OPER 732: Optimization under Uncertainty
Lecture 2: Stochastic Programming Models and Value of Stochastic Solutions

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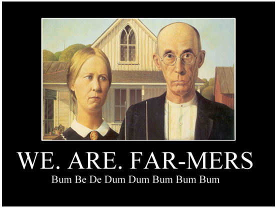
Aug. 30, 2017

Outline

We are farmers!

- You can grow Wheat, Corn, or Beans on 500 acres.
- You need 200 tons of wheat and 240 tons of corn to feed your cattle
- You can grow these food on your land or buy from a super farmer.



Market price

- You can sell your product at the price of:
  - Wheat: $\$170$/ton
  - Corn: $\$150$/ton
  - Beans: $\$36$/ton for the first 6000 tons, and $\$10$/ton after that
- You can buy product from the super farmer at the price of:
  - Wheat: $\$238$/ton
  - Corn: $\$210$/ton

Problem data

	Wheat	Corn	Beans
Yield (T/acre)	2.5	3	20
Production Cost ($/acre)	150	230	260
Selling price	170	150	36/10 (before/after 6000)
Purchase price	238	210	-
Cattle demand	200	240	-

Suppose you have 500 acres of land for growing crops, what is your planting plan?

LP Formulation

Decision variables

- $x_{W,C,B}$: acres of wheat, corn, beans planted
- $w_{W,C,B}$: tons of wheat, corn, beans sold at regular price
- e_B: tons of beans sold at lower price after 6000
- $y_{W,C}$: tons of wheat, corn purchased

Suppose the yield is deterministic: optimal objective value is 118600

$$
\begin{aligned}
\max\ & 170w_W + 150w_C + 36w_B + 10e_B - 150x_W - 230x_C - 260x_B \\
& - 238y_W - 210y_C \\
\text{s.t.}\ & x_W + x_C + x_B \le 500 \\
& 2.5x_W + y_W - w_W = 200 \\
& 3x_C + y_C - w_C = 240 \\
& 20x_B - w_B - e_B = 0 \\
& w_B \le 6000 \\
& \text{All variables } \ge 0
\end{aligned}
$$

Yield is stochastic!

Suppose there are 3 possible outcomes, each with probability $1/3$:

- Good weather: $1.2 \times (2.5, 3, 20)$
- Bad weather: $0.8 \times (2.5, 3, 20)$
- Normal weather: $(2.5, 3, 20)$

Question: which variables are first-stage? Which are second-stage recourse variables?

Decision process: implement first-stage decisions $x \to$ observe the yield outcome $\xi \to$ make recourse actions according to the outcome $w(\xi), e(\xi), y(\xi)$

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Come up with a stochastic LP model

1. For first-stage decision variables, these are determined regardless of which scenario happens, just $x_{W,C,B}$
2. For second-stage decision variables, these are dependent on which scenario happens, so we make copies of these variables, one for each scenario: e.g., $w_{W,C,B} \to w^1_{W,C,B}, w^2_{W,C,B}, w^3_{W,C,B}$

In the objective, associate the second-stage cost/profit in each scenario with the corresponding probability: e.g.

$$
\begin{aligned}
&170w_W + 150w_C + 36w_B \\
&\to \frac{1}{3}(170w^1_W + 150w^1_C + 36w^1_B) + \frac{1}{3}(170w^2_W + 150w^2_C + 36w^2_B) \\
&+ \frac{1}{3}(170w^3_W + 150w^3_C + 36w^3_B)
\end{aligned}
$$

Come up with a stochastic LP model (continued)

Two types of constraints:

- First-stage constraints: only involve first-stage variables
- Second-stage constraints: involve second-stage variables

$$
\begin{aligned}
&x_W + x_C + x_B \leq 500 \;\Rightarrow x_W + x_C + x_B \leq 500 \\
&2.5x_W + y_W - w_W = 200 \;\Rightarrow a^k x_W + y_W^k - w_W^k = 200, \forall k = 1,2,3 \\
&3x_C + y_C - w_C = 240 \;\Rightarrow b^k x_C + y_C^k - w_C^k = 240, \forall k = 1,2,3 \\
&20x_B - w_B - e_B = 0 \;\Rightarrow c^k x_B - w_B^k - e_B^k = 0, \forall k = 1,2,3 \\
&w_B \leq 6000 \;\Rightarrow w_B^k \leq 6000, \forall k = 1,2,3
\end{aligned}
$$

where $(a^1, b^1, c^1) = (2.5, 3, 20)$, $(a^2, b^2, c^2) = (3, 3.6, 24)$, $(a^3, b^3, c^3) = (2, 2.4, 16)$

