编码器预积分公式推导

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目录

1　预积分残差的设计

预积分的残差设计为如下形式

$$\begin{bmatrix} \boldsymbol{r}_p \\ \boldsymbol{r}_q \\ \boldsymbol{r}_{bg} \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}^*_{wb_i}(\boldsymbol{p}_{wb_j} - \boldsymbol{p}_{wb_i}) - \boldsymbol{\alpha}_{b_i b_j} \\ 2 * (\boldsymbol{q}^*_{b_i b_j} \otimes \boldsymbol{q}^*_{wb_i} \otimes \boldsymbol{q}_{wb_j})_{xyz} \\ \boldsymbol{b}^g_j - \boldsymbol{b}^g_i \end{bmatrix} \tag{1.1}$$

2　预积分方差的递推

2.1　连续时间误差微分方程

2.1.1　$\delta\dot{\boldsymbol{\theta}}_t^{b_k}$ 的微分推导

简便起见，把 $\delta\dot{\boldsymbol{\theta}}_t^{b_k}$ 写作 $\delta\dot{\boldsymbol{\theta}}$

1) 写出不考虑误差的微分方程

$$\dot{\boldsymbol{q}}_t = \frac{1}{2}\boldsymbol{q}_t \otimes \begin{bmatrix} 0 \\ \boldsymbol{\omega}_t - \boldsymbol{b}^g_t \end{bmatrix} \tag{2.1}$$

2) 写出考虑误差的微分方程

$$\dot{\tilde{\boldsymbol{q}}}_t = \frac{1}{2}\tilde{\boldsymbol{q}}_t \otimes \begin{bmatrix} 0 \\ \tilde{\boldsymbol{\omega}}_t - \tilde{\boldsymbol{b}}^g_t \end{bmatrix} \tag{2.2}$$

3) 写出带误差的值与理想值之间的关系

$$\begin{aligned} \tilde{\boldsymbol{q}}_t &= \boldsymbol{q}_t \otimes \delta\boldsymbol{q} \\ \tilde{\boldsymbol{\omega}}_t &= \boldsymbol{\omega}_t + \boldsymbol{n}_\omega \\ \tilde{\boldsymbol{b}}^g_t &= \boldsymbol{b}^g_t + \delta\boldsymbol{b}^g_t \end{aligned} \tag{2.3}$$

其中

$$\delta\boldsymbol{q} = \begin{bmatrix} \cos\left(\frac{|\delta\theta|}{2}\right) \\ \frac{\delta\boldsymbol{\theta}}{|\delta\theta|}\sin\left(\frac{|\delta\theta|}{2}\right) \end{bmatrix} \approx \begin{bmatrix} 1 \\ \frac{\delta\boldsymbol{\theta}}{2} \end{bmatrix} \tag{2.4}$$

4) 将 3) 中的关系带入 2)

$$(\boldsymbol{q}_t \dot{\otimes} \delta\boldsymbol{q}) = \frac{1}{2}\boldsymbol{q}_t \otimes \delta\boldsymbol{q} \otimes \begin{bmatrix} 0 \\ \boldsymbol{\omega}_t + \boldsymbol{n}_\omega - \boldsymbol{b}^g_t - \delta\boldsymbol{b}^g_t \end{bmatrix} \tag{2.5}$$

其中

$$(\boldsymbol{q}_t \dot{\otimes} \delta\boldsymbol{q}) = \dot{\boldsymbol{q}}_t \otimes \delta\boldsymbol{q} + \boldsymbol{q}_t \otimes \delta\dot{\boldsymbol{q}} \tag{2.6}$$

