Huy-Phan Nhat

2024-07-14

Essential Linear Algebra - Titu Andresscu

Contents

1 Matrix Algebra

1.1 Vectors, Matrices, and Basic Operations on Them

1.1.1 Summary

1.1.2 Exercises

Problem 1.1.6

Let F be a field.

a. Prove that if $A, B \in M_n(\mathbb{F})$ are diagonal matrices, then $A + cB$ is a diagonal matrix for any $c \in \mathbb{F}$.
b. Prove that the same result holds if we replace diagonal with upper-triangular.
c. Prove that any matrix $A \in M_n(\mathbb{F})$ can be written as the sum of an upper-triangular matrix and of a lower-triangular matrix. Is there such a unique writing?

Solution.

a. Since A and B are both diagonal matrices,

$$A + cB = \begin{bmatrix} a_{11} + cb_{11} & 0 & \cdots & 0 \\ 0 & a_{22} + cb_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} + cb_{nn} \end{bmatrix}$$

$\Longrightarrow A + cB$ is also a diagonal matrix.

b. Since A and B are both upper-triangular matrices,

$$A + cB = \begin{bmatrix} a_{11} + cb_{11} & a_{12} + cb_{12} & \cdots & a_{1n} + cb_{1n} \\ 0 & a_{22} + cb_{22} & \cdots & a_{2n} + cb_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} + cb_{nn} \end{bmatrix}$$

$\Longrightarrow A + cB$ is also a diagonal matrix.

c. We always have

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = B + C,$$

where B is an lower-triangular matrix of b_{ij}-entries, C a upper-triangular matrix of cd_{ij}-entries.

Now let's suppose that this way of writing isn't unique; in other words, we assume that $A = E + F(E, F \in M_n(\mathbb{F}))$ and E,F are upper-triangular and lower-triangular matrices, respectively. Thus $E + G = B + C \Longleftrightarrow O_n = (B - E) + (C - G)$. This means that $b_{ij} = e_{ij}, c_{ij} = g_{ij}$, which implies that A can be written as the sum of two unique matrices: one upper-triangular, the other lower-triangular.

□

Problem 1.1.7

a. How many distinct matrices are there $M_{m,n}(\mathbb{F}_2)$?
b. How many of these matrices are diagonal?
c. How many of these matrices are upper-triangular?

Solution.

a. Each entry in an $m \times n$ matrix can take on two values: 0 or 1. Since there are $m \times n$ entries, the toal number of distinct $m \times n$ matrices are 2^{mn} for $m, n \in \mathbb{N}^*$. Thus the total number of distinct matrices is $\sum_{i=1}^{m} \sum_{j=1}^{n} 2^{ij}$.
b. A diagonal matrix A has to be $M_n(\mathbb{F}_2)$, and $a_{ij} = 0, i \neq j$. We only need to consider the a_{ii}-entries, of which there are n entries. Using the same reasoning in (a), we get the numer the number of $n \times n$ diagonal matrices: 2^n. Since n runs from $1 \longrightarrow n$, we get the total number of distinct diagonal matrices: $\sum_{i=1}^{n} 2^i$.
c. Similar to (b), we get the number of distinct upper-triangular matrices: $\sum_{i=1}^{n} 2^{\frac{i(i+1)}{2}}$.

□

1.2 Matrices as Linear Maps

1.2.1 Summary

1.2.2 Problems for Practice

Problem 1.2.2

Consider the map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ defined by

$$f(x, y, z) = (x - 2y + 2z, y - z + x, x, z).$$

Prove that f is linear and describe the matrix associated with f.

Problem 1.2.3

a. Consider the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(x, y) = \left(x^2, y^2\right).$$

Is this map linear?

b. Answer the same question with the field $\mathbb{R}$ replaced with $\mathbb{F}^2$.

Solution.

a. Let $X, Y \in \mathbb{R}^2 \wedge \alpha, \beta \in \mathbb{R}$. We then have

$$f(\alpha X + \beta Y) = \alpha^2 f(X) + \beta^2 f(Y) \neq \alