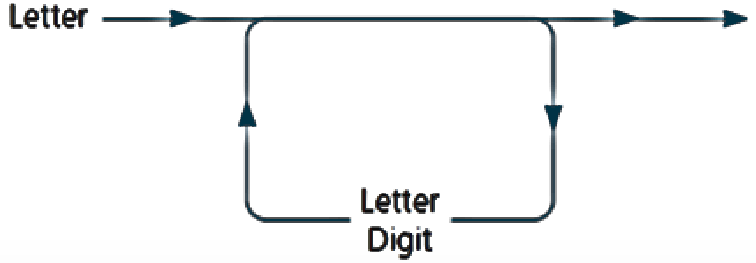
The Basics of Grammars

- Special symbols
  - x | y means x or y
  - xy (and sometimes x • y) means x followed by y
  - < word > means any instance of word, where word is a symbol that must be defined elsewhere in the grammar.

```
C++ Identifiers = {string s : s is a legal C++ identifier}
```

The Basics of Grammars

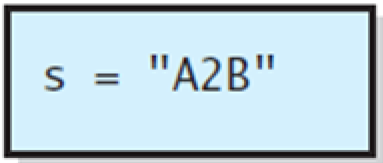
- A syntax diagram for C++ identifiers



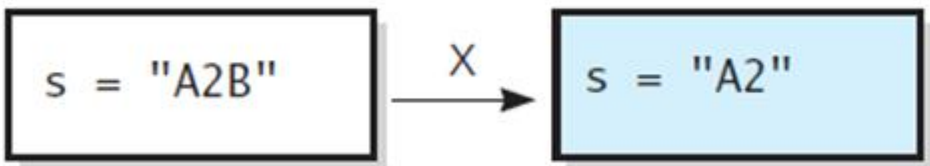

```
// Returns true if s is a legal C++ identifier; otherwise returns false.
isId(s: string): boolean
{
   if (s is of length 1)                          // Base case
      if (s is a letter)
         return true
      else
         return false
   else if (the last character of s is a letter or a digit)
      return isId(s minus its last character)  // Point X
   else
      return false
}
```

Recognition Algorithm for Identifiers

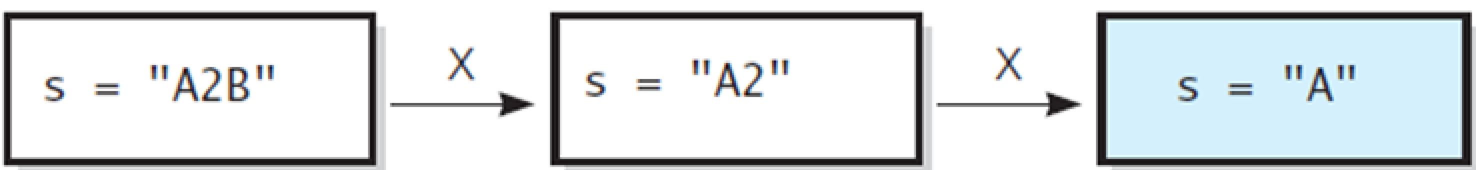
The initial call is made and the function begins execution.



At point X, a recursive call is made and the new invocation of `isId` begins executi



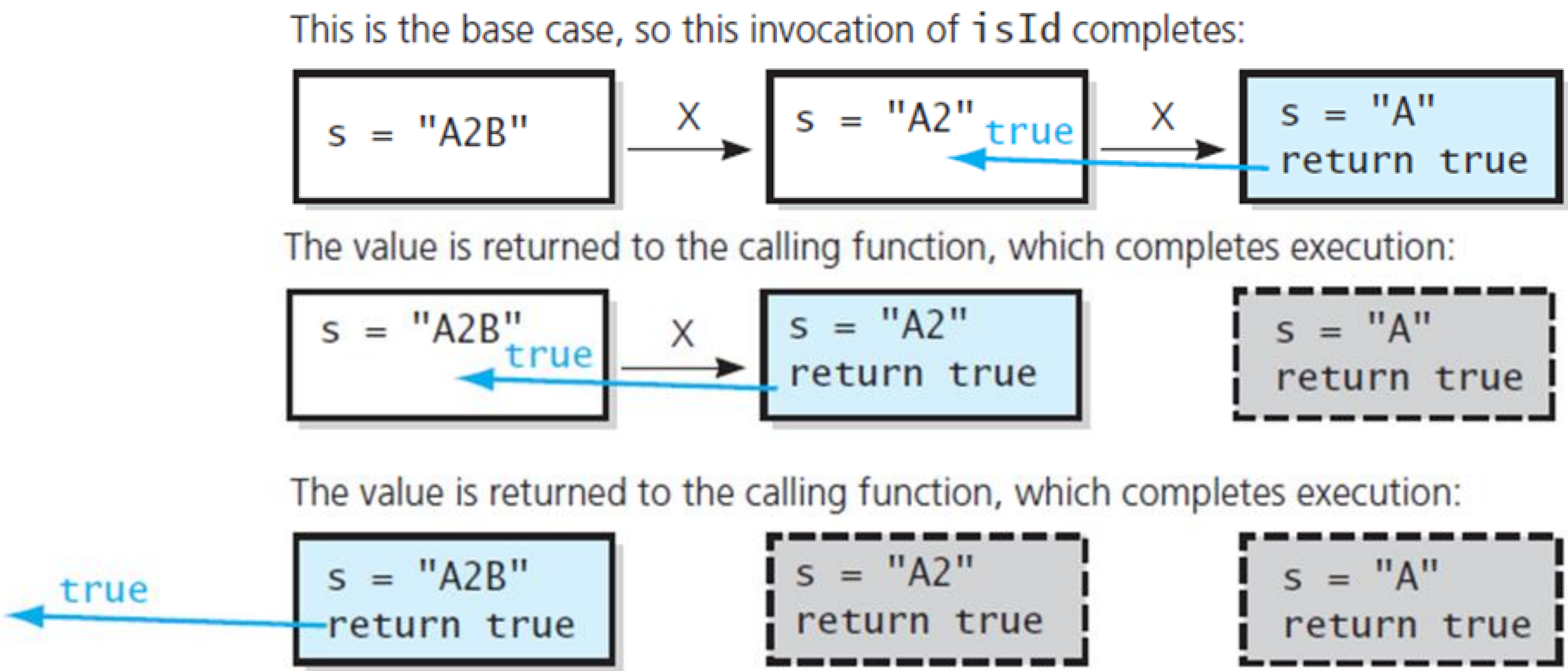
At point X, a recursive call is made and the new invocation of `isId` begins executi