5) 把 1) 中的关系带入 4)

$$\begin{aligned} (\boldsymbol{q}_t \dot{\otimes} \delta\boldsymbol{q}) &= \dot{\boldsymbol{q}}_t \otimes \delta\boldsymbol{q} + \boldsymbol{q}_t \otimes \delta\dot{\boldsymbol{q}} \\ &= \frac{1}{2}\boldsymbol{q}_t \otimes \begin{bmatrix} 0 \\ \boldsymbol{\omega}_t - \boldsymbol{b}^g_t \end{bmatrix} \otimes \delta\boldsymbol{q} + \boldsymbol{q}_t \otimes \delta\dot{\boldsymbol{q}} \\ &= \frac{1}{2}\boldsymbol{q}_t \otimes \delta\boldsymbol{q} \otimes \begin{bmatrix} 0 \\ \boldsymbol{\omega}_t + \boldsymbol{n}_\omega - \boldsymbol{b}^g_t - \delta\boldsymbol{b}^g_t \end{bmatrix} \end{aligned} \tag{2.7}$$

6) 化简方程

首先把 5) 中最后两行左乘 $(\boldsymbol{q}_t)^{-1}$ 并移项可得

$$
\begin{aligned}
\delta\dot{\boldsymbol{q}} =& \frac{1}{2}\delta\boldsymbol{q}\otimes\begin{bmatrix} 0 \\ \boldsymbol{\omega}_t+\boldsymbol{n}_\omega-\boldsymbol{b}_t^g-\delta\boldsymbol{b}_t^g \end{bmatrix} \\
&-\frac{1}{2}\begin{bmatrix} 0 \\ \boldsymbol{\omega}_t-\boldsymbol{b}_t^g \end{bmatrix}\otimes\delta\boldsymbol{q}
\end{aligned} \tag{2.8}
$$

四元数相乘可以转换成矩阵与向量相乘，令

$$
\begin{aligned}
\boldsymbol{\omega}_1 &= \boldsymbol{\omega}_t+\boldsymbol{n}_\omega-\boldsymbol{b}_t^g-\delta\boldsymbol{b}_t^g \\
\boldsymbol{\omega}_2 &= \boldsymbol{\omega}_t-\boldsymbol{b}_t^g
\end{aligned} \tag{2.9}
$$

则

$$
\begin{aligned}
\delta\dot{\boldsymbol{q}} &= \frac{1}{2}\begin{bmatrix} 0 \\ \boldsymbol{\omega}_1 \end{bmatrix}_R\delta\boldsymbol{q}-\frac{1}{2}\begin{bmatrix} 0 \\ \boldsymbol{\omega}_2 \end{bmatrix}_L\delta\boldsymbol{q} \\
&= \frac{1}{2}\begin{bmatrix} 0 & (\boldsymbol{\omega}_2-\boldsymbol{\omega}_1)^T \\ (\boldsymbol{\omega}_1-\boldsymbol{\omega}_2) & -[\boldsymbol{\omega}_1+\boldsymbol{\omega}_2]_\times \end{bmatrix}\delta\boldsymbol{q}
\end{aligned} \tag{2.10}
$$

由于

$$
\delta\dot{\boldsymbol{q}} = \begin{bmatrix} 0 \\ \frac{\delta\dot{\boldsymbol{\theta}}}{2} \end{bmatrix} \tag{2.11}
$$

把它代入上式，又可以得到

$$
\begin{aligned}
\delta\dot{\boldsymbol{\theta}} =& -[\boldsymbol{\omega}_1+\boldsymbol{\omega}_2]_\times\frac{\delta\boldsymbol{\theta}}{2}+(\boldsymbol{\omega}_1-\boldsymbol{\omega}_2) \\
=& -[2\boldsymbol{\omega}_t+\boldsymbol{n}_\omega-2\boldsymbol{b}_t^g-\delta\boldsymbol{b}_t^g]_\times\frac{\delta\boldsymbol{\theta}}{2} \\
&+\boldsymbol{n}_\omega-\delta\boldsymbol{b}_t^g
\end{aligned} \tag{2.12}
$$

忽略其中的二阶小项，可得

$$
\delta\dot{\boldsymbol{\theta}} = -[\boldsymbol{\omega}_t-\boldsymbol{b}_t^g]_\times\delta\boldsymbol{\theta}+\boldsymbol{n}_\omega-\delta\boldsymbol{b}_t^g \tag{2.13}
$$

2.1.2 $\delta\boldsymbol{\alpha}_{b_k b_{k+1}}$ 的推导

由于编码器的模型是离散时间下，因此这里推导离散时间下的误差方程。

1) 写出不考虑误差的方程

$$
\boldsymbol{\alpha}_{b_i b_{k+1}} = \boldsymbol{\alpha}_{b_i b_k}+\boldsymbol{q}_{b_i b_k}\boldsymbol{\phi}_k \tag{2.14}
$$

2) 写出考虑误差的方程

$$
\tilde{\boldsymbol{\alpha}}_{b_i b_{k+1}} = \tilde{\boldsymbol{\alpha}}_{b_i b_k}+\tilde{\boldsymbol{q}}_{b_i b_k}\tilde{\boldsymbol{\phi}}_k \tag{2.15}
$$

3) 写出带误差的值与理想值之间的关系

$$
\tilde{\boldsymbol{\alpha}}_{b_i b_{k+1}} = \boldsymbol{\alpha}_{b_i b_{k+1}}+\delta\boldsymbol{\alpha}_{b_i b_{k+1}} \tag{2.16}
$$

$$
\tilde{\boldsymbol{\alpha}}_{b_i b_k} = \boldsymbol{\alpha}_{b_i b_k}+\delta\boldsymbol{\alpha}_{b_i b_k} \tag{2.17}
$$

$$
\tilde{\boldsymbol{R}}_{b_i b_k} = \boldsymbol{R}_{b_i b_k}exp([\delta\boldsymbol{\theta}_k]_\times) = \boldsymbol{R}_{b_i b_k}(I+[\delta\boldsymbol{\theta}_k]_\times) \tag{2.18}
$$

$$
\tilde{\boldsymbol{\phi}}_k = \boldsymbol{\phi}_k+\boldsymbol{n}_{\phi k} \tag{2.19}
$$

4) 将 3) 中的关系带入 2)

$$\boldsymbol{\alpha}_{b_i b_{k+1}} + \delta\boldsymbol{\alpha}_{b_i b_{k+1}} = \boldsymbol{\alpha}_{b_i b_k} + \delta\boldsymbol{\alpha}_{b_i b_k} + \boldsymbol{R}_{b_i b_k}(I + [\delta\boldsymbol{\theta}_k]_\times)(\boldsymbol{\phi}_k + \boldsymbol{n}_{\phi k}) \tag{2.20}$$

5) 把 1) 中的关系带入 4)

$$\boldsymbol{\alpha}_{b_i b_k} + \boldsymbol{R}_{b_i b_k}\boldsymbol{\phi}_k + \delta\boldsymbol{\alpha}_{b_i b_{k+1}} = \boldsymbol{\alpha}_{b_i b_k} + \delta\boldsymbol{\alpha}_{b_i b_k} + \boldsymbol{R}_{b_i b_k}(I + [\delta\boldsymbol{\theta}_k]_\times)(\boldsymbol{\phi}_k + \boldsymbol{n}_{\phi k}) \tag{2.21}$$

6) 化简方程

$$\begin{aligned} \delta\boldsymbol{\alpha}_{b_i b_{k+1}} &= \delta\boldsymbol{\alpha}_{b_i b_k} + \boldsymbol{R}_{b_i b_k}[\delta\boldsymbol{\theta}_k]_\times\boldsymbol{\phi}_k + \boldsymbol{R}_{b_i b_k}\boldsymbol{n}_{\phi k} \\ &= \delta\boldsymbol{\alpha}_{b_i b_k} - \boldsymbol{R}_{b_i b_k}[\boldsymbol{\phi}_k]_\times\delta\boldsymbol{\theta}_k + \boldsymbol{R}_{b_i b_k}\boldsymbol{n}_{\phi k} \end{aligned} \tag{2.22}$$

2.1.3 $\delta\dot{\boldsymbol{b}}_k^g$ 的微分推导

简便起见，把 $\delta\dot{\boldsymbol{b}}_k^g$ 写作 $\delta\dot{\boldsymbol{b}}^g$

1) 写出不考虑误差的微分方程

$$\dot{\boldsymbol{b}}^g = 0 \tag{2.23}$$

2) 写出考虑误差的微分方程

$$\dot{\tilde{\boldsymbol{b}}}^g = \boldsymbol{n}_{b^g} \tag{2.24}$$

3) 写出带误差的值与理想值之间的关系

$$\tilde{\boldsymbol{b}}^g = \boldsymbol{b}^g + \delta\boldsymbol{b}^g \tag{2.25}$$

4) 将 3) 中的关系带入 2)

$$\dot{\boldsymbol{b}}^g + \delta\dot{\boldsymbol{b}}^g = \boldsymbol{n}_{b^g} \tag{2.26}$$

5) 把 1) 中的关系带入 4)

$$0 + \delta\dot{\boldsymbol{b}}^g = \boldsymbol{n}_{b^g} \tag{2.27}$$

6) 化简方程

$$\delta\dot{\boldsymbol{b}}^g = \boldsymbol{n}_{b^g} \tag{2.28}$$

2.2 离散时间误差递推方程

离散时间的误差递推方程可表示为

$$\boldsymbol{X}_{k+1} = \boldsymbol{F}_k\boldsymbol{X}_k + \boldsymbol{G}_k\boldsymbol{N}_k \tag{2.29}$$

其中

$$\boldsymbol{X}_{k+1} = \begin{bmatrix} \delta\boldsymbol{\alpha}_{k+1} \\ \delta\boldsymbol{\theta}_{k+1} \\ \delta\boldsymbol{b}_{k+1}^g \end{bmatrix} \quad \boldsymbol{X}_k = \begin{bmatrix} \delta\boldsymbol{\alpha}_k \\ \delta\boldsymbol{\theta}_k \\ \delta\boldsymbol{b}_k^g \end{bmatrix} \quad \boldsymbol{N}_k = \begin{bmatrix} \boldsymbol{n}_{\phi_k} \\ \boldsymbol{n}_{\omega_k} \\ \boldsymbol{n}_{\omega_{k+1}} \\ \boldsymbol{n}_{b_k^g} \end{bmatrix} \tag{2.30}$$

2.2.1 $\delta\boldsymbol{\theta}_{k+1}$ 的求解

根据连续时间的微分方程公式 (2.13)，可得离散时间下

$$\delta\dot{\boldsymbol{\theta}}_k = -\left[\frac{\boldsymbol{\omega}_k + \boldsymbol{\omega}_{k+1}}{2} - \boldsymbol{b}_k^g\right]_\times \delta\boldsymbol{\theta}_k + \boldsymbol{n}_\omega - \delta\boldsymbol{b}_k^g \tag{2.31}$$

从而有

$$\delta\boldsymbol{\theta}_{k+1} = \left[\boldsymbol{I} - \left[\frac{\boldsymbol{\omega}_k + \boldsymbol{\omega}_{k+1}}{2} - \boldsymbol{b}_k^g\right]_\times \delta t\right]\delta\boldsymbol{\theta}_k + \delta t\frac{\boldsymbol{n}_{\omega_k} + \boldsymbol{n}_{\omega_{k+1}}}{2} - \delta t\delta\boldsymbol{b}_k^g \tag{2.32}$$

简化为

$$\delta\boldsymbol{\theta}_{k+1} = \left[\boldsymbol{I} - [\overline{\boldsymbol{\omega}}]_\times\delta t\right]\delta\boldsymbol{\theta}_k + \frac{\delta t}{2}\boldsymbol{n}_{\omega_k} + \frac{\delta t}{2}\boldsymbol{n}_{\omega_{k+1}} - \delta t\delta\boldsymbol{b}_k^g \tag{2.33}$$

其中

$$\overline{\boldsymbol{\omega}} = \frac{\boldsymbol{\omega}^{b_k} + \boldsymbol{\omega}^{b_{k+1}}}{2} - \boldsymbol{b}_k^g \tag{2.34}$$

2.2.2 $\delta\boldsymbol{\alpha}_{k+1}$ 的求解

重新书写公式 (2.22) 如下

$$\begin{aligned}\delta\boldsymbol{\alpha}_{k+1} =&\delta\boldsymbol{\alpha}_k \\ &- \boldsymbol{R}_k[\boldsymbol{\phi}_k]_\times\delta\boldsymbol{\theta}_k \\ &+ \boldsymbol{R}_k\boldsymbol{n}_{\phi k}\end{aligned} \tag{2.35}$$

2.2.3 $\delta\boldsymbol{b}_{k+1}^g$ 的求解

根据连续时间的微分方程公式 (2.28)，可得陀螺仪零偏的离散时间传递模型为

$$\delta\boldsymbol{b}_{k+1}^g = \delta\boldsymbol{b}_k^g + \boldsymbol{n}_{\boldsymbol{b}_k^g}\delta t \tag{2.36}$$

2.3 方差传递

由上一节的推导结果，便可以写出公式 (2.29) 中的矩阵

$$\begin{aligned}\boldsymbol{F}_k &= \begin{bmatrix}\boldsymbol{I} & -\boldsymbol{R}_k[\boldsymbol{\phi}_k]_\times & \boldsymbol{0} \\ 0 & \boldsymbol{I} - [\overline{\boldsymbol{\omega}}]_\times\delta t & -\boldsymbol{I}\delta t \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}\end{bmatrix} \\ \boldsymbol{G}_k &= \begin{bmatrix}\boldsymbol{R}_k & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{2}\boldsymbol{I}\delta t & \frac{1}{2}\boldsymbol{I}\delta t & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}\delta t\end{bmatrix}\end{aligned} \tag{2.37}$$

由公式 (2.29) 可得到方差的传递形式为

$$\boldsymbol{P}_{i,k+1} = \boldsymbol{F}_k\boldsymbol{P}_{i,k}\boldsymbol{F}_k^T + \boldsymbol{G}_k\boldsymbol{Q}\boldsymbol{G}_k^T \tag{2.38}$$

3 预积分残差对各状态量扰动的雅可比

残差表达形式如公式 (1.1)，待优化的变量是 $[\boldsymbol{p}_{wb_i}\ \boldsymbol{q}_{wb_i}, \boldsymbol{b}_i^g]$ 和 $[\boldsymbol{p}_{wb_j}, \boldsymbol{q}_{wb_j}, \boldsymbol{b}_j^g]$，但在实际使用中，往往都是使用扰动量，因此，实际是对以下变量求雅可比：$[\delta\boldsymbol{p}_{wb_i}, \delta\boldsymbol{\theta}_{b_ib_i'}, \delta\boldsymbol{b}_i^g]$ 和 $[\delta\boldsymbol{p}_{wb_j}, \delta\boldsymbol{\theta}_{b_jb_j'}, \delta\boldsymbol{b}_j^g]$。

3.1 位置残差的雅可比

3.1.1 对 $\delta \boldsymbol{p}_{wb_i}$ 的雅可比

$$\frac{\partial \boldsymbol{r}_p}{\partial \delta \boldsymbol{p}_{wb_i}} = -\boldsymbol{R}_{b_i w} \tag{3.1}$$

3.1.2 对 $\delta \boldsymbol{p}_{wb_j}$ 的雅可比

$$\frac{\partial \boldsymbol{r}_p}{\partial \delta \boldsymbol{p}_{wb_j}} = \boldsymbol{R}_{b_i w} \tag{3.2}$$

