Solving Interval-Valued Returns Mean Absolute Deviation Portfolio Selection Model

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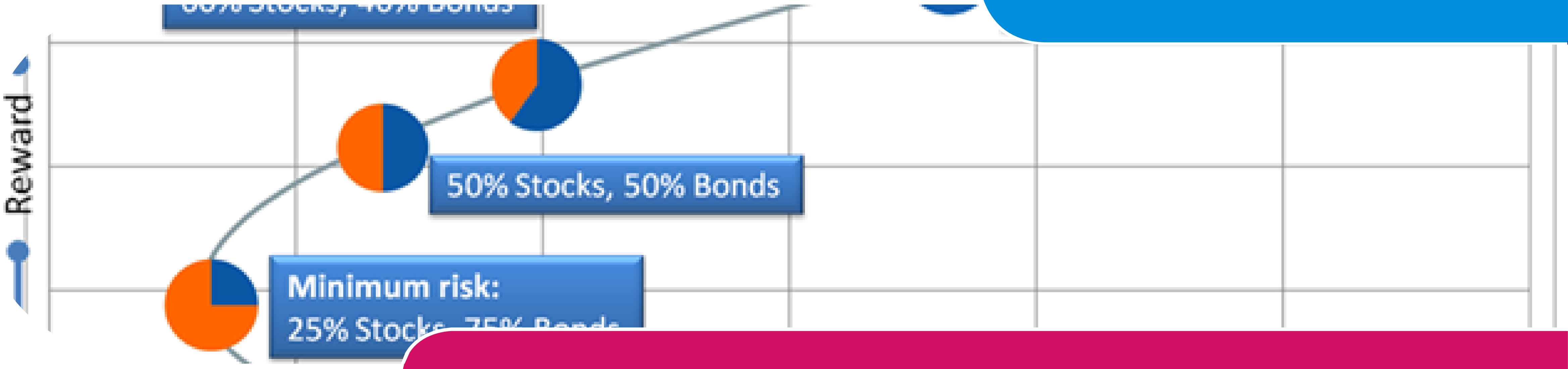
1. Introduction : Portfolio Construction

Minimize risk **subject to threshold return**

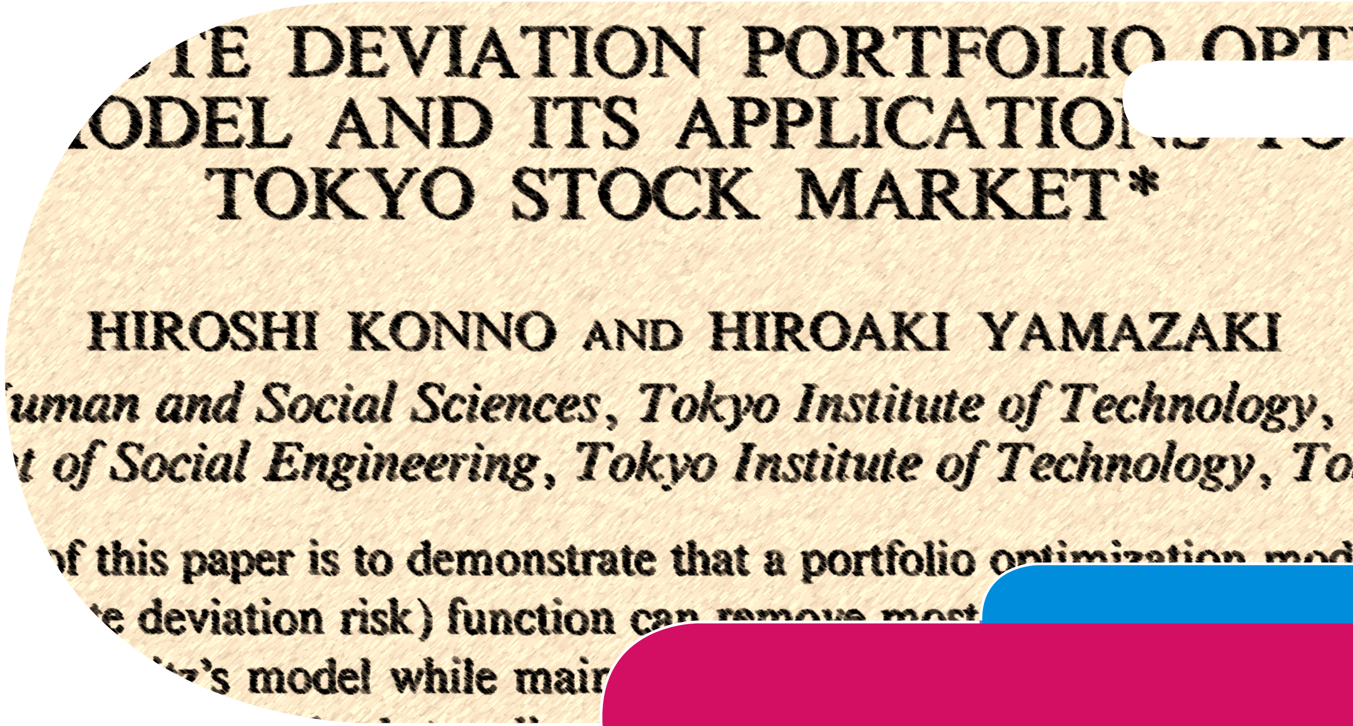


Risk reflects the deviation of the ***exact return (random)*** from the ***expected return (nonrandom)***.

1. Introduction : Software Packages

The *[mean absolute deviation (MAD)]* is employed as a risk of the portfolio.

- This model was proposed by Konno and Yamazaki **in 1991**
- MAD risk = $E[|E[R_{portfolio}] - R_{portfolio}|]$
- MAD risk is an L^1-norm
- The MAD model has appeared in many software packages
  - Commercial: Financial Toolbox in MATLAB software
  - Non-commercial: portfolio.optimization and PortfolioOptim in R statistical software


TE DEVIATION PORTFOLIO OPT
ODEL AND ITS APPLICATIO
TOKYO STOCK MARKET*

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of this paper is to demonstrate that a portfolio optimization mod
e deviation risk) function can remove most
's model while main

$$\text{minimize} \quad \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(\tilde{r}_{i,t} - \boxed{r_i})w_i\right|$$

$$\text{subject to} \quad \sum_{i=1}^{n}\boxed{r_i}w_i \geq \theta,$$

threshold rate of portfolio return (predetermined)

$$\sum_{i=1}^{n} w_i = 1,$$

weight of asset *i*

$$w_i \geq 0, \quad i = 1, 2, \ldots, n$$

Interval MAD Model : $r_i = E\left[R_{i,T+1}\right]$ *is in a return interval* $[\underline{r}_i, \overline{r}_i]$

1. Introduction : Parametric LP

Charitopoulos, Papageorgion and Dua (2017)

- Multi-parametric linear programming under global uncertainty
- Apply KKT conditions to obtain the exact parametric solution with symbolic manipulation
- **Pro :** Clear picture
- **Con :** As more financial assets are included, the algebraic expression becomes more complicated

Witmann-Hohlbein and Pistikopoulos (2013)

- On the global solution of multi-parametric mixed integer programming problems
- Transform LHS uncertainty to RHS uncertainty
- Suppose r = parameter and x = decision variable
- The nonlinear term is rx
- $(r - r_{min})(x - x_{min}) \geq 0$
- $(r - r_{max})(x - x_{max}) \geq 0$
- Only RHS contains parameter
- **Pro :** Optimal solution is expressed in terms of r
- **Con :** Feasibility may be violated and CR partition is required

NO NUMERICAL BOUNDS

2. Interval LP : Overview

Notation

- **Interval vector/matrix :** $\boldsymbol{Y} = [\underline{Y}, \bar{Y}]$ contains infinitely many scenarios (unless *degenerate*)
- **Center :** Y^c
- **Radius :** Y^Δ
- **1-vector :** e and e_i

Feasibility and Solvability : $\boldsymbol{A}x = \boldsymbol{b}$

- **Feasible / Feasibility :** nonnegative solution
- **Solvable :** unrestricted solution
- **Strongly / Strong :** all scenarios
- **Weakly / Weak :** at least one scenario

Interval LP

- **Standard form :** min $\boldsymbol{c}^T x$ subject to $\boldsymbol{A}x = \boldsymbol{b}, x \geq 0$
- **Objective :** compute the set of all *weakly optimal solutions*
- **Range of optimal values :** Minimum and maximum may not exist
- **Basis stability :** same choice of decision variables (no matter what)

2. Interval LP : Feasibility

Theorem 2.4. A system $\boldsymbol{A}x = \boldsymbol{b}$ is strongly feasible if and only if for each $y \in \{\pm 1\}^m$ the system

$$(A^c - \operatorname{diag}(y)A^\Delta)x = b^c + \operatorname{diag}(y)b^\Delta$$

has a nonnegative solution.

Theorem 2.5. The weakly feasible solution set to the system $\boldsymbol{A}x = \boldsymbol{b}$ is given by the set $\{x \mid \underline{A}x \le \overline{b},\ \overline{A}x \ge \underline{b},\ x \ge 0\}$.

Theorem 2.6. The weakly feasible solution set to the system $\boldsymbol{A}x \le \boldsymbol{b}$ is given by the set $\{x \mid \underline{A}x \le \overline{b},\ x \ge 0\}$.

2. Interval LP : Solvability

Theorem 2.4. A system $\boldsymbol{A}x = \boldsymbol{b}$ is strongly feasible if and only if for each $y \in \{\pm 1\}^m$ the system

$$(A^c - \operatorname{diag}(y)A^\Delta)x = b^c + \operatorname{diag}(y)b^\Delta$$

has a nonnegative solution.

2. Interval LP : Minimum Optimal Value

Theorem 2.8. If the minimum of optimal objective values of the interval linear program (2.2) exists, then it is identical to the optimal value of the following linear program:

$$
\begin{aligned}
\text{minimize} \quad & \underline{c}^{\mathsf{T}} x \\
\text{subject to} \quad & \underline{A}x \le \overline{b}, \\
& \overline{A}x \ge \underline{b}, \\
& x \ge 0.
\end{aligned}
\tag{2.3}
$$

2. Interval LP : Maximum Optimal value

Theorem 2.9. If the maximum of optimal objective values of the interval linear program (2.2) exists, then it is identical to

$$\max_{z \in \{\pm 1\}^m} f_z$$

where f_z is the optimal value to the following linear program:

$$\begin{aligned} \text{maximize} \quad & (b^c + \operatorname{diag}(z) b^{\Delta})^{\intercal} y \\ \text{subject to} \quad & (A^c - \operatorname{diag}(z) A^{\Delta})^{\intercal} y \le \overline{c}, \\ & \operatorname{diag}(z) y \ge 0. \end{aligned} \tag{2.4}$$

2. Interval LP : Basis Stability

Theorem 2.11. For the interval linear program (2.2), a basis B is optimal if and only if the following conditions hold:

1. For every $A \in \boldsymbol{A}$ we have A_B is a nonsingular matrix;
2. For every $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ we have $A_B^{-1} b \geq 0$;
3. For every $A \in \boldsymbol{A}$ and $c \in \boldsymbol{c}$ we have $c_N^\intercal - c_B^\intercal A_B^{-1} A_N \geq 0^\intercal$.

2. Interval LP : Basis Stability

Theorem 2.12. The interval linear program (2.2) is nondegenerate B-stable with the optimal basis B if and only if for each $y \in \{\pm 1\}^m$ and $z \in Z_B$ where

$$Z_B = \{z \in \mathbb{R}^n \mid |z_i| = 1 \text{ for } i \in B,\ z_i = 1 \text{ for } i \notin B\},$$

the deterministic linear program

$$\begin{aligned} \text{minimize} \quad & (c^c + \operatorname{diag}(z)c^\Delta)^\mathsf{T} x \\ \text{subject to} \quad & (A^c - \operatorname{diag}(y)A^\Delta \operatorname{diag}(z))x = b^c + \operatorname{diag}(y)b^\Delta, \\ & x \geq 0 \end{aligned}$$

has a nondegenerate basic optimal solution with basic variables x_i where $i \in B$.

3. Lower Returns : SDF

Asset Pricing

- X = payoff (random variable)
- $p(X)$ = price (real-valued)
- **Complete financial market :** X is in L^2-space
- **Law of one price :** $p(X)$ is continuous (i.e. bounded) linear operator
- Then there is an SDF M such that $p(X) = < M, X >$

Rate of Return

- **No arbitrage :** $p(X) > 0$ whenever X is nonnegative and $X = 0$ not almost surely
- Then $1 = < M, X/p(X) > = < M, R > = E[MR]$
- Note that R is arbitrary but M stays the same

Distributions

- **Real-world :** $E_T\left[M_{T+1}R_{i,T+1}\right] = 1$
- **Risk-neutral :** $E_T^*\left[R_{i,T+1}\right]/R_{f,T} = 1$

Negative Correlation Condition (NCC)

- **Definition :** $Cov_T(M_{T+1}R_{i,T+1}, R_{i,T+1}) \leq 0$

Lower Bound on Stock Return

$$\underline{\mathbb{E}_T[R_{i,T+1}]} = R_{f,T} + \frac{2}{S_{i,T}^2}\left[\int_0^{F_{i,T}} \nu_{i,T}^{\text{put}}(K)\,dK + \int_{F_{i,T}}^{\infty} \nu_{i,T}^{\text{call}}(K)\,dK\right]$$

3. Lower Returns : NCC Assessment

Algorithm 3.1: Assess the negative correlation condition (NCC)

Input: The historical market returns $\{R_{m,t}\}_{t=1}^{T}$ and the historical returns $\{r_{i,t}\}_{t=1}^{T}$ of stock i

Output: Whether or not the NCC holds for the given stock i

1: Compute $\mu_i \leftarrow \frac{1}{T}\sum_{t=1}^{T} R_{i,t}$

2: Compute $\mu_m \leftarrow \frac{1}{T}\sum_{t=1}^{T} R_{m,t}$

3: Compute $\delta_i \leftarrow \sum_{t=1}^{T}(R_{i,t}-\mu_i)^2 \Big/ \sum_{t=1}^{T}(R_{m,t}-\mu_m)(R_{i,t}-\mu_i)$

4: **if** $\delta_i \in [1, 10]$ **then**

5: **for** $\gamma \leftarrow 2$ to 10 **do**

6: Let $\rho = \mathrm{Corr}(R_{m,T+1}^{-\gamma} R_{i,T+1}, R_{i,T+1})$

7: Test whether $\rho \le 0$ against $\rho > 0$ at a significance level of 0.05

3. Lower Returns : NCC Assessment

```
        if the null hypothesis is rejected then
            return "The NCC does not hold"
        end if
        if γ = 10 then
            return "The NCC holds"
        end if
    end for
else
    return "The NCC does not hold"
end if
```

3. Lower Returns : Computation

Algorithm 3.2: Lower bound on the expected stock return

Input: The closing stock price S_T at time T, the risk-free rate of return $R_{f,T}$ at time $T+1$ and the sets $\mathcal{P}$ and $\mathcal{C}$ of put and call options on the given stock with the same maturity $T+1$

Output: The lower bound on the expected stock return at time $T+1$ (if computable)

```
Execute Algorithm 3.1 to assess the NCC for the given stock
if the NCC holds then
    Consider the following criteria [9] for a reliable estimate:
        The volume is less than 20 contracts in total
        K_min > 0.8 S_T
        K_max < 1.2 S_T
    if at least one of the criteria above holds then
        return "Uncomputable"
    end if
```

3. Lower Returns : Computation

    if $|\mathcal{P} \cup \mathcal{C}| < 20$ **then**
        return "Uncomputable"
    else
        Sort all elements in $\mathcal{P}$ and $\mathcal{C}$ in ascending order of their strike prices
        for all $e \in \mathcal{P} \cup \mathcal{C}$ **do**
            Calculate the option price $v \leftarrow$ (closing bid price + closing ask price)/2
        end for
        Apply the numerical scheme for the formula (3.4) to obtain the bound
        return the calculated result
    end if
else
    return "Uncomputable"
end if

3. Upper Returns : Computation

Concept

- **Assumption :** buy-and-hold strategy in 6 months
- **Overestimation :** reinvestment in one period
- **Estimates :** reinvest every month and consider the past 3 semiannual periods

4. Portfolio Problem : Overview

General Portfolio Model

$$\begin{aligned}
\text{minimize} \quad & \varrho_T(\mathbb{E}_T[R_{T+1}] - R_{T+1}) \\
\text{subject to} \quad & \mathbb{E}_T[R_{T+1}] \geq \theta, \\
& R_{T+1} = w_1 R_{1,T+1} + w_2 R_{2,T+1} + \ldots + w_n R_{n,T+1}, \\
& w_1 + w_2 + \ldots + w_n = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

Risk Measures

Definiton 4.1 (Subadditive Risk Measure [19]). Let $\mathcal{M}$ be the set of random variables representing portfolio losses. A risk measure $\varrho : \mathcal{M} \to \mathbb{R}$ is called *subadditive* if

$$\varrho(L + S) \leq \varrho(L) + \varrho(S)$$

for all $L, S \in \mathcal{M}$.

Proposition 4.2. The following risk measures are subadditive:

1. The standard deviation (SD) risk measure $\varrho^{\mathrm{SD}}(L) = \sqrt{\mathbb{E}[L^2]}$;
2. The mean absolute deviation (MAD) risk measure $\varrho^{\mathrm{MAD}}(L) = \mathbb{E}[|L|]$.

$$\text{minimize} \quad \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(\tilde{r}_{i,t} - \boxed{r_i})w_i\right|$$

$$\text{subject to} \quad \sum_{i=1}^{n}\boxed{r_i}w_i \geq \theta,$$

threshold rate of portfolio return (predetermined)

$$\sum_{i=1}^{n} w_i = 1,$$

weight of asset i $\longrightarrow$ $w_i \geq 0, \quad i = 1, 2, \ldots, n$

Interval MAD Model **:** *$r_i = E\left[R_{i,T+1}\right]$ is in a return interval $[\underline{r}_i , \overline{r}_i]$*

4. Portfolio Problem : Interval MAD

Not an LP model

historical rate of return
of asset i at time t (known)

$$\begin{aligned} \text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T} \boxed{\left|\sum_{i=1}^{n}(\tilde{r}_{i,t} - r_i)w_i\right|} \\ \text{subject to} \quad & \sum_{i=1}^{n} r_i w_i \geq \theta, \\ & \sum_{i=1}^{n} w_i = 1, \\ & w_i \geq 0, \quad i = 1, 2, \ldots, n \end{aligned}$$

An LP model but with redundant constraints

$$\begin{aligned} \text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T} \boxed{d_t} \\ \text{subject to} \quad & \boxed{d_t} + \sum_{i=1}^{n}(\tilde{r}_{i,t} - r_i)w_i \geq 0, \quad t = 1, 2, \ldots, T, \\ & \boxed{d_t} - \sum_{i=1}^{n}(\tilde{r}_{i,t} - r_i)w_i \geq 0, \quad t = 1, 2, \ldots, T, \\ & \sum_{i=1}^{n} r_i w_i \geq \theta, \\ & \sum_{i=1}^{n} w_i = 1, \\ & w_i \geq 0, \quad i = 1, 2, \ldots, n, \\ & d_t \geq 0, \quad t = 1, 2, \ldots, T. \end{aligned}$$

4. Portfolio Problem : Interval MAD

An LP model with redundant constraints

$$
\begin{aligned}
\text{minimize}\quad & \boxed{\frac{1}{T}\sum_{t=1}^{T} d_t} \\
\text{subject to}\quad & d_t + \sum_{i=1}^{n}(\widetilde{r}_{i,t} - r_i)w_i \ge 0, \quad t = 1,2,\ldots,T, \\
& d_t - \sum_{i=1}^{n}(\widetilde{r}_{i,t} - r_i)w_i \ge 0, \quad t = 1,2,\ldots,T, \\
& \sum_{i=1}^{n} r_i w_i \ge \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \ge 0, \quad i = 1,2,\ldots,n, \\
& d_t \ge 0, \quad t = 1,2,\ldots,T.
\end{aligned}
$$

add surplus variable $2u_t$

add surplus variable $2v_t$

add surplus variable s (excessive return)

At most $2T + 2$ optimal assets

An LP model with full-rank constraint matrix

$$
\begin{aligned}
\text{minimize}\quad & \underbrace{\boxed{\frac{1}{T}\sum_{t=1}^{T}(u_t + v_t)}}_{c^T x} \\
\text{subject to}\quad & -u_t + v_t + \sum_{i=1}^{n}\widetilde{r}_{i,t} w_i - s = \theta, \quad t = 1,2,\ldots,T, \\
& \sum_{i=1}^{n} \boxed{r_i} w_i - s = \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \ge 0, \quad i = 1,2,\ldots,n, \\
& u_t, v_t \ge 0, \quad t = 1,2,\ldots,T, \\
& s \ge 0
\end{aligned}
$$