Trace of **`isId("A2B")`**

Recognition Algorithm for Identifiers

- Trace of **`isId("A2B")`**



Two Simple Languages

- Palindromes

$$Palindromes = \{\text{string } s : s \text{ reads the same left to right as right to left}\}$$

- Grammar for the language of palindromes

$$\begin{aligned} &<pal> \; = \text{empty string} \mid <ch> \mid \text{a} <pal> \text{a} \mid \text{b} <pal> \text{b} \mid \ldots \mid \text{Z} <pal> \text{Z} \\ &<ch> \; = \text{a} \mid \text{b} \mid \ldots \mid \text{z} \mid \text{A} \mid \text{B} \mid \ldots \mid \text{Z} \end{aligned}$$

- Recursive definition of palindrome
  - The first and last characters of *s* are the same
  - *s* minus its first and last characters is a palindrome

Two Simple Languages

- A recognition algorithm for palindromes

```
// Returns true if the string s of letters is a palindrome; otherwise returns false.
isPalindrome(s: string): boolean

   if (s is the empty string or s is of length 1)
      return true
   else if (s's first and last characters are the same letter)
      return isPalindrome(s minus its first and last characters)
   else
      return false
```

Algebraic Expressions

- Compiler must recognize and evaluate algebraic expressions

```
y = x + z * (w / k + z * (7 * 6));
```

  - Determine if legal expression
  - If legal, evaluate expression

Kinds of Algebraic Expressions

- Infix expressions
  - Every binary operator appears between its operands

- This convention necessitates …
  - Associativity rules
  - Precedence rules
  - Use of parentheses

$$a + b * c$$

$$(a + b) * c$$

Kinds of Algebraic Expressions

- Prefix expression
  - Operator appears before its operands

$$a + b \text{ equivalent to } + a\, b$$

- Postfix expressions
  - Operator appears after its operands

$$a + b \text{ equivalent to } a\, b\, +$$

Prefix Expressions

- Grammar that defines language of all prefix expressions

$$
\begin{aligned}
<prefix> &= <identifier> \mid <operator> <prefix> <prefix> \\
<operator> &= + \mid - \mid * \mid / \\
<identifier> &= a \mid b \mid \ldots \mid z
\end{aligned}
$$

- Recursive algorithm that recognizes whether string is a prefix expression
  - Check if first character is an operator
  - Remainder of string consists of two consecutive prefix expressions

Prefix Expressions

```
// Finds the end of a prefix expression, if one exists.
// Precondition: The substring of strExp from the index first through the end of
// the string contains no blank characters.
// Postcondition: Returns the index of the last character in the prefix expression that
// begins at index first of strExp, or –1 if no such prefix expression exists.
endPre(strExp: string, first: integer): integer
{
   last = strExp.length() - 1
   if (first < 0 or first > last)
      return -1

   ch = character at position first of strExp
   if (ch is an identifier)
      return first            // Index of last character in simple prefix expression
   else if (ch is an operator)
   {
```

endPre determines the end of a prefix expression

Prefix Expressions

```
      return first          // Index of last character in simple prefix expression
   else if (ch is an operator)
   {
      // Find the end of the first prefix expression
      endPos = endPre(strExp, first + 1)     // Point X
     // If the end of the first prefix expression was found, find the end of the second
     // prefix expression
     if (endPos > -1)
        return endPre(strExp, endPos + 1) // Point Y
     else
        return -1
   }
   else
      return -1
}
```

endPre determines the end of a prefix expression

```
int apply(char op, int a, int b) {
    switch (op) {
        case '+': return a + b;
        case '-': return a - b;
        case '/': return a / b;
        case '*': return a * b;
        default: throw bad_operator(op);
    }
}

int expression(char *&input) {
    char op = *input++;

    if (isdigit(op))
        return op - '0';

    int a = expression(input);
    int b = expression(input);
    return apply(op, a, b);
}



int main() {
    char *input="+/42-43";
    std::cout << expression(input);
    return 0;
}
```

Prefix Expressions

Trace of `endPre( "+*ab-cd",0)`

The initial call endPre("+*ab-cd", 0) is made, and endPre begins execution:

```
first = 0
last  = 6
```

First character of strExp is +, so at point *X*, a recursive call is made and the new invocation of endPre begins execution:

```
first = 0                              first = 1
last  = 6                  X           last  = 6
X: endPre("+*ab-cd", 1)  ----->
```