Optimal solution

		Wheat	Corn	Beans
Scenario	Decison	170	80	250
1	Production	510	288	6000
1	Sales	310	48	6000
1	Purchase	0	0	0
2	Production	425	240	5000
2	Sales	225	0	5000
2	Purchase	0	0	0
3	Production	340	192	4000
3	Sales	140	0	4000
3	Purchase	0	48	0

Solution: expected profit $= 108390$

Outline

Value of Stochastic Solutions (VSS)

- Q: Who to compare with?
  - Compare to “an average person” (mean-value problem)
- Q: What to compare?
  - Only compare the first-stage solution (decision)!
  - The second-stage solution is always “easy”, since we already know what happened (deterministic decision)
- Q: We know the optimal objective value of the mean-value model is 118600, so $VSS = 118600 - 108390 = 10210$?
  - Wrong! This is not a fair comparison: we should implement the mean-value solution, and calculate the corresponding expected profit

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Calculating VSS

1. Plug in only the first-stage solution $\hat{x}_W = 120, \hat{x}_C = 80, \hat{x}_B = 300$
2. Calculate the profit made in each scenario $k = 1, 2, 3$:

$$
\begin{aligned}
P^k = \max\ & 170w_W + 150w_C + 36w_B + 10e_B - 150\hat{x}_W - 230\hat{x}_C - 260\hat{x}_B \\
& - 238y_W - 210y_C \\
\text{s.t.}\ & a^k\hat{x}_W + y_W - w_W = 200 \\
& b^k\hat{x}_C + y_C - w_C = 240 \\
& c^k\hat{x}_B - w_B - e_B = 0 \\
& w_B \leq 6000
\end{aligned}
$$

   and we get: $P^1 = 118600, P^2 = 55120, P^3 = 148000$
3. Expected profit $= \frac{1}{3}(P^1 + P^2 + P^3) = 107240$
4. VSS $= 108390 - 107240 = 1150$

This is the amount of money that an average farmer can earn after taking this course!

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Expected Value of Perfect Information (EVPI)

Perfect Information

- Not only know the probability distribution
- But also know exactly what is going to happen

1. If we have the perfect information, solve:

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2. We get: $\bar{P}^1 = 118600, \bar{P}^2 = 59950, \bar{P}^3 = 167667$. So with perfect information, the expected profit $= \frac{1}{3}(\bar{P} + \bar{P}^2 + \bar{P}^3) = 115406$
3. $EVPI = 115406 - 108390 = 7016$

This is the gap between a wise person (who knows stochastic programming) and a very wise person (who knows the future)

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2. We get: $\bar{P}^1 = 118600, \bar{P}^2 = 59950, \bar{P}^3 = 167667$. So with perfect information, the expected profit $= \frac{1}{3}(\bar{P} + \bar{P}^2 + \bar{P}^3) = 115406$
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VSS

- Optimal objective value of a stochastic program:

$$z_S = \min_{x \in X} \mathbb{E}[F(x, \xi)]$$

- Optimal solution of the mean-value problem:

$$x_{MV} \in \arg\min_{x \in X} F(x, \mathbb{E}[\xi])$$

- The expected performance of the mean-value solution $z_{MV} = \mathbb{E}[F(x_{MV}, \xi)]$
- VSS = $z_{MV} - z_S$

Theorem

VSS $= z_{MV} - z_S \geq 0$

Always better to use stochastic programming solutions!

EVPI

- The expected value of a stochastic programmer's solution:

$$z_S = \min_{x \in X} \mathbb{E}[F(x, \xi)]$$

- The expected value of the prophet's solution

$$z_{PI} = \mathbb{E}[\min_{x \in X} F(x, \xi)]$$

Theorem

EVPI $= z_S - z_{PI} \geq 0$

The limit of a stochastic programmer is a prophet.

Summary

- The first two-stage stochastic LP model
- VSS, EVPI

- Reading:
  - Appendix math review (pp 94-97, 449-450 on textbook)
  - LP review handout
  - Chapter 1.1, 2.1, 2.2, 2.3
  - No class on Monday, Sept. 4th (Labor Day)
- Homework 1 is out, due on Sept. 8th at 11:59pm