$c^T x$

interval vector of expected asset returns

$A(\boldsymbol{r} := [\underline{r}, \overline{r}])x = b(\theta)$

Strongly Feasible

constant vector

interval-valued expected return of asset *i*

$x \ge 0$

At most $T + 2$ optimal assets

5. Bilevel (Lui 2011) : Inner Program

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T} d_t \\
\text{subject to} \quad & d_t + \sum_{i=1}^{n} \tilde{r}_{i,t} w_i - \sum_{i=1}^{n} \boxed{r_i w_i} \geq 0, \quad t = 1, 2, \ldots, T, \\
& d_t - \sum_{i=1}^{n} \tilde{r}_{i,t} w_i + \sum_{i=1}^{n} \boxed{r_i w_i} \geq 0, \quad t = 1, 2, \ldots, T, \\
& \sum_{i=1}^{n} \boxed{r_i w_i} \geq \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& d_t \geq 0, \quad t = 1, 2, \ldots, T
\end{aligned}
$$

where

$$
\underline{r}_i \leq r_i \leq \overline{r}_i.
$$

5. Bilevel (Lui 2011) : Lowest-Risk Portfolio

1-level (with nonlinear terms)

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T} d_t \\
\text{subject to} \quad & d_t + \sum_{i=1}^{n} \tilde{r}_{i,t} w_i - \sum_{i=1}^{n} \boxed{r_i w_i} \geq 0, \quad t = 1, 2, \ldots, T, \\
& d_t - \sum_{i=1}^{n} \tilde{r}_{i,t} w_i + \sum_{i=1}^{n} \boxed{r_i w_i} \geq 0, \quad t = 1, 2, \ldots, T, \\
& \sum_{i=1}^{n} \boxed{r_i w_i} \geq \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& d_t \geq 0, \quad t = 1, 2, \ldots, T, \\
& \boxed{\underline{r}_i \leq r_i \leq \overline{r}_i,} \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

1-level (without nonlinear terms)

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T} d_t \\
\text{subject to} \quad & d_t + \sum_{i=1}^{n} \tilde{r}_{i,t} w_i - \sum_{i=1}^{n} \boxed{\eta_i} \geq 0, \quad t = 1, 2, \ldots, T, \\
& d_t - \sum_{i=1}^{n} \tilde{r}_{i,t} w_i + \sum_{i=1}^{n} \boxed{\eta_i} \geq 0, \quad t = 1, 2, \ldots, T, \\
& \sum_{i=1}^{n} \boxed{\eta_i} \geq \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& \boxed{-\underline{r}_i w_i + \eta_i \geq 0,} \quad i = 1, 2, \ldots, n, \\
& \boxed{\overline{r}_i w_i - \eta_i \geq 0,} \quad i = 1, 2, \ldots, n, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& d_t \geq 0, \quad t = 1, 2, \ldots, T, \\
& \eta_i \geq 0, \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

5. Bilevel (Lui 2011) : Highest-Risk Portfolio

1-level Dual of inner (with nonlinear terms)

$$
\begin{array}{lll}
\text{maximize} & \theta y_{2T+1} + y_{2T+2} & \\
\text{subject to} & y_t + y_{T+t} \leq \frac{1}{T}, & t \leq T, \\
& \sum_{t=1}^{T} (\tilde{r}_{i,t} - \boxed{r_i) y_t} - \sum_{t=1}^{T} (\tilde{r}_{i,t} - \boxed{r_i) y_{T+t}} + r_i y_{2T+1} + y_{2T+2} \leq 0, & i \leq n, \\
& y_k \geq 0, & k \leq 2T+1, \\
& \boxed{\underline{r}_i \leq r_i \leq \overline{r}_i,} & i \leq n.
\end{array}
$$

1-level (without nonlinear terms)

$$
\begin{array}{lll}
\text{maximize} & \theta y_{2T+1} + y_{2T+2} & \\
\text{subject to} & y_t + y_{T+t} \leq \frac{1}{T}, & t \leq T, \\
& \sum_{t=1}^{T} [\tilde{r}_{i,t} (y_t - y_{T+t}) - \boxed{\xi_{i,t}} + \boxed{\xi_{i,T+t}}] + \xi_{i,2T+1} + y_{2T+2} \leq 0, & \\
& \boxed{\begin{array}{l} \underline{r}_i y_k - \xi_{i,k} \leq 0, \\ -\overline{r}_i y_k + \xi_{i,k} \leq 0 \end{array}} & \begin{array}{l} k \leq 2T+1, \\ k \leq 2T+1, \end{array} \\
& y_k \geq 0, & k \leq 2T+1, \\
& \xi_{i,k} \geq 0, & k \leq 2T+1.
\end{array}
$$

OVERRELAXATION implies LOOSE BOUND

6. Proposed Algorithms

STEP 1 Validate a given threshold rate of return

- A threshold should be between -100% and the maximum of lower bounds on expected asset returns

$$\underline{\theta} = 0 \quad \text{and} \quad \overline{\theta} = \max_{1 \le i \le n} \{\underline{r}_i\}$$

6. Proposed Algorithms

STEP 2 Assess basis stability of interval MAD problem

- Run LP solver at the center of return interval and obtain nonbasic variables
- If interval LP is B-stable, then
  - The nonbasic variables bind at 0 for every return in the interval
  - All assets in a set of such nonbasic variables are unattractive
  - This is an improvement of STEP 3

Argument 6.2. Basis Stability

Proposition 6.3. Let B and N denote the nondegenerate optimal basis and nonbasis respectively when the MAD portfolio optimization problem (4.8) is solved with a restriction of the interval return $\boldsymbol{r}$ at its center r^c. If every problem derived from (4.8) with the constraint $\boldsymbol{A}(r) = b$ when r_B arbitrarily locates at their end points and r_N lies in the set $\{\underline{r}_N, \overline{r}_N\}$ still has the same optimal basis B with nondegeneracy. Then $w_N^* = 0$ in almost every scenario.

