SOLVING INTERVAL-VALUED RETURNS MEAN ABSOLUTE DEVIATION PORTFOLIO SELECTION MODEL UNDER BASIS STABILITY

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การหาผลเฉลยของตัวแบบการจัดพอร์ตการลงทุนโดยใช้ค่าเบี่ยงเบนเฉลี่ยสัมบูรณ์
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This work discusses a portfolio selection model with interval linear programming arising from the uncertainty of future rates of return and the disagreement over their estimates. The risk of the overall portfolio is proposed as an objective function to obtain a well-diversified portfolio with a certain threshold rate of return. The optimization problem employs mean absolute deviation as a risk measure for the sake of risk diversification and time complexity. The possible ranges of optimal portfolio returns and associated risks are derived. The duality theory contributes to an enclosure of the optimal portfolios. When the bounds on future rates of return are sufficiently tight, the ambiguity of optimal portfolio compositions can significantly be reduced by basis stability. A theoretical framework for the study of the interval linear programming is also provided. The use of this method is illustrated with the historical returns of S&P 500 stocks, for which the negative correlation condition empirically holds, with a 6-month investment horizon from November 2018 to April 2019. The historical data is collected monthly over the past four years from November 2014 to October 2018. Compared to the bilevel optimization method, our proposed algorithms produce better results in both empirical and theoretical aspects.

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งานชิ้นนี้ศึกษาตัวแบบการจัดพอร์ตการลงทุนแบบช่วงที่มีสาเหตุมาจากความไม่แน่นอนของผลตอบแทนในอนาคต โดยมีฟังก์ชันจุดประสงค์คือความเสี่ยง เพื่อหาพอร์ตที่มีการกระจายความเสี่ยงที่เหมาะสม ภายใต้อัตราผลตอบแทนขั้นต่ำที่กำหนดไว้ ปัญหาการหาค่าเหมาะสมที่สุดนี้ใช้ค่าเฉลี่ยเบี่ยงเบนสัมบูรณ์เป็นมาตรวัดความเสี่ยง โดยคำนึงถึงคุณสมบัติของการกระจายความเสี่ยงและเวลาที่ใช้ประมวลผล คำตอบของช่วงการจัดพอร์ตที่เหมาะสมพร้อมทั้งความเสี่ยงสามารถหาได้จากการอาศัยทฤษฎีควบคู่ ในกรณีที่ข้อมูลเกี่ยวกับผลตอบแทนของสินทรัพย์มีความแม่นยำมากขึ้น แนวคิดของเสถียรภาพเชิงมูลฐานสามารถทำให้ช่วงการจัดพอร์ตแคบลง นอกจากนี้งานชิ้นนี้ยังได้อธิบายกรอบแนวคิดเชิงทฤษฎีของกำหนดการเชิงเส้นแบบช่วงเช่นกัน ผลการศึกษาวิจัยครั้งนี้ถูกนำไปประยุกต์ใช้กับการลงทุนหุ้น S&P 500 ที่สอดคล้องกับความสัมพันธ์สหสัมพัทธ์เชิงลบ โดยมีระยะเวลาลงทุน 6 เดือน ตั้งแต่เดือนพฤศจิกายน 2561 ถึงเดือนเมษายน 2562 อาศัยข้อมูลย้อนหลัง 4 ปี ตั้งแต่เดือนพฤศจิกายน 2557 ถึงเดือนตุลาคม 2561 พบว่าขั้นตอนวิธีที่นำเสนอครั้งนี้ให้ผลลัพธ์ที่ดีกว่า ทั้งในเชิงประจักษ์และเชิงทฤษฎี เมื่อเทียบกับแบบจำลองการหาค่าเหมาะสมที่สุดแบบสองชั้น

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Nomenclature

$\widetilde{r}_{i,t}$	observed rate of return of asset i during time t
$r_i = [\underline{r}_i, \overline{r}_i]$	expected rate of return of asset i in the future (at least $\underline{r}_i$ but no more than $\overline{r}_i$) based on all available information
θ	threshold rate of portfolio return
$\{w_i\}_{i=1}^n$	portfolio composition of n assets
τ	attainable portfolio return
τ^*	optimal portfolio return
$\hat{r}_i$	attainable rate of return of asset i in a bilevel portfolio
μ	parameter for adjustment of upper bounds on expected asset returns

CHAPTER I

INTRODUCTION

Common stock trading has historically appeared since the sixteenth century [1]. Unlike fixed-income securities such as Treasury bonds, corporate bonds, certificates of deposit (CDs) and preferred stocks, the returns of common stocks fluctuate based on the performance of their issuing companies and also on many macroeconomic factors. Investors are chiefly risk-averse. They maximize returns while minimizing associated risks. One of the best ways to deal with this problem is bundling these risky assets together in the hope that the upturn in some stock returns will offset the downturn in the others. This leads to the advent of portfolio.

There are numerous risk measurements of the deviation of realized returns from expected returns. In 1952, Markowitz [2] employed variance or equivalently standard deviation (L^2-norm). However, the required calculation of the covariance matrix may be inexact. With additional portfolio constraints, the L^2-risk model may also take up significantly high computational time. In 1991, Konno and Yamazaki [3] used mean absolute deviation (L^1-norm) and identified the optimal portfolio with a linear program solver which is faster than a nonlinear program solver.

Nonetheless, another problem arises when the exactness of estimation of expected future asset returns is of interest. Various distributions of stock returns have long been investigated [4, 5, 6, 7, 8, 9, 10]. Many estimators of expected returns such as the arithmetic average of historical returns and the CAPM model [11] are also widely used. Hence, these parameters should be represented by ranges of values, not single numerical values. Extensive research studies on the mean absolute deviation (MAD) portfolio model as well as its long-lasting appearance in a wide variety of commercial and non-commercial software packages such as Financial Toolbox™ in MATLAB® software and the packages portfolio.optimization and PortfolioOptim in the R statistical software are also attributed to our investigation into the classical MAD portfolio selection problem with interval-valued returns.

At first glance, the interval model can be solved by the method of parametric linear programming when each interval of expected return is parametrized. The exact parametric solution of a linear program is found via the first-order Karush-Kuhn-Tucker (KKT) conditions with symbolic manipulation [12]. Its breakthrough advantage is to illustrate the direct impact of the expected returns of assets on the optimal portfolio. As more financial assets are included in a portfolio, the symbolic expression becomes more sophisticated and impede interpretation. To alleviate this routine, the parametric solution can instead be approximated by transforming the multiparametric linear programming (mp-LP) problem with left-hand-side (LHS) uncertainty to the mp-LP problem with right-hand-side (RHS) uncertainty with the help of McCormick-type relaxations of bilinear terms [13]. Although its optimal solution function is simplified and becomes more user-friendly, certain of precedented feasibilities may be violated as a consequence of relaxation procedures. Both techniques also have a number of shortcomings. They require a partition of the parametric space into several regions, known as critical regions (CRs). This time-consuming process significantly contributes to depleted memory and system resources of a computer in a high-dimensional portfolio problem. In addition, they do not provide numerical bounds which are realistic and useful for the design and implementation of a policy and also for a decision on investment.

In this work, the interval linear programming is therefore employed as a primary tool for addressing the optimization problem. The model is reformulated for subsequent investigation. The possible ranges of returns and risks across optimal portfolios are provided. An enclosure of optimal portfolio compositions is also suggested. When the return bounds are sufficiently tight, the enclosure can significantly be improved by basis stability at our suggestion. All results are illustrated through an example of the S&P 500 stocks satisfying the NCC compared to the optimal portfolios obtained by the bilevel optimization. Our proposed method gives a more accurate and precise enclosure of optimal portfolios, supported by both empirical evidence and theoretical framework.

The rest is organized as follows. Chapter 2 lays out the theoretical backgrounds on the interval linear programming. Chapter 3 suggests the methods for computing the lower and upper bounds on expected stock returns. The reader may skip this chapter without losing any understanding of proposed algorithms in subsequent chapters. Chapter 4 formulates the MAD portfolio model. Chapter 5 discusses the bilevel portfolio optimization and its limitation. Chapter 6 addresses the portfolio optimization problem with the interval-valued returns. Chapter 7 illustrates numerical results on the S&P 500 stocks. Chapter 8 concludes the work.

CHAPTER II

INTERVAL LINEAR PROGRAMMING

Traditionally, the linear programming involves the problem with deterministic coefficients. Due to the disagreement of measurements for the expected asset returns, intervals are employed as representatives. The theory of interval linear programming is extensively used in Chapter 6 for deriving the optimal portfolios. This chapter paves its theoretical frameworks [14].

The standard linear programming with a decision variable $x \in \mathbb{R}^n$ is usually given by

$$\begin{aligned} \text{minimize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0 \end{aligned} \tag{2.1}$$

where the deterministic parameters $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are called the technological matrix, the right-hand-side vector and the objective coefficient vector, respectively. In addition, the theory of linear programming is applicable when the matrix A presumably has a full row rank. These parameters are normally obtained from the given data. However, with inexact data, they can take on a wide range of values. When their lower and upper bounds are known, they can be viewed as closed intervals. The notations from [14] will be used throughout thus thesis.

An interval matrix (or vector) $\boldsymbol{Y}$ is defined by

$$\boldsymbol{Y} = [\underline{Y}, \overline{Y}] = \{Y \mid \underline{Y} \leq Y \leq \overline{Y}\},$$

each element of which are compared componentwise. The matrices $\underline{Y}$ and $\overline{Y}$ are its lower and upper bound respectively. Its center and radius matrices are given by

$$Y^c = \frac{1}{2}(\underline{Y} + \overline{Y})$$

$$Y^\Delta = \frac{1}{2}(\overline{Y} - \underline{Y})$$

correspondingly. The symbols e and e_i denote the vector, each element of which equals 1, and a unit vector whose the i^{th} component is 1 respectively. For each $x \in \mathbb{R}^n$, its sign vector $\operatorname{sgn}(x)$ is defined by

$$(\operatorname{sgn}(x))_i = \begin{cases} 1 & \text{if } x_i \geq 0, \\ -1 & \text{if } x_i < 0. \end{cases}$$

2.1 Feasibility

The system $Ax = b$ in the standard linear program (2.1) is said to be *feasible* if it possesses at least one nonnegative solution, called *a feasible solution*. When A and b are uncertain, the interval system $\boldsymbol{A}x = \boldsymbol{b}$ is said to be *weakly (strongly) feasible* if the deterministic system $Ax = b$ is feasible for some (all) $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. Each solution is called *a weakly (strongly) feasible solution.* When the equality is replaced by the inequality, the term *feasible* can be defined in the same manner. The trichotomy theorem and the Farkas lemma provided below bring about a criterion for assessing the strong feasibility of the system of interval equations.

Theorem 2.1 (Strong Duality Theorem or Trichotomy Theorem [15])**.** Exactly one of the following statements holds:

1. Both primal and dual problems are feasible and both possess the same optimal value.
2. Both primal and dual problems are infeasible.
3. Exactly one of the problems is infeasible and the other problem has an unbounded objective value.

Theorem 2.2 (Farkas Lemma)**.** The system $Ax = b$ is feasible if and only if for any p we have $A^{\intercal}p \geq 0$ implies $b^{\intercal}p \geq 0$.

Proof. Let $Ax = b$ be feasible with $A^{\intercal}p \geq 0$. With its nonnegative solution x_0, it follows that

$$b^{\intercal}p = (Ax_0)^{\intercal}p = x_0^{\intercal}(A^{\intercal}p) \geq 0.$$

Conversely, suppose the system $Ax = b$ is infeasible. Consider the primal problem

$$\begin{aligned} \text{maximize} \quad & 0^{\intercal}x \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0 \end{aligned}$$

with its dual problem

$$\begin{aligned} \text{minimize} \quad & b^{\intercal}y \\ \text{subject to} \quad & A^{\intercal}y \geq 0. \end{aligned}$$

By the supposition, the primal problem is infeasible. According to the trichotomy theorem, the dual problem has an unbounded objective value. Note that 0 is its feasible solution and its objective value becomes zero. On grounds of the unboundedness, there must be p such that $A^\intercal p \geq 0$ and $b^\intercal p < 0$. $\square$

Remark 2.3. The Farkas lemma can be restated as follows: the system $Ax = b$ is feasible if and only if for any p we have $p^\intercal A \geq 0$ implies $p^\intercal b \geq 0$.

Theorem 2.4. A system $\boldsymbol{A}x = \boldsymbol{b}$ is strongly feasible if and only if for each $y \in \{\pm 1\}^m$ the system

$$(A^c - \operatorname{diag}(y)A^\Delta)x = b^c + \operatorname{diag}(y)b^\Delta$$

has a nonnegative solution.

Proof. It suffices to verify the converse statement. Let $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. For any p with $p^\intercal A \geq 0$, there exists $y \in \{\pm 1\}^m$ such that $-p^\intercal \operatorname{diag}(y) = |p|^\intercal$. By the supposition,

$$(A^c - \operatorname{diag}(y)A^\Delta)x_y = b + \operatorname{diag}(y)b^\Delta$$

for some nonnegative solution x_y. Then

$$\begin{aligned}
p^\intercal b &= p^\intercal[(A^c - \operatorname{diag}(y)A^\Delta)x_y - \operatorname{diag}(y)b^\Delta] \\
&= (p^\intercal A)x_y - p^\intercal[(A - A^c)x_y + \operatorname{diag}(y)(A^\Delta x_y + b^\Delta)] \\
&= (p^\intercal A)x_y - p^\intercal(A - A^c)x_y + |p|^\intercal(A^\Delta x_y + b^\Delta) \\
&\geq (p^\intercal A)x_y - |p|^\intercal A^\Delta x_y + |p|^\intercal(A^\Delta x_y + b^\Delta) \\
&= (p^\intercal A)x_y + |p|^\intercal b^\Delta \\
&\geq 0.
\end{aligned}$$

By the Farkas lemma, the system $Ax = b$ is feasible. $\square$

Theorem 2.5. The weakly feasible solution set to the system $\boldsymbol{A}x = \boldsymbol{b}$ is given by the set $\{x \mid \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0\}$.

Proof. Assume $x \geq 0$ is a weakly feasible solution to the system $\boldsymbol{A}x = \boldsymbol{b}$. That is, $Ax = b$ for some $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. Then

$$\begin{aligned}
\underline{A}x \leq Ax = b \leq \overline{b} \\
\overline{A}x \geq Ax = b \geq \underline{b}.
\end{aligned}$$

Conversely, suppose $x \geq 0$ satisfies the properties $\underline{A}x \leq \overline{b}$ and $\overline{A}x \geq \underline{b}$. Then

$$(A^c - A^\Delta)x = \underline{A}x \leq \overline{b} = b^c + b^\Delta$$
$$(A^c + A^\Delta)x = \overline{A}x \geq \underline{b} = b^c - b^\Delta.$$

It follows that

$$|A^c x - b^c| \leq A^\Delta x + b^\Delta.$$

The inequality implies

$$A^c x - b^c = \text{diag}(y)(A^\Delta x + b^\Delta)$$

for some y whose each entry is not greater than 1 in magnitude. Simplifying the equation yields

$$(A^c - \text{diag}(y)A^\Delta)x = b^c + \text{diag}(y)b^\Delta.$$

Since $|y| \leq e$, we have $A^c - \text{diag}(y)A^\Delta \in \boldsymbol{A}$ and $b^c + \text{diag}(y)b^\Delta \in \boldsymbol{b}$. □

Theorem 2.6. The weakly feasible solution set to the system $\boldsymbol{A}x \leq \boldsymbol{b}$ is given by the set $\{x \mid \underline{A}x \leq \overline{b},\, x \geq 0\}$.

Proof. Assume $x \geq 0$ is a weakly feasible solution to the system $\boldsymbol{A}x \leq \boldsymbol{b}$. Then

$$Ax \leq b$$

for some $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. It follows that

$$\underline{A}x \leq Ax \leq b \leq \overline{b}.$$

Conversely, suppose $x \geq 0$ satisfies the property $\underline{A}x \leq \overline{b}$. Since $\underline{A} \in \boldsymbol{A}$ and $\overline{b} \in \boldsymbol{b}$, it can be concluded that $x \geq 0$ satisfies $\boldsymbol{A}x = \boldsymbol{b}$. □

2.2 Solvability

Solvability is identical to feasibility except the condition of nonnegativity. This section only covers the criterion of weak solvability of inequality constraint. Technically, the system $Ax \leq b$ is said to be *solvable* if it possesses at least one solution. When A and b are uncertain, the interval system $\boldsymbol{A}x \leq \boldsymbol{b}$ is said to be *weakly (strongly) solvable* if the deterministic system $Ax \leq b$ is solvable for some (all) $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$.

Theorem 2.7. The weakly solvable solution set to the system $\boldsymbol{A}x \leq \boldsymbol{b}$ is given by the set $\{x \mid (A^c - A^\Delta \text{diag}(z))x \leq \overline{b},\, \text{diag}(z)x \geq 0,\, z \in \{\pm 1\}^n\}$.

Proof. By the assumption of solvability, x is unrestricted. Consider each orthant of the space $\mathbb{R}^n$. Let $z = \operatorname{sgn}(x)$ and $w = \operatorname{diag}(z)x$. Then

$$w = \operatorname{diag}(z)x \geq 0.$$

With $x = \operatorname{diag}(z)w$, the system $\boldsymbol{A}x \leq \boldsymbol{b}$ becomes $(\boldsymbol{A}\operatorname{diag}(z))w \leq \boldsymbol{b}$. According to Theorem 2.6, this interval system of inequalities is feasible only when

$$(\underline{A\operatorname{diag}(z)})w \leq \overline{b}.$$

By the investigation of the identity

$$\begin{aligned}(A^c - A^{\Delta}\operatorname{diag}(z))x &= (A^c - A^{\Delta}\operatorname{diag}(z))(\operatorname{diag}(z)w) \\ &= (A^c\operatorname{diag}(z) - A^{\Delta})w \\ &= (\underline{A\operatorname{diag}(z)})w,\end{aligned}$$

the explicit description of the weakly solvable solution set is derived. □

2.3 Range of Optimal Values

Consider an interval linear programming problem

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathsf{T}}x \\ \text{subject to} \quad & \boldsymbol{A}x = \boldsymbol{b}, \\ & x \geq 0. \end{aligned} \tag{2.2}$$

Usually, the primary concern is to compute the set of all *weakly optimal solutions*, each of which is optimal to at least one deterministic linear program (2.1) with the parameters $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$. Arising from the uncertainties, there is a wide range of these optimal values. Its minimum and maximum are not necessarily existent [16]. For example, consider the interval linear program

$$\begin{aligned} \text{minimize} \quad & -x \\ \text{subject to} \quad & [0,1]x = 1, \\ & x \geq 0. \end{aligned}$$

A weakly feasible solution is $x = 1/t$ where $t \in (0, 1]$. The corresponding optimal value $-1/t$ is not bounded below and therefore has no minimum value.

Assume the minimum and maximum exist. Both can be obtained by the minimin and maximin problems. Alternatively, the following theorems provide how to calculate the extreme values without the aid of nonlinear programming.

Theorem 2.8. If the minimum of optimal objective values of the interval linear program (2.2) exists, then it is identical to the optimal value of the following linear program:

$$\begin{aligned} \text{minimize} \quad & \underline{c}^\mathsf{T} x \\ \text{subject to} \quad & \underline{A}x \le \overline{b}, \\ & \overline{A}x \ge \underline{b}, \\ & x \ge 0. \end{aligned} \tag{2.3}$$

Proof. This theorem holds because the weakly feasible solution set to the interval linear program (2.2) and the constraint set given in the linear program (2.3) are identical as a result of Theorem 2.5. □

Theorem 2.9. If the maximum of optimal objective values of the interval linear program (2.2) exists, then it is identical to

$$\max_{z \in \{\pm 1\}^m} f_z$$

where f_z is the optimal value to the following linear program:

$$\begin{aligned} \text{maximize} \quad & (b^c + \operatorname{diag}(z) b^\Delta)^\mathsf{T} y \\ \text{subject to} \quad & (A^c - \operatorname{diag}(z) A^\Delta)^\mathsf{T} y \le \overline{c}, \\ & \operatorname{diag}(z) y \ge 0. \end{aligned} \tag{2.4}$$

Proof. By the supposition, the optimal objective values of the primal problem (2.2) and the corresponding dual problem are equal. The dual problem is given by

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{b}^\mathsf{T} y \\ \text{subject to} \quad & \boldsymbol{A}^\mathsf{T} y \le \boldsymbol{c}. \end{aligned} \tag{2.5}$$

Since y is unrestricted, each orthant of $\mathbb{R}^m$ is considered. Let $z = \operatorname{sgn}(y)$ and $w = \operatorname{diag}(z) y$. Then

$$w = \operatorname{diag}(z) y \ge 0.$$

With $y = \operatorname{diag}(z) w$, the interval linear program (2.5) becomes

$$\begin{aligned} \text{maximize} \quad & (\operatorname{diag}(z)\boldsymbol{b})^\mathsf{T} w \\ \text{subject to} \quad & (\operatorname{diag}(z)\boldsymbol{A})^\mathsf{T} w \le \boldsymbol{c}, \\ & w \ge 0. \end{aligned} \tag{2.6}$$

The maximum of the optimal value is

$$\begin{aligned}(\overline{\mathrm{diag}(z)b})^\intercal w &= (\mathrm{diag}(z)b^c + b^\Delta)^\intercal w\\ &= (b^c + \mathrm{diag}(z)b^\Delta)^\intercal(\mathrm{diag}(z)w)\\ &= (b^c + \mathrm{diag}(z)b^\Delta)^\intercal y.\end{aligned}$$

By Theorem 2.6, the constraint can be interpreted as the weakly feasible solution set to the system

$$\begin{aligned}(A^c - \mathrm{diag}(z)A^\Delta)^\intercal y &= (A^c - \mathrm{diag}(z)A^\Delta)^\intercal(\mathrm{diag}(z)w)\\ &= (\mathrm{diag}(z)A^c - A^\Delta)^\intercal w\\ &= (\underline{\mathrm{diag}(z)A})^\intercal w\\ &\leq \overline{c}.\end{aligned}$$

The maximum of the optimal value of the original interval linear program (2.2) is simply obtained by maximizing the optimal value of the program (2.6) over all possible orthants of the space $\mathbb{R}^m$. □

2.4 Basis Stability

If the interval linear program (2.2) is weakly feasible, then it can have many weakly optimal solutions. Their numerical values may not be of interest. However, the perception of which basic variables are optimal may instead be of value. When the program has the same optimal basis despite the perturbations of the coefficients in $\boldsymbol{A}, \boldsymbol{b}$ and $\boldsymbol{c}$, it emphasizes the importance of all characteristics declared by basic variables. This phenomenon is called *basis stability*.

Denote by B and N respectively the sets of basic and nonbasic variables, also called basis and nonbasis for clarity and brevity. To distinguish between basic and nonbasic components of the coefficients A, b and c, the use of subscripts B and N is suggested.

Definiton 2.10 (B-stable [17]). For a given basis B, the interval linear program is called *B-stable* if B serves as an optimal basis for each scenario of interval values. The program is called nondegenerate B-stable if each scenario has a nondegenerate optimal basic solution with the basis B.

The subsequent theorems [17, 18] illustrate the verification of B-stability. Their proofs are omitted here for convenience.

Theorem 2.11. For the interval linear program (2.2), a basis B is optimal if and only if the following conditions hold:

1. For every $A \in \boldsymbol{A}$ we have A_B is a nonsingular matrix;
2. For every $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ we have $A_B^{-1} b \geq 0$;
3. For every $A \in \boldsymbol{A}$ and $c \in \boldsymbol{c}$ we have $c_N^\intercal - c_B^\intercal A_B^{-1} A_N \geq 0^\intercal$.

Theorem 2.12. The interval linear program (2.2) is nondegenerate B-stable with the optimal basis B if and only if for each $y \in \{\pm 1\}^m$ and $z \in Z_B$ where

$$Z_B = \{z \in \mathbb{R}^n \mid |z_i| = 1 \text{ for } i \in B,\ z_i = 1 \text{ for } i \notin B\},$$

the deterministic linear program

$$\begin{aligned} \text{minimize} \quad & (c^c + \operatorname{diag}(z) c^\Delta)^\intercal x \\ \text{subject to} \quad & (A^c - \operatorname{diag}(y) A^\Delta \operatorname{diag}(z)) x = b^c + \operatorname{diag}(y) b^\Delta, \\ & x \geq 0 \end{aligned}$$

has a nondegenerate basic optimal solution with basic variables x_i where $i \in B$.

CHAPTER III

BOUND ON EXPECTED STOCK RETURNS

This thesis represents expected asset returns by intervals. In Chapter 7, only stocks which empirically satisfy the negative correlation condition (NCC) serve as the examples of financial assets because their lower bounds on expected returns are computable [19]. They fluctuate in nature, increasing with CAPM beta [11] and decreasing with momentum [20]. The upper bounds can be gauged by the reinvestment returns during previous holding periods. A brief theoretical overview of the lower bounds on expected stock returns [19, 21] is included here.

3.1 Stochastic Discount Factor

The main purpose of this section is to provide an existence proof of stochastic discount factor as defined subsequently in the setting of continuous events as an extension of discrete events [22]. In a complete financial market, an asset is characterized by a random variable of payoff $X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ belonging to an L^2-space. Assume a real-valued price function p is a continuous, or equivalently bounded, linear operator on the Hilbert space of payoffs. This property is called *law of one price*. By the Riesz–Fréchet representation theorem, there exists a unique payoff M such that

$$p(X) = < M, X > = \mathbb{E}[MX]$$

for any payoff X. The stochastic process M is called a *stochastic discount factor (SDF)* because it discounts all payoffs of assets to their prices.

Under the assumption of no arbitrage, $p(X)$ is always positive whenever X is a nonnegative-valued payoff and $X = 0$ not almost surely. Since this thesis involves empirical data with T historical observations, the subscripts T and $T+1$ of current time and future time are introduced. Denote by $\mathbb{E}_T[\,\cdot\,]$ an expectation function conditioned on all available information by time T. For any individual asset i, a stochastic discount factor M_{T+1} discounts a stochastic rate of return

$$R_{i,T+1} = \frac{X_{i,T+1}}{p_T(X_{i,T+1})}$$

from time T to $T+1$ as if it were equivalent to investing in a risk-free asset with a deterministic rate of return $R_{f,T}$ fixed at time T. Mathematically,

$$\mathbb{E}_T[M_{T+1}R_{i,T+1}] = 1 \tag{3.1}$$

under a real-world distribution or

$$\frac{1}{R_{f,T}}\mathbb{E}^*_T[R_{i,T+1}] = 1 \tag{3.2}$$

under a risk-neutral distribution denoted by the superscript $*$. The equations (3.1) and (3.2) are used to interchange these two notations as follows:

$$\mathbb{E}^*_T[R_{i,T+1}] = R_{f,T}\mathbb{E}_T[M_{T+1}R_{i,T+1}]. \tag{3.3}$$

3.2 Estimation of Lower Bound on Return

Definiton 3.1 (Negative Correlation Condition)**.** An asset i satisfies *the negative correlation condition (NCC)* if

$$\mathrm{Cov}_T(M_{T+1}R_{i,T+1}, R_{i,T+1}) \leq 0.$$

Under the assumption of NCC, the lower bound on expected return can be obtained by employing the following theorems.

Theorem 3.2. An asset i for which the NCC holds possesses its lower bound on expected return via the inequality

$$\mathbb{E}_T[R_{i,T+1}] \geq R_{f,T} + \frac{\mathrm{Var}^*_T[R_{i,T+1}]}{R_{f,T}}$$

where $\mathrm{Var}^*_T(\cdot)$ is a risk-neutral variance.

Proof. Consider

$$\begin{aligned}
\text{R.H.S.} - \text{L.H.S.} &= R_{f,T} + \frac{\mathbb{E}^*_T[R^2_{i,T+1}] - (\mathbb{E}^*_T[R_{i,T+1}])^2}{R_{f,T}} - \mathbb{E}_T[R_{i,T+1}] \\
&= R_{f,T} + \frac{R_{f,T}\mathbb{E}_T[M_{T+1}R^2_{i,T+1}] - R^2_{f,T}}{R_{f,T}} - \mathbb{E}_T[R_{i,T+1}] \\
&= R_{f,T} + (\mathbb{E}_T[M_{T+1}R^2_{i,T+1}] - R_{f,T}) - \mathbb{E}_T[R_{i,T+1}] \\
&= \mathbb{E}_T[M_{T+1}R^2_{i,T+1}] - \mathbb{E}_T[M_{T+1}R_{i,T+1}]\mathbb{E}_T[R_{i,T+1}] \\
&= \mathrm{Cov}_T(M_{T+1}R_{i,T+1}, R_{i,T+1}) \\
&\leq 0.
\end{aligned}$$

The second equality is followed by the equations (3.2) and (3.3). The inequality is obtained by the definition of NCC. $\square$

Theorem 3.3. Let $S_{i,T}$ and $F_{i,T}$ denote the price of asset i at time T and its forward price at time T for delivery at time $T+1$ respectively. Then the risk-neutral conditional variance $\mathrm{Var}_T^*[R_{i,T+1}]$ can be found by the identity

$$\frac{\mathrm{Var}_T^*[R_{i,T+1}]}{R_{f,T}} = \frac{2}{S_{i,T}^2}\left[\int_0^{F_{i,T}} \nu_{i,T}^{\mathrm{put}}(K)\, dK + \int_{F_{i,T}}^{\infty} \nu_{i,T}^{\mathrm{call}}(K)\, dK\right]$$

across European call and put options on asset i with different strike prices K but the same maturity $T+1$. Both are priced at $\nu_{i,T}^{\mathrm{put}}(K)$ and $\nu_{i,T}^{\mathrm{call}}(K)$ under the assumption of no arbitrage.

Proof. Consider

$$\begin{aligned}
\frac{1}{R_{f,T}}\mathrm{Var}_T^*[R_{i,T+1}] &= \frac{1}{S_{i,T}^2}\left[\frac{1}{R_{f,T}}\mathbb{E}_T^*[S_{i,T+1}^2] - \frac{1}{R_{f,T}}(\mathbb{E}_T^*[S_{i,T+1}])^2\right] \\
&= \frac{1}{S_{i,T}^2}\left[2\int_0^{\infty}\frac{1}{R_{f,T}}\mathbb{E}_T^*[(S_{i,T+1}-K)^+]\, dK - \frac{F_{i,T}^2}{R_{f,T}}\right] \\
&= \frac{1}{S_{i,T}^2}\left[2\int_0^{\infty}\nu_{i,T}^{\mathrm{call}}(K)\, dK - \frac{F_{i,T}^2}{R_{f,T}}\right].
\end{aligned}$$

By the put-call parity formula [23, 24, 25],

$$\begin{aligned}
\int_0^{\infty}\nu_{i,T}^{\mathrm{call}}(K)\, dK &= \int_0^{F_{i,T}}\nu_{i,T}^{\mathrm{call}}(K)\, dK + \int_{F_{i,T}}^{\infty}\nu_{i,T}^{\mathrm{call}}(K)\, dK \\
&= \int_0^{F_{i,T}}\left[\nu_{i,T}^{\mathrm{put}}(K) + \frac{1}{R_{f,T}}(F_{i,T}-K)\right] dK + \int_{F_{i,T}}^{\infty}\nu_{i,T}^{\mathrm{call}}(K)\, dK.
\end{aligned}$$

Simplifying the integral completes the proof. □

The lower bound on expected return of asset i for which the NCC holds is given by Theorems 3.2 and 3.3 as follows:

$$\underline{\mathbb{E}_T[R_{i,T+1}]} = R_{f,T} + \frac{2}{S_{i,T}^2}\left[\int_0^{F_{i,T}} \nu_{i,T}^{\mathrm{put}}(K)\, dK + \int_{F_{i,T}}^{\infty} \nu_{i,T}^{\mathrm{call}}(K)\, dK\right]. \tag{3.4}$$

The second term of this lower bound is approximated via the following numerical scheme. Assume there are N_{put} put options with strike prices not greater than $F_{i,T}$ and N_{call} call options with strike prices not less than $F_{i,T}$. All options are labeled in ascending order of their corresponding strike prices, i.e.

$$K_1^{\mathrm{put}} < \ldots < K_{N_{\mathrm{put}}}^{\mathrm{put}} < F_{i,T} < K_1^{\mathrm{call}} < \ldots < K_{N_{\mathrm{call}}}^{\mathrm{call}}.$$

The numerical estimate of the bracket on the right-hand side of the equation (3.4) is given by

$$\sum_{i=1}^{N_{\text{put}}-1} \nu_i^{\text{put}}(K_{i+1}^{\text{put}} - K_i^{\text{put}}) + \sum_{i=1}^{N_{\text{call}}-1} \nu_i^{\text{call}}(K_{i+1}^{\text{call}} - K_i^{\text{call}}) + (K_1^{\text{call}} - K_{N_{\text{put}}}^{\text{put}}) \min\{\nu_{N_{\text{put}}}^{\text{put}}, \nu_1^{\text{call}}\}.$$

Although the lower bound is theoretically approximated by the European options, in practice the American options are used for individual stocks in the U.S. financial market.

3.2.1 Assessment of Negative Correlation Condition

3.2.1.1 Theoretical Framework

This section covers how to determine whether or not the negative correlation condition (NCC) holds for a particular stock. In this work, an investor believes in the MAD model to construct an optimal portfolio by minimizing risk when no short sales are allowed ($w_i \geq 0$). Assume multiple agents who manipulate the stock market maximize their satisfactions such as return via the utility function u with the properties that $u' > 0$ and $u'' < 0$ when short sales are allowed ($w_i \in \mathbb{R}$). That is, they wants to maximize the term

$$\mathbb{E}_T\left[u\left(\sum_i w_i R_{i,T+1}\right)\right] = \mathbb{E}_T\left[u\left(\sum_i w_i R_{i,T+1}\right)\right] - \lambda\left(\sum_i w_i - 1\right)$$

where $\lambda > 0$ is a Lagrange multiplier. The first-order condition yields

$$\mathbb{E}_T\left[u'(R_{m,T+1})R_{i,T+1}\right] = \lambda$$

for all stock i in the market portfolio

$$R_{m,T+1} = \sum_i w_i R_{i,T+1}.$$

Then

$$\mathbb{E}_T\left[\frac{u'(R_{m,T+1})}{\lambda} R_{i,T+1}\right] = 1.$$

Hence the stochastic discount factor M_{T+1} can be expressed as

$$M_{T+1} = \frac{u'(R_{m,T+1})}{\lambda}.$$

Definiton 3.4 (Coefficient of Relative Risk Aversion [26, 27])**.** Let u be a utility function. *A coefficient of relative risk aversion* at wealth w is defined by

$$\gamma(w) = -w\frac{u''(w)}{u'(w)}.$$

A higher value of γ implies more risk aversion. This is useful for empirical testing of the negative correlation condition on a particular stock as illustrated subsequently.

Theorem 3.5. Up to the first-order Taylor approximation, the negative correlation condition (NCC) holds for asset i with the property that $\mathrm{Cov}_T(R_{m,T+1}, R_{i,T+1}) > 0$ if

$$\gamma(\mathbb{E}_T[R_{m,T+1}]) \geq \frac{\mathrm{Var}_T[R_{i,T+1}]}{\mathrm{Cov}_T(R_{m,T+1}, R_{i,T+1})}.$$

Proof. The NCC holds when

$$\text{L.H.S.} := \mathrm{Cov}_T(u'(R_{m,T+1})R_{i,T+1}, R_{i,T+1}) \leq 0 \cdot \lambda = 0.$$

The marginal utility u' as a multivariate function of individual stock returns is approximated by the Taylor expansion around expected stock returns:

$$\begin{aligned} u'(R_{m,T+1}) &= u'\left(\sum_j w_j R_{j,T+1}\right) \\ &\approx u'\left(\sum_j w_j \mathbb{E}_T[R_{j,T+1}]\right) \\ &\quad + \sum_j u''\left(\sum_j w_j \mathbb{E}_T[R_{j,T+1}]\right) w_j (R_{j,T+1} - \mathbb{E}_T[R_{j,T+1}]) \\ &= u'(\mathbb{E}_T[R_{m,T+1}]) + u''(\mathbb{E}_T[R_{m,T+1}])(R_{m,T+1} - \mathbb{E}_T[R_{m,T+1}]). \end{aligned}$$

Then

$$\begin{aligned} \text{L.H.S.} &\approx u'(\mathbb{E}_T[R_{m,T+1}])\mathrm{Var}_T[R_{i,T}] \\ &\quad + \mathbb{E}_T[R_{m,T+1}]u''(\mathbb{E}_T[R_{m,T+1}])\mathrm{Cov}_T(R_{m,T+1}, R_{i,T+1}) \\ &= u'(\mathbb{E}_T[R_{m,T+1}])[\mathrm{Var}_T[R_{i,T+1}] \\ &\quad - \gamma(\mathbb{E}_T[R_{m,T+1}])\mathrm{Cov}_T(R_{m,T+1}, R_{i,T+1})]. \end{aligned}$$

Since $u' > 0$ and $\mathrm{Cov}_T(R_{m,T+1}, R_{i,T+1})] > 0$, the desired inequality is obtained. ∎

Remark 3.6. By Theorem 3.5, the NCC holds when the relative risk aversion is high compared with the ratio of the asset return's variance to its covariance with the market return. For convenience of NCC testing, this ratio is denoted by

$$\delta_{i,T} = \frac{\mathrm{Var}_T[R_{i,T+1}]}{\mathrm{Cov}_T(R_{m,T+1}, R_{i,T+1})}.$$

The result from Theorem 3.5 can be restated as

$$\gamma(\mathbb{E}_T[R_{m,T+1}]) \geq \delta_{i,T}.$$

3.2.1.2 Empirical Assessment

The subscript T of the term $\delta_{i,T}$ can be omitted because the present time T is fixed. The NCC holds when $\gamma \geq \delta_i$. For a positive value of δ_i, the return of an asset is positively correlated to that of the market. Hence, an asset is highly likely to satisfy the NCC when the ratio δ_i is relatively low, but still positive. The estimate $\hat{\delta}_i$ between 1 and 10 is claimed to guarantee the property of NCC [19].

To ensure this testing, a parametric statistical procedure is taken into consideration. With the Pearson correlation coefficient $\hat{\rho} = \mathrm{Corr}(M_T R_{i,T+1}, R_{i,T+1})$, the null hypothesis $H_0 : \rho \leq 0$ against the alternative hypothesis $H_1 : \rho > 0$ should not be rejected by a standard t-test. In this work, the constant relative risk aversion (CRRA) model [26, 27] is applied as a particular choice of the stochastic discount factor M_T.

The CRRA utility function is given by

$$u(w) = \begin{cases} \dfrac{w^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 0 \text{ and } \gamma \neq 1, \\ \ln \gamma & \text{if } \gamma = 1 \end{cases}$$

where γ is the degree of relative risk aversion. The stochastic discount factor becomes

$$M_T^{\mathrm{CRRA}} = \frac{u'(R_{m,T+1})}{\lambda} = \frac{R_{m,T+1}^{-\gamma}}{\lambda}$$

where $\gamma > 0$ and $\gamma \neq 1$. The Pearson correlation coefficient can be written as

$$\hat{\rho} = \mathrm{Corr}\left(\frac{R_{m,T+1}^{-\gamma}}{\lambda} \cdot R_{i,T+1}, R_{i,T+1}\right) = \mathrm{Corr}(R_{m,T+1}^{-\gamma} R_{i,T+1}, R_{i,T+1}).$$

On the purpose of NCC assessment, the values of γ should discretely vary from 2 to 10.

3.2.1.3 Algorithms

Algorithms 3.1 and 3.2 recapitulate the criteria used for assessing the NCC of a particular stock and the procedures for computing a lower bound on its expected future return respectively.

Algorithm 3.1: Assess the negative correlation condition (NCC)

Input: The historical market returns $\{R_{m,t}\}_{t=1}^{T}$ and the historical returns $\{r_{i,t}\}_{t=1}^{T}$ of stock i

Output: Whether or not the NCC holds for the given stock i

Compute $\mu_i \leftarrow \frac{1}{T}\sum_{t=1}^{T} R_{i,t}$
Compute $\mu_m \leftarrow \frac{1}{T}\sum_{t=1}^{T} R_{m,t}$
Compute $\delta_i \leftarrow \sum_{t=1}^{T}(R_{i,t}-\mu_i)^2 \Big/ \sum_{t=1}^{T}(R_{m,t}-\mu_m)(R_{i,t}-\mu_i)$
if $\delta_i \in [1, 10]$ **then**
  for $\gamma \leftarrow 2$ to 10 **do**
    Let $\rho = \mathrm{Corr}(R_{m,T+1}^{-\gamma} R_{i,T+1}, R_{i,T+1})$
    Test whether $\rho \le 0$ against $\rho > 0$ at a significance level of 0.05
    if the null hypothesis is rejected **then**
      return “The NCC does not hold”
    end if
    if $\gamma = 10$ **then**
      return “The NCC holds”
    end if
  end for
else
  return “The NCC does not hold”
end if

In Algorithm 3.1, the expected return is estimated from the sample mean of historical returns to calculate the Pearson correlation coefficient. Despite the inaccuracy of estimating, this algorithm is still reliable because it primarily serves as the guideline to compute the lower bound (not the exact value) of the expected return of an individual asset.

Algorithm 3.2: Lower bound on the expected stock return

Input: The closing stock price S_T at time T, the risk-free rate of return $R_{f,T}$ at time $T+1$ and the sets $\mathcal{P}$ and $\mathcal{C}$ of put and call options on the given stock with the same maturity $T+1$

Output: The lower bound on the expected stock return at time $T+1$ (if computable)

1: Execute Algorithm 3.1 to assess the NCC for the given stock
2: **if** the NCC holds **then**
3:   Consider the following criteria [19] for a reliable estimate:
4:       The volume is less than 20 contracts in total
5:       $K_{min} > 0.8S_T$
6:       $K_{max} < 1.2S_T$
7:   **if** at least one of the criteria above holds **then**
8:     **return** "Uncomputable"
9:   **end if**
10:   **if** $|\mathcal{P} \cup \mathcal{C}| < 20$ **then**
11:     **return** "Uncomputable"
12:   **else**
13:     Sort all elements in $\mathcal{P}$ and $\mathcal{C}$ in ascending order of their strike prices
14:     **for all** $e \in \mathcal{P} \cup \mathcal{C}$ **do**
15:       Calculate the option price $v \leftarrow$ (closing bid price + closing ask price)/2
16:     **end for**
17:     Apply the numerical scheme for the formula (3.4) to obtain the bound
18:     **return** the calculated result
19:   **end if**
20: **else**
21:   **return** "Uncomputable"
22: **end if**

In Algorithm 3.2, the condition $|\mathcal{P} \cup \mathcal{C}| < 20$ in line 10 is used to ensure that there are sufficient number of options so that the numerical scheme in line 17 is not overly coarse.

3.3 Estimation of Upper Bound on Return

With the buy-and-hold strategy, the investor cannot reinvest the capital gains during the holding period. The returns from reinvestment during the previous k periods (for a suitable choice of $1 \leq k \leq T$) therefore serves as an appropriate upper bound on expected returns. Note that this method requires no assumption of negative correlation condition (NCC).

3.4 Stock Market Indexes

In the global financial system, there are various types of stock market indexes based on different weightings and components that have been changed over time. The main purpose is to indicate the performance of the overall economy as well as some particular sectors and to serve as portfolio benchmarks [20]. For example, the Standard & Poor's 500 (S&P 500) index comprises 505 common stocks issued by 500 large-cap companies operating in the United States. It is commonly used as a broad market indicator of U.S. equities. On the other hand, the Dow Jones Industrial Average (DJIA) includes 30 companies in the United States. Higher-priced stocks have larger impacts on the index's movements. Nevertheless, the analysis in this work is restricted to S&P 500 stocks based on the assumption that the S&P 500 represents the entire stock market due to its broadness. In fact, it covers all 11 different U.S. sectors: communication services, consumer discretionary, consumer staples, energy, financials, health care, industrials, information technology, materials, real estate and utilities.

3.5 Examples

In this thesis, all historical data were gathered in October 2018 from Yahoo! Finance via the R statistical software. For an individual stock i listed in the S&P 500 index, the sequence of monthly log-returns $\vartheta_{i,t}^{k}$ (where $1 \leq t \leq 8$ and $1 \leq k \leq 6$) is observed over the past four years from November 2014 to October 2018. Its historical return during the 6-month period t becomes

$$\widetilde{r}_{i,t} = \exp\left[\sum_{k=1}^{6} \vartheta_{i,t}^{k}\right].$$

The 6-month U.S. Treasury bill quoted on October 12, 2018 is used as the risk-free rate $R_{f,T}$ which equals 2.38%.

Among 505 constitutes, 12 companies have not been listed in the S&P 500 index throughout the entire period and 2 additional companies have no option expiring in April 2019. It turns out that 243 out of 491 remaining stocks empirically follow the NCC condition according to Algorithm 3.1. Finally, 61 stocks in total as provided in Table 3.1 pass the screens established in Algorithm 3.2. Interestingly, two U.S. sectors are excluded: communication services and consumer staples. Their lower bounds for expected returns accumulated from November 2018 to April 2019 are therefore computable. To calculate their upper bounds as stated in Section 3.3 , the reinvestments over the past three periods are considered. In other words, the parameter $k = 3$ is selected. It can be simply calculated by

$$\overline{\mathbb{E}_T[R_{i,T+1}]} = \max_{6 \le t \le 8} \left\{ \exp\left[\sum_{k=1}^{6} \max(\vartheta_{i,t}^k, 0) \right] \right\}.$$

All historical returns and estimates are reported in Tables 3.2 and 3.3 respectively.

Table 3.1: Tickers, company names and sectors of S&P 500 stocks exhibiting the negative correlation condition

Ticker	Company	Sector
AAPL	Apple Inc.	Information Technology
ADBE	Adobe Systems Incorporated	Information Technology
ADSK	Autodesk Inc.	Information Technology
AET	Aetna Inc.	Health Care
ALGN	Align Technology Inc.	Health Care
ALL	Allstate Corporation	Financials
AMGN	Amgen Inc.	Health Care
AMT	American Tower Corporation	Real Estate
AVGO	Broadcom Limited	Information Technology
AXP	American Express Company	Financials
BLK	BlackRock Inc.	Financials
CCL.U	Carnival Corporation	Consumer Discretionary
CI	Cigna Corporation	Health Care
CMCSA	Comcast Corporation Class A	Consumer Discretionary
COG	Cabot Oil & Gas Corporation	Energy
COL	Rockwell Collins Inc.	Industrials
CTSH	Cognizant Technology Solutions Corporation Class A	Information Technology

Table 3.1: Tickers, company names and sectors of S&P 500 stocks exhibiting the negative correlation condition (continued)

Ticker	Company name	Sector
D	Dominion Energy Inc	Utilities
DFS	Discover Financial Services	Financials
DIS	Walt Disney Company	Consumer Discretionary
DRI	Darden Restaurants Inc.	Consumer Discretionary
DUK	Duke Energy Corporation	Utilities
DVA	DaVita Inc.	Health Care
EBAY	eBay Inc.	Information Technology
ECL	Ecolab Inc.	Materials
EOG	EOG Resources Inc.	Energy
ETN	Eaton Corp. Plc	Industrials
FDX	FedEx Corporation	Industrials
FMC	FMC Corporation	Materials
IBM	International Business Machines Corporation	Information Technology
INTC	Intel Corporation	Information Technology
INTU	Intuit Inc.	Information Technology
ISRG	Intuitive Surgical Inc.	Health Care
IVZ	Invesco Ltd.	Financials
KMX	CarMax Inc.	Consumer Discretionary

Table 3.1: Tickers, company names and sectors of S&P 500 stocks exhibiting the negative correlation condition (continued)

Ticker	Company name	Sector
LLL	L3 Technologies Inc.	Industrials
LRCX	Lam Research Corporation	Information Technology
MA	Mastercard Incorporated Class A	Information Technology
MAR	Marriott International Inc. Class A	Consumer Discretionary
MCHP	Microchip Technology Incorporated	Information Technology
MMM	3M Company	Industrials
MPC	Marathon Petroleum Corporation	Energy
MRO	Marathon Oil Corporation	Energy
MS	Morgan Stanley	Financials
MSFT	Microsoft Corporation	Information Technology
NUE	Nucor Corporation	Materials
PHM	PulteGroup Inc.	Consumer Discretionary
PX	Praxair Inc.	Materials
ROK	Rockwell Automation Inc.	Industrials
STI	SunTrust Banks Inc.	Financials
SWK	Stanley Black & Decker Inc.	Industrials
TROW	T. Rowe Price Group	Financials
TRV	Travelers Companies Inc.	Financials

Table 3.1: Tickers, company names and sectors of S&P 500 stocks exhibiting the negative correlation condition (continued)

Ticker	Company name	Sector
TXN	Texas Instruments Incorporated	Information Technology
UPS	United Parcel Service Inc. Class B	Industrials
VRTX	Vertex Pharmaceuticals Incorporated	Health Care
WFC	Wells Fargo & Company	Financials
WM	Waste Management Inc.	Industrials
WY	Weyerhaeuser Company	Real Estate
ZION	Zions Bancorporation	Financials
ZTS	Zoetis Inc. Class A	Health Care

Table 3.2: Historical S&P 500 stock returns over 6-month periods from November 2014 to October 2018

Ticker	11/14 - 04/15	05/15 - 10/15	11/15 - 04/16	05/16 - 10/16	11/16 - 04/17	05/17 - 10/17	11/17 - 04/18	05/18 - 10/18
AAPL	11.30%	-2.86%	3.58%	15.99%	4.36%	10.18%	0.49%	-2.82%
ADBE	92.80%	16.51%	31.88%	8.08%	24.13%	48.04%	56.00%	23.64%
ADSK	14.75%	3.30%	-8.96%	5.66%	29.77%	39.58%	29.30%	7.59%
AET	24.63%	-2.78%	-6.32%	13.81%	18.88%	19.82%	-4.33%	-3.95%
ALGN	16.53%	9.05%	-29.80%	-3.52%	16.71%	-27.32%	-13.30%	19.07%
ALL	22.35%	-17.57%	-16.16%	36.24%	-12.28%	-14.26%	37.13%	-11.65%
AMGN	-21.22%	0.78%	3.27%	27.21%	64.77%	43.15%	32.35%	-16.53%
AMT	13.18%	-20.28%	-5.51%	-22.11%	24.82%	12.63%	-11.60%	-24.38%
AVGO	-14.33%	-15.99%	6.48%	7.39%	22.53%	32.80%	6.30%	-1.77%
AXP	-19.29%	-6.19%	67.45%	103.66%	83.96%	-17.37%	-1.00%	141.36%
BLK	22.74%	10.50%	0.44%	8.05%	8.17%	22.85%	24.78%	7.97%
CCL.U	10.57%	-11.70%	-34.86%	-27.40%	11.37%	42.03%	2.16%	30.90%
CI	-2.32%	-2.41%	-8.25%	2.28%	8.56%	14.68%	-1.69%	-0.08%
CMCSA	19.84%	16.35%	-14.30%	-12.03%	17.29%	25.64%	8.13%	-11.99%
COG	14.59%	-25.12%	20.18%	1.27%	17.56%	-15.69%	-12.33%	10.49%
COL	22.50%	2.31%	12.26%	10.14%	3.93%	-2.32%	27.34%	19.52%
CTSH	9.16%	16.44%	-3.56%	18.81%	14.81%	21.70%	14.26%	8.27%

Table 3.2: Historical S&P 500 stock returns over 6-month periods from November 2014 to October 2018 (continued)

Ticker	11/14 - 04/15	05/15 - 10/15	11/15 - 04/16	05/16 - 10/16	11/16 - 04/17	05/17 - 10/17	11/17 - 04/18	05/18 - 10/18
D	27.98%	9.95%	12.84%	3.09%	9.15%	-19.79%	3.24%	9.86%
DFS	3.24%	1.31%	1.82%	2.28%	16.71%	15.14%	-12.54%	-6.80%
DIS	-1.09%	4.97%	12.90%	2.59%	1.34%	11.35%	-4.61%	5.69%
DRI	-4.68%	-14.91%	1.97%	1.81%	17.92%	21.06%	-2.92%	9.55%
DUK	4.10%	-13.24%	-3.76%	9.44%	2.30%	7.97%	18.32%	3.72%
DVA	4.48%	4.49%	5.65%	19.57%	6.04%	2.70%	-4.85%	-1.07%
EBAY	12.47%	-0.02%	-14.64%	-8.59%	-8.99%	-0.08%	23.51%	23.54%
ECL	10.01%	-0.68%	0.00%	-2.89%	14.19%	7.35%	-8.67%	0.38%
EOG	7.43%	8.64%	-0.36%	3.75%	8.98%	4.08%	9.39%	5.04%
ETN	-1.20%	4.47%	10.80%	-2.45%	7.88%	5.45%	-3.82%	1.86%
FDX	3.42%	-31.36%	6.26%	8.39%	56.17%	26.81%	-14.14%	5.39%
FMC	6.51%	5.01%	5.56%	1.04%	-7.21%	-9.72%	-15.76%	-0.39%
IBM	-4.38%	-11.16%	19.22%	-7.66%	3.05%	15.07%	-15.15%	-6.22%
INTC	12.54%	-3.38%	8.52%	3.87%	17.48%	28.26%	-11.42%	29.76%
INTU	9.31%	-12.42%	8.52%	-1.53%	9.86%	18.66%	10.38%	-5.02%
ISRG	-8.27%	-7.47%	5.71%	-0.04%	3.10%	-0.43%	-18.17%	-2.13%
IVZ	-9.67%	-6.35%	11.06%	15.70%	6.48%	5.99%	-0.21%	25.58%
KMX	13.37%	-7.69%	-15.31%	4.35%	10.53%	4.28%	-13.73%	7.18%

Table 3.2: Historical S&P 500 stock returns over 6-month periods from November 2014 to October 2018 (continued)

Ticker	11/14 - 04/15	05/15 - 10/15	11/15 - 04/16	05/16 - 10/16	11/16 - 04/17	05/17 - 10/17	11/17 - 04/18	05/18 - 10/18
LLL	9.40%	2.66%	2.10%	0.02%	11.82%	9.68%	11.09%	-0.71%
LRCX	-5.39%	10.00%	4.06%	4.11%	25.43%	8.97%	4.65%	12.78%
MA	8.35%	-10.57%	-0.46%	5.01%	17.71%	20.36%	13.90%	-3.54%
MAR	-9.37%	-12.71%	26.48%	1.42%	-28.89%	-37.02%	4.82%	-2.64%
MCHP	3.54%	20.82%	3.70%	5.44%	8.56%	-10.53%	-12.63%	9.50%
MMM	4.62%	-8.73%	8.37%	6.48%	26.35%	23.58%	-3.37%	-0.17%
MPC	20.03%	24.58%	5.90%	0.55%	19.88%	7.56%	-4.90%	-15.37%
MRO	5.68%	-4.08%	-8.71%	-1.98%	37.44%	26.54%	14.40%	-15.49%
MS	-2.05%	0.15%	-1.28%	3.73%	26.63%	7.31%	9.29%	-12.76%
MSFT	8.44%	5.10%	-24.56%	11.54%	16.86%	17.28%	25.39%	3.84%
NUE	2.04%	5.30%	19.69%	0.07%	-1.50%	-8.02%	-5.89%	11.21%
PHM	-8.61%	-20.63%	21.12%	7.78%	-3.37%	-13.48%	34.66%	-14.02%
PX	-2.74%	8.53%	9.42%	-8.77%	10.98%	-1.44%	-22.52%	4.64%
ROK	6.23%	10.81%	-5.51%	-1.58%	26.19%	18.05%	2.33%	5.79%
STI	58.40%	-16.27%	-13.49%	15.15%	29.63%	14.16%	-23.80%	0.33%
SWK	6.03%	0.05%	0.53%	8.36%	25.60%	5.98%	10.95%	-7.50%
TROW	-1.17%	19.85%	8.55%	8.55%	-7.63%	-15.66%	-11.91%	-16.24%
TRV	-6.05%	-26.95%	-48.63%	22.78%	-8.19%	25.12%	46.99%	-5.61%

Table 3.2: Historical S&P 500 stock returns over 6-month periods from November 2014 to October 2018 (continued)

