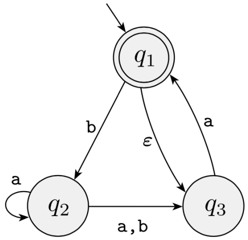
# Lec 03. Equivalence of DFA and NFA

Eunjung Kim

# Formal definition of NFA

## Nondeterministic FA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$

- $Q$ a finite set called the states,
- $\Sigma$ a finite set called the alphabet,
- $\delta$ a function from $Q \times \Sigma_\epsilon$ to $2^Q$ called the transition function,
- $q_0 \in Q$ the start state,
- $F \subseteq Q$ the set of accept states.

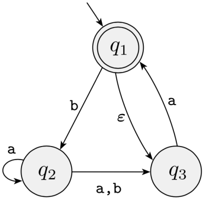



Write a transition table of this NFA.

# Language recognized by NFA

## NFA $N$ accepts $w$ if

1. $w$ can be written as $y_1, \ldots, y_m$ with $y_i \in \Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ such that there exists a sequence of states $r_0, \ldots, r_m$ satisfying the following:
   - $r_0 = q_0$,
   - $r_{i+1} \quad (??) \quad \delta(q_i, y_i)$,
   - $r_m \in F$.
2. Equivalently, there exists an accepting computation history starting with the (initial) configuration $(q_0, w)$.




Write a computation tree for $w = baabbaa$. How many accepting paths?

# Language recognized by NFA

## Definition

- Let $M$ be a nondeterministic finite automaton.
- A string $w \in \Sigma^*$ is accepted by $M$ if there exists an accepting computation history.
- $L(M)$ denotes the set of all strings accepted by $M$.
- A language $A$ is said to be recognized by $M$ if $A = L(M)$.

# Equivalence of NFA and DFA

**NFA and DFA own the same computational power**

For every NFA, there exists a deterministic finite automaton which recognizes the same language (a.k.a. equivalent DFA).

Proof outline.

- Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA.
- We want to construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$ such that $L(N) = L(M)$.
- Define $Q' := 2^Q$, i.e. the collection of all subsets of $Q$.
- Let us define $\delta'$, $q_0' \in 2^Q$ and $F' \subseteq 2^Q$,

# Proof: constructing DFA $M$, when no $\epsilon$-transition

- $q_0' = \{q_0\}$.
- transition function $\delta'$ from $2^Q \times \Sigma$ to $2^Q$:
  for every $R \in 2^Q$ ($R$ is a subset of $Q$) and every symbol $a \in \Sigma$,

$$\delta'(R, a) := \bigcup_{r \in R} \delta(r, a)$$

- Define $F' \subseteq 2^Q$ as the collection of all subsets of $Q$ containing at least one accept state of $N$.

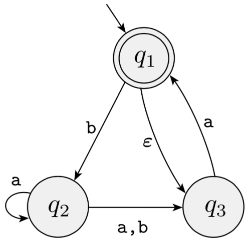
# PROOF: CONSTRUCTING DFA $M$ WITH $\epsilon$-TRANSITION

- How to define the initial state for DFA: from a state $q \in Q$ of NFA $N$, any other state $q'$ that can be reached by reading a string $\epsilon$, can be aggregated with $q$ to form a single state in DFA.
- Define $ext(q) \subseteq Q$ as the set of all states $q'$ of $N$ such that there is a directed path from $q$ to $q'$ in (the state diagram of) $N$ each of whose arcs carries the label $\epsilon$. Extend the definition $ext(X) := \bigcup_{q \in X} ext(q)$.
- transition function $\delta'$ from $2^Q \times \Sigma_\epsilon$ to $2^Q$:
  for every $R \in 2^Q$ ($R$ is a subset of $Q$) and every symbol $a \in \Sigma$,

$$\delta'(R, a) := ext(\bigcup_{r \in R} \delta(r, a))$$

- Define $q'_0 := ext(q_0) \in 2^Q$. Note that $q'_0$ corresponds to a subset of $Q$.
- Define $F' \subseteq 2^Q$ as the family of all subsets of $Q$ containing at least one accept state of $N$.

# Constructing DFA $M$ from NFA $N$, example



# Proof: $L(N) \subseteq L(M)$

Strategy: from an accepting computation history of $N$ on $w$, build an accepting computation history of $M$ on $w$.

Let $w = y_1 y_2 \cdots y_s$ for $y_i \in \Sigma$.

- Let $\pi = (q_0, w = w_0), (r_0, w_0), \ldots, (r_i, w_i), \ldots, (r_s, w_s = \epsilon)$ be an accepting computation history of $N$ for $w$ such that

  $r_i$ is reachable from $r_{i-1}$ via a walk (in the transition diagram of $N$) labelled by $y_i \circ \epsilon^*$.

- Observe: $r_0 \in ext(\{q_0\})$ and $r_i \in ext(\delta(r_{i-1}, y_i))$ for every $i \in [s]$ and $r_s \in F$.

# Proof: $L(N) \subseteq L(M)$

Strategy: from an accepting computation history of $N$ on $w$, build an accepting computation history of $M$ on $w$.

Let $w = y_1 y_2 \cdots y_s$ for $y_i \in \Sigma$.

- Let $\pi = (q_0, w = w_0), (r_0, w_0), \ldots, (r_i, w_i), \ldots, (r_s, w_s = \epsilon)$ be an accepting computation history of $N$ for $w$ such that

  $r_i$ is reachable from $r_{i-1}$ via a walk (in the transition diagram of $N$) labelled by $y_i \circ \epsilon^*$.

