FORMAL LANGUAGES AND AUTOMATA, 2025 FALL SEMESTER

Lec 04. Regular expression

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Regular expression

We want to compactly describe the 'pattern' of the following languages using union, concatenation and Kleene star operations.

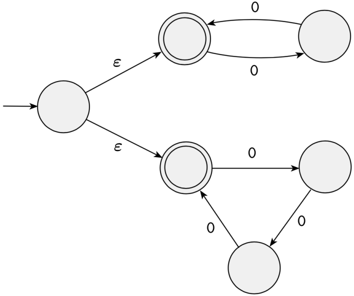


Figure 1.33, Sipser 2012.

1. The set of all $0, 1$ strings with exactly one 1's.
2. The set of all $0, 1$ strings with at least one 1's.
3. The set of strings over Σ of even length.

Formal definition of Regular expression

Regular expression over a finite alphabet Σ

Regular expression over Σ is a string consisting of symbols of Σ, parenthesis ( ), and the operators $\cup, \circ, ^*$ that can be generated as follows.

- Each symbol $x \in \Sigma \cup \{\epsilon\}$ is a regular expression.
- $\emptyset$ is a regular expression.
- $(R_1 \cup R_2)$ is a regular expression if R_1 and R_2 are regular expressions.
- $(R_1 \circ R_2)$ is a regular expression if R_1 and R_2 are regular expressions.
- R^* is a regular expression if R is a regular expression

Examples of Regular expression

Assume $\Sigma = \{0, 1\}$. Which language does the regular expression describe?

1. 0^*10^*.
2. $\Sigma^*1\Sigma^*$.
3. $1^*(01^+)^*$.
4. $(\Sigma\Sigma)^*$.
5. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$.
6. $1^*\emptyset = \emptyset$.
7. $\emptyset^* = \{\epsilon\}$.

Regular language

Value of a regular expression

For a regular expression R, the set of all strings which <u>can be generated following the expression</u> is denoted by $\mathcal{L}(R)$, said to be <u>the language of R</u>.

$\mathcal{L}(R)$ is also called the value of R, or the language described by R.

For two regular expressions R_1 and R_2, the value of union / concatenation / kleene star is...

- $L(R_1 \cup R_2) = L(R_1) \cup L(R_2)$.
- $L(R_1 \circ R_2) = L(R_1) \circ L(R_2)$.
- $L(R_1^\star) = L(R_1)^\star$.

Equivalence of regular expression and finite automata

Regular expression=NFA

A language $A \subseteq \Sigma^*$ is described by a regular expression if and only it is a regular language, i.e. recognized by some NFA.

One direction can be proved easily using what we learnt in the previous lecture. Which direction is it?

Equivalence proof: easy direction

Equivalence Theorem, Part I

If a language $A \subseteq \Sigma^*$ is described by a regular expression, then there is a (nondeterminist) finite automata M such that $L(M) = A$.

Proof idea: inductively build an NFA from NFAs accepting each symbol, $\{\epsilon\}$ or $\emptyset$ by applying each regular operations (union, concatenation, Kleene star).

Equivalence proof: easy direction

Constructing NFA recognizing the language $(\{\epsilon\} \cup \{a\} \cup \{ab\})^\star$.

Constructing NFA recognizing the language $(0 \cup 1)^\star 10$.

Equiv proof: the other direction

Equivalence Theorem, Part II

If a language $A \subseteq \Sigma^*$ is recognized by a finite automata A, then there is a regular expression R such that $\mathcal{L}(R) = A$.

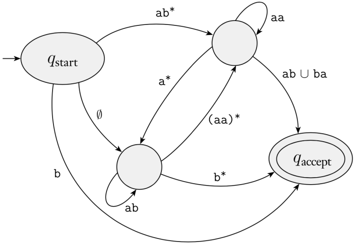
Proof idea: we shall see a procedure which converts DFA (or NFA) recognizing A into a regular expression.

Equiv proof: the other direction

Generalized Nondeterministic Finite Automata

- Generalized NFA is a NFA in which each arc carries a regular expression as a label.
- There are a unique source q_{start} as an initial state and a unique sink q_{accept} as an accept state.
- Between any other states there are arcs in both ways, including loops.

Figure 1.61, Sipser 2012.



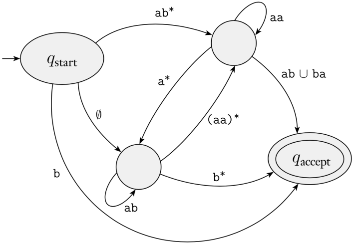
Equiv proof: the other direction

A formal description of GNFA

- Generalized NFA is a 5-tuple $(Q, \Sigma, \delta, q_{start}, q_{accept})$, where
- the transition function maps $(Q \setminus q_{accept}) \times (Q \setminus q_{start})$ to the set of all regular expressions over Σ.

In the transition diagram, we often omit an arc which carries $\emptyset$.

Figure 1.61, Sipser 2012.



EQUIV PROOF: THE OTHER DIRECTION

A GNFA ACCEPTS A STRING w

If w can be written as $w_1 w_2 \cdots w_\ell$, $w_i \in \Sigma^*$, and there is a sequence of states $r_0, \ldots, r_\ell$ such that

- r_0 is the initial state of GNFA,
- w_i is in the value of $\delta(r_{i-1}, r_i)$, i.e. $w_i \in L(\delta(r_{i-1}, r_i))$, and
- r_ℓ is the accept state of GNFA.

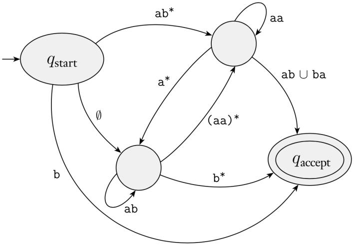
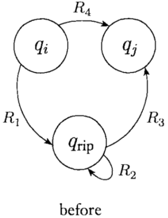


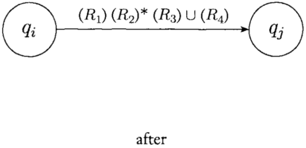
Figure 1.61, Sipser 2012.

EQUIV PROOF: THE OTHER DIRECTION

Proof idea.

- Initial NFA can be seen as GNFA, possibly after a simple modification to ensure a unique accept state, no incoming/outgoing arc from the unique initial/accept state.
- Reduce the number of states of GNFA inductively by eliminating a state of $Q - \{q_{start}, q_{accet}\}$ one by one, each elimination yielding an equivalent GNFA.
- The final GNFA with two states q_{start} and q_{accept} carries a single regular expression, a desired end product.
- How to eliminate a state q_k and update the label on (q_i, q_j):





Example: from NFA to regular expression

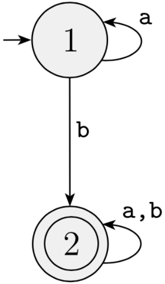


Figure 1.67 (a), Sipser 2012.

Equiv proof: the other direction

Proof by induction. Let G' be a GNFA obtained by eliminating state q_k from GNFA G. It suffices to prove that G accepts a string w if and only if G' does.

- Let $r_0, \ldots, r_\ell$ be a sequence of states appearing in the accepting computation history for $w = w_1 \cdots w_\ell$ in G.
- If q_k does not appear in this sequence, done.

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- If not, consider a maximal subsequence $r_a, \ldots, r_b$ of contiguous occurrences of q_k. Note that $1 \leq a \leq b < \ell$.

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- Observe: $w_a \in L(\delta(r_{a-1}, r_a))$, $w_{i+1} \in L(\delta(r_i, r_{i+1})) = L(\delta(q_k, q_k))$ for every $i \in [a, b-1]$, and $w_{b+1} \in L(\delta(r_b, r_{b+1}))$.

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- That is: $w_a \cdots w_{b+1}$ is described by

  $$\delta(r_{a-1}, r_a) \cdot \delta(q_k, q_k)^\star \cdot \delta(q_k, r_{b+1})$$

  which is in the union expression $\delta'(r_{a-1}, r_{b+1})$ in the G'.

Equiv proof: the other direction

Conversely,

- Let $r_0, \ldots, r_\ell$ be a sequence of states appearing in the accepting computation history for $w = w_1 \cdots w_\ell$ in G'.
- Each w_i in is the language of $\delta'(r_{i-1}, r_i)$.
- $\delta'(r_{i-1}, r_i) = \delta(r_{i-1}, r_i) \cup \delta(r_{i-1}, q_k) \cdot \delta(q_k, q_k)^\star \cdot \delta(q_k, r_i)$.
- Therefore, $w_i \in \delta(r_{i-1}, r_i)$ or
  w_i can be written as $x_1 \cdots, x_m$ such that
  1. $x_1 \in \delta(r_{i-1}, q_k)$
  2. $x_2 \in \delta(q_k, r_i)$

  or

  1. $x_1 \in \delta(r_{i-1}, q_k)$
  2. $x_j \in \delta(q_k, q_k)$ for each $2 \leq j < m$
  3. $x_m \in \delta(q_k, r_i)$
- Now replace the computation history in G' by...... to obtain a computation history in G.

Example: from NFA to regular expression

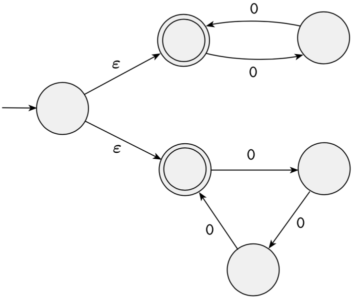

Figure 1.33, Sipser 2012.