Ticker	11/14 - 04/15	05/15 - 10/15	11/15 - 04/16	05/16 - 10/16	11/16 - 04/17	05/17 - 10/17	11/17 - 04/18	05/18 - 10/18
TXN	5.95%	3.39%	-0.21%	9.54%	20.97%	9.67%	3.10%	-3.40%
UPS	-12.25%	-22.49%	-10.59%	13.04%	44.94%	29.02%	6.02%	-8.25%
VRTX	9.45%	1.18%	-32.39%	-10.05%	55.95%	23.61%	4.74%	15.99%
WFC	1.31%	8.54%	9.36%	11.69%	10.84%	12.90%	-1.07%	8.78%
WM	1.28%	-9.93%	7.01%	-1.28%	4.25%	-6.27%	-20.19%	15.83%
WY	-6.94%	-6.92%	9.51%	-6.82%	13.16%	6.02%	2.42%	-23.82%
ZION	-2.17%	1.52%	-4.34%	17.04%	24.28%	16.06%	17.84%	-12.33%
ZTS	19.54%	-3.17%	9.35%	1.64%	17.38%	13.74%	30.81%	7.91%

Table 3.3: Estimates of expected S&P 500 stock returns accrued in the next six months from November 2018 to April 2019

Ticker	Lower bound on expected return	Upper bound on expected return
AAPL	2.78%	12.34%
ADBE	3.87%	72.42%
ADSK	6.39%	42.83%
AET	3.59%	32.08%
ALGN	7.87%	27.13%
ALL	3.46%	37.13%
AMGN	3.38%	43.63%
AMT	4.06%	29.77%
AVGO	3.99%	32.80%
AXP	2.70%	183.92%
BLK	4.35%	28.81%
CCL.U	4.16%	43.80%
CI	4.17%	29.73%
CMCSA	4.38%	26.60%
COG	5.79%	24.64%
COL	4.85%	34.54%
CTSH	4.46%	25.29%
D	3.45%	20.01%
DFS	5.84%	16.70%
DIS	2.52%	18.28%
DRI	4.80%	24.71%
DUK	2.99%	34.17%
DVA	9.61%	9.94%
EBAY	3.74%	41.57%
ECL	3.72%	12.39%
EOG	4.62%	21.94%
ETN	3.72%	12.38%
FDX	2.87%	30.83%
FMC	5.14%	14.19%
IBM	2.67%	18.51%
INTC	3.68%	35.48%

Table 3.3: Estimates of expected S&P 500 stock returns accrued in the next six months from November 2018 to April 2019 (continued)

Ticker	Lower bound on expected return	Upper bound on expected return
INTU	3.67%	31.06%
ISRG	6.55%	12.91%
IVZ	6.74%	34.41%
KMX	6.15%	13.74%
LLL	3.79%	19.84%
LRCX	7.17%	17.30%
MA	3.42%	21.20%
MAR	5.17%	50.39%
MCHP	5.90%	11.20%
MMM	3.16%	26.74%
MPC	4.46%	12.15%
MRO	9.89%	36.60%
MS	2.77%	18.41%
MSFT	2.52%	35.59%
NUE	4.00%	15.96%
PHM	6.98%	44.69%
PX	5.66%	13.29%
ROK	5.52%	18.97%
STI	3.82%	30.92%
SWK	5.03%	17.42%
TROW	3.45%	10.36%
TRV	3.43%	61.75%
TXN	3.56%	14.33%
UPS	3.39%	31.11%
VRTX	8.97%	35.71%
WFC	2.75%	13.71%
WM	3.57%	25.01%
WY	4.13%	11.92%
ZION	6.17%	22.85%
ZTS	4.07%	31.31%

CHAPTER IV

PORTFOLIO SELECTION PROBLEM

Generally, a portfolio can include any types of financial assets. However, only stocks are considered here because their lower bounds on expected returns are computable (see Chapter 3 for more details)

4.1 General Portfolio Model

Suppose that, at time T, a risk-averse investor wants to invest proportion w_i of capital in asset i (where $i = 1, 2, \ldots, n$) of a portfolio with the buy-and-hold strategy to maximize the overall rate of return R_{T+1} attainable at the next time $T + 1$. During this investment period, the rate of return of each asset i can be represented by a random variable $R_{i,T+1}$. The overall rate of return of the portfolio becomes

$$R_{T+1} = w_1 R_{1,T+1} + w_2 R_{2,T+1} + \ldots + w_n R_{n,T+1}.$$

When no short sales are allowed during the time between T and $T + 1$, the proportion invested in each asset i cannot be negative: $w_i \geq 0$.

Arising from the return unpredictability, the investor faces risk across the entire portfolio. Intuitively, risk occurs when the realized return falls below the expected return. It must depend on the loss function occurred at time $T + 1$ given by

$$L_{i,T+1} = \mathbb{E}_T[R_{i,T+1}] - R_{i,T+1},$$

where $\mathbb{E}_T[R_{i,T+1}]$ denotes the expected return of asset i at time $T+1$ based on all previous trading information by time T. Loss is a random variable to which risk assigns a real number for comparative purposes [9]. The notation of risk $\varrho_T(L) \in \mathbb{R}$ is introduced and it will be discussed in Section 4.2.

To derive the optimal portfolio, the risk threshold is also required. Nonetheless, this parameter is relatively abstract, and it is of less importance compared with the overall return. There is also no consensus about this parameter value because various types of risk measures are available [28, 29].

Instead, the risk-averse investor decides to minimize risk with a given threshold (or guaranteed) rate of return θ. This is in accordance with the primary purpose of a portfolio: smoothing out the firm-specific volatilities of its included assets. This strategy is technically called *diversification*. Yet, macroeconomic-level risks such as deflation and political instability cannot be mitigated by a portfolio method. Return from an investment must compensate with associated risk. This leads to the principle of risk-return tradeoff [20].

Using the historical rate of return $\widetilde{r}_{i,t}$ for asset i at time $t = 1, 2, \ldots, T$ to estimate the expected values $\mathbb{E}_T[R_{i,T+1}]$, the optimal asset weight w_i^* is achievable. To put it more concretely, this problem can be formulated as follows:

$$
\begin{aligned}
\text{minimize} \quad & \varrho_T(\mathbb{E}_T[R_{T+1}] - R_{T+1}) \\
\text{subject to} \quad & \mathbb{E}_T[R_{T+1}] \geq \theta, \\
& R_{T+1} = w_1 R_{1,T+1} + w_2 R_{2,T+1} + \ldots + w_n R_{n,T+1}, \\
& w_1 + w_2 + \ldots + w_n = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{4.1}
$$

4.2 Risk Measure

As mentioned above, the risk concerned in this work reflects the deviation of the realized return from its expected value. Although it seems abstract, this concept contributes to the formulation of many portfolio problems.

Definiton 4.1 (Subadditive Risk Measure [9]). Let $\mathcal{M}$ be the set of random variables representing portfolio losses. A risk measure $\varrho : \mathcal{M} \to \mathbb{R}$ is called *subadditive* if

$$\varrho(L + S) \leq \varrho(L) + \varrho(S)$$

for all $L, S \in \mathcal{M}$.

Subadditivity implies investing in well-diversified assets lessens portfolio risk. The following proposition gives some important examples of subadditive risk measures.

Proposition 4.2. The following risk measures are subadditive:

1. The standard deviation (SD) risk measure $\varrho^{\text{SD}}(L) = \sqrt{\mathbb{E}[L^2]}$;
2. The mean absolute deviation (MAD) risk measure $\varrho^{\text{MAD}}(L) = \mathbb{E}[|L|]$.

Proof. Let L and S be financial losses. Then

$$\begin{aligned}\varrho^{\mathrm{SD}}(L+S) &= \sqrt{\mathbb{E}[(L+S)^2]} \\ &= \sqrt{\mathbb{E}[L^2] + 2\mathbb{E}[LS] + \mathbb{E}[S^2]} \\ &\leq \sqrt{\mathbb{E}[L^2] + 2\sqrt{\mathbb{E}[L^2]\mathbb{E}[S^2]} + \mathbb{E}[S^2]} \\ &= \sqrt{\mathbb{E}[L^2]} + \sqrt{\mathbb{E}[S^2]} \\ &= \varrho^{\mathrm{SD}}(L) + \varrho^{\mathrm{SD}}(S)\end{aligned}$$

and

$$\begin{aligned}\varrho^{\mathrm{MAD}}(L+S) &= \mathbb{E}[|L+S|] \\ &\leq \mathbb{E}[|L| + |S|] \\ &= \mathbb{E}[|L|] + \mathbb{E}[|S|] \\ &= \varrho^{\mathrm{MAD}}(L) + \varrho^{\mathrm{MAD}}(S).\end{aligned}$$

By definition 4.1, both risk measures are subadditive. □

4.3 MAD Portfolio Model

4.3.1 Development of Model with Real-Valued Returns

The MAD risk measure is approximated as follows [3]:

$$\begin{aligned}\varrho_T^{\mathrm{MAD}}(L) &= \varrho_T^{\mathrm{MAD}}(\mathbb{E}_T[R_{T+1}] - R_{T+1}) \\ &= \mathbb{E}_T[|\mathbb{E}_T[R_{T+1}] - R_{T+1}|] \\ &= \mathbb{E}_T[|R_{T+1} - \mathbb{E}_T[R_{T+1}]|] \\ &= \mathbb{E}_T\left[\left|\sum_{i=1}^{n} R_{i,T+1}w_i - \mathbb{E}_T\left[\sum_{i=1}^{n} R_{i,T+1}w_i\right]\right|\right] \\ &= \mathbb{E}_T\left[\left|\sum_{i=1}^{n}(R_{i,T+1} - \mathbb{E}_T[R_{i,T+1}])w_i\right|\right] \\ &\approx \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(\widetilde{r}_{i,t} - \mathbb{E}_T[R_{i,T+1}])w_i\right|.\end{aligned}$$

Define

$$\begin{aligned}r_i &= \mathbb{E}_T[R_{i,T+1}] \\ a_{i,t} &= \widetilde{r}_{i,t} - r_i\end{aligned}$$

where $i = 1, 2, \ldots, n$ and $t = 1, 2, \ldots, T$.

The portfolio optimization problem (4.1) reads

$$\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n} a_{i,t}w_i\right| \\
\text{subject to} \quad & \sum_{i=1}^{n} r_i w_i \geq \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n
\end{aligned} \tag{4.2}$$

which is equivalent to the linear program

$$\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T} d_t \\
\text{subject to} \quad & d_t + \sum_{i=1}^{n} a_{i,t}w_i \geq 0, \quad t = 1, 2, \ldots, T, \\
& d_t - \sum_{i=1}^{n} a_{i,t}w_i \geq 0, \quad t = 1, 2, \ldots, T, \\
& \sum_{i=1}^{n} r_i w_i \geq \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& d_t \geq 0, \quad t = 1, 2, \ldots, T.
\end{aligned} \tag{4.3}$$

It contains $2T + 2$ basic variables which may be degenerate. Hence, an investor should invest in at most $2T + 2$ assets.

To improve this upper bound on the number of invested assets, add the surplus variables $2u_t$ and $2v_t$ to the first two constraints [30]:

$$\begin{aligned}
d_t + \sum_{i=1}^{n} a_{i,t}w_i - 2u_t = 0, \\
d_t - \sum_{i=1}^{n} a_{i,t}w_i - 2v_t = 0,
\end{aligned}$$

where $u_t, v_t \geq 0$ for $t = 1, 2, \ldots, T$. Then

$$\begin{aligned}
d_t &= u_t + v_t, \\
\sum_{i=1}^{n} a_{i,t}w_i - u_t + v_t &= 0.
\end{aligned}$$

This leads to the minimization problem

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T}(u_t + v_t) \\
\text{subject to} \quad & -u_t + v_t + \sum_{i=1}^{n} a_{i,t} w_i = 0, \quad t = 1, 2, \ldots, T, \\
& \sum_{i=1}^{n} r_i w_i \geq \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& u_t, v_t \geq 0, \quad t = 1, 2, \ldots, T.
\end{aligned} \tag{4.4}
$$

The number of basic variables is reduced to $T+2$. This implied the investor should include at most $T+2$ assets in his portfolio.

4.3.2 Model with Interval-Valued Returns

All parameters in the linear program (4.4) are certain without ambiguity except the conditional expected return

$$r_i = \mathbb{E}_T[R_{i,T+1}],$$

which is also appeared in the term $a_{i,t}$. In general, the return process is nonstationary. Its distribution over time is also arguable [9]. As a result, the expected return should not be predicted by the average rate of historical returns. In this thesis, the parameter is represented by the interval

$$\boldsymbol{r}_i = [\underline{r}_i, \overline{r}_i].$$

There are several methods of obtaining the lower and upper bounds of the returns. This work follows the guidelines suggested in Chapter 3.

It is very unlikely to calculate an optimal portfolio w_i^* in accordance with every possible outcome of expected return r_i in the interval $\boldsymbol{r}_i = [\underline{r}_i, \overline{r}_i]$. In this work, the optimal portfolios satisfying at least one scenario by computing the weakly optimal solution set. A threshold rate of return θ in the optimization problem (4.4) becomes an equality constraint by adding a surplus variable s to the second constraint. This contributes to the multiparametric linear programming (mp-LP) problem

$$\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T}(u_t + v_t) \\
\text{subject to} \quad & -u_t + v_t + \sum_{i=1}^{n} a_{i,t} w_i = 0, \quad t = 1, 2, \ldots, T, \\
& \sum_{i=1}^{n} r_i w_i - s = \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& u_t, v_t \geq 0, \quad t = 1, 2, \ldots, T, \\
& s \geq 0
\end{aligned} \tag{4.5}$$

where $r_i \in \boldsymbol{r}_i = [\underline{r}_i, \overline{r}_i]$ also embedded in the term $a_{i,t} = \widetilde{r}_{i,t} - r_i$.

To eliminate the dependence of $a_{i,t}$ and r_i in the portfolio model (4.5), the second constraint is added to the first counterpart. It leads to the following mp-LP problem

$$\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T}(u_t + v_t) \\
\text{subject to} \quad & -u_t + v_t + \sum_{i=1}^{n} \widetilde{r}_{i,t} w_i - s = \theta, \quad t = 1, 2, \ldots, T, \\
& \sum_{i=1}^{n} r_i w_i - s = \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& u_t, v_t \geq 0, \quad t = 1, 2, \ldots, T, \\
& s \geq 0
\end{aligned} \tag{4.6}$$

where $r_i \in \boldsymbol{r}_i = [\underline{r}_i, \overline{r}_i]$, which can equivalently be transformed into the interval linear program

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T}(u_t+v_t) \\
\text{subject to} \quad & -u_t+v_t+\sum_{i=1}^{n}\widetilde{r}_{i,t}w_i - s = \theta, \quad t=1,2,\dots,T, \\
& \sum_{i=1}^{n}\boldsymbol{r}_i w_i - s = \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \geq 0, \quad i=1,2,\dots,n, \\
& u_t, v_t \geq 0, \quad t=1,2,\dots,T, \\
& s \geq 0
\end{aligned}
\tag{4.7}
$$

with the replacement of the expected return parameter r_i by the interval $\boldsymbol{r}_i = [\underline{r}_i, \overline{r}_i]$. For simplicity, the problem with the objective function $\sum_{t=1}^{T}(u_t+v_t)$ is investigated instead.

Denote

$$
\begin{aligned}
u &= \begin{bmatrix} u_1 & \cdots & u_T \end{bmatrix}^{\mathsf{T}} \\
v &= \begin{bmatrix} v_1 & \cdots & v_T \end{bmatrix}^{\mathsf{T}} \\
w &= \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^{\mathsf{T}} \\
\widetilde{R} &= \begin{bmatrix} \widetilde{r}_{1,1} & \cdots & \widetilde{r}_{1,t} & \cdots & \widetilde{r}_{1,T} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widetilde{r}_{i,1} & \cdots & \widetilde{r}_{i,t} & \cdots & \widetilde{r}_{i,T} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widetilde{r}_{n,1} & \cdots & \widetilde{r}_{n,t} & \cdots & \widetilde{r}_{n,T} \end{bmatrix} \\
r &= \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix}^{\mathsf{T}} \\
\boldsymbol{r} &= \begin{bmatrix} \boldsymbol{r}_1 & \cdots & \boldsymbol{r}_n \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} [\underline{r}_1, \overline{r}_1] & \cdots & [\underline{r}_n, \overline{r}_n] \end{bmatrix}^{\mathsf{T}}.
\end{aligned}
$$

The optimization problem (4.7) in the matrix notation becomes

$$
\begin{aligned}
\text{minimize} \quad & c^{\mathsf{T}} x \\
\text{subject to} \quad & \boldsymbol{A} x = b, \\
& x \geq 0
\end{aligned}
\tag{4.8}
$$

where

$$x = \begin{bmatrix} u_{1\times T}^{\mathsf{T}} & v_{1\times T}^{\mathsf{T}} & s_{1\times 1} & w_{1\times n}^{\mathsf{T}} \end{bmatrix}_{(2T+1+n)\times 1}^{\mathsf{T}}$$

$$c = \begin{bmatrix} 1_{1\times T}^{\mathsf{T}} & 1_{1\times T}^{\mathsf{T}} & 0_{1\times 1} & 0_{1\times n}^{\mathsf{T}} \end{bmatrix}_{(2T+1+n)\times 1}^{\mathsf{T}}$$

$$\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r}) = \begin{bmatrix} -I_{T\times T} & I_{T\times T} & -1_{T\times 1} & \widetilde{R}_{T\times n}^{\mathsf{T}} \\ 0_{1\times T}^{\mathsf{T}} & 0_{1\times T}^{\mathsf{T}} & -1_{1\times 1} & \boldsymbol{r}_{1\times n}^{\mathsf{T}} \\ 0_{1\times T}^{\mathsf{T}} & 0_{1\times T}^{\mathsf{T}} & 0_{1\times 1} & 1_{1\times n}^{\mathsf{T}} \end{bmatrix}_{(T+2)\times(2T+1+n)}$$

$$b = b(\theta) = \begin{bmatrix} \theta_{1\times T}^{\mathsf{T}} & \theta_{1\times 1} & 1_{1\times 1} \end{bmatrix}_{(T+2)\times 1}^{\mathsf{T}}.$$

In this thesis, the notations $\boldsymbol{A}$ and $\boldsymbol{A}(\boldsymbol{r})$ are used interchangeably depending upon the emphasis on the parameter $\boldsymbol{r}$, so do the notations b and $b(\theta)$. Note that every matrix belonging to the interval matrix $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r})$ has a full row rank. In this program, an investor must initialize the threshold rate of return θ for the portfolio in order to determine whether or not this goal can be fulfilled and, if any, to compute a weakly optimal asset weight $w_{n\times 1}^*(r)$ with its attainable rate of return $\tau^* = \tau(w^*(r))$.

CHAPTER V

BILEVEL PORTFOLIO OPTIMIZATION

The main purpose of this chapter is to provide the background knowledge on the bilevel method to solve the MAD model (4.3) where the expected return r_i of individual asset i is represented by the interval $[\underline{r}_i, \overline{r}_i]$. This scheme is compared to our proposed method in Chapter 6 and the result is illustrated in Chapter 7. Lui [31] constructs the pair of two-level mathematical programs and restates the MAD model (4.3) without employing the notation $a_{i,t}$ as follows:

$$\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T} d_t \\
\text{subject to} \quad & d_t + \sum_{i=1}^{n} \widetilde{r}_{i,t} w_i - \sum_{i=1}^{n} r_i w_i \geq 0, \quad t = 1, 2, \ldots, T, \\
& d_t - \sum_{i=1}^{n} \widetilde{r}_{i,t} w_i + \sum_{i=1}^{n} r_i w_i \geq 0, \quad t = 1, 2, \ldots, T, \\
& \sum_{i=1}^{n} r_i w_i \geq \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& d_t \geq 0, \quad t = 1, 2, \ldots, T
\end{aligned} \tag{5.1}$$

where

$$\underline{r}_i \leq r_i \leq \overline{r}_i.$$

This is the inner-level program computing the optimal portfolio and its risk under a given set of expected return r_i. Since r_i varies from $\underline{r}_i$ to $\overline{r}_i$, there are many optimal portfolios. The outer-level program determines the value of each r_i in the suitable range and produces the corresponding optimal portfolio. It can be either minimization (low risk) or maximization (high risk) problem.

5.1 Lowest-Risk Bilevel Portfolio

An optimal portfolio with the lowest risk is obtained by the program

$$\min_{r_i} \min_{w_i, d_t} \frac{1}{T} \sum_{t=1}^{T} d_t$$

with the same constraint specified in the program (5.1). Since both the inner-level program and the outer-level program are minimization problems, they can be combined into a one-level program:

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T} d_t \\
\text{subject to} \quad & d_t + \sum_{i=1}^{n} \widetilde{r}_{i,t} w_i - \sum_{i=1}^{n} r_i w_i \geq 0, \quad t = 1, 2, \ldots, T, \\
& d_t - \sum_{i=1}^{n} \widetilde{r}_{i,t} w_i + \sum_{i=1}^{n} r_i w_i \geq 0, \quad t = 1, 2, \ldots, T, \\
& \sum_{i=1}^{n} r_i w_i \geq \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& d_t \geq 0, \quad t = 1, 2, \ldots, T, \\
& \underline{r}_i \leq r_i \leq \overline{r}_i, \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

This is a nonlinear program due to the product term $r_i w_i$. Substitute

$$\eta_i = r_i w_i \tag{5.2}$$

and multiply the last inequality by the term w_i to obtain the linear program

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{T}\sum_{t=1}^{T} d_t \\
\text{subject to} \quad & d_t + \sum_{i=1}^{n} \widetilde{r}_{i,t} w_i - \sum_{i=1}^{n} \eta_i \geq 0, \quad t = 1, 2, \ldots, T, \\
& d_t - \sum_{i=1}^{n} \widetilde{r}_{i,t} w_i + \sum_{i=1}^{n} \eta_i \geq 0, \quad t = 1, 2, \ldots, T, \\
& \sum_{i=1}^{n} \eta_i \geq \theta, \\
& \sum_{i=1}^{n} w_i = 1, \\
& -\underline{r}_i w_i + \eta_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& \overline{r}_i w_i - \eta_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n, \\
& d_t \geq 0, \quad t = 1, 2, \ldots, T, \\
& \eta_i \geq 0, \quad i = 1, 2, \ldots, n.
\end{aligned}
\tag{5.3}
$$

The optimal portfolio with the lowest risk

$$\frac{1}{T}\sum_{t=1}^{T} d_t^*$$

has the asset weight w_i^* and it attains a rate of return

$$\sum_{i=1}^{n} \eta_i^*.$$

This occurs when an individual asset i has an exact rate of return

$$\hat{r}_i = \frac{\eta_i^*}{w_i^*} \in [\underline{r}_i, \overline{r}_i]. \tag{5.4}$$

When the optimal weight w_i^* becomes 0, the term η_i^* binds at 0. The computed return $\hat{r}_i$ is therefore not a number. This does not indicate a flaw in this bilevel model. The optimality indeed happens for some unknown expected return r_i between $\underline{r}_i$ and $\overline{r}_i$. It is simply because division by zero in the equation (5.2) is undefined.

5.2 Highest-Risk Bilevel Portfolio

An optimal portfolio with the highest risk is obtained by the program

$$\max_{r_i} \min_{w_i, d_t} \frac{1}{T}\sum_{t=1}^{T} d_t$$

with the same constraint specified in the program (5.1). Both the inner-level program and the outer-level program have different operations. The dual problem of the inner-level program (5.1) is therefore considered:

$$\begin{aligned}
\text{maximize} \quad & \theta y_{2T+1} + y_{2T+2} \\
\text{subject to} \quad & y_t + y_{T+t} \le \frac{1}{T}, \quad t \le T, \\
& \sum_{t=1}^{T} (\widetilde{r}_{i,t} - r_i) y_t - \sum_{t=1}^{T} (\widetilde{r}_{i,t} - r_i) y_{T+t} + r_i y_{2T+1} + y_{2T+2} \le 0, \quad i \le n, \\
& y_k \ge 0, \quad k \le 2T+1.
\end{aligned}$$

Now both programs have the same maximization operation. They can be combined into a one-level program:

$$
\begin{aligned}
\text{maximize} \quad & \theta y_{2T+1} + y_{2T+2} \\
\text{subject to} \quad & y_t + y_{T+t} \leq \frac{1}{T}, \quad t \leq T, \\
& \sum_{t=1}^{T} (\widetilde{r}_{i,t} - r_i) y_t - \sum_{t=1}^{T} (\widetilde{r}_{i,t} - r_i) y_{T+t} + r_i y_{2T+1} + y_{2T+2} \leq 0, \quad i \leq n, \\
& y_k \geq 0, \quad k \leq 2T+1, \\
& \underline{r}_i \leq r_i \leq \overline{r}_i, \quad i \leq n.
\end{aligned}
$$

This is a nonlinear program due to the product term $r_i y_k$ where $k \neq 2T+1$. Substitute

$$\xi_{i,k} = r_i y_k,$$

where $i = 1, \ldots, n$ and $k = 1, \ldots, 2T+1$, and multiply the last inequality by the term y_k to obtain the linear program

$$
\begin{aligned}
\text{maximize} \quad & \theta y_{2T+1} + y_{2T+2} \\
\text{subject to} \quad & y_t + y_{T+t} \leq \frac{1}{T}, \quad t \leq T, \\
& \sum_{t=1}^{T} [\widetilde{r}_{i,t}(y_t - y_{T+t}) - \xi_{i,t} + \xi_{i,T+t}] + \xi_{i,2T+1} + y_{2T+2} \leq 0, \\
& \underline{r}_i y_k - \xi_{i,k} \leq 0, \quad k \leq 2T+1, \\
& -\overline{r}_i y_k + \xi_{i,k} \leq 0 \quad k \leq 2T+1, \\
& y_k \geq 0, \quad k \leq 2T+1, \\
& \xi_{i,k} \geq 0, \quad k \leq 2T+1.
\end{aligned}
\tag{5.5}
$$

The optimal portfolio with the highest risk is attained when the expected return of an individual asset becomes

$$\hat{r}_i = \frac{\xi^*_{i,k}}{y^*_k} \in [\underline{r}_i, \overline{r}_i] \tag{5.6}$$

for any $k = 1, 2, \ldots, 2T+1$. The corresponding portfolio composition can be obtained by the dual solution w_i of the second constraint in the program (5.5) with the assistance of linear program solvers such as CPLEX® optimizer.

5.3 Discussion

Although there is no sophisticated theoretical background behind the bilevel portfolio models, their optimal portfolios are not universal. They merely account for the two extreme risks.

Furthermore, the validity of the bilevel portfolio model (5.5) accounting for the highest risk is not guaranteed because each individual asset's expected return obtained by the condition (5.6) may be nonconstant across different values of index $k \leq 2T + 1$ as illustrated in Chapter 7. This is a common phenomenon by its nature of relaxation technique. Nonetheless, the bilevel portfolio model (5.3) accounting for the lowest risk is always valid because the resultant portfolio becomes optimal under the realistic scenario as specified in the condition (5.4).

CHAPTER VI

UNCERTAIN PORTFOLIO OPTIMIZATION

The derivation and statement of all proposed algorithms central to this work are included in this chapter. The optimal investment strategy under the disagreement of measurements for expected asset returns involve three primary steps:

1. validating a particular choice of threshold portfolio returns
2. describing overall optimal asset allocation
3. computing a range of optimal returns from the portfolio investment.

Moreover, two portfolios may contain a different set of financial assets. The evaluation of their attractiveness may require the additional following step: calculating a range of optimal risks. This is mainly to compare the degree of risks incurred by two different portfolios.

6.1 Range of Threshold Portfolio Returns

Intuitively, a threshold (or guaranteed) rate of return θ is valid only when a portfolio under any circumstances can generate beyond the rate of return θ. However, an investor must determine the possible values of these thresholds beforehand. In other words, the parameter θ enables the system $\boldsymbol{A}x = b(\theta)$ strongly feasible (see Section 2.1). The following proposition is employed to establish a criterion for validating a choice of $\theta \in [\underline{\theta}, \overline{\theta}]$.

Proposition 6.1. For the interval linear program (4.8), the system of constraints $\boldsymbol{A}(\boldsymbol{r})x = b$ is strongly feasible if and only if each of the systems $\boldsymbol{A}(\underline{r})x = b$ and $\boldsymbol{A}(\overline{r})x = b$ contains a nonnegative solution.

Proof. According to Theorem 2.4, the system

$$\begin{aligned}(A^c - z_{T+1}A^{\Delta})x &= (A^c - \operatorname{diag}(z)A^{\Delta})x\\ &= b^c + \operatorname{diag}(z)b^{\Delta}\\ &= b + \operatorname{diag}(z)0\\ &= b\end{aligned}$$

must have a nonnegative solution for every $z \in \{\pm 1\}^{T+2}$. The term z_{T+1} can take on the values 1 and -1 which correspond to the expressions $\boldsymbol{A}(\underline{r})$ and $\boldsymbol{A}(\overline{r})$ respectively on the left hand side. $\square$

As a result, the parameters $\underline{\theta}$ and $\overline{\theta}$ can be found by minimizing and maximizing the objective function θ respectively subject to each constraint separately specified in Proposition 6.1. Intuitively, every threshold between these two extreme values is attainable. The soundness of Proposition 6.1 is confirmed by the following proposition.

Proposition 6.2. Any portfolio with a guaranteed rate of return θ between $\underline{\theta}$ and $\overline{\theta}$ is always feasible.

Proof. Suppose the two portfolios attain the threshold rates of return $\underline{\theta}$ and $\overline{\theta}$. By Proposition 6.1,

$$\boldsymbol{A}(\underline{r})x^{(1)} = b(\underline{\theta}) \quad \text{and} \quad \boldsymbol{A}(\underline{r})x^{(2)} = b(\overline{\theta})$$

for some nonnegative solutions $x^{(1)}$ and $x^{(2)}$. Since θ is between $\underline{\theta}$ and $\overline{\theta}$, it follows that

$$\theta = t\underline{\theta} + (1-t)\overline{\theta}$$

for some $0 \leq t \leq 1$. Consider

$$\begin{aligned} \boldsymbol{A}(\underline{r})(tx^{(1)} + (1-t)x^{(2)}) &= t\boldsymbol{A}(\underline{r})x^{(1)} + (1-t)\boldsymbol{A}(\underline{r})x^{(2)} \\ &= tb(\underline{\theta}) + (1-t)b(\overline{\theta}) \\ &= b(t\underline{\theta} + (1-t)\overline{\theta}) \\ &= b(\theta). \end{aligned}$$

Hence, the system $\boldsymbol{A}(\underline{r})x = b(\theta)$ has a nonnegative solution, and so does the system $\boldsymbol{A}(\overline{r})x = b(\theta)$ in the similar manner. Proposition 6.1 implies there always exists a portfolio with the guaranteed rate of return $\theta \in [\underline{\theta}, \overline{\theta}]$. $\square$

The algorithm for computing the range of threshold portfolio returns is suggested below.

Algorithm 6.1: Determine the range of threshold portfolio returns

Input: The portfolio selection model (4.8)

Output: The minimum threshold $\underline{\theta}$ and the maximum threshold $\overline{\theta}$

1: Compute $\underline{\theta}_1 \leftarrow \min\{\theta \mid \boldsymbol{A}(\underline{r})x - b(\theta) = 0,\ x \geq 0,\ \theta \geq 0\}$

2: Compute $\underline{\theta}_2 \leftarrow \min\{\theta \mid \boldsymbol{A}(\overline{r})x - b(\theta) = 0,\ x \geq 0,\ \theta \geq 0\}$

3: Set $\underline{\theta} \leftarrow \max\{\underline{\theta}_1, \underline{\theta}_2\}$

4: Compute $\overline{\theta}_1 \leftarrow \max\{\theta \mid \boldsymbol{A}(\underline{r})x - b(\theta) = 0,\ x \geq 0,\ \theta \geq 0\}$

5: Compute $\overline{\theta}_2 \leftarrow \max\{\theta \mid \boldsymbol{A}(\overline{r})x - b(\theta) = 0,\ x \geq 0,\ \theta \geq 0\}$

6: Set $\overline{\theta} \leftarrow \min\{\overline{\theta}_1, \overline{\theta}_2\}$

7: **if** $\underline{\theta} \leq \overline{\theta}$ **then**

8: **return** $\underline{\theta}$ and $\overline{\theta}$

9: **else**

10: **return** "No suitable choice for a threshold portfolio return"

11: **end if**

Algorithm 6.1 can be executed in either exponential or polynomial worst-case time complexities with the dual-simplex method or Karmarkar's algorithm [32, 33]. Although it is clear that the minimum and maximum thresholds are given by

$$\underline{\theta} = 0 \quad \text{and} \quad \overline{\theta} = \max_{1 \leq i \leq n} \{\underline{r}_i\},$$

this algorithm is applicable to any linear programming model for portfolio selection with additional constraints [34] on transaction costs, investment thresholds and decision dependency, for instance.

6.2 Enclosure of Optimal Portfolios

6.2.1 Bounds on Optimal Compositions

For the interval MAD model (4.8), a (weakly) optimal weight w_i^* of asset i minimizes a portfolio risk subject to the constraint $\boldsymbol{A}x = b$ and it depends on a scenario of asset return $r_i \in \boldsymbol{r}_i = [\underline{r}_i, \overline{r}_i]$ lying inside the interval matrix $\boldsymbol{A}$. Since there are many choices of r_i's, optimal solutions should be represented in terms of an enclosure in which all weakly optimal solutions (satisfying at least one scenario) locate. Any portfolio outside this region is completely nonoptimal and therefore unattractive. The term *enclosure* is coined here to emphasize that any portfolio inside this region is not necessarily, yet sufficiently, optimal.

For any given scenario $A \in \boldsymbol{A}$, an optimal solution $x \in \mathbb{R}^{2T+1+n}$ satisfies

$$Ax = b,\ x \geq 0,\ A^\intercal y \leq c,\ c^\intercal x = b^\intercal y$$

for some $y \in \mathbb{R}^{T+2}$ as a result of the duality theory. A superset of optimal solutions [35] is described by

$$\boldsymbol{A}x = b,\ x \geq 0,\ \boldsymbol{A}^\intercal y \leq c,\ c^\intercal x = b^\intercal y.$$

According to Theorems 2.5 and 2.7, a weak solution x to the region above must meet all of the following requirements:

$$\begin{aligned}
\underline{A}x &\leq b \\
-\overline{A}x &\leq -b \\
x &\geq 0 \\
(A^c - z_{T+1}A^\Delta)^\intercal y = (A^c - \operatorname{diag}(z)A^\Delta)^\intercal y &\leq c \\
c^\intercal x &= b^\intercal y \\
\operatorname{diag}(z)y &\geq 0
\end{aligned}$$

where $z \in \{\pm 1\}^{T+2}$. This representation can be decomposed into a union of two regions between

$$\underline{A}x \leq b,\ -\overline{A}x \leq -b,\ x \geq 0, \underline{A}^\intercal y \leq c,\ c^\intercal x = b^\intercal y,\ e_{T+1}^\intercal y \geq 0$$

and

$$\underline{A}x \leq b,\ -\overline{A}x \leq -b,\ x \geq 0, \overline{A}^\intercal y \leq c,\ c^\intercal x = b^\intercal y,\ -e_{T+1}^\intercal y \geq 0.$$

Note that both regions may not simultaneously be feasible. Each enclosure is formed by the interval hull as stated in Algorithm 6.2.

6.2.2 Range of Optimal Returns

With the inexact data of the expected return of an individual asset, the optimal portfolios can have a wide range of returns. They should attain at least the given threshold rate θ as set in the portfolio selection problems (4.4) and (4.7). The highest achievable portfolio return is still questionable. A bound for the surplus variable s directly provides the true extent of the portfolio returns $\boldsymbol{\tau}$. More formally,

$$\tau = r_1 w_1 + \ldots + r_n w_n = \theta + s.$$

Similarly at optimality, $\tau^* = \theta + s^*$. Algorithm 6.2 also provides how to compute the range of optimal returns by examining each interval hull of the variable s in the weak solution set for optimal portfolios as given in Subsection 6.2.1. An optimal portfolio usually generates an investment return beyond the threshold θ as elaborated in Chapter 7.

6.2.3 Algorithm for Computing Optimal Weights and Returns

An enclosure of optimal asset weights and their returns can be found by the following algorithm.

Algorithm 6.2: Describe optimal portfolios and determine their returns

---

Input: The portfolio selection model (4.8) with the threshold portfolio return θ

Output: Enclosure $\mathcal{E}$ of optimal portfolios and the range $\mathcal{R}$ of their returns

1: Set $\mathcal{E}_1, \mathcal{E}_2, \mathcal{R}_1, \mathcal{R}_2 \leftarrow \emptyset$

2: Let $\mathcal{P}^{(1)}$ be the linear program $\min\{0^\intercal x + 0^\intercal y \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0,$ $(\boldsymbol{A}(\underline{r}))^\intercal y \leq c,\ c^\intercal x = b^\intercal y,\ e_{T+1}^\intercal y \geq 0\}$

3: **if** the program $\mathcal{P}^{(1)}$ is feasible **then**

4:     **for** $i \leftarrow 0$ **to** n **do**

5:         Compute $\underline{w}_i^{(1)} \leftarrow \min\{e_{2T+1+i}^\intercal x \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0,$ $(\boldsymbol{A}(\underline{r}))^\intercal y \leq c,\ c^\intercal x = b^\intercal y,\ e_{T+1}^\intercal y \geq 0\}$

6:         Compute $\overline{w}_i^{(1)} \leftarrow \max\{e_{2T+1+i}^\intercal x \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0,$ $(\boldsymbol{A}(\underline{r}))^\intercal y \leq c,\ c^\intercal x = b^\intercal y,\ e_{T+1}^\intercal y \geq 0\}$

7:     **end for**

8:     Set $\mathcal{E}_1 \leftarrow [\underline{w}_1^{(1)}, \overline{w}_1^{(1)}] \times \ldots \times [\underline{w}_n^{(1)}, \overline{w}_n^{(1)}]$

9:     Set $\mathcal{R}_1 \leftarrow \theta + [\underline{w}_0^{(1)}, \overline{w}_0^{(1)}]$

10: **end if**

11: Let $\mathcal{P}^{(2)}$ be the linear program $\min\{0^\intercal x + 0^\intercal y \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0,$ $(\boldsymbol{A}(\overline{r}))^\intercal y \leq c,\ c^\intercal x = b^\intercal y,\ -e_{T+1}^\intercal y \geq 0\}$

12: **if** the program $\mathcal{P}^{(2)}$ is feasible **then**

13:     **for** $i \leftarrow 0$ **to** n **do**

14:         Compute $\underline{w}_i^{(2)} \leftarrow \min\{e_{2T+1+i}^\intercal x \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0,$ $(\boldsymbol{A}(\overline{r}))^\intercal y \leq c,\ c^\intercal x = b^\intercal y,\ -e_{T+1}^\intercal y \geq 0\}$

15:         Compute $\overline{w}_i^{(2)} \leftarrow \max\{e_{2T+1+i}^\intercal x \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0,$ $(\boldsymbol{A}(\overline{r}))^\intercal y \leq c,\ c^\intercal x = b^\intercal y,\ -e_{T+1}^\intercal y \geq 0\}$

16:     **end for**

17:     Set $\mathcal{E}_2 \leftarrow [\underline{w}_1^{(2)}, \overline{w}_1^{(2)}] \times \ldots \times [\underline{w}_n^{(2)}, \overline{w}_n^{(2)}]$

18:     Set $\mathcal{R}_2 \leftarrow \theta + [\underline{w}_0^{(2)}, \overline{w}_0^{(2)}]$

19: **end if**

20: Compute $\mathcal{E} \leftarrow$ Interval hull of $\mathcal{E}_1 \cup \mathcal{E}_2$

21: Compute $\mathcal{R} \leftarrow$ Interval hull of $\mathcal{R}_1 \cup \mathcal{R}_2$

22: **return** $\mathcal{E}$ and $\mathcal{R}$

At least one of linear programs $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ must be feasible as stated in lines 3 and 12. Most linear program solvers such as Optimization Toolbox™ in MATLAB® software and CPLEX® optimizer can directly test the feasibility with ease. Another suggestion is to add artificial variables to all constraints.

6.2.4 Suggested Optimal Asset Allocation

Proposition 6.3. Let B and N denote the nondegenerate optimal basis and nonbasis respectively when the MAD portfolio optimization problem (4.8) is solved with a restriction of the interval return $\boldsymbol{r}$ at its center r^c. If every problem derived from (4.8) with the constraint $\boldsymbol{A}(r) = b$ when r_B arbitrarily locates at their end points and r_N lies in the set $\{\underline{r}_N, \overline{r}_N\}$ still has the same optimal basis B with nondegeneracy. Then $w_N^* = 0$ in almost every scenario.

Proof. The sufficient and necessary condition for basis stability is stated in Theorem 2.12. It suffices to consider the system

$$(A^c - \mathrm{diag}(z)A^\Delta \mathrm{diag}(p))x = b^c + \mathrm{diag}(z)b^\Delta = b$$

where $z \in \{\pm 1\}^{T+2}$ and $p \in \{\pm 1\}^{T+1+n}$ with $p_N = 1^\intercal$ or $p_N = -1^\intercal$. Note that

$$A^c - \mathrm{diag}(z)A^\Delta \mathrm{diag}(p) = A^c - \begin{bmatrix} 0_{T\times(2T+1)} & 0_{T\times n} \\ 0_{1\times(2T+1)} & z_{T+1}\cdot\left(p_{2T+1+i}\cdot r_i^\Delta\right)_{1\times n} \\ 0_{1\times(2T+1)} & 0_{1\times n} \end{bmatrix}.$$

The coefficients of those assets lying in the nonbasis N must have the same sign. □

This proposition can be restated in the form of algorithm as follows.

Algorithm 6.3: Stability of optimal assets (B-stable)

Input: The portfolio selection model (4.8) with the threshold portfolio return θ

Output: Whether or not optimal assets are preserved

Compute $x \leftarrow \operatorname{argmin}\{c^\intercal x \mid \boldsymbol{A}(r^c)x = b,\, x \geq 0\}$
Construct $B = \{i \mid x_i > 0\}$
Construct $N = \{i \mid x_i = 0\}$
if $|B| \neq T + 2$ **then**
  return "Inconclusive"
end if
Set $\mathcal{O} = \{r \mid r_i \in \{\underline{r}_i, \overline{r}_i\},\, r_N \in \{\underline{r}_N, \overline{r}_N\}\}$
for all $r \in \mathcal{O}$ **do**
  Compute $x \leftarrow \operatorname{argmin}\{c^\intercal x \mid \boldsymbol{A}(r)x = b,\, x \geq 0\}$
  Set $I \leftarrow \{i \mid x_i > 0\}$
  if $I \neq B$ **then**
    return "Inconclusive"
  end if
end for
return "Optimal assets are stable"

6.2.5 Range of Optimal Risks

For any valid level of threshold portfolio return θ, an investor may want to evaluate the risk of an optimal portfolio in order to ensure that it is not beyond a stipulated risk tolerance and to compare different portfolio investments. Due to the uncertainty of the expected asset return $\boldsymbol{r}$, all possible risks are embedded in a closed interval as a result of the following proposition.

Proposition 6.4. Both minimum and maximum of optimal portfolio risks attained by the MAD model (4.8) exist.

Proof. Obviously, the lowest optimal portfolio risk equals the minimum value of $c^\intercal x$ over the weakly feasible solution set to the constraint $\boldsymbol{A}x = b$ in the MAD model (4.8). Whenever each asset i has an expected return of $r_i \in [\underline{r}_i, \overline{r}_i]$, the primal problem (4.8) always possesses an optimal value due to the assumption of strong feasibility imposed on a threshold portfolio return as stated in Subsection 6.1. The strong duality theorem 2.1 implies its dual program which is a maximum problem always have an optimal value. With a dual variable y, the maximum value of $b^\intercal y$ over the weakly feasible solution set to the dual constraint becomes the highest optimal portfolio risk. $\square$

According to Theorems 2.8 and 2.9, the lowest risk $\underline{\varrho}$ and the highest risk $\overline{\varrho}$ are obtained by the formula

$$\begin{aligned}\underline{\varrho} &= \frac{1}{T} \cdot \min\{\underline{c}^\intercal x \mid \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0\} \\ \overline{\varrho} &= \frac{1}{T} \cdot \max\{\varrho_z \mid z \in \{\pm 1\}^{T+2}\}\end{aligned}$$

where

$$\varrho_z = \max\{b^\intercal y \mid (A^c - z_{T+1}A^\Delta)^\intercal y \leq c,\ \mathrm{diag}(z)y \geq 0\}.$$

Yet, z_{T+1} takes on a value of either -1 or 1. The other components of z are unrestricted. The constraint $\mathrm{diag}(z)y \geq 0$ can be reduced to $z_{T+1}y_{T+1} \geq 0$. Hence, the range of portfolio risks can be calculated by the following procedure.

Algorithm 6.4: Determine the range of optimal portfolio risks

---

Input: The portfolio selection model (4.8) with T observations and the threshold portfolio return θ

Output: The lowest risk $\underline{\varrho}$ and the highest risk $\overline{\varrho}$

1: Compute $\underline{\varrho} \leftarrow (1/T) \cdot \min\{c^\intercal x \mid \boldsymbol{A}(\underline{r})x \leq b,\ \boldsymbol{A}(\overline{r})x \geq b,\ x \geq 0\}$

2: Calculate $\overline{\varrho}_1 \leftarrow (1/T) \cdot \max\{b^\intercal y \mid (\boldsymbol{A}(\underline{r}))^\intercal y \leq c,\ e_{T+1}^\intercal y \geq 0\}$

3: Calculate $\overline{\varrho}_2 \leftarrow (1/T) \cdot \max\{b^\intercal y \mid (\boldsymbol{A}(\overline{r}))^\intercal y \leq c,\ -e_{T+1}^\intercal y \geq 0\}$

4: Compare $\overline{\varrho} = \max\{\overline{\varrho}_1, \overline{\varrho}_2\}$

5: **return** $\underline{\varrho}$ and $\overline{\varrho}$

Algorithm 6.4 is simple and useful for comparison only within the same class of risk measures. The MAD measure is consistently employed throughout this work.

6.3 Summary of Algorithms

In conclusion, a risk-averse investor wants to obtain the optimal portfolio $\{w_i^*\}_{i=1}^n$ consisting of n assets, each of which has expected return represented by an interval. Arising from the uncertainty, the single-valued optimal portfolio may be nonexistent. To overcome this difficulty, the enclosure in which the optimal weights must lie is suggested in Algorithms 6.2 and 6.3. With the historical returns of each asset, the uncertain future rates and the minimum overall rate of return expected from the portfolio, the summary of algorithms is provided below.

Algorithm 6.5: MAD portfolio selection model

Input: Interval of future asset returns $\{[\underline{r}_i, \overline{r}_i]\}_{i=1}^{n}$, historical returns $\{\widetilde{r}_{i,t}\}_{t=1}^{T}$ and threshold portfolio return θ

Output: Enclosures of optimal portfolios $\mathcal{E}$ and returns $\mathcal{R}$

1: Obtain the interval matrix $\boldsymbol{A}$ and the vectors b, c based on the program (4.8)
2: Compute the minimum $\underline{\theta}$ and the maximum $\overline{\theta}$ by Algorithm 6.1
3: **if** $\theta \in [\underline{\theta}, \overline{\theta}]$ **then**
4:     Apply Algorithm 6.3 to determine whether the program (4.8) is B-stable
5:     **if** the program (4.8) is B-stable **then**
6:         Add the constraints $x_i = 0$ for i in the nonbasis N
7:     **end if**
8:     Apply Algorithm 6.2 to obtain the enclosures $\mathcal{E}$ and $\mathcal{R}$ of optimal portfolios and their returns
9:     **return** $\mathcal{E}$ and $\mathcal{R}$
10: **else**
11:     **return** "Infeasible portfolio"
12: **end if**

CHAPTER VII

EXAMPLES AND NUMERICAL RESULTS

The results developed in previous chapters are illustrated through specific examples and numerical results here. The bilevel portfolio model (5.5) with the highest risk and the rest of the linear optimization problems are solved by the dual simplex method implemented in CPLEX® and MATLAB® softwares respectively. The results demonstrate the effect of diversification on portfolio risk. An optimal risk tends to decrease as more stocks are included in a portfolio when compared under the same value of the adjustment parameter $\mu = 0$. The relaxation behavior of the bilevel model (5.5) as discussed in Section 5.3 is also consistently emphasized. In reporting tables, the symbols N/A and * stand for not applicable and any numerical value in a given return interval, respectively.

7.1 Adjusted Bounds on Expected Stock Returns

To demonstrate the usefulness of Algorithm 6.3 for obtaining a tighter enclosure of portfolio investments, the bounds on expected stock returns should be sufficiently accurate. However, the estimates in Table 3.3 are overly inexact by the nature of methodology for computing the upper bounds. Reinvestment rates during the buy-and-hold investment period completely overestimate the unknown parameter of expected returns due to the impossibility of reinvestment strategy. Therefore, the upper bounds are adjusted to

$$\overline{r}^{adj}(\mu) = \frac{\mu}{1+\mu} \cdot \underline{r} + \frac{1}{1+\mu} \cdot \overline{r}. \tag{7.1}$$

depending upon the parameter $\mu \in [0, +\infty)$. No adjustment occurs when $\mu = 0$. The return bounds become more precise with an increasing value of μ. These adjusted returns are illustrated in Table 7.1 across three different values of the parameter $\mu = 0, 7, 27$.

Table 7.1: Adjusted bounds on expected S&P 500 stock returns accrued in the next six months from November 2018 to April 2019 with the parameters $\mu = 0, 7, 27$

Ticker	Lower bound on return	Adjusted upper bounds on return		
		$\mu = 27$	$\mu = 7$	$\mu = 0$
AAPL	2.78%	3.12%	3.97%	12.34%
ADBE	3.87%	6.32%	12.44%	72.42%

Table 7.1: Adjusted bounds on expected S&P 500 stock returns accrued in the next six months from November 2018 to April 2019 with the parameters $\mu = 0, 7, 27$ (continued)

Ticker	Lower bound on return	Adjusted upper bounds on return		
		$\mu = 27$	$\mu = 7$	$\mu = 0$
ADSK	6.39%	7.69%	10.94%	42.83%
AET	3.59%	4.61%	7.15%	32.08%
ALGN	7.87%	8.56%	10.28%	27.13%
ALL	3.46%	4.66%	7.67%	37.13%
AMGN	3.38%	4.81%	8.41%	43.63%
AMT	4.06%	4.97%	7.27%	29.77%
AVGO	3.99%	5.02%	7.59%	32.80%
AXP	2.70%	9.17%	25.35%	183.92%
BLK	4.35%	5.22%	7.41%	28.81%
CCL.U	4.16%	5.58%	9.12%	43.80%
CI	4.17%	5.08%	7.36%	29.73%
CMCSA	4.38%	5.18%	7.16%	26.60%
COG	5.79%	6.46%	8.14%	24.64%
COL	4.85%	5.91%	8.56%	34.54%
CTSH	4.46%	5.20%	7.06%	25.29%
D	3.45%	4.04%	5.52%	20.01%
DFS	5.84%	6.23%	7.20%	16.70%
DIS	2.52%	3.09%	4.49%	18.28%
DRI	4.80%	5.51%	7.29%	24.71%
DUK	2.99%	4.10%	6.89%	34.17%
DVA	9.61%	9.62%	9.65%	9.94%
EBAY	3.74%	5.09%	8.47%	41.57%
ECL	3.72%	4.03%	4.81%	12.39%
EOG	4.62%	5.24%	6.79%	21.94%
ETN	3.72%	4.03%	4.81%	12.38%
FDX	2.87%	3.87%	6.37%	30.83%
FMC	5.14%	5.46%	6.27%	14.19%
IBM	2.67%	3.24%	4.65%	18.51%
INTC	3.68%	4.81%	7.65%	35.48%
INTU	3.67%	4.65%	7.09%	31.06%

Table 7.1: Adjusted bounds on expected S&P 500 stock returns accrued in the next six months from November 2018 to April 2019 with the parameters $\mu = 0, 7, 27$ (continued)

Ticker	Lower bound on return	Adjusted upper bounds on return		
		$\mu = 27$	$\mu = 7$	$\mu = 0$
ISRG	6.55%	6.78%	7.35%	12.91%
IVZ	6.74%	7.73%	10.20%	34.41%
KMX	6.15%	6.42%	7.10%	13.74%
LLL	3.79%	4.37%	5.80%	19.84%
LRCX	7.17%	7.53%	8.43%	17.30%
MA	3.42%	4.06%	5.64%	21.20%
MAR	5.17%	6.79%	10.83%	50.39%
MCHP	5.90%	6.09%	6.56%	11.20%
MMM	3.16%	4.00%	6.11%	26.74%
MPC	4.46%	4.74%	5.43%	12.15%
MRO	9.89%	10.84%	13.23%	36.60%
MS	2.77%	3.33%	4.72%	18.41%
MSFT	2.52%	3.70%	6.65%	35.59%
NUE	4.00%	4.42%	5.49%	15.96%
PHM	6.98%	8.33%	11.70%	44.69%
PX	5.66%	5.93%	6.61%	13.29%
ROK	5.52%	6.00%	7.20%	18.97%
STI	3.82%	4.79%	7.21%	30.92%
SWK	5.03%	5.48%	6.58%	17.42%
TROW	3.45%	3.70%	4.32%	10.36%
TRV	3.43%	5.51%	10.72%	61.75%
TXN	3.56%	3.94%	4.90%	14.33%
UPS	3.39%	4.38%	6.86%	31.11%
VRTX	8.97%	9.92%	12.31%	35.71%
WFC	2.75%	3.14%	4.12%	13.71%
WM	3.57%	4.34%	6.25%	25.01%
WY	4.13%	4.41%	5.11%	11.92%
ZION	6.17%	6.77%	8.26%	22.85%
ZTS	4.07%	5.04%	7.47%	31.31%

7.2 Examples of 2-Asset Portfolios

Two portfolios consisting of 2 assets are considered with unadjusted return bounds (at the parameter $\mu = 0$). The first invests in MPC and DVA stocks as shown in Table 7.2. The other invests in ALL and DVA stocks as also shown in Table 7.3. Algorithm 6.1 implies an investor cannot have a prior expectation of more than 9.61% return from investments (i.e. $\theta \leq 1.0961$).

With a no-loss guarantee (the threshold rate of return $\theta = 1$ is selected), both tables provide details on various investment strategies: the overall enclosures from Algorithm 6.2, the tight enclosures suggested in Algorithms 6.3 and 6.5, the optimal portfolios at a particular scenario based on the model (4.4) and the bilevel portfolios obtained by the models (5.3) and (5.5). Adding ALL stock to DVA stock is more risky compared to adding MPC stock. A portfolio of MPC and DVA stocks yields a return of ranging from 7.63% to 10.68%, and both stocks must be included. A portfolio of ALL and DVA stocks guarantee a return of 8.37% but no more than 9.94%, and over 79.82% of the capital must be allocated to DVA stock.

It is also revealed that applying bilevel optimization to tackle a portfolio problem contributes to misleading results as previously discussed in Section 5.3. To illustrate, the bilevel portfolio investing in MPC and DVA stocks has the highest risk of 0.0658 which does not lie between 0.0525 and 0.0593 as reported in the overall enclosure. The same problem also arises in the same type of optimal portfolio investing in ALL and DVA stocks. The erratic behavior of the bilevel portfolio model (5.5) can be explained by the inconsistency of the attainable rate of an individual stock's return $\xi^*_{i,t}/y^*_t$ based on the formula (5.6) as shown in Tables 7.4 and 7.5.

Furthermore, both bilevel portfolios do not serve as a bound on optimal portfolio composition. In Table 7.2, when MPC and DVA stocks attain the rates of return 12.15% and 9.94%, an investor should allocate exactly 33.51% of the capital to MPC stock. This proportion is not between 24.51% and 24.79% as formed by the bilevel portfolios. However, our proposed method yields a more universally efficient bound on optimal investments, at least 13.71% but no more than 38.49%.

Our novel approach of tightening enclosure completely outperforms the rest of methods for optimal portfolio investments in ALL and DVA stocks. Despite up to 33.76% and 0.33% disagreements over the expected returns on both stocks, a unique optimal portfolio composition is suggested: investing all money in DVA stock. This agrees on all compositions under the specific scenario when ALL and DVA stocks attain 3.46% and 9.94% returns and also under bilevel optimization. The exactness of optimal portfolio composition does not contradict its corresponding high risk because risk simply measures the deviation of exact return observed in the future from its expected value.