3.1.3 对 $\delta \boldsymbol{\theta}_{b_i b_i'}$ 的雅可比

$$\begin{aligned}
\frac{\partial \boldsymbol{r}_p}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} &= \frac{\partial (\boldsymbol{q}_{wb_i} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta \boldsymbol{\theta}_{b_i b_i'} \end{bmatrix})^{-1} (\boldsymbol{p}_{wb_j} - \boldsymbol{p}_{wb_i})}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} \\
&= \frac{\partial (\boldsymbol{R}_{wb_i} \exp([\delta \boldsymbol{\theta}_{b_i b_i'}]_\times))^{-1} (\boldsymbol{p}_{wb_j} - \boldsymbol{p}_{wb_i})}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} \\
&= \frac{\partial \exp([-\delta \boldsymbol{\theta}_{b_i b_i'}]_\times) \boldsymbol{R}_{b_i w} (\boldsymbol{p}_{wb_j} - \boldsymbol{p}_{wb_i})}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} \\
&= \frac{\partial (\boldsymbol{I} - [\delta \boldsymbol{\theta}_{b_i b_i'}]_\times) \boldsymbol{R}_{b_i w} (\boldsymbol{p}_{wb_j} - \boldsymbol{p}_{wb_i})}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} \\
&= \frac{\partial (-[\delta \boldsymbol{\theta}_{b_i b_i'}]_\times) \boldsymbol{R}_{b_i w} (\boldsymbol{p}_{wb_j} - \boldsymbol{p}_{wb_i})}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} \\
&= [\boldsymbol{R}_{b_i w} (\boldsymbol{p}_{wb_j} - \boldsymbol{p}_{wb_i})]_\times
\end{aligned} \tag{3.3}$$

3.1.4 对 $\delta \boldsymbol{\theta}_{b_j b_j'}$ 的雅可比

$$\frac{\partial \boldsymbol{r}_p}{\partial \delta \boldsymbol{\theta}_{b_j b_j'}} = \boldsymbol{0} \tag{3.4}$$

3.1.5 对 $\delta \boldsymbol{b}_i^g$ 的雅可比

$$\begin{aligned}
\frac{\partial \boldsymbol{r}_p}{\partial \delta \boldsymbol{b}_i^g} &= -\frac{\partial \boldsymbol{\alpha}_{b_i b_j}}{\partial \delta \boldsymbol{b}_i^g} \\
&= -\boldsymbol{J}^{\alpha}_{b_i^g}
\end{aligned} \tag{3.5}$$

其中 $\boldsymbol{J}^{\alpha}_{b_i^g}$ 是由公式 (2.29) 递推得到，即

$$\boldsymbol{J}^{\alpha}_{b_i^g} = (\boldsymbol{F}_{j,j-1} \boldsymbol{F}_{j-1,j-2} \cdots \boldsymbol{F}_{k+1,k} \cdots \boldsymbol{F}_{i+1,i})^{\alpha}_{b_i^g} \tag{3.6}$$

3.1.6 对 $\delta \boldsymbol{b}_j^g$ 的雅可比

$$\frac{\partial \boldsymbol{r}_p}{\partial \delta \boldsymbol{b}_i^g} = \mathbf{0} \tag{3.7}$$

3.2 姿态残差的雅可比

3.2.1 对 $\delta \boldsymbol{p}_{wb_i}$ 的雅可比

$$\frac{\partial \boldsymbol{r}_q}{\partial \delta \boldsymbol{p}_{wb_i}} = \mathbf{0} \tag{3.8}$$

3.2.2 对 $\delta \boldsymbol{p}_{wb_j}$ 的雅可比

$$\frac{\partial \boldsymbol{r}_q}{\partial \delta \boldsymbol{p}_{wb_j}} = \mathbf{0} \tag{3.9}$$