- Observe: $r_0 \in ext(\{q_0\})$ and $r_i \in ext(\delta(r_{i-1}, y_i))$ for every $i \in [s]$ and $r_s \in F$.
- Let $Q_0 = q'_0$. Inductively for each $i \in [s-1]$, let

$$Q_i := \delta'(Q_{i-1}, y_i).$$

- Now we have a computation history for $w = y_1 y_2 \cdots y_s$ in DFA $M$

$$\pi' = (Q_0, w_0 = w), (Q_1, w_1), \cdots (Q_t, w_s = \epsilon).$$

# Proof: $L(N) \subseteq L(M)$

Strategy: from an accepting computation history of $N$ on $w$, build an accepting computation history of $M$ on $w$.

Let $w = y_1 y_2 \cdots y_s$ for $y_i \in \Sigma$.

- Let $\pi = (q_0, w = w_0), (r_0, w_0), \ldots, (r_i, w_i), \ldots, (r_s, w_s = \epsilon)$ be an accepting computation history of $N$ for $w$ such that

  $r_i$ is reachable from $r_{i-1}$ via a walk (in the transition diagram of $N$) labelled by $y_i \circ \epsilon^*$.

- Observe: $r_0 \in ext(\{q_0\})$ and $r_i \in ext(\delta(r_{i-1}, y_i))$ for every $i \in [s]$ and $r_s \in F$.

- Let $Q_0 = q_0'$. Inductively for each $i \in [s-1]$, let

$$Q_i := \delta'(Q_{i-1}, y_i).$$

- Now we have a computation history for $w = y_1 y_2 \cdots y_s$ in DFA $M$

$$\pi' = (Q_0, w_0 = w), (Q_1, w_1), \cdots (Q_t, w_s = \epsilon).$$

# **Proof:** $L(N) \subseteq L(M)$

- It remains to see $Q_s$ is an accept state of $M$, i.e. the subset $Q_s \subseteq Q$ contains at least one accept state of $N$.
- This is because $r_0 \in ext(q_0) = Q_0$ and inductively $r_i \in ext(\delta(r_{i-1}, y_i)) \subseteq ext(\delta(Q_{i-1}, y_i) = \delta'(Q_i)$.

# PROOF: $L(M) \subseteq L(N)$

- Let $\pi' = (R_0, w = w_0), \ldots, (R_i, w_i), \ldots, (R_s, w_s = \epsilon)$ be an accepting computation history of $M$ on $w$. By definition of computation history $R_i = \delta'(R_{i-1}, y_i)$, where $y_i \in \Sigma$ is the leading symbol of $w_{i-1}$.
- We construct an accepting computation history of $N$ by following the sequence $\pi'$ backwardly.
- Observe: for each $q \in R_i \subseteq Q$, there exists a state $q' \in R_{i-1}$ such that from $q'$ to $q$ there is a computation history via a string consisting of $y_i$ followed by $\epsilon$'s.
- Now starting from $q_f \in R_s \subseteq Q$, we concatenate computation histories witnessed by the previous observation.

# Closure under regular operation

## Union operation

Let $A_1$ and $A_2$ be two languages recognized by NFA $N_1$ and $N_2$ respectively. Then $A_1 \cup A_2$ is recognized by some NFA.

# Closure under regular operation

## Concatenation operation

Let $A_1$ and $A_2$ be two languages recognized by NFA $N_1$ and $N_2$ respectively. Then $A_1 \circ A_2$ is recognized by some NFA.

# Closure under regular operation

## Kleene star operation

Let $A$ a languages recognized by NFA $N$. Then $A^*$ is recognized by some NFA.

# Closure under complementation

## Complementation operation

Let $A$ a languages recognized by NFA $N$. Then $\bar{A}$, that is, $\Sigma^* - A$ is recognized by some NFA.

- For a regular language $L$, we can obtain a DFA recognizing the complement of $L$.
- ...using the trick...
- Can we use the same trick for NFA in general?

# Closure under intersection

## Intersection operation

Let $A_1$ and $A_2$ be two languages recognized by NFA $N_1$ and $N_2$ respectively. Then $A_1 \cap A_2$ is recognized by some NFA.

- Use the expression that $A_1 \cap A_2 =$??????.
- Combine the above (which ones?) operations on NFAs...
- Direct way with two DFAs $M_1$ and $M_2$ by simulating both automata simultaneously.

# Closure under intersection

- Direct way with two DFAs $M_1$ and $M_2$ by simulating both automata simultaneously.

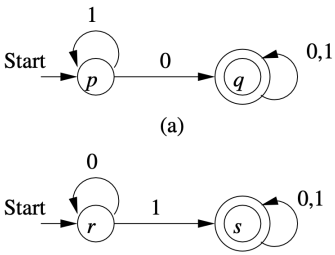



Figure 4.4 (a)-(b), Hopcroft et al. 2014.

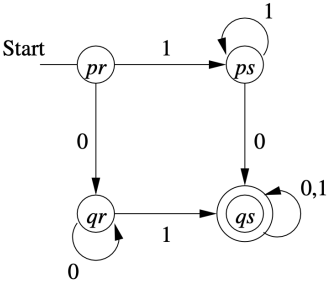



Figure 4.4 (c), Hopcroft et al. 2014.