Table 7.2: Optimal 2-asset portfolios investing in MPC and DVA stocks with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
MPC	13.71%	38.49%	N/A	N/A	12.15%	33.51%	4.46%	24.79%	N/A	24.51%
DVA	61.51%	86.29%	N/A	N/A	9.94%	66.49%	9.61%	75.21%	N/A	75.49%
Optimal return	7.63%	10.68%	N/A	N/A		10.68%		8.34%		N/A
Optimal risk	0.0525	0.0593	N/A	N/A		0.0593		0.0525		0.0658

Table 7.3: Optimal 2-asset portfolios investing in ALL and DVA stocks with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
ALL	0.00%	20.18%	0.00%	0.00%	3.46%	0.00%	*	0.00%	N/A	0.00%
DVA	79.82%	100.00%	100.00%	100.00%	9.94%	100.00%	9.61%	100.00%	N/A	100.00%
Optimal return	8.37%	9.94%	9.61%	9.94%		9.94%		9.61%		N/A
Optimal risk	0.0747	0.0772	0.0747	0.0772		0.0772		0.0747		0.0776

Table 7.4: Contradictory attainable rates of return of MPC and DVA stocks in their 2-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
MPC	12.15%	12.15%	12.15%	*	*	12.15%	12.15%	12.15%	*	4.46%	*	4.46%	4.46%	*	*	*	*
DVA	9.94%	9.94%	9.94%	*	*	9.94%	9.94%	9.94%	*	9.61%	*	9.61%	9.61%	*	*	*	*

Table 7.5: Contradictory attainable rates of return of ALL and DVA stocks in their 2-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
ALL	37.13%	37.13%	37.13%	*	37.13%	37.13%	37.13%	37.13%	*	*	*	3.46%	*	*	*	*	*
DVA	9.94%	9.94%	9.94%	*	9.94%	9.94%	9.94%	9.94%	*	*	*	9.61%	*	*	*	*	*

7.3 Examples of 9-Asset Portfolios

A portfolio including D, DVA, ETN, KMX, MCHP, MPC, PX, TROW and WY stocks across 9 different U.S. sectors is considered. The initial value of the parameter θ cannot exceed 9.61%. When the upper bounds on their expected returns are unadjusted, optimal portfolio weights returns are illustrated in Table 7.6 with returns of between 3.55% and 11.26%. The enclosure cannot be improved because the data are overly inexact. As a result, the upper return bounds are adjusted with the parameter $\mu = 7$. The new optimal portfolio composition is shown in Table 7.7. A tightened enclosure suggests investing in only 5 out of 9 stocks (D, DVA, ETN, KMX and MCHP) and reports attainable returns of between 5.81% and 6.82%. The problematic issue of bilevel optimization also arises as seen in Tables 7.8 and 7.9.

Table 7.6: Optimal 9-asset portfolios investing in D, DVA, ETN, KMX, MCHP, MPC, PX, TROW and WY stocks with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
D	0.00%	69.93%	N/A	N/A	20.01%	0.00%	3.45%	11.16%	N/A	0.00%
DVA	0.00%	83.86%	N/A	N/A	9.61%	55.76%	9.61%	26.66%	N/A	75.49%
ETN	0.00%	100.00%	N/A	N/A	12.38%	13.00%	3.72%	38.16%	N/A	0.00%
KMX	0.00%	62.29%	N/A	N/A	13.74%	0.00%	6.15%	11.23%	N/A	0.00%
MCHP	0.00%	74.88%	N/A	N/A	11.20%	0.00%	5.90%	12.79%	N/A	0.00%
MPC	0.00%	48.88%	N/A	N/A	12.15%	31.25%	*	0.00%	N/A	24.51%
PX	0.00%	63.30%	N/A	N/A	13.29%	0.00%	*	0.00%	N/A	0.00%
TROW	0.00%	50.96%	N/A	N/A	10.36%	0.00%	*	0.00%	N/A	0.00%
WY	0.00%	55.16%	N/A	N/A	11.92%	0.00%	*	0.00%	N/A	0.00%
Optimal return	3.55%	11.26%	N/A	N/A		10.77%		5.81%		N/A
Optimal risk	0.0273	0.0565	N/A	N/A		0.0554		0.0273		0.0658

Table 7.7: Optimal 9-asset portfolios investing in D, DVA, ETN, KMX, MCHP, MPC, PX, TROW and WY stocks with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 7$

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
D	0.00%	20.59%	11.16%	15.89%	5.52%	15.82%	3.45%	11.16%	N/A	13.52%
DVA	0.00%	38.30%	26.66%	31.09%	9.61%	31.02%	9.61%	26.66%	N/A	28.87%
ETN	27.00%	78.86%	32.55%	38.16%	4.81%	32.64%	3.72%	38.16%	N/A	35.36%
KMX	0.00%	27.62%	6.57%	11.23%	7.10%	6.64%	6.15%	11.23%	N/A	8.90%
MCHP	0.00%	20.51%	12.79%	13.90%	6.56%	13.88%	5.90%	12.79%	N/A	13.34%
MPC	0.00%	16.50%	0.00%	0.00%	5.43%	0.00%	*	0.00%	N/A	0.00%
PX	0.00%	7.74%	0.00%	0.00%	6.61%	0.00%	*	0.00%	N/A	0.00%
TROW	0.00%	18.38%	0.00%	0.00%	4.32%	0.00%	*	0.00%	N/A	0.00%
WY	0.00%	6.87%	0.00%	0.00%	5.11%	0.00%	*	0.00%	N/A	0.00%
Optimal return	3.91%	6.82%	5.81%	6.82%		6.81%		5.81%		N/A
Optimal risk	0.0273	0.0305	0.0273	0.0305		0.0304		0.0273		0.0331

Table 7.8: Contradictory attainable rates of return of D, DVA, ETN, KMX, MCHP, MPC, PX, TROW and WY stocks in their 9-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
D	20.01%	20.01%	20.01%	*	*	20.01%	20.01%	20.01%	*	3.45%	*	3.45%	3.45%	*	*	*	*
DVA	9.94%	9.94%	9.94%	*	*	9.94%	9.94%	9.94%	*	9.61%	*	9.61%	9.61%	*	*	*	*
ETN	12.38%	12.38%	12.38%	*	*	12.38%	12.38%	12.38%	*	3.72%	*	3.72%	3.72%	*	*	*	*
KMX	13.74%	13.74%	13.74%	*	*	13.74%	13.74%	13.74%	*	6.15%	*	6.15%	6.15%	*	*	*	*
MCHP	11.20%	11.20%	11.20%	*	*	11.20%	11.20%	11.20%	*	5.90%	*	5.90%	5.90%	*	*	*	*
MPC	12.15%	12.15%	12.15%	*	*	12.15%	12.15%	12.15%	*	4.46%	*	4.46%	4.46%	*	*	*	*
PX	13.29%	13.29%	13.29%	*	*	13.29%	13.29%	13.29%	*	5.66%	*	5.66%	5.66%	*	*	*	*
TROW	10.36%	10.36%	10.36%	*	*	10.36%	10.36%	10.36%	*	3.45%	*	3.45%	3.45%	*	*	*	*
WY	11.92%	11.92%	11.92%	*	*	11.92%	11.92%	11.92%	*	4.13%	*	4.13%	4.13%	*	*	*	*

Table 7.9: Contradictory attainable rates of return of D, DVA, ETN, KMX, MCHP, MPC, PX, TROW and WY stocks in their 9-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 7$

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
D	5.52%	5.52%	5.52%	5.52%	*	5.52%	5.52%	5.52%	3.45%	3.45%	3.45%	3.45%	3.45%	*	*	*	*
DVA	9.65%	9.65%	9.65%	9.65%	*	9.65%	9.65%	9.65%	9.61%	9.61%	9.61%	9.61%	9.61%	*	*	*	*
ETN	4.81%	4.81%	4.81%	4.81%	*	4.81%	4.81%	4.81%	3.72%	3.72%	3.72%	3.72%	3.72%	*	*	*	*
KMX	7.10%	7.10%	7.10%	7.10%	*	7.10%	7.10%	7.10%	6.15%	6.15%	6.15%	6.15%	6.15%	*	*	*	*
MCHP	6.56%	6.56%	6.56%	6.56%	*	6.56%	6.56%	6.56%	5.90%	5.90%	5.90%	5.90%	5.90%	*	*	*	*
MPC	5.43%	5.43%	5.43%	5.43%	*	5.43%	5.43%	5.43%	4.46%	4.46%	4.46%	4.46%	4.46%	*	*	*	*
PX	6.61%	6.61%	6.61%	6.61%	*	6.61%	6.61%	6.61%	5.66%	5.66%	5.66%	5.66%	5.66%	*	*	*	*
TROW	4.32%	4.32%	4.32%	4.32%	*	4.32%	4.32%	4.32%	3.45%	3.45%	3.45%	3.45%	3.45%	*	*	*	*
WY	5.11%	5.11%	5.11%	5.11%	*	5.11%	5.11%	5.11%	4.13%	4.13%	4.13%	4.13%	4.13%	*	*	*	*

7.4 Examples of 61-Asset Portfolios

A portfolio including all 61 S&P 500 stocks empirically exhibiting the negative correlation condition (NCC) is considered. The initial value of the parameter θ cannot exceed 9.81%. When the upper bounds on their expected returns are unadjusted, optimal portfolio compositions are illustrated in Table 7.10 with a wide range of returns, between 3.55% and 11.26%. The enclosure cannot be improved because the data are overly inexact. As a result, the upper return bounds are adjusted with the parameter $\mu = 27$. The new optimal portfolio weights are shown in Table 7.11. A tightened enclosure suggests investing in only 9 out of 61 stocks: ADBE, AMGN, AXP, CCL.U, DIS, EBAY, MAR, TROW and TRV. Returns of between 5.81% and 6.82% are obtained. The problematic issue of bilevel optimization also arises as seen in Tables 7.12 and 7.13.

According to Table 7.11, the bilevel portfolio with the lowest risk seems to oppose our tight enclosure because the first recommends investing 3.40% in AAPL stock whereas the latter suggests none. This is due to the assumption of unique optimal solution which is reasonably relaxed as noted in Subsection 6.2.4. The tight enclosure is merely a suggestion. All combinations of optimal portfolio compositions are reported in the overall enclosure.

Table 7.10: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
AAPL	0.00%	99.51%	N/A	N/A	2.78%	0.00%	2.78%	10.94%	N/A	0.00%
ADBE	0.00%	52.83%	N/A	N/A	3.87%	6.46%	*	0.00%	N/A	0.00%
ADSK	0.00%	68.30%	N/A	N/A	6.39%	0.00%	*	0.00%	N/A	0.00%
AET	0.00%	71.71%	N/A	N/A	3.59%	0.00%	*	0.00%	N/A	0.00%
ALGN	0.00%	49.90%	N/A	N/A	7.87%	0.00%	*	0.00%	N/A	0.00%
ALL	0.00%	45.48%	N/A	N/A	3.46%	0.00%	*	0.00%	N/A	0.00%
AMGN	0.00%	39.23%	N/A	N/A	3.38%	8.57%	*	0.00%	N/A	0.00%
AMT	0.00%	58.66%	N/A	N/A	4.06%	0.00%	*	0.00%	N/A	0.00%
AVGO	0.00%	65.57%	N/A	N/A	3.99%	0.00%	*	0.00%	N/A	0.00%
AXP	0.00%	22.90%	N/A	N/A	2.70%	2.66%	*	0.00%	N/A	0.00%
BLK	0.00%	95.72%	N/A	N/A	4.35%	0.00%	*	0.00%	N/A	0.00%
CCL.U	0.00%	46.15%	N/A	N/A	4.16%	21.04%	*	0.00%	N/A	0.00%
CI	0.00%	95.62%	N/A	N/A	4.17%	0.00%	*	0.00%	N/A	0.00%
CMCSA	0.00%	80.99%	N/A	N/A	4.38%	0.00%	*	0.00%	N/A	0.00%
COG	0.00%	60.66%	N/A	N/A	5.79%	0.00%	*	0.00%	N/A	0.00%

Table 7.10: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$ (continued)

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
COL	0.00%	86.61%	N/A	N/A	4.85%	0.00%	*	0.00%	N/A	0.00%
CTSH	0.00%	99.45%	N/A	N/A	4.46%	0.00%	*	0.00%	N/A	0.00%
D	0.00%	88.52%	N/A	N/A	3.45%	0.00%	3.45%	2.54%	N/A	0.00%
DFS	0.00%	87.24%	N/A	N/A	5.84%	0.00%	*	0.00%	N/A	0.00%
DIS	0.00%	100.00%	N/A	N/A	2.52%	3.24%	2.52%	29.43%	N/A	0.00%
DRI	0.00%	79.42%	N/A	N/A	4.80%	0.00%	*	0.00%	N/A	0.00%
DUK	0.00%	93.12%	N/A	N/A	2.99%	0.00%	*	0.00%	N/A	0.00%
DVA	0.00%	94.71%	N/A	N/A	9.61%	0.00%	*	0.00%	N/A	75.49%
EBAY	0.00%	70.01%	N/A	N/A	3.74%	0.79%	3.74%	19.33%	N/A	0.00%
ECL	0.00%	96.98%	N/A	N/A	3.72%	0.00%	*	0.00%	N/A	0.00%
EOG	0.00%	100.00%	N/A	N/A	4.62%	0.00%	*	0.00%	N/A	0.00%
ETN	0.00%	100.00%	N/A	N/A	3.72%	0.00%	*	0.00%	N/A	0.00%
FDX	0.00%	50.03%	N/A	N/A	2.87%	0.00%	*	0.00%	N/A	0.00%
FMC	0.00%	93.25%	N/A	N/A	5.14%	0.00%	5.14%	3.66%	N/A	0.00%
IBM	0.00%	75.48%	N/A	N/A	2.67%	0.00%	*	0.00%	N/A	0.00%

Table 7.10: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$ (continued)

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
INTC	0.00%	71.62%	N/A	N/A	3.68%	0.00%	*	0.00%	N/A	0.00%
INTU	0.00%	90.36%	N/A	N/A	3.67%	0.00%	*	0.00%	N/A	0.00%
ISRG	0.00%	87.82%	N/A	N/A	6.55%	0.00%	*	0.00%	N/A	0.00%
IVZ	0.00%	82.65%	N/A	N/A	6.74%	0.00%	6.74%	4.51%	N/A	0.00%
KMX	0.00%	71.42%	N/A	N/A	6.15%	0.00%	*	0.00%	N/A	0.00%
LLL	0.00%	100.00%	N/A	N/A	3.79%	0.00%	*	0.00%	N/A	0.00%
LRCX	0.00%	96.23%	N/A	N/A	7.17%	0.00%	*	0.00%	N/A	0.00%
MA	0.00%	83.29%	N/A	N/A	3.42%	0.00%	*	0.00%	N/A	0.00%
MAR	0.00%	59.85%	N/A	N/A	5.17%	16.31%	5.17%	0.86%	N/A	0.00%
MCHP	0.00%	88.43%	N/A	N/A	5.90%	0.00%	*	0.00%	N/A	0.00%
MMM	0.00%	78.88%	N/A	N/A	3.16%	0.00%	*	0.00%	N/A	0.00%
MPC	0.00%	75.35%	N/A	N/A	4.46%	0.00%	*	0.00%	N/A	24.51%
MRO	0.00%	63.14%	N/A	N/A	9.89%	0.00%	*	0.00%	N/A	0.00%
MS	0.00%	84.26%	N/A	N/A	2.77%	0.00%	2.77%	18.41%	N/A	0.00%
MSFT	0.00%	74.46%	N/A	N/A	2.52%	0.00%	*	0.00%	N/A	0.00%

Table 7.10: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$ (continued)

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
NUE	0.00%	93.03%	N/A	N/A	4.00%	0.00%	*	0.00%	N/A	0.00%
PHM	0.00%	59.07%	N/A	N/A	6.98%	0.00%	*	0.00%	N/A	0.00%
PX	0.00%	87.55%	N/A	N/A	5.66%	0.00%	*	0.00%	N/A	0.00%
ROK	0.00%	89.90%	N/A	N/A	5.52%	0.00%	*	0.00%	N/A	0.00%
STI	0.00%	36.45%	N/A	N/A	3.82%	0.00%	*	0.00%	N/A	0.00%
SWK	0.00%	90.45%	N/A	N/A	5.03%	0.00%	*	0.00%	N/A	0.00%
TROW	0.00%	78.34%	N/A	N/A	3.45%	39.09%	3.45%	10.32%	N/A	0.00%
TRV	0.00%	41.58%	N/A	N/A	3.43%	1.85%	*	0.00%	N/A	0.00%
TXN	0.00%	99.64%	N/A	N/A	3.56%	0.00%	*	0.00%	N/A	0.00%
UPS	0.00%	50.65%	N/A	N/A	3.39%	0.00%	*	0.00%	N/A	0.00%
VRTX	0.00%	49.16%	N/A	N/A	8.97%	0.00%	*	0.00%	N/A	0.00%
WFC	0.00%	100.00%	N/A	N/A	2.75%	0.00%	*	0.00%	N/A	0.00%
WM	0.00%	83.23%	N/A	N/A	3.57%	0.00%	*	0.00%	N/A	0.00%
WY	0.00%	75.48%	N/A	N/A	4.13%	0.00%	*	0.00%	N/A	0.00%
ZION	0.00%	67.82%	N/A	N/A	6.17%	0.00%	*	0.00%	N/A	0.00%

Table 7.10: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$ (continued)

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
ZTS	0.00%	84.70%	N/A	N/A	4.07%	0.00%	*	0.00%	N/A	0.00%
Optimal return	2.52%	32.28%	N/A	N/A		3.85%		3.26%		N/A
Optimal risk	0.0000	0.0535	N/A	N/A		0.0000		0.0000		0.0658

Table 7.11: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
AAPL	0.00%	23.50%	0.00%	0.00%	2.78%	0.00%	2.78%	3.40%	N/A	0.00%
ADBE	0.00%	9.87%	6.46%	7.17%	3.87%	6.46%	*	0.00%	N/A	0.00%
ADSK	0.00%	29.86%	0.00%	0.00%	6.39%	0.00%	*	0.00%	N/A	0.00%
AET	0.00%	14.52%	0.00%	0.00%	3.59%	0.00%	*	0.00%	N/A	0.00%
ALGN	0.00%	13.75%	0.00%	0.00%	7.87%	0.00%	*	0.00%	N/A	0.00%
ALL	0.00%	12.39%	0.00%	0.00%	3.46%	0.00%	*	0.00%	N/A	0.00%
AMGN	0.00%	12.73%	8.57%	8.63%	3.38%	8.57%	*	0.00%	N/A	0.00%
AMT	0.00%	12.16%	0.00%	0.00%	4.06%	0.00%	*	0.00%	N/A	0.00%
AVGO	0.00%	26.39%	0.00%	0.00%	3.99%	0.00%	*	0.00%	N/A	0.00%
AXP	0.00%	5.74%	2.66%	2.89%	2.70%	2.66%	*	0.00%	N/A	0.00%
BLK	0.00%	41.27%	0.00%	0.00%	4.35%	0.00%	*	0.00%	N/A	18.33%
CCL.U	0.00%	21.49%	16.47%	21.04%	4.16%	21.04%	*	0.00%	N/A	0.00%
CI	0.00%	34.06%	0.00%	0.00%	4.17%	0.00%	*	0.00%	N/A	0.00%
CMCSA	0.00%	33.23%	0.00%	0.00%	4.38%	0.00%	*	0.00%	N/A	0.00%
COG	0.00%	11.15%	0.00%	0.00%	5.79%	0.00%	*	0.00%	N/A	0.00%

Table 7.11: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$ (continued)

