OPER 732: Optimization under Uncertainty
Lecture 6: L-shaped Method

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This lecture

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Benders decomposition for LP

Cutting planes algorithm:

- Iteratively solve the relaxation problem with a cut pool $\mathcal{P}$, which includes all cuts that have been added
- Generate a valid inequality if the relaxation solution $(\hat{x}, \hat{\theta})$ violates any, and update the relaxation problem by adding this cut

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Questions on the cutting plane algorithm:

1. Can we make sure that $\forall (\hat{x}, \hat{\theta})$ that $\hat{\theta} < f(\hat{x})$, we can generate a valid inequality to cut it off?
2. How to generate such a linear inequality?
3. Will this procedure stop finitely?

Case 1: Optimality cut

If $f(\hat{x})$ is finite, according to LP duality:

$$\begin{aligned} f(\hat{x}) &= \min\{q^\top y \mid Wy = h - T\hat{x}, y \geq 0\} \\ &= \max\{(h - T\hat{x})^\top u \mid W^\top u \leq q\} \\ &= (h - T\hat{x})^\top \hat{u} \end{aligned}$$

We have a valid inequality:

$$\theta \geq (h - Tx)^\top \hat{u},$$

that is violated by $(\hat{x}, \hat{\theta})$, and add it to the cut pool $\mathcal{P}$

Question 1 and 2 are solved, by strong duality

Case 2: Feasibility cut

If $f(\hat{x}) = +\infty$, then $\hat{x}$ is not feasible, i.e., $\nexists y \geq 0$ that $Wy = h - T\hat{x}$. Let us solve a feasibility problem:

$$\begin{aligned} z(\hat{x}) := \min_{y \in \mathbb{R}^{n_2}_+} \ & \sum_{i=1}^{n_2} (\xi_i + \eta_i) \\ \text{s.t. } & Wy + \xi - \eta = h - T\hat{x} \\ & \xi, \eta \in \mathbb{R}^{n_2}_+ \end{aligned}$$

$z(\hat{x}) > 0$ since $\hat{x}$ is not feasible. The corresponding dual problem is:

$$\begin{aligned} z(\hat{x}) = \max_{v \in \mathbb{R}^{m_2}} \ & (h - T\hat{x})^\top v \\ \text{s.t. } & W^\top v \leq 0 \\ & -1 \leq v_i \leq 1, \ \forall i = 1, 2, \ldots, m_2 \end{aligned}$$

The optimal $\hat{v}$ represents an extreme ray in the recession cone $\{v \in \mathbb{R}^{m_2} \mid W^\top v \leq 0\}$

Feasibility cut (continued)

Feasibility cut

$$\hat{v}^\top (h - Tx) \leq 0$$

- This is a valid inequality for all feasible x, why?
- This inequality is violated by current $\hat{x}$ solution

Benders decomposition: a loop

1. Suppose the current set of optimality cuts is $\mathcal{O}$, and set of feasibility cuts is $\mathcal{F}$. Obtain a relaxation solution $(\hat{x}, \hat{\theta})$ by solving:

$$
\begin{aligned}
\min_{x \in \mathbb{R}^{n_1}_{+}} \quad & c^\top x + \theta \\
\text{s.t.} \quad & Ax = b \\
& E_l x + \theta \geq e_l, \ \forall l \in \mathcal{O} \\
& D_l x \geq d_l, \ \forall l \in \mathcal{F},
\end{aligned}
$$

2. Check if $\hat{x}$ is feasible by solving: $z(\hat{x}) = \max\{(h - T\hat{x})^\top v \mid W^\top v \leq 0, v \in [-1, 1]^{m_2}\}$. If $z(\hat{x}) > 0$, add a feasibility cut to $\mathcal{F}$, and return to Step 1
3. Check if $\theta \geq f(\hat{x})$, by solving $f(\hat{x}) = \min\{q^\top y \mid Wy = h - T\hat{x}, y \geq 0\}$. If yes, we claim $\hat{x}$ is the optimal solution. Otherwise, add an optimality cut to $\mathcal{O}$, and return to Step 1

An example

$$
\begin{aligned}
\min\ & 1.5x + 2y_1 + y_2 \\
\text{s.t. } & x + y_1 + y_2 \geq 4 \\
& x + y_1 \geq 2 \\
& x, y_1, y_2 \geq 0
\end{aligned}
$$

Finiteness of Benders decomposition

1. Finiteness of feasibility cuts: $\hat{v}$ of the feasibility cut $\hat{v}^\top(h - Tx) \leq 0$ is an extreme ray in the recession cone $\{v \in \mathbb{R}^{m_2} \mid W^\top v \leq 0\}$

   # extreme rays of a polyhedron is finite $\Rightarrow$ a finite number of feasibility cuts are sufficient to ensure feasibility

2. Finiteness of optimality cuts: $\hat{u}$ of the optimality cut $\theta \geq \hat{u}^\top(h - Tx)$ is an extreme point of polyhedron $\{u \mid W^\top u \leq q\}$

   # extreme points of a polyhedron is finite $\Rightarrow$ a finite number of optimality cuts are sufficient to define the epigraph

Cutting plane algorithm

Benders decomposition is a cutting plane algorithm for solving LPs. Which algorithm will you choose to iteratively solve the relaxation LP? Why?