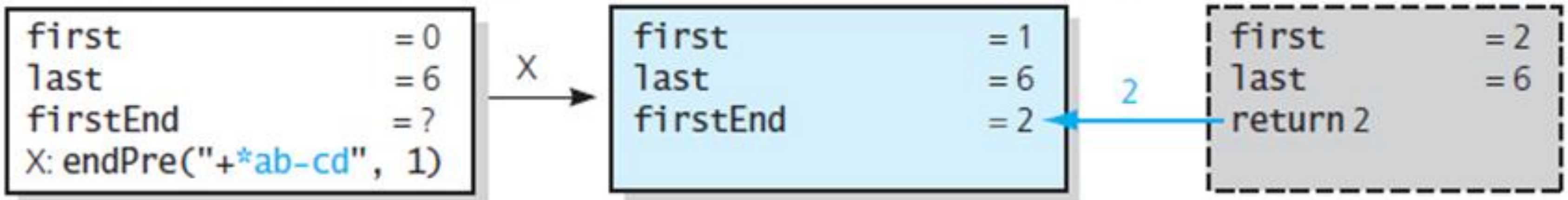
Next character of strExp is *, so at point *X*, a recursive call is made and the new invocation of endPre begins execution:

```
first    = 0                           first    = 1                           first = 2
last     = 6               X           last     = 6               X           last  = 6
firstEnd = ?             ----->        firstEnd = ?             ----->
X: endPre("+*ab-cd", 1)                X: endPre("+*ab-cd", 2)
```

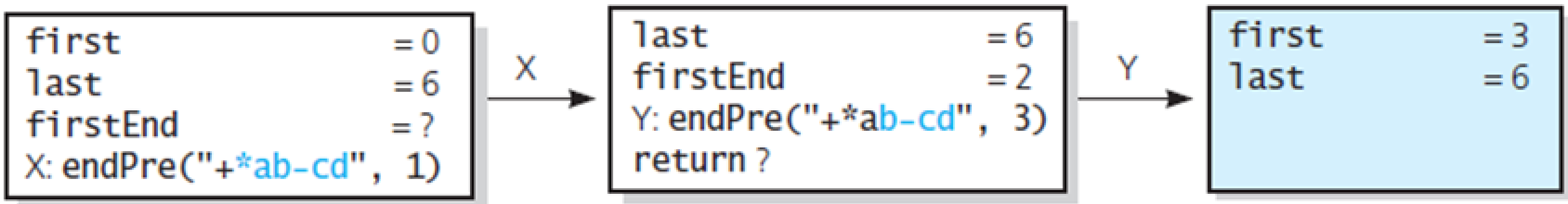
Prefix Expressions

Trace of **endPre("+*ab-cd",0)**

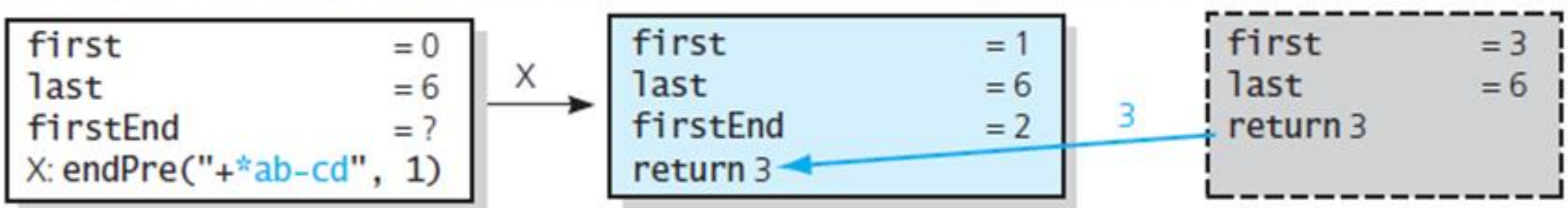
Next character of **strExp** is a, which is a base case. The current invocation of **endPre** completes execution and returns its value:



Because **firstEnd** > –1, a recursive call is made from point *Y* and the new invocation of **endPre** begins execution:



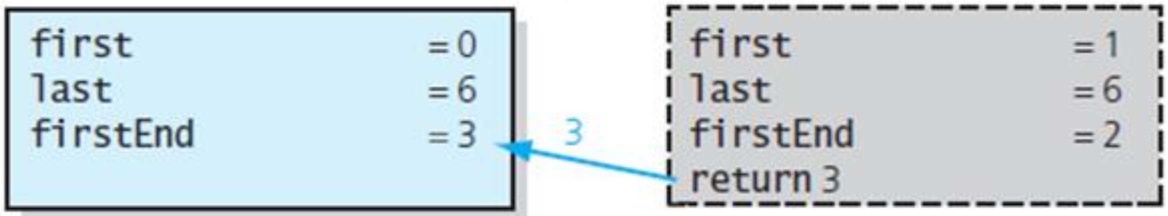
Next character of **strExp** is b, which is a base case. The current invocation of **endPre** completes execution and returns its value:



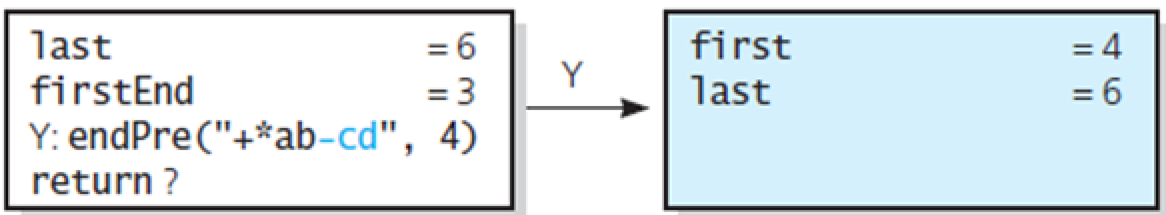
Prefix Expressions

Trace of **endPre("+*ab-cd",0)**

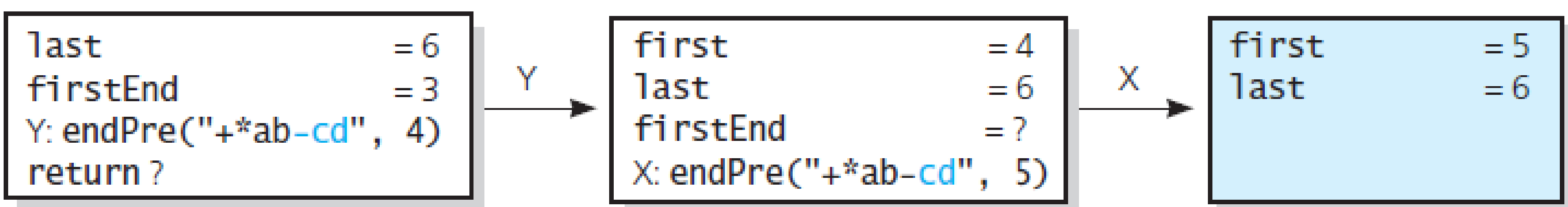
The current invocation of **endPre** completes execution and returns its value:



Because `firstEnd` > –1, a recursive call is made from point *Y* and the new invocation of **endPre** begins execution:



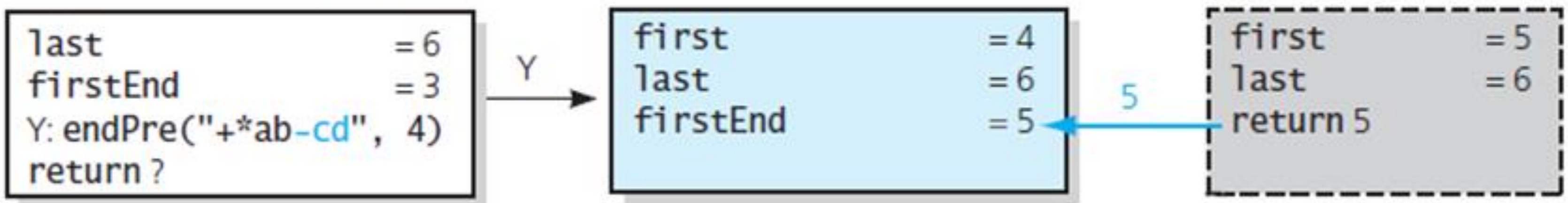
Next character of **strExp** is -, so at point *X*, a recursive call is made and the new invocation of **endPre** begins execution:



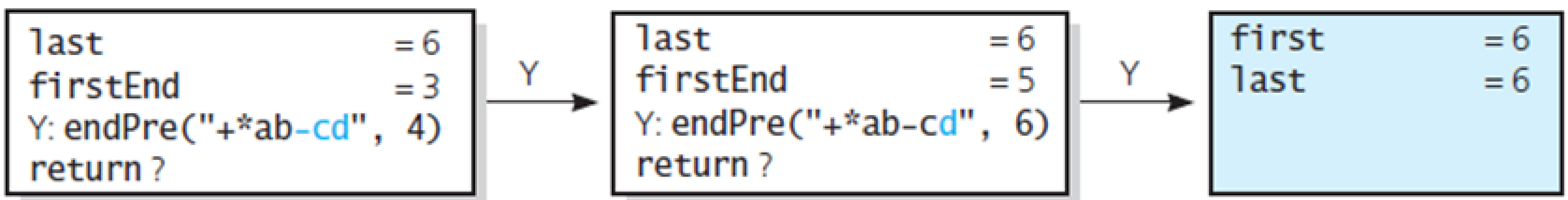
Prefix Expressions

Trace of **`endPre( "+*ab-cd",0)`**

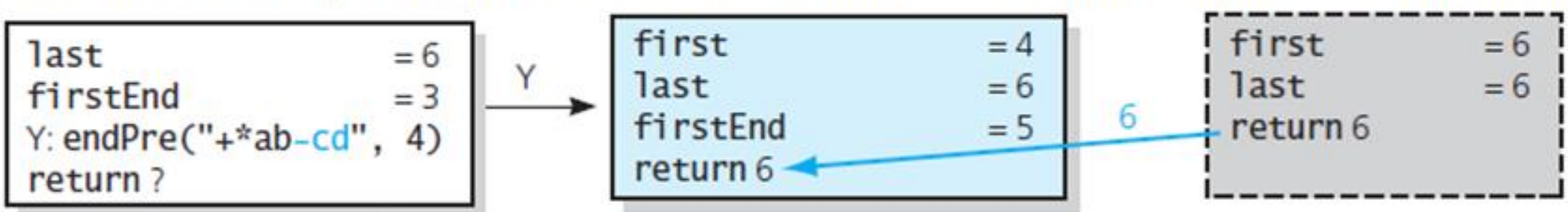
Next character of **strExp** is c, which is a base case. The current invocation of **endPre** completes execution and returns its value:



Because **firstEnd** > –1, a recursive call is made from point *Y* and the new invocation of **endPre** begins execution:



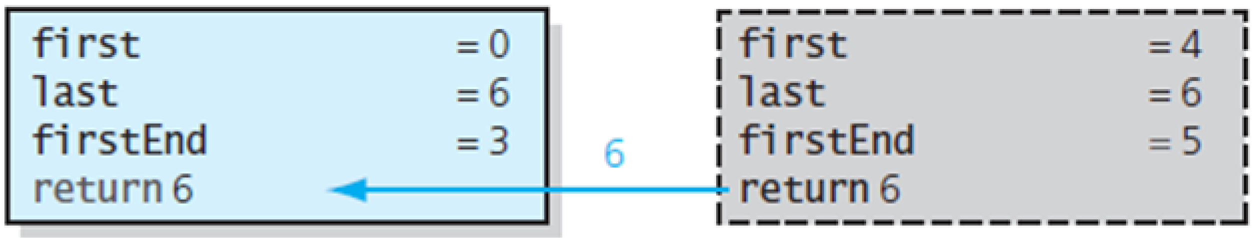
Next character of **strExp** is d, which is a base case. The current invocation of **endPre** completes execution and returns its value:



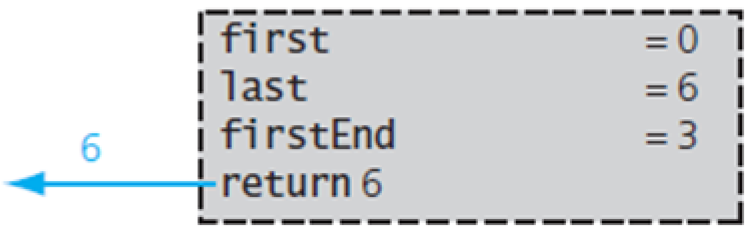
Prefix Expressions

Trace of `endPre( "+*ab-cd",0)`

The current invocation of **endPre** completes execution and returns its value:



The current invocation of **endPre** completes execution and returns its value to the original call to **endPre:**



Prefix Expressions

```
// Sees whether an expression is a prefix expression.
// Precondition: strExp contains a string with no blank characters.
// Postcondition: Returns true if the expression is in prefix form; otherwise returns false.
isPrefix(strExp: string): boolean
{
   lastChar = endPre(strExp, 0)
   return (lastChar >= 0) and (lastChar == strExp.length() - 1)
}
```

A recognition algorithm for prefix expressions

Prefix Expressions

```
// Returns the value of a given prefix expression.
// Precondition: strExp is a string containing a valid prefix expression with no blanks.
evaluatePrefix(strExp: string): float
{
   strLength = the length of strExp
   if (strLength == 1)
      return value of the identifier // Base case—single identifier
   else
   {
      op = strExp[0]              // strExp begins with an operator

      // Find the end of the first prefix expression—will be the first operand
      endFirst = endPre(strExp, 1)

      // Recursively evaluate this first prefix expression
```

An algorithm to evaluate a prefix expression

Prefix Expressions

```
        endFirst = endPre(strExp, 1)

        // Recursively evaluate this first prefix expression
        operand1 = evaluatePrefix(strExp[1..endFirst]);

        // Recursively evaluate the second prefix expression—will be the second operand
        endSecond = strLength - endFirst + 1
        operand2 = evaluatePrefix(strExp[endFirst + 1..endSecond])

        // Evaluate the prefix expression
        return operand1 op operand2
    }
}
```

An algorithm to evaluate a prefix expression

Postfix Expressions

Grammar that defines the language of all postfix expressions

$$\langle postfix \rangle = \langle identifier \rangle | \langle postfix \rangle \langle postfix \rangle \langle operator \rangle$$
$$\langle operator \rangle = + | - | * | /$$
$$\langle identifier \rangle = a | b | \ldots | z$$

- An algorithm that converts a prefix expression to postfix form

```
if (exp is a single letter)
   return exp
else
   return postfix(prefix1) • postfix(prefix2) • <operator>
```

Postfix Expressions

```
// Converts a prefix expression to postfix form.
// Precondition: The string preExp is a valid prefix expression with no blanks.
// Postcondition: Returns the equivalent postfix expression.
convertPreToPost(preExp: string): string
{
   preLength = the length of preExp
   ch = first character in preExp
   postExp = an empty string

   if (ch is a lowercase letter)
      // Base case—single identifier
      postExp = postExp • ch              // Append to end of postExp
   else // ch is an operator
   {
      // pre has the form <operator> <prefix1> <prefix2>
      endFirst = endPre(preExp, 1)        // Find the end of prefix1

      // Recursively convert prefix1 into postfix form
      postExp = postExp • convert(preExp[1..endFirst])

      // Recursively convert prefix2 into postfix form
      postExp = postExp • convert(preExp[endFirst + 1..preLength - 1])

      postExp = postExp • ch              // Append the operator to the end of postExp
   }
   return postExp
}
```

- Recursive algorithm that converts a prefix expression to postfix form

Fully Parenthesized Expressions

- Grammar for language of fully parenthesized algebraic expressions

$$
\begin{aligned}
<infix> &= <identifier> \mid (<infix> <operator> <infix>) \\
<operator> &= + \mid - \mid * \mid / \\
<identifier> &= a \mid b \mid \ldots \mid z
\end{aligned}
$$

- Most programming languages support definition of algebraic expressions
  - Includes both precedence rules for operators and rules of association

Backtracking

- Strategy for guessing at a solution and …
  - Backing up when an impasse is reached
  - Retracing steps in reverse order
  - Trying a new sequence of steps

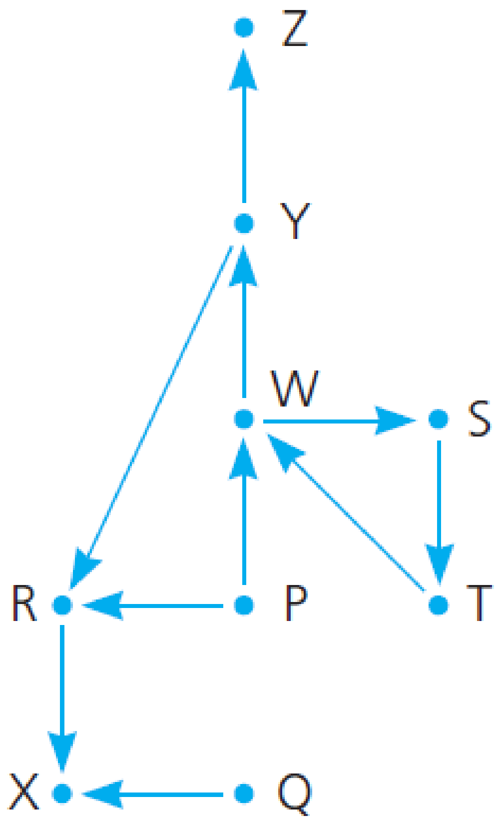
- Combine recursion and backtracking to solve problems