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
COL	0.00%	28.41%	0.00%	0.00%	4.85%	0.00%	*	0.00%	N/A	0.00%
CTSH	0.00%	44.69%	0.00%	0.00%	4.46%	0.00%	4.46%	13.03%	N/A	0.00%
D	0.00%	18.70%	0.00%	0.00%	3.45%	0.00%	3.45%	2.06%	N/A	0.00%
DFS	0.00%	24.20%	0.00%	0.00%	5.84%	0.00%	*	0.00%	N/A	0.00%
DIS	0.00%	54.96%	3.24%	9.69%	2.52%	3.24%	2.52%	27.15%	N/A	0.00%
DRI	0.00%	24.78%	0.00%	0.00%	4.80%	0.00%	*	0.00%	N/A	0.00%
DUK	0.00%	29.60%	0.00%	0.00%	2.99%	0.00%	*	0.00%	N/A	0.00%
DVA	0.00%	23.49%	0.00%	0.00%	9.61%	0.00%	*	0.00%	N/A	20.38%
EBAY	0.00%	31.81%	0.79%	5.04%	3.74%	0.79%	3.74%	14.00%	N/A	10.38%
ECL	0.00%	25.32%	0.00%	0.00%	3.72%	0.00%	*	0.00%	N/A	0.00%
EOG	0.00%	57.91%	0.00%	0.00%	4.62%	0.00%	*	0.00%	N/A	0.00%
ETN	0.00%	55.75%	0.00%	0.00%	3.72%	0.00%	*	0.00%	N/A	20.98%
FDX	0.00%	10.12%	0.00%	0.00%	2.87%	0.00%	*	0.00%	N/A	0.00%
FMC	0.00%	34.67%	0.00%	0.00%	5.14%	0.00%	5.14%	15.20%	N/A	0.00%
IBM	0.00%	27.89%	0.00%	0.00%	2.67%	0.00%	*	0.00%	N/A	0.00%

Table 7.11: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$ (continued)

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
INTC	0.00%	25.58%	0.00%	0.00%	3.68%	0.00%	*	0.00%	N/A	0.00%
INTU	0.00%	27.68%	0.00%	0.00%	3.67%	0.00%	*	0.00%	N/A	0.00%
ISRG	0.00%	24.12%	0.00%	0.00%	6.55%	0.00%	*	0.00%	N/A	0.00%
IVZ	0.00%	36.34%	0.00%	0.00%	6.74%	0.00%	6.74%	0.65%	N/A	0.00%
KMX	0.00%	13.46%	0.00%	0.00%	6.15%	0.00%	*	0.00%	N/A	0.00%
LLL	0.00%	40.81%	0.00%	0.00%	3.79%	0.00%	*	0.00%	N/A	0.00%
LRCX	0.00%	29.66%	0.00%	0.00%	7.17%	0.00%	*	0.00%	N/A	8.15%
MA	0.00%	26.98%	0.00%	0.00%	3.42%	0.00%	*	0.00%	N/A	0.00%
MAR	0.00%	28.33%	12.21%	16.31%	5.17%	16.31%	5.17%	5.84%	N/A	0.00%
MCHP	0.00%	37.56%	0.00%	0.00%	5.90%	0.00%	*	0.00%	N/A	0.00%
MMM	0.00%	24.19%	0.00%	0.00%	3.16%	0.00%	*	0.00%	N/A	0.00%
MPC	0.00%	30.92%	0.00%	0.00%	4.46%	0.00%	*	0.00%	N/A	0.00%
MRO	0.00%	15.20%	0.00%	0.00%	9.89%	0.00%	*	0.00%	N/A	0.00%
MS	0.00%	20.18%	0.00%	0.00%	2.77%	0.00%	2.77%	18.66%	N/A	0.00%
MSFT	0.00%	31.90%	0.00%	0.00%	2.52%	0.00%	*	0.00%	N/A	0.00%

Table 7.11: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$ (continued)

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
NUE	0.00%	59.00%	0.00%	0.00%	4.00%	0.00%	*	0.00%	N/A	9.19%
PHM	0.00%	18.85%	0.00%	0.00%	6.98%	0.00%	*	0.00%	N/A	3.16%
PX	0.00%	35.60%	0.00%	0.00%	5.66%	0.00%	*	0.00%	N/A	0.00%
ROK	0.00%	30.62%	0.00%	0.00%	5.52%	0.00%	*	0.00%	N/A	0.00%
STI	0.00%	6.48%	0.00%	0.00%	3.82%	0.00%	*	0.00%	N/A	0.00%
SWK	0.00%	22.69%	0.00%	0.00%	5.03%	0.00%	*	0.00%	N/A	0.00%
TROW	0.00%	44.13%	36.34%	39.09%	3.45%	39.09%	*	0.00%	N/A	3.58%
TRV	0.00%	15.62%	1.56%	1.85%	3.43%	1.85%	*	0.00%	N/A	0.00%
TXN	0.00%	34.12%	0.00%	0.00%	3.56%	0.00%	*	0.00%	N/A	0.00%
UPS	0.00%	14.89%	0.00%	0.00%	3.39%	0.00%	*	0.00%	N/A	0.00%
VRTX	0.00%	12.71%	0.00%	0.00%	8.97%	0.00%	*	0.00%	N/A	0.00%
WFC	0.00%	54.03%	0.00%	0.00%	2.75%	0.00%	*	0.00%	N/A	5.85%
WM	0.00%	20.52%	0.00%	0.00%	3.57%	0.00%	*	0.00%	N/A	0.00%
WY	0.00%	20.84%	0.00%	0.00%	4.13%	0.00%	*	0.00%	N/A	0.00%
ZION	0.00%	24.57%	0.00%	0.00%	6.17%	0.00%	*	0.00%	N/A	0.00%

Table 7.11: Optimal 61-asset portfolios investing in all S&P 500 stocks exhibiting the negative correlation condition with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$ (continued)

Ticker	Overall enclosure		Tight enclosure		Specific scenario		Bilevel with low risk		Bilevel with high risk	
	Min	Max	Min	Max	Return	Weight	Return	Weight	Return	Weight
ZTS	0.00%	26.76%	0.00%	0.00%	4.07%	0.00%	*	0.00%	N/A	0.00%
Optimal return	3.00%	6.43%	3.85%	4.87%		3.85%		3.60%		N/A
Optimal risk	0.0000	0.0000	0.0000	0.0000		0.0000		0.0000		0.0026

Table 7.12: Contradictory attainable rates of return of all S&P 500 stocks exhibiting the negative correlation condition in their 61-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$							
	1, 2, 3	4, 5	6, 7, 8	9	10	11	12, 13	14, 15, 16, 17
AAPL	12.34%	*	12.34%	*	2.78%	*	2.78%	*
ADBE	72.42%	*	72.42%	*	3.87%	*	3.87%	*
ADSK	42.83%	*	42.83%	*	6.39%	*	6.39%	*
AET	32.08%	*	32.08%	*	3.59%	*	3.59%	*
ALGN	27.13%	*	27.13%	*	7.87%	*	7.87%	*
ALL	37.13%	*	37.13%	*	3.46%	*	3.46%	*
AMGN	43.63%	*	43.63%	*	3.38%	*	3.38%	*
AMT	29.77%	*	29.77%	*	4.06%	*	4.06%	*
AVGO	32.80%	*	32.80%	*	3.99%	*	3.99%	*
AXP	183.92%	*	183.92%	*	2.70%	*	2.70%	*
BLK	28.81%	*	28.81%	*	4.35%	*	4.35%	*
CCL.U	43.80%	*	43.80%	*	4.16%	*	4.16%	*
CI	29.73%	*	29.73%	*	4.17%	*	4.17%	*
CMCSA	26.60%	*	26.60%	*	4.38%	*	4.38%	*
COG	24.64%	*	24.64%	*	5.79%	*	5.79%	*

Table 7.12: Contradictory attainable rates of return of all S&P 500 stocks exhibiting the negative correlation condition in their 61-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$ (continued)

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$							
	1, 2, 3	4, 5	6, 7, 8	9	10	11	12, 13	14, 15, 16, 17
COL	34.54%	*	34.54%	*	4.85%	*	4.85%	*
CTSH	25.29%	*	25.29%	*	4.46%	*	4.46%	*
D	20.01%	*	20.01%	*	3.45%	*	3.45%	*
DFS	16.70%	*	16.70%	*	5.84%	*	5.84%	*
DIS	18.28%	*	18.28%	*	2.52%	*	2.52%	*
DRI	24.71%	*	24.71%	*	4.80%	*	4.80%	*
DUK	34.17%	*	34.17%	*	2.99%	*	2.99%	*
DVA	9.94%	*	9.94%	*	9.61%	*	9.61%	*
EBAY	41.57%	*	41.57%	*	3.74%	*	3.74%	*
ECL	12.39%	*	12.39%	*	3.72%	*	3.72%	*
EOG	21.94%	*	21.94%	*	4.62%	*	4.62%	*
ETN	12.38%	*	12.38%	*	3.72%	*	3.72%	*
FDX	30.83%	*	30.83%	*	2.87%	*	2.87%	*
FMC	14.19%	*	14.19%	*	5.14%	*	5.14%	*
IBM	18.51%	*	18.51%	*	2.67%	*	2.67%	*

Table 7.12: Contradictory attainable rates of return of all S&P 500 stocks exhibiting the negative correlation condition in their 61-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$ (continued)

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$							
	1, 2, 3	4, 5	6, 7, 8	9	10	11	12, 13	14, 15, 16, 17
INTC	35.48%	*	35.48%	*	3.68%	*	3.68%	*
INTU	31.06%	*	31.06%	*	3.67%	*	3.67%	*
ISRG	12.91%	*	12.91%	*	6.55%	*	6.55%	*
IVZ	34.41%	*	34.41%	*	6.74%	*	6.74%	*
KMX	13.74%	*	13.74%	*	6.15%	*	6.15%	*
LLL	19.84%	*	19.84%	*	3.79%	*	3.79%	*
LRCX	17.30%	*	17.30%	*	7.17%	*	7.17%	*
MA	21.20%	*	21.20%	*	3.42%	*	3.42%	*
MAR	50.39%	*	50.39%	*	5.17%	*	5.17%	*
MCHP	11.20%	*	11.20%	*	5.90%	*	5.90%	*
MMM	26.74%	*	26.74%	*	3.16%	*	3.16%	*
MPC	12.15%	*	12.15%	*	4.46%	*	4.46%	*
MRO	36.60%	*	36.60%	*	9.89%	*	9.89%	*
MS	18.41%	*	18.41%	*	2.77%	*	2.77%	*
MSFT	35.59%	*	35.59%	*	2.52%	*	2.52%	*

Table 7.12: Contradictory attainable rates of return of all S&P 500 stocks exhibiting the negative correlation condition in their 61-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$ (continued)

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$							
	1, 2, 3	4, 5	6, 7, 8	9	10	11	12, 13	14, 15, 16, 17
NUE	15.96%	*	15.96%	*	4.00%	*	4.00%	*
PHM	44.69%	*	44.69%	*	6.98%	*	6.98%	*
PX	13.29%	*	13.29%	*	5.66%	*	5.66%	*
ROK	18.97%	*	18.97%	*	5.52%	*	5.52%	*
STI	30.92%	*	30.92%	*	3.82%	*	3.82%	*
SWK	17.42%	*	17.42%	*	5.03%	*	5.03%	*
TROW	10.36%	*	10.36%	*	3.45%	*	3.45%	*
TRV	61.75%	*	61.75%	*	3.43%	*	3.43%	*
TXN	14.33%	*	14.33%	*	3.56%	*	3.56%	*
UPS	31.11%	*	31.11%	*	3.39%	*	3.39%	*
VRTX	35.71%	*	35.71%	*	8.97%	*	8.97%	*
WFC	13.71%	*	13.71%	*	2.75%	*	2.75%	*
WM	25.01%	*	25.01%	*	3.57%	*	3.57%	*
WY	11.92%	*	11.92%	*	4.13%	*	4.13%	*
ZION	22.85%	*	22.85%	*	6.17%	*	6.17%	*

Table 7.12: Contradictory attainable rates of return of all S&P 500 stocks exhibiting the negative correlation condition in their 61-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on unadjusted return bounds at the parameter $\mu = 0$ (continued)

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$							
	1, 2, 3	4, 5	6, 7, 8	9	10	11	12, 13	14, 15, 16, 17
ZTS	31.31%	*	31.31%	*	4.07%	*	4.07%	*

Table 7.13: Contradictory attainable rates of return of all S&P 500 stocks exhibiting the negative correlation condition in their 61-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$		
	1, 2, 3, 4, 5, 6, 7, 8	9, 10, 11, 12, 13, 14, 15, 16	17
AAPL	3.12%	2.78%	*
ADBE	6.32%	3.87%	*
ADSK	7.69%	6.39%	*
AET	4.61%	3.59%	*
ALGN	8.56%	7.87%	*
ALL	4.66%	3.46%	*
AMGN	4.81%	3.38%	*
AMT	4.97%	4.06%	*
AVGO	5.02%	3.99%	*
AXP	9.17%	2.70%	*
BLK	5.22%	4.35%	*
CCL.U	5.58%	4.16%	*
CI	5.08%	4.17%	*
CMCSA	5.18%	4.38%	*
COG	6.46%	5.79%	*

Table 7.13: Contradictory attainable rates of return of all S&P 500 stocks exhibiting the negative correlation condition in their 61-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$ (continued)

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$		
	1, 2, 3, 4, 5, 6, 7, 8	9, 10, 11, 12, 13, 14, 15, 16	17
COL	5.91%	4.85%	*
CTSH	5.20%	4.46%	*
D	4.04%	3.45%	*
DFS	6.23%	5.84%	*
DIS	3.09%	2.52%	*
DRI	5.51%	4.80%	*
DUK	4.10%	2.99%	*
DVA	9.62%	9.61%	*
EBAY	5.09%	3.74%	*
ECL	4.03%	3.72%	*
EOG	5.24%	4.62%	*
ETN	4.03%	3.72%	*
FDX	3.87%	2.87%	*
FMC	5.46%	5.14%	*
IBM	3.24%	2.67%	*

Table 7.13: Contradictory attainable rates of return of all S&P 500 stocks exhibiting the negative correlation condition in their 61-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$ (continued)

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$		
	1, 2, 3, 4, 5, 6, 7, 8	9, 10, 11, 12, 13, 14, 15, 16	17
INTC	4.81%	3.68%	*
INTU	4.65%	3.67%	*
ISRG	6.78%	6.55%	*
IVZ	7.73%	6.74%	*
KMX	6.42%	6.15%	*
LLL	4.37%	3.79%	*
LRCX	7.53%	7.17%	*
MA	4.06%	3.42%	*
MAR	6.79%	5.17%	*
MCHP	6.09%	5.90%	*
MMM	4.00%	3.16%	*
MPC	4.74%	4.46%	*
MRO	10.84%	9.89%	*
MS	3.33%	2.77%	*
MSFT	3.70%	2.52%	*

Table 7.13: Contradictory attainable rates of return of all S&P 500 stocks exhibiting the negative correlation condition in their 61-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$ (continued)

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$		
	1, 2, 3, 4, 5, 6, 7, 8	9, 10, 11, 12, 13, 14, 15, 16	17
NUE	4.42%	4.00%	*
PHM	8.33%	6.98%	*
PX	5.93%	5.66%	*
ROK	6.00%	5.52%	*
STI	4.79%	3.82%	*
SWK	5.48%	5.03%	*
TROW	3.70%	3.45%	*
TRV	5.51%	3.43%	*
TXN	3.94%	3.56%	*
UPS	4.38%	3.39%	*
VRTX	9.92%	8.97%	*
WFC	3.14%	2.75%	*
WM	4.34%	3.57%	*
WY	4.41%	4.13%	*
ZION	6.77%	6.17%	*

Table 7.13: Contradictory attainable rates of return of all S&P 500 stocks exhibiting the negative correlation condition in their 61-asset bilevel portfolio accounting for the highest risk with a no-loss guarantee based on adjusted return bounds at the parameter $\mu = 27$ (continued)

Ticker i	Computed values of $\hat{r}_i = \xi^*_{i,k}/y^*_k$ for each index value $k = 1, 2, \ldots, 17$		
	1, 2, 3, 4, 5, 6, 7, 8	9, 10, 11, 12, 13, 14, 15, 16	17
ZTS	5.04%	4.07%	*

CHAPTER VIII

CONCLUDING REMARKS

Throughout this thesis, we have developed a framework for describing bounds on optimal portfolios in the mean absolute deviation portfolio selection model introduced in 1991 by Konno and Yamazaki but with the incorporation of uncertainty during an estimation of expected future asset returns. Proposed algorithms utilize historical asset returns, ranges of future returns and minimum portfolio return, and they return bounds on optimal portfolio risks and returns in addition to bounds on optimal portfolio compositions. As the estimates become more precise, the computed bounds can significantly be improved.

In many works of literature, discrete optimal compositions are suggested based on each approach or underlying parameter without a sense of continuum as it should have been. We also detect a minor flaw in the bilevel portfolio optimization method. Limitations of our study include the relaxation of feasible regions to derive an overall enclosure and the ignorance of alternative optimal solutions to obtain a tight enclosure. An overall enclosure may contain excess of nonoptimal outcomes, but the computed bounds are still correct. A tight bound, if any, may omit certain possibilities. Therefore, further investigation is required.

This framework is not limited to the MAD model. Indeed, it is applicable to all LP solvable problems including multicommodity network flows and pricing policies. Our single-period problem can be generalized to a multiperiod problem which entails the repetition of the same procedures suggested in this work. However, uncertainty in optimal sequential policies is amplified as a compensation.

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APPENDICES

APPENDIX A

LIST OF PUBLICATIONS

Parts of this work are (or planned to be) disseminated and published in the following articles.

Book Chapter and Conference Paper

S. Chaiyakan and P. Thipwiwatpotjana, “Mean Absolute Deviation Portfolio Frontiers with Interval-Valued Returns,” in Integrated Uncertainty in Knowledge Modelling and Decision Making (Lecture Notes in Computer Science, Vol. 11471), ch. 19, pp. 222-234, 2019.

Academic Journal

S. Chaiyakan and P. Thipwiwatpotjana, “Bounds on Mean Absolute Deviation Portfolios under Interval-Valued Expected Future Asset Returns,” (submitted for consideration for publication in Computational Management Science).

Biography

Songkomkrit Chaiyakan was born in Hatyai, Thailand, on August 12, 1991. He had been studying Mathematics and Applied Mathematics-Economics at Brown University, United States of America, from 2011 to 2013. In 2014, he transferred to a university in Thailand and received the Bachelor of Science (B.Sc.) degree in Mathematics from Prince of Songkla University, Thailand, in 2017. Currently, he is pursuing the Master of Science (M.Sc.) program in Applied Mathematics and Computational Science at Chulalongkorn University, Thailand.

Regarding work experience, he served as a homework grader for two undergraduate-level courses in calculus and microeconomics at Brown University from September 2012 to May 2013. He also worked as an academic officer at Learn Corporation from June 2019 to November 2019. At Chulalongkorn University, he has served as a teaching assistant for two graduate-level courses in mathematical programming and real analysis in addition to three undergraduate-level courses in calculus and stochastic processes from January 2018 to April 2020.