3.2.3 对 $\delta \boldsymbol{\theta}_{b_i b_i'}$ 的雅可比

$$\begin{aligned}
\frac{\partial \boldsymbol{r}_q}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} &= \frac{\partial 2 \left[\boldsymbol{q}_{b_j b_i} \otimes \left(\boldsymbol{q}_{b_i w} \otimes \boldsymbol{q}_{wb_j} \right) \right]_{xyz}}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} \\
&= \frac{\partial 2 \left[\boldsymbol{q}_{b_i b_j}^* \otimes \left(\boldsymbol{q}_{wb_i} \otimes \begin{bmatrix} 1 \\ \frac{1}{2} \delta \boldsymbol{\theta}_{b_i b_i'} \end{bmatrix} \right)^* \otimes \boldsymbol{q}_{wb_j} \right]_{xyz}}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} \\
&= \frac{\partial -2 \left[\left(\boldsymbol{q}_{b_i b_j}^* \otimes \left(\boldsymbol{q}_{wb_i} \otimes \begin{bmatrix} 1 \\ \frac{1}{2} \delta \boldsymbol{\theta}_{b_i b_i'} \end{bmatrix} \right)^* \otimes \boldsymbol{q}_{wb_j} \right)^* \right]_{xyz}}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} \\
&= \frac{\partial -2 \left[\boldsymbol{q}_{wb_j}^* \otimes \left(\boldsymbol{q}_{wb_i} \otimes \begin{bmatrix} 1 \\ \frac{1}{2} \delta \boldsymbol{\theta}_{b_i b_i'} \end{bmatrix} \right) \otimes \boldsymbol{q}_{b_i b_j} \right]_{xyz}}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}}
\end{aligned} \tag{3.10}$$

上式可以化简为

$$\begin{aligned}
\frac{\partial \boldsymbol{r}_q}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} &= -2 \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \end{bmatrix} \frac{\partial \boldsymbol{q}_{wb_j}^* \otimes \left(\boldsymbol{q}_{wb_i} \otimes \begin{bmatrix} 1 \\ \frac{1}{2} \delta \boldsymbol{\theta}_{b_i b_i'} \end{bmatrix} \right) \otimes \boldsymbol{q}_{b_i b_j}}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} \\
&= -2 \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \end{bmatrix} \frac{\partial \left[\boldsymbol{q}_{wb_j}^* \otimes \boldsymbol{q}_{wb_i} \right]_L \left[\boldsymbol{q}_{b_i b_j} \right]_R \begin{bmatrix} 1 \\ \frac{1}{2} \delta \boldsymbol{\theta}_{b_i b_i'} \end{bmatrix}}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} \\
&= -2 \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \end{bmatrix} \left[\boldsymbol{q}_{wb_j}^* \otimes \boldsymbol{q}_{wb_i} \right]_L \left[\boldsymbol{q}_{b_i b_j} \right]_R \begin{bmatrix} \mathbf{0} \\ \frac{1}{2} \boldsymbol{I} \end{bmatrix}
\end{aligned} \tag{3.11}$$

3.2.4 对 $\delta\boldsymbol{\theta}_{b_j b_j'}$ 的雅可比

$$\begin{aligned}\frac{\partial \boldsymbol{r}_q}{\partial \delta\boldsymbol{\theta}_{b_j b_j'}} &= \frac{\partial 2\left[\boldsymbol{q}_{b_i b_j}^* \otimes \boldsymbol{q}_{wb_i}^* \otimes \boldsymbol{q}_{wb_j} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta\boldsymbol{\theta}_{b_j b_j'} \end{bmatrix}\right]_{xyz}}{\partial \delta\boldsymbol{\theta}_{b_j b_j'}} \\ &= \frac{\partial 2\left[\left[\boldsymbol{q}_{b_i b_j}^* \otimes \boldsymbol{q}_{wb_i}^* \otimes \boldsymbol{q}_{wb_j}\right]_L \begin{bmatrix} 1 \\ \frac{1}{2}\delta\boldsymbol{\theta}_{b_j b_j'} \end{bmatrix}\right]_{xyz}}{\partial \delta\boldsymbol{\theta}_{b_j b_j'}} \\ &= 2\begin{bmatrix} \mathbf{0} & \boldsymbol{I} \end{bmatrix}\left[\boldsymbol{q}_{b_i b_j}^* \otimes \boldsymbol{q}_{wb_i}^* \otimes \boldsymbol{q}_{wb_j}\right]_L \begin{bmatrix} \mathbf{0} \\ \frac{1}{2}\boldsymbol{I} \end{bmatrix}\end{aligned} \tag{3.12}$$