Dual simplex!

- Dual multipliers remain feasible after adding cuts
- Hot startusing dual multipliers from the previous step

Cutting plane algorithm

Benders decomposition is a cutting plane algorithm for solving LPs. Which algorithm will you choose to iteratively solve the relaxation LP? Why?

Dual simplex!

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L-shaped method: Bender decomposition in stochastic programs

$$\min_{x\in\mathbb{R}^{n_1}_+} c^\top x + \sum_{k\in N} p_k Q_k(x)$$
$$\text{s.t. } Ax = b,$$

where

$$Q_k(x) = \min_{y^k\in\mathbb{R}^{n_2}_+} \{(q^k)^\top y^k \mid W y^k = h^k - T^k x\}$$

Let $\mathcal{Q}(x) = \sum_{k\in N} p_k Q_k(x)$, assume θ_0 is its lowerbound. The initial relaxation problem:

$$\min_{x\in\mathbb{R}^{n_1}_+} c^\top x + \theta$$
$$\text{s.t. } Ax = b$$
$$\theta \geq \theta_0$$

L-shaped method: Master problem

$$
\begin{aligned}
\min_{x \in \mathbb{R}^{n_1}_+} \ & c^\top x + \theta \\
\text{s.t. } & Ax = b \\
& E_l x + \theta \geq e_l, \ \forall l \in \mathcal{O} \\
& D_l x \geq d_l, \ \forall l \in \mathcal{F}.
\end{aligned}
$$

Master problem is a relaxation of the original problem, constructed by all optimality cuts and feasibility cuts generated so far.

Let $(\hat{x}, \hat{\theta})$ be the solution to the master problem.

Subproblem: separable for each scenario k

Given a master problem solution $(\hat{x}, \hat{\theta})$, evaluate $\mathcal{Q}(\hat{x}) = \sum_{k \in N} p_k Q_k(\hat{x})$:

- Solve each problem $Q_k(\hat{x})$ separately for each scenario k
- Use the solution to construct optimality and feasibility cuts

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Feasibility cuts: if for some k that $Q_k(\hat{x}) = +\infty$, we generate a feasibility cut:

$$(h^k - T^k x)^\top v^k \leq 0,$$

and add it to the set of feasibility cuts $\mathcal{F}$ in the master problem.

Subproblem: separable for each scenario k

If $\forall k \in N$, $Q_k(\hat{x}) < \infty$ (so $\hat{x}$ is feasible), we look for an optimality cut:

$$\theta \geq \sum_{k \in N} p_k (h^k - T^k x)^\top u^k,$$

- For each $k \in N$, u^k is the optimal dual multiplier for problem $Q_k(\hat{x})$.
- We aggregate these dual information to construct a single cut.
- If this inequality is violated by the current $(\hat{x}, \hat{\theta})$, we add it to the set of optimality cuts $\mathcal{O}$ in the master problem.
- Otherwise, we claim $\hat{x}$ is the optimal solution.

An example

$$
\begin{aligned}
\min\ & \mathbb{E}_\xi(y_1 + y_2) \\
\text{s.t. } & 0 \le x \le 10 \\
& y_1 - y_2 = \xi - x \\
& y_1, y_2 \ge 0,
\end{aligned}
$$

where ξ takes the values $1, 2, 4$ with probability $1/3$.

Multicut: disaggregated optimality cuts

Introduce a separate θ_k variable for each scenario $k \in N$:

$$
\begin{aligned}
\min_{x \in \mathbb{R}_+^{n_1}} \; & c^\top x + \sum_{k \in N} p_k \theta_k \\
\text{s.t. } & Ax = b \\
& D_l x \geq d_l, \; \forall l \in \mathcal{F} \\
& E_{l(k)} x + \theta_k \geq e_{l(k)}, \; \forall l(k) \in \mathcal{O}_k, \; \forall k \in N
\end{aligned}
$$

Optimality cuts are generated one for each scenario k, if violated by the relaxation solution $(\hat{x}, \theta_k)$.

Single cut or multicut?

	Single cut	Multicut
Information for the first-stage	less detailed	more detailed
# of iterations	more	less
Size of problem formulation	less	more

A rule of thumb: the multicut approach is more effective when # of scenarios N is not significantly larger than # first-stage constraints m_1.

Reading this week: Section 5.1 of textbook.