Argument 6.2. Basis Stability

Proof. The sufficient and necessary condition for basis stability is stated in Theorem 2.12. It suffices to consider the system

$$(A^c - \mathrm{diag}(z)A^{\Delta}\mathrm{diag}(p))x = b^c + \mathrm{diag}(z)b^{\Delta} = b$$

where $z \in \{\pm 1\}^{T+2}$ and $p \in \{\pm 1\}^{T+1+n}$ with $p_N = 1^{\intercal}$ or $p_N = -1^{\intercal}$. Note that

$$A^c - \mathrm{diag}(z)A^{\Delta}\mathrm{diag}(p) = A^c - \begin{bmatrix} 0_{T\times(2T+1)} & 0_{T\times n} \\ 0_{1\times(2T+1)} & z_{T+1}\cdot\left(p_{2T+1+i}\cdot r_i^{\Delta}\right)_{1\times n} \\ 0_{1\times(2T+1)} & 0_{1\times n} \end{bmatrix}.$$

The coefficients of those assets lying in the nonbasis N must have the same sign. $\square$

Algorithm 6.2. Basis Stability

Algorithm 6.3: Stability of optimal assets (B-stable)

Input: The portfolio selection model (4.8) with the threshold portfolio return θ

Output: Whether or not optimal assets are preserved

Compute $x \leftarrow \operatorname{argmin}\{c^\intercal x \mid \boldsymbol{A}(r^c)x = b,\ x \geq 0\}$
Construct $B = \{i \mid x_i > 0\}$
Construct $N = \{i \mid x_i = 0\}$
if $|B| \neq T + 2$ **then**
    return "Inconclusive"
end if
Set $\mathcal{O} = \{r \mid r_i \in \{\underline{r}_i, \overline{r}_i\},\ r_N \in \{\underline{r}_N, \overline{r}_N\}\}$
for all $r \in \mathcal{O}$ **do**
    Compute $x \leftarrow \operatorname{argmin}\{c^\intercal x \mid \boldsymbol{A}(r)x = b,\ x \geq 0\}$
    Set $I \leftarrow \{i \mid x_i > 0\}$
    if $I \neq B$ **then**
        return "Inconclusive"
    end if
end for
return "Optimal assets are stable"

6. Proposed Algorithms

STEP 3 Find an enclosure of optimal portfolios

- Find an enclosure of optimal asset allocation
- Find a range of optimal portfolio returns

Argument 6.3. Optimal Portfolios

For any given scenario $A \in \boldsymbol{A}$, an optimal solution $x \in \mathbb{R}^{2T+1+n}$ satisfies

$$Ax = b,\ x \geq 0,\ A^{\intercal} y \leq c,\ c^{\intercal} x = b^{\intercal} y$$

for some $y \in \mathbb{R}^{T+2}$ as a result of the duality theory. A superset of optimal solutions [28] is described by

$$\boldsymbol{A}x = b,\ x \geq 0,\ \boldsymbol{A}^{\intercal} y \leq c,\ c^{\intercal} x = b^{\intercal} y.$$

Argument 6.3. Optimal Portfolios

According to Theorems 2.5 and 2.7, a weak solution x to the region above must meet all of the following requirements:

$$\begin{aligned} \underline{A}x &\leq b \\ -\overline{A}x &\leq -b \\ x &\geq 0 \\ (A^c - z_{T+1}A^{\Delta})^{\intercal} y = (A^c - \operatorname{diag}(z)A^{\Delta})^{\intercal} y &\leq c \\ c^{\intercal}x &= b^{\intercal}y \\ \operatorname{diag}(z)y &\geq 0 \end{aligned}$$

where $z \in \{\pm 1\}^{T+2}$. This representation can be decomposed into a union of two regions between

$$\underline{A}x \leq b,\ -\overline{A}x \leq -b,\ x \geq 0, \underline{A}^{\intercal} y \leq c,\ c^{\intercal}x = b^{\intercal}y,\ e_{T+1}^{\intercal} y \geq 0$$

and

$$\underline{A}x \leq b,\ -\overline{A}x \leq -b,\ x \geq 0, \overline{A}^{\intercal} y \leq c,\ c^{\intercal}x = b^{\intercal}y,\ -e_{T+1}^{\intercal} y \geq 0.$$

Algorithm 6.3. Optimal Portfolios

Algorithm 6.2: Describe optimal portfolios and determine their returns

Input: The portfolio selection model (4.8) with the threshold portfolio return θ

Output: Enclosure $\mathcal{E}$ of optimal portfolios and the range $\mathcal{R}$ of their returns

Set $\mathcal{E}_1, \mathcal{E}_2, \mathcal{R}_1, \mathcal{R}_2 \leftarrow \emptyset$