3.2.5 对 $\delta\boldsymbol{b}_i^g$ 的雅可比

$$\begin{aligned}\frac{\partial \boldsymbol{r}_q}{\partial \delta\boldsymbol{b}_i^g} &= \frac{\partial 2\left[\left(\boldsymbol{q}_{b_i b_j} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\boldsymbol{J}_{b_i^g}^q \delta\boldsymbol{b}_i^g \end{bmatrix}\right)^* \otimes \boldsymbol{q}_{wb_i}^* \otimes \boldsymbol{q}_{wb_j}\right]_{xyz}}{\partial \delta\boldsymbol{b}_i^g} \\ &= \frac{\partial -2\left[\left(\left(\boldsymbol{q}_{b_i b_j} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\boldsymbol{J}_{b_i^g}^q \delta\boldsymbol{b}_i^g \end{bmatrix}\right)^* \otimes \boldsymbol{q}_{wb_i}^* \otimes \boldsymbol{q}_{wb_j}\right)^*\right]_{xyz}}{\partial \delta\boldsymbol{b}_i^g} \\ &= \frac{\partial -2\left[\mathbf{q}_{wb_j}^* \otimes \mathbf{q}_{wb_i} \otimes \left(\mathbf{q}_{b_i b_j} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\mathbf{J}_{b_i^g}^q \delta\mathbf{b}_i^g \end{bmatrix}\right)\right]_{xyz}}{\partial \delta\mathbf{b}_i^g} \\ &= -2\begin{bmatrix} 0 & \mathbf{I} \end{bmatrix}\left[\mathbf{q}_{wb_j}^* \otimes \mathbf{q}_{wb_i} \otimes \mathbf{q}_{b_i b_j}\right]_L \begin{bmatrix} \mathbf{0} \\ \frac{1}{2}\mathbf{J}_{b_i^g}^q \end{bmatrix}\end{aligned} \tag{3.13}$$

3.2.6 对 $\delta\boldsymbol{b}_j^g$ 的雅可比

$$\frac{\partial \delta\boldsymbol{r}_q}{\partial \delta\boldsymbol{b}_j^g} = \mathbf{0} \tag{3.14}$$

3.3 陀螺仪零偏残差的雅可比

3.3.1 对 $\delta\boldsymbol{p}_{wb_i}$ 的雅可比

$$\frac{\partial \delta\boldsymbol{r}_{bg}}{\partial \delta\boldsymbol{p}_{wb_i}} = \mathbf{0} \tag{3.15}$$

3.3.2 对 $\delta \boldsymbol{p}_{wb_j}$ 的雅可比

$$\frac{\partial \delta \boldsymbol{r}_{bg}}{\partial \delta \boldsymbol{p}_{wb_j}} = \mathbf{0} \tag{3.16}$$

3.3.3 对 $\delta \boldsymbol{\theta}_{b_i b_i'}$ 的雅可比

$$\frac{\partial \delta \boldsymbol{r}_{bg}}{\partial \delta \boldsymbol{\theta}_{b_i b_i'}} = \mathbf{0} \tag{3.17}$$

3.3.4 对 $\delta \boldsymbol{\theta}_{b_j b_j'}$ 的雅可比

$$\frac{\partial \delta \boldsymbol{r}_{bg}}{\partial \delta \boldsymbol{\theta}_{b_j b_j'}} = \mathbf{0} \tag{3.18}$$

3.3.5 对 $\delta \boldsymbol{b}_i^g$ 的雅可比

$$\frac{\partial \delta \boldsymbol{r}_{bg}}{\partial \delta \boldsymbol{b}_i^g} = -\boldsymbol{I} \tag{3.19}$$

3.3.6 对 $\delta \boldsymbol{b}_j^g$ 的雅可比

$$\frac{\partial \delta \boldsymbol{r}_{bg}}{\partial \delta \boldsymbol{b}_j^g} = \boldsymbol{I} \tag{3.20}$$