OPER 732: Optimization under Uncertainty Lecture 4: Multistage Stochastic Programming Models

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Introduction

1. Multistage stochastic program is still a challenging and hot topic in stochastic programming
2. Closely related to Approximate Dynamic Programming (ADP), and Machine Learning (ML)
3. Fun things just begin!

Chris needs stochastic programming!

My wife and I have to afford Chris to college ($\$G$ in total) after T years,

- Our money is $\$b > 0$, we have a set I of investment choices: stock, real estate, bond, ...
  - The returns of these investments are random
- We have a set $\mathcal{T} = \{1, 2, \ldots, T\}$ of investment periods
  - In each period, we can redistribute our asset. This is called "decision stage".
  - Two stage: $T = 2$, multistage: $T > 2$
  - We are able to adjust our investment portfolio based on the behavior of market
- If we exceed the goal of $\$G$, we can enjoy our lives by enjoying the interest rate of q (assuming it is still positive at that time....)
- Otherwise, we have to borrow money at a rate $r > q$
- Ignore transaction costs and taxes

Timeline of decisions

1. At time $t = 0$, we make our initial investment, by maximizing the expected return over all periods
2. At time $t = 1$, the random returns take place, and we adjust our investment (recourse) based on the return information at $t = 1$
3. At time $t = 2$, the random returns take place again, we adjust our investment (recourse) based on the return information at $t = 1$ and $t = 2$
4. $\ldots$

- Let $\mathbf{x}^t = \{x_{it}\}_{i \in I}$ be our investment decisions at time $t \in \mathcal{T}$
- Let $\xi^t = \{\xi_{it}\}_{i \in I}$ be the random returns at time $t \in \mathcal{T}$

$$\mathbf{x}^0 \to \xi^1 \to \mathbf{x}^1(\xi^1, \mathbf{x}^0) \to \xi^2 \to \mathbf{x}^2(\xi^1, \xi^2, \mathbf{x}^0, \mathbf{x}^1) \to \ldots$$
$$\to \mathbf{x}^T(\xi^1, \ldots, \xi^T, \mathbf{x}^0, \ldots, \mathbf{x}^{T-1})$$

If the returns were deterministic...

Suppose the returns over time $\{\xi_{it}\}_{i\in I, t\in\mathcal{T}}$ are fixed numbers, $\{\hat{\xi}_{it}\}_{i\in I, t\in\mathcal{T}}$, the multi-period investment problem is an LP:

$$\begin{aligned}
&\max\ qy - rw \\
&\text{s.t.}\ \sum_{i\in I} x_{i0} = b \\
&\qquad \sum_{i\in I} x_{it} = \sum_{i\in I} \hat{\xi}_{it} x_{i,t-1},\ \forall t = 1, 2, \ldots, T \\
&\qquad \sum_{i\in I} x_{iT} - y + w = G \\
&\qquad x_{it} \geq 0,\ \forall i \in I, t \in \mathcal{T} \\
&\qquad y, w \geq 0
\end{aligned}$$

- x_{it}: amount of investment on i at time t
- y: amount of money left after time T
- w: amount of money in short after time T

If you are a super prophet...

Given a set N of possible scenarios, each scenario k happens with probability p_k. Suppose all the random returns $\{\xi_{it}\}_{i\in I, t\in\mathcal{T}}$ are known in advance (recall how we calculate $EVPI$ for two-stage problem):

$$
\begin{aligned}
\max \quad & \sum_{k\in N} p_k(qy_k - rw_k) \\
\text{s.t.} \quad & \sum_{i\in I} x_{i0k} = b, \forall k \in N \\
& \sum_{i\in I} x_{itk} = \sum_{i\in I} \xi_{itk} x_{i,t-1,k}, \ \forall t = 1, 2, \ldots, T, \forall k \in N \\
& \sum_{i\in I} x_{iTk} - y_k + w_k = G, \forall k \in N \\
& x_{itk} \geq 0, \ \forall i \in I, t \in \mathcal{T}, \forall k \in N \\
& y_k, w_k \geq 0, \forall k \in N
\end{aligned}
$$

Be realistic: you are just a (smart) human

If scenario 1 and 2 are exactly the same up to time t

- In other words, $\xi_{i\tau 1} = \xi_{i\tau 2}, \forall \tau = 1, 2, \ldots, t, \forall i \in I$
- Your decisions should be identical for scenario 1 and 2 up to time t
- Information needs time to evolve, there is not enough information to distinguish between scenario 1 and 2 before time t
- You decision should be compatible with information

Prophet's spell: nonanticipativity

Decisions can only depend on the information that has been resolved, i.e.,

$$x_{itk} = x_{itk'}, \ \forall i \in I, \forall t \in T, \forall k' \in S_k^t,$$

where S_k^t is the set of scenarios that are not distinguishable from k up to time t (including k)

