OPER 732: Optimization under Uncertainty
Lecture 3: Properties of two-stage stochastic linear programs

Department of Statistical Sciences and Operations Research
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Sept. 6, 2017

Outline

1. Properties of two-stage stochastic LP

A generic two-stage stochastic linear program

Two-stage SLP, the overall problem:

$$\begin{aligned} \min\ & c^\top x + \mathbb{E}[q(\xi)^\top y(\xi)] \\ \text{s.t. } & Ax = b \\ & T(\xi)x + W(\xi)y(\xi) = h(\xi) \\ & x \in \mathbb{R}_+^{n_1}, y(\xi) \in \mathbb{R}_+^{n_2} \end{aligned}$$

The second-stage cost function

$$\begin{aligned} Q(x,\xi) := \min\ & q(\xi)^\top y(\xi) \\ \text{s.t. } & W(\xi)y(\xi) = h(\xi) - T(\xi)x \\ & y(\xi) \in \mathbb{R}_+^{n_2} \end{aligned}$$

- $T(\xi)$: Technology matrix, $W(\xi)$: Recourse matrix
- If W matrix is deterministic, we call it fixed recourse
- We assume fixed recourse for now

Q: Is newsvendor problem a fixed-recourse stochastic program?

Two-stage SLP with discrete random variables

Given a set N of scenarios, each scenario $k \in N$ happens with probability $p_k > 0$:

$$
\begin{aligned}
\min\ & c^\top x + \sum_{k \in N} p_k (q_k)^\top y_k \\
\text{s.t. } & Ax = b \\
& T^k x + W y_k = h_k,\ \forall k \in N \\
& x \in \mathbb{R}_+^{n_1}, y_k \in \mathbb{R}_+^{n_2},\ \forall k \in N
\end{aligned}
$$

This is called the extensive form of a two-stage stochastic LP

Deterministic Equivalent

$$
\begin{array}{llllr}
\min & c^\top x + p_1(q_1)^\top y_1 + & p_2(q_2)^\top y_2 + \cdots + & p_N(q_N)^\top y_N & \\
& Ax & & & = b \\
& T_1 x + W y_1 & & & = h_1 \\
& T_2 x + & W y_2 & & = h_2 \\
& \vdots & & & \\
& T_N x & + & W y_N & = h_N
\end{array}
$$

- Good news: it's just an LP
- Bad news: it may be a HUGE LP
  - $n_1 + n_2 \times N$ variables
  - $m_1 + m_2 \times N$ constraints
- Not so bad news: luckily, the matrix of the LP has a special structure (block-angular)

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Solution methods of the extended form

$$
\begin{array}{llllll}
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\end{array}
$$

- If you work in Argonne: solve two-stage SLPs using this extended formulation
  - Perform "decomposition" at the linear algebra level
  - Petra and Anitescu (2012): *A preconditioning technique for Schur complement systems arising in stochastic optimization*
- Other people: perform "decomposition" at the optimization level

Another perspective on two-stage SP

Two-stage SLP using only first-stage variables x:

$$\min_{x \in \mathbb{R}^{n_1}_+} \; c^\top x + \sum_{k \in N} p_k Q_k(x),$$

where Q_k is the second-stage cost function for each scenario $k \in N$:

$$Q_k(x) = \min_{y_k \in \mathbb{R}^{n_2}_+} \{(q_k)^\top y_k \mid W y_k = h_k - T_k x\}$$

- Your decision is essentially just x
- y_k, recourse variables, are determined by x and scenario $k \Rightarrow$ auxiliary variables

Questions to ask:

- What is the feasible region of this program?
- What does the objective function look like? Is it linear?
- Is this program convex?

Before we answer these question, let's review some background

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Linear Programming

Consider a linear programming (LP) of the form

$$\max\{c^\top x \mid Ax = b, x \geq 0\}$$

- What is a basis?
- What is a basic feasible solution?
- What is an extreme point solution?
- What is an extreme ray?
- What is a simplex pivot?

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LP duality

Given an LP:

$$(P) \;\; z^P := \max\{c^\top x \mid Ax = b, x \geq 0\}$$

Its dual is:

$$(D) \;\; z^D := \min\{\pi^\top b \mid \pi^\top A \geq c^\top, \pi \text{ free}\}$$

What can we say about z^P and z^D?

- If (P) is feasible and bounded?
- If (P) is infeasible?
- If (P) is unbounded?

What is complementary slackness?

A convention: if a minimization(maximization) problem is infeasible, we say its objective value is $+\infty(-\infty)$

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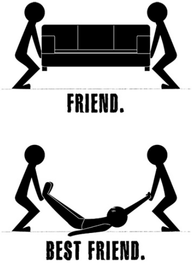
Convexity

In optimization, convexity is your best friend

- A set S is convex, if $x, y \in S$, then $\alpha x + (1-\alpha)y \in S, \forall \alpha \in [0,1]$
- A function f is convex, if:
  (1) Its domain (where the value of f is finite) S is a convex set
  (2) $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \forall \alpha \in [0,1]$
  $\forall x, y \in S$
- f is convex if and only if its epigraph $epi(f) = \{(x,t) \mid t \geq f(x)\}$ is convex



Two-stage SLP: properties

Two-stage SLP using only first-stage variables x:

$$\min_{x \in \mathbb{R}^{n_1}_+} \; c^\top x + \sum_{k \in N} p_k Q_k(x)$$

Feasible region K:

- $K_1 := \{x \in \mathbb{R}^{n_1}_+ \mid Ax = b\}$ (First-stage constraints)
- $K_2(k) := \{x \mid \exists y^k \in \mathbb{R}^{n_2}_+ \text{ s.t. } Wy^k = h^k - T^k x\}$ (Second-stage constraints)
- $K = K_1 \cap (\cap_{k \in N} K_2(k))$

Is feasible region K convex?

Yes, if $K_2(k)$ is convex, $\forall k \in N$

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Convexity of $K_2(k)$

Proof by definition.

$Q_k(x)$: piecewise linear convex function

What does $Q_k(x)$ look like?

Example

Three scenarios, equally likely:

$$\min \{2y \mid y \geq \xi - x, y \in [0, 1]\}$$

where $\xi^1 = 1, \xi^2 = 0, \xi^3 = -1$.

Proof:

- Convexity: definition
- Piecewise linear:
  - Basis
  - Optimality condition for a basis
  - Number of optimal bases is finite

Special recourse structures

Feasible region K:

- $K_1 := \{x \in \mathbb{R}_+^{n_1} \mid Ax = b\}$
- $K_2(k) := \{x \mid \exists y_k \in \mathbb{R}_+^{n_2}$ s.t. $Wy_k = h_k - T_k x\}$
- $K = K_1 \cap K_2$, where $K_2 = \cap_{k \in N} K_2(k)$

1. Relative complete recourse: $K_1 \subseteq K_2$ (every solution that is feasible for the first stage, is also feasible for the second stage)
2. Complete recourse: $K_2 = \mathbb{R}^{n_1}$ (the second-stage problem is always feasible regardless of x)
3. Simple recourse: $W = [I, -I]$. In this case, we can simply "observe" the value of $Q_k(x)$ without "solving" it

Summary

- VSS and EVPI
- Two-stage SLP and its properties
- Reading: B& L, Chapter 3.1

- Homework 1 due on Friday, Sept. 8th
- Next time: multistage stochastic programming models