Searching for an Airline Route

- Must find a path from some point of origin to some destination point
- Program to process customer requests to fly
  - From some origin city
  - To some destination city
- Use three input text files
  - names of cities served
  - Pairs of city names, flight origins and destinations
  - Pairs of names, request origins, destinations

Backtracking

Flight map for HPAir



Backtracking

- A recursive strategy

To fly from the origin to the destination:

```
Select a city C adjacent to the origin
Fly from the origin to city C
if (C is the destination city)
    Terminate— the destination is reached
else
    Fly from city C to the destination
```

Searching for an Airline Route

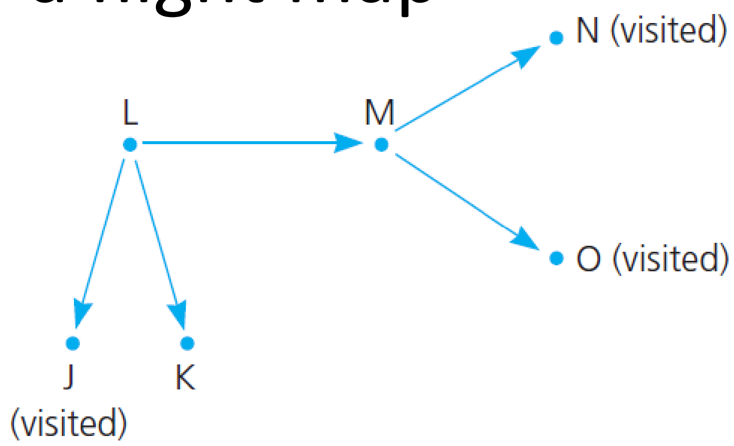
- Possible outcomes of applying the previous strategy
  1. Eventually reach destination city and can conclude that it is possible to fly from origin to destination.
  2. Reach a city C from which there are no departing flights.
  3. Go around in circles.
- Use backtracking to recover from a wrong choice (2 or 3)

Backtracking

```
// Discovers whether a sequence of flights from originCity to destinationCity exists.
searchR(originCity: City, destinationCity: City): boolean

   Mark originCity as visited
   if (originCity is destinationCity)
      Terminate—the destination is reached
   else
      for (each unvisited city C adjacent to originCity)
         searchR(C, destinationCity)
```

A piece of a flight map



Searching for an Airline Route

```
// Reads flight information into the flight map.
+readFlightMap(cityFileName: string, flightFileName: string): void

// Displays flight information.
+displayFlightMap(): void

// Displays the names of all cities that HPAir serves.
+displayAllCities(): void

// Displays all cities that are adjacent to a given city.
+displayAdjacentCities(aCity: City): void

// Marks a city as visited.
+markVisited(aCity: City): void

// Clears marks on all cities.
```

- ADT flight map operations

Searching for an Airline Route

```
+markVisited(aCity: City): void

// Clears marks on all cities.
+unvisitAll(): void

// Sees whether a city was visited.
+isVisited(aCity: City): boolean

// Inserts a city adjacent to another city in a flight map.
+insertAdjacent(aCity: City, adjCity: City): void

// Returns the next unvisited city, if any, that is adjacent to a given city.
// Returns a sentinel value if no unvisited adjacent city was found.
+getNextCity(fromCity: City): City

// Tests whether a sequence of flights exists between two cities.
+isPath(originCity: City, destinationCity: City): boolean
```

- ADT flight map operations

Searching for an Airline Route

```
/** Tests whether a sequence of flights exists between two cities.
 @pre  originCity and destinationCity both exist in the flight map.
 @post  Cities visited during the search are marked as visited
     in the flight map.
 @param originCity  The origin city.
 @param destinationCity  The destination city.
 @return  True if a sequence of flights exists from originCity
    to destinationCity; otherwise returns false. */
bool Map::isPath(City originCity, City destinationCity)
{
   // Mark the current city as visited
   markVisited(originCity);

   bool foundDestination = (originCity == destinationCity);
   if (!foundDestination)
   {
      // Try a flight to each unvisited city
```

C++ implementation of searchR

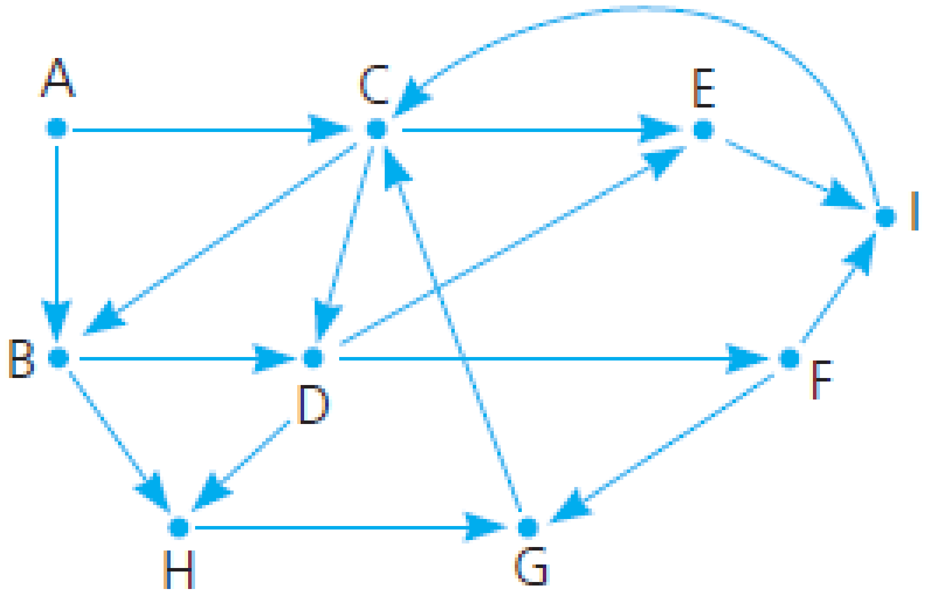
Searching for an Airline Route

```
      {
         // Try a flight to each unvisited city
         City nextCity = getNextCity(originCity);
         while (!foundDestination && (nextCity != NO_CITY))
         {
            foundDestination = isPath(nextCity, destinationCity);
            if (!foundDestination)
               nextCity = getNextCity(originCity);
         }  // end while
      }  // end if

      return foundDestination;
   }  // end isPath
```

C++ implementation of searchR

Searching for an Airline Route



Flight map for Checkpoint Question

The Eight Queens Problem

- Chessboard contains 64 squares
  – Form eight rows and eight column
- Problem asks you to place eight queens on the chessboard …
  – So that no queen can attack any other queen.
- Note there are 4,426,165,368 ways to arrange eight queens on a chessboard
  – Exhausting to check all possible ways

The Eight Queens Problem

- However, each row and column can contain exactly one queen.
  - Attacks along rows or columns are eliminated, leaving only 8! = 40,320 arrangements

- Solution now more feasible

The Eight Queens Problem

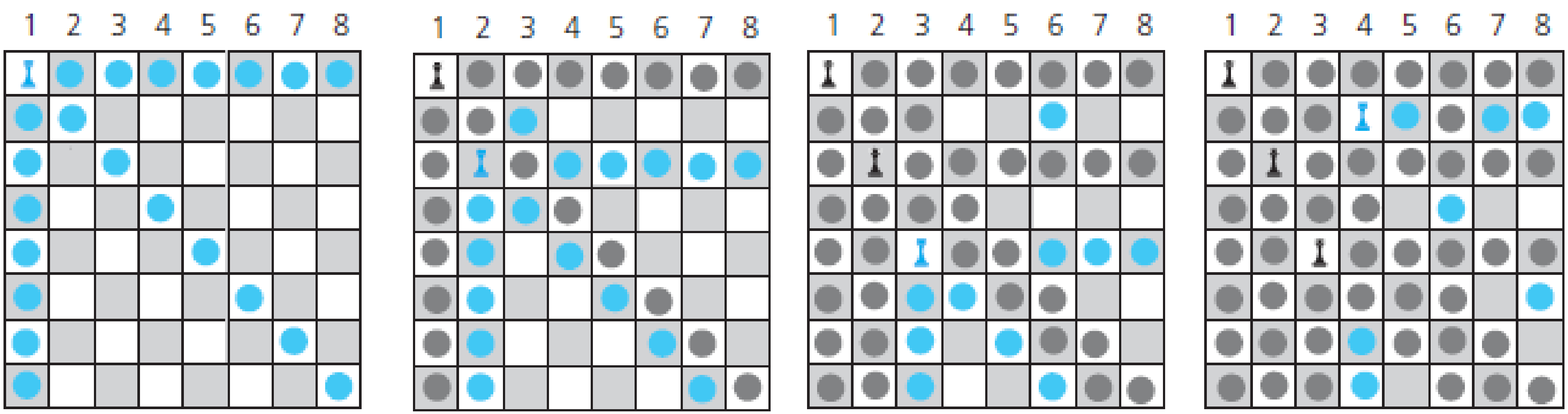
Placing one queen at a time in each column, and the placed queens' range of attack:

(a) the first queen in column 1;

(b) the second queen in column 2;

(c) the third queen in column 3;

(d) the fourth queen in column 4;



(a) (b) (c) (d)

The Eight Queens Problem

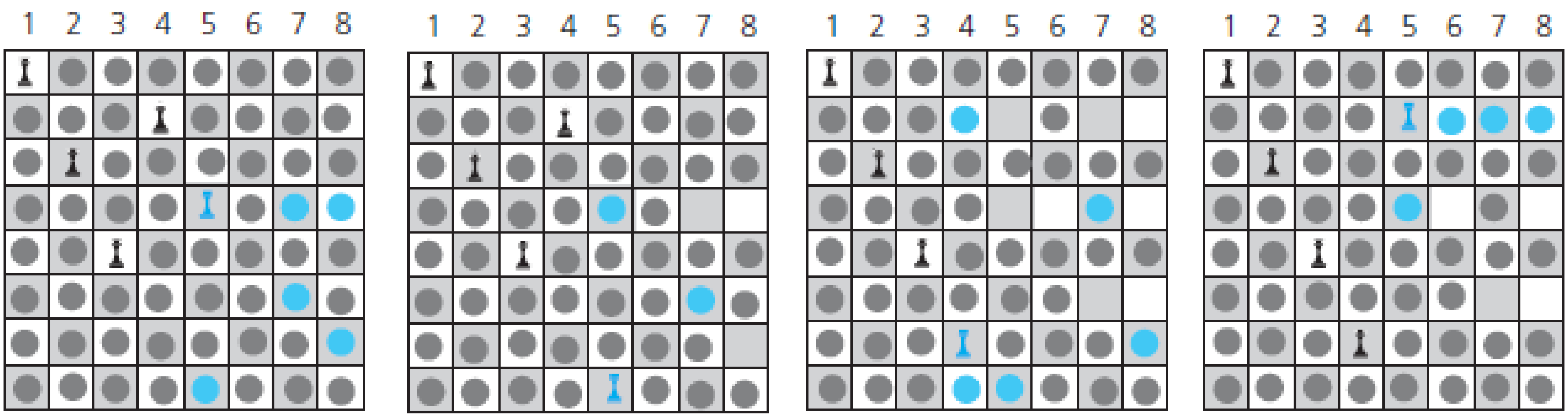
Placing one queen at a time in each column, and the placed queens' range of attack:

(e) five queens can attack all of column 6;

(f) backtracking to column 5 to try another square for queen;

(g) backtracking to column 4 to try another square for the queen;

(h) considering column 5 again



(e) (f) (g) (h)

The Eight Queens Problem

```
// Places queens in eight columns.
placeQueens(queen: Queen, row: integer, column: integer): boolean
{
   if (column > BOARD_SIZE)
      The problem is solved
   else
   {
      while (unconsidered squares exist in the given column and
               the problem is unsolved)
      {
         Find the next square in the given column that is
          not under attack by a queen in an earlier column
         if (such a square exists)
         {
            Place a queen in the square
```

Pseudocode describes the algorithm for placing queens

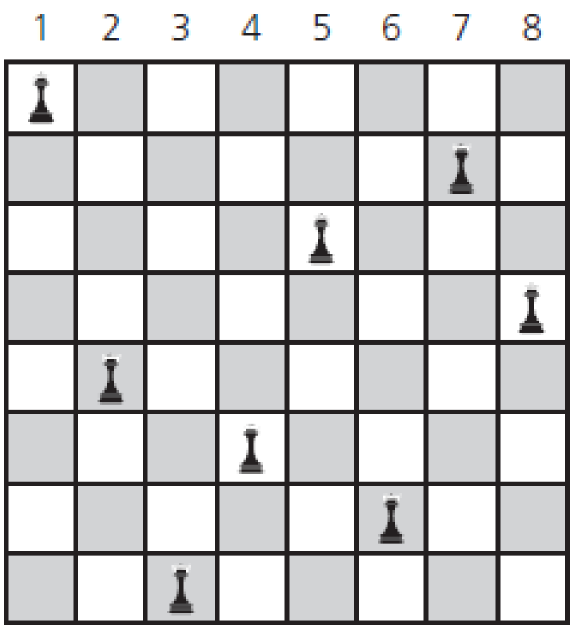
The Eight Queens Problem

```
            Place a queen in the square

            // Try next column
            if (!placeQueens(a new queen, firstRow, column + 1))
            {
               // No queen is possible in the next column
               Delete the new queen
               Move the last queen that was placed on the board
                  to the next row in that column
            }
         else
         {
            // Delete the new queen
            return true
         }
      }
    }
  }
}
```

Pseudocode describes the algorithm for placing queens

The Eight Queens Problem



A solution to the Eight Queens problem

- Recursion solves a problem by
  - Specifying a solution to one or more base cases
  - Then demonstrating how to derive solution to problem of arbitrary size
  - From solutions to smaller problems of same type.

- Can use induction to prove
  - Recursive algorithm either is correct
  - Or performs certain amount of work

Correctness of the Recursive Factorial Function

- A recursive function that computes the factorial of a nonnegative integer *n*

```
fact(n: integer): integer

    if (n is 0)
        return 1
    else
        return n * fact(n - 1)
```

Correctness of Recursive Factorial Function

- Inductive hypothesis

$$factorial(k) = k! = k \times (k-1) \times (k-2) \times \ldots \times 2 \times 1$$

- Inductive conclusion

$$factorial(k+1) = (k+1) \times k \times (k-1) \times (k-2) \times \ldots \times 2 \times 1$$

The Cost of Towers of Hanoi

- Recall solution to the Towers of Hanoi problem

```
solveTowers(count, source, destination, spare)

   if (count is 1)
      Move a disk directly from source to destination
   else
   {
      solveTowers(count - 1, source, spare, destination)
      solveTowers(1, source, destination, spare)
      solveTowers(count - 1, spare, destination, source)
   }
```

The Cost of Towers of Hanoi

- Claim

$$moves(N) = 2^N - 1 \quad \text{for all } N \geq 1$$

- Basis
  – Show that the property is true for N = 1
- Inductive hypothesis
  – Assume that property is true for N = k
- Inductive conclusion
  – Show that property is true for N = k + 1

The End

CS 302 - Data Structures

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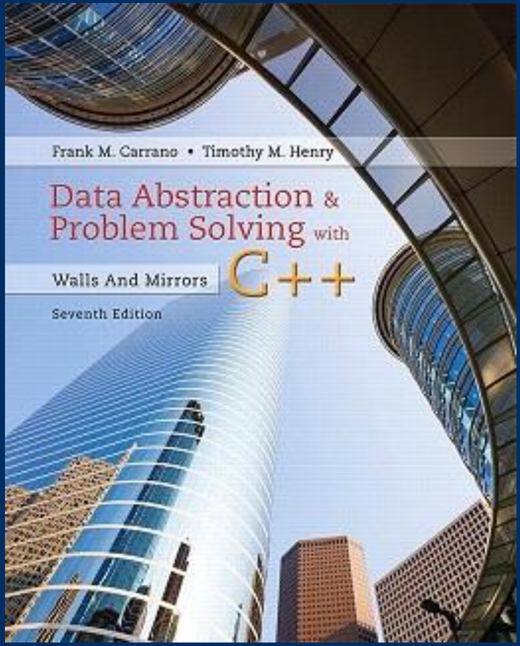

Modified from authors' slides

Out of the box

Reverse Linked List

- https://leetcode.com/problems/reverse-linked-list/description/

Reverse Words in a String

- https://leetcode.com/problems/reverse-words-in-a-string/description/