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Extended formulation for multistage stochastic programs with recourse

$$
\begin{aligned}
\max \quad & \sum_{k \in N} p_k (q y_k - r w_k) \\
\text{s.t.} \quad & \sum_{i \in I} x_{i0} = b \\
& \sum_{i \in I} x_{itk} = \sum_{i \in I} \xi_{itk} x_{i,t-1,k}, \ \forall t = 1, 2, \ldots, T, \forall k \in N \\
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& x_{itk} \geq 0, \ \forall i \in I, t \in \mathcal{T}, \forall k \in N \\
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Scenario-tree

Suppose the returns are discrete random variables that have a finite number of outcomes S_t at each stage $t = 1, 2, \ldots, T$

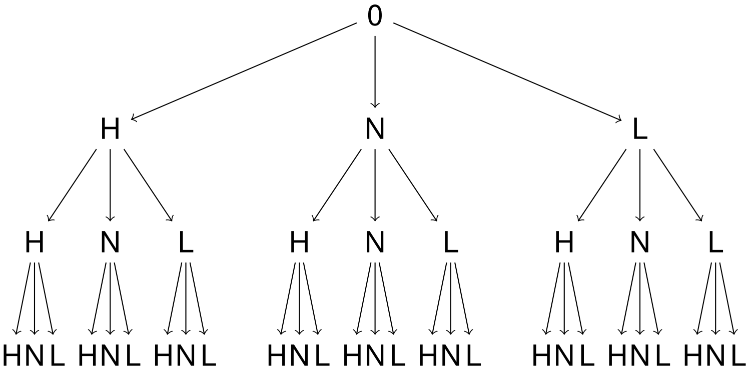
Scenario-tree: an example

- Suppose there are 3 possible outcomes for the return of an asset: High, Normal, and Low
- Suppose the randomness is stage-wise independent

After 3 stages:

Scenario	Stage 1	Stage 2	Stage 3
1	H	H	H
2	H	H	N
3	H	H	L
...	...	...	...
27	L	L	L

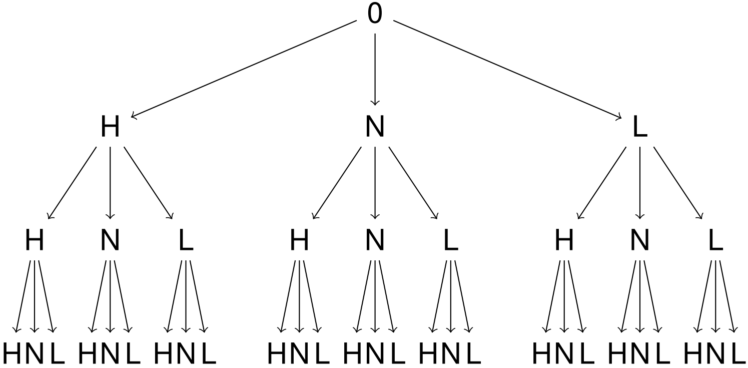
Scenario-tree



Q: How to define a scenario in this tree? A path from root to leaf!

e.g. Scenario 1: $H \rightarrow H \rightarrow H$

Scenario-tree



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e.g. Scenario 1: $H \to H \to H$

How many NAC sets?

Recall: S_k^t is the set of scenarios that are not distinguishable from k up to time t (We call it a NAC set)

- How many NAC sets are there?
  $N^T \times (T-1)!$
- After removing the redundent ones?
  $(N^T - 1)/(N-1)!$

Curse of dimensionality: The formulation size blows up as # time periods T increases

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A Dam Example

What a dam problem!

- Williams Island Dam is located in Richmond, VA on James River
- We need to decide how much water to release from the dam in each of the following 3 month (we release water once a month)
- Currently the river is 150mm below flood level (-150)

What is random? Rain and evaporation!

Month	Outcome 1	Outcome 2	Outcome 3	Outcome 4
1	+100	-75	+200	+250
2	+100	-75	+200	+250
3	+150	-50	+250	+400
probability	0.4	0.3	0.2	0.1

We assume the outcome random variables are stage-wise independent and follow the same probability distribution each month

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A dam problem

- If a flood occurs (flood level > 0), the damage to the city is $\$20000/mm$ above flood level
- If water level is too low (below $-250mm$), we need to import water into the river with a cost of $\$10000/mm$
- We use dam to make money! For each $1mm$ released in a month, the city earns $\$6000$

- Q: What model are you going to use? Two-stage or multi-stage stochastic program?
- Q: What is the decision making process?

A dam scenario-tree