Let $\mathcal{P}^{(1)}$ be the linear program $\min\{0^\mathsf{T}x + 0^\mathsf{T}y \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0,$
$(\boldsymbol{A}(\underline{r}))^\mathsf{T}y \leq c,\ c^\mathsf{T}x = b^\mathsf{T}y,\ e_{T+1}^\mathsf{T}y \geq 0\}$

if the program $\mathcal{P}^{(1)}$ is feasible **then**

  for $i \leftarrow 0$ **to** n **do**

    Compute $\underline{w}_i^{(1)} \leftarrow \min\{e_{2T+1+i}^\mathsf{T}x \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0,$
$(\boldsymbol{A}(\underline{r}))^\mathsf{T}y \leq c,\ c^\mathsf{T}x = b^\mathsf{T}y,\ e_{T+1}^\mathsf{T}y \geq 0\}$

    Compute $\overline{w}_i^{(1)} \leftarrow \max\{e_{2T+1+i}^\mathsf{T}x \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0,$
$(\boldsymbol{A}(\underline{r}))^\mathsf{T}y \leq c,\ c^\mathsf{T}x = b^\mathsf{T}y,\ e_{T+1}^\mathsf{T}y \geq 0\}$

  end for

  Set $\mathcal{E}_1 \leftarrow [\underline{w}_1^{(1)}, \overline{w}_1^{(1)}] \times \ldots \times [\underline{w}_n^{(1)}, \overline{w}_n^{(1)}]$

  Set $\mathcal{R}_1 \leftarrow \theta + [\underline{w}_0^{(1)}, \overline{w}_0^{(1)}]$

end if

Algorithm 6.3. Optimal Portfolios (ctd.)

Let $\mathcal{P}^{(2)}$ be the linear program $\min\{0^\intercal x + 0^\intercal y \mid \boldsymbol{A}(\underline{r})x \le b,\ \boldsymbol{A}(\overline{r})x \ge b,\ x \ge 0,$
$(\boldsymbol{A}(\overline{r}))^\intercal y \le c,\ c^\intercal x = b^\intercal y,\ -e_{T+1}^\intercal y \ge 0\}$
if the program $\mathcal{P}^{(2)}$ is feasible **then**
  for $i \leftarrow 0$ **to** n **do**
    Compute $\underline{w}_i^{(2)} \leftarrow \min\{e_{2T+1+i}^\intercal x \mid \boldsymbol{A}(\underline{r})x \le b,\ \boldsymbol{A}(\overline{r})x \ge b,\ x \ge 0,$
$(\boldsymbol{A}(\overline{r}))^\intercal y \le c,\ c^\intercal x = b^\intercal y,\ -e_{T+1}^\intercal y \ge 0\}$
    Compute $\overline{w}_i^{(2)} \leftarrow \max\{e_{2T+1+i}^\intercal x \mid \boldsymbol{A}(\underline{r})x \le b,\ \boldsymbol{A}(\overline{r})x \ge b,\ x \ge 0,$
$(\boldsymbol{A}(\overline{r}))^\intercal y \le c,\ c^\intercal x = b^\intercal y,\ -e_{T+1}^\intercal y \ge 0\}$
  end for
  Set $\mathcal{E}_2 \leftarrow [\underline{w}_1^{(2)}, \overline{w}_1^{(2)}] \times \ldots \times [\underline{w}_n^{(2)}, \overline{w}_n^{(2)}]$
  Set $\mathcal{R}_2 \leftarrow \theta + [\underline{w}_0^{(2)}, \overline{w}_0^{(2)}]$
end if
Compute $\mathcal{E} \leftarrow$ Interval hull of $\mathcal{E}_1 \cup \mathcal{E}_2$
Compute $\mathcal{R} \leftarrow$ Interval hull of $\mathcal{R}_1 \cup \mathcal{R}_2$
return $\mathcal{E}$ and $\mathcal{R}$

6. Proposed Algorithms

STEP 4 Find a range of optimal portfolio risks (optional)

- To compare two different portfolios
- Other factors such as return should also be concerned
    - This is because risk simply measure deviation of portfolio return from its expected value

Argument 6.4. Portfolio Risks

Both minimum and maximum of optimal portfolio risks exist

- **Strong feasibility :** Every (min) LP problem always has an optimal value
- **Strong duality theorem :** Every (max) dual LP problem always has an optimal value
- **Minimum of optimal portfolio risks :** Existent because return vectors are compact
- **Maximum of optimal portfolio risks :** Existent because return vectors are compact

Argument 6.4. Portfolio Risks

According to Theorems 2.8 and 2.9, the lowest risk $\underline{\varrho}$ and the highest risk $\overline{\varrho}$ are obtained by the formula

$$\underline{\varrho} = \frac{1}{T} \cdot \min\{\underline{c}^{\mathsf{T}} x \mid \underline{A}x \le \overline{b},\ \overline{A}x \ge \underline{b},\ x \ge 0\}$$
$$\overline{\varrho} = \frac{1}{T} \cdot \max\{\varrho_z \mid z \in \{\pm 1\}^{T+2}\}$$

where

$$\varrho_z = \max\{b^{\mathsf{T}} y \mid (A^c - z_{T+1} A^{\Delta})^{\mathsf{T}} y \le c,\ \mathrm{diag}(z) y \ge 0\}.$$

Yet, z_{T+1} takes on a value of either -1 or 1. The other components of z are unrestricted. The constraint $\mathrm{diag}(z)y \ge 0$ can be reduced to $z_{T+1} y_{T+1} \ge 0$. Hence, the range of portfolio risks can be calculated by the following procedure.

Algorithm 6.4. Portfolio Risks

Algorithm 6.4: Determine the range of optimal portfolio risks

Input: The portfolio selection model (4.8) with T observations and the threshold portfolio return θ

Output: The lowest risk $\underline{\varrho}$ and the highest risk $\overline{\varrho}$

1: Compute $\underline{\varrho} \leftarrow (1/T) \cdot \min\{c^\intercal x \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0\}$

2: Calculate $\overline{\varrho}_1 \leftarrow (1/T) \cdot \max\{b^\intercal y \mid (\boldsymbol{A}(\underline{r}))^\intercal y \leq c,\ e_{T+1}^\intercal y \geq 0\}$

3: Calculate $\overline{\varrho}_2 \leftarrow (1/T) \cdot \max\{b^\intercal y \mid (\boldsymbol{A}(\overline{r}))^\intercal y \leq c,\ -e_{T+1}^\intercal y \geq 0\}$

4: Compare $\overline{\varrho} = \max\{\overline{\varrho}_1, \overline{\varrho}_2\}$

5: **return** $\underline{\varrho}$ and $\overline{\varrho}$

7. Comparison

Numerical results will be illustrated through an example of the S&P 500 stocks, for which their lower bounds on expected returns can be derived (see Kadan and Tang 2017).

6. PROPOSED METHOD

- **Tool :** Interval linear programming
- An **enclosure** of optimal portfolios is obtained at very best.
- **PRO :** ***Rule out*** nonoptimal portfolios.
- **CON :** Enclosure is ***inexact***.

5. BILEVEL OPTIMIZATION (Lui 2011)

- **Tool :** Linear bilevel program
- **Two** optimal portfolios (with feasibility guaranteed) are suggested.
- **PRO :** Portfolios are ***explicitly suggested***.
- **CON :** One of these two is ***infeasible***.

7. Adjusted Return Bounds

$$\overline{r}^{adj}(\mu) = \frac{\mu}{1+\mu} \cdot \underline{r} + \frac{1}{1+\mu} \cdot \overline{r}$$

Example 7.1. Two-Asset Portfolio

Symbols	Lower bound on expected return	Upper bound on expected return
MPC	4.46%	12.15%
DVA	9.61%	9.94%

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
MPC	13.71%	38.49%	N/A	N/A	12.15%	33.51%	4.46%	24.79%	N/A	24.51%
DVA	61.51%	86.29%	N/A	N/A	9.94%	66.49%	9.61%	75.21%	N/A	75.49%
Optimal return	7.63%	10.68%	N/A	N/A		10.68%		8.34%		N/A
Optimal risk	0.0525	0.0593	N/A	N/A		0.0593		0.0525		0.0658

Example 7.1. Two-Asset Portfolio

Symbols	Lower bound on expected return	Upper bound on expected return
MPC	4.46%	12.15%
DVA	9.61%	9.94%

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
MPC	12.15%	12.15%	12.15%	NaN	NaN	12.15%	12.15%	12.15%	NaN	4.46%	NaN	4.46%	4.46%	NaN	NaN	NaN	NaN
DVA	9.94%	9.94%	9.94%	NaN	NaN	9.94%	9.94%	9.94%	NaN	9.61%	NaN	9.61%	9.61%	NaN	NaN	NaN	NaN

Example 7.2. Two-Asset Portfolio

Symbols	Lower bound on expected return	Upper bound on expected return
ALL	3.46%	37.13%
DVA	9.61%	9.94%

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
ALL	0.00%	20.18%	0.00%	0.00%	3.46%	0.00%	NaN	0.00%	N/A	0.00%
DVA	79.82%	100.00%	100.00%	100.00%	9.94%	100.00%	9.61%	100.00%	N/A	100.00%
Optimal return	8.37%	9.94%	9.61%	9.94%		9.94%		9.61%		N/A
Optimal risk	0.0747	0.0772	0.0747	0.0772		0.0772		0.0747		0.0776

Example 7.2. Two-Asset Portfolio

Symbols	Lower bound on expected return	Upper bound on expected return
ALL	3.46%	37.13%
DVA	9.61%	9.94%

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
ALL	37.13%	37.13%	37.13%	NaN	37.13%	37.13%	37.13%	37.13%	NaN	NaN	NaN	3.46%	NaN	NaN	NaN	NaN	NaN
DVA	9.94%	9.94%	9.94%	NaN	9.94%	9.94%	9.94%	9.94%	NaN	NaN	NaN	9.61%	NaN	NaN	NaN	NaN	NaN

Example 7.3.
Alternative Optimal Solution

Table 7.11: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
AAPL	0.00%	23.50%	0.00%	0.00%	2.78%	0.00%	2.78%	3.40%	N/A	0.00%
ADBE	0.00%	9.87%	6.46%	7.17%	3.87%	6.46%	NaN	0.00%	N/A	0.00%
ADSK	0.00%	29.86%	0.00%	0.00%	6.39%	0.00%	NaN	0.00%	N/A	0.00%
AET	0.00%	14.52%	0.00%	0.00%	3.59%	0.00%	NaN	0.00%	N/A	0.00%
ALGN	0.00%	13.75%	0.00%	0.00%	7.87%	0.00%	NaN	0.00%	N/A	0.00%
ALL	0.00%	12.39%	0.00%	0.00%	3.46%	0.00%	NaN	0.00%	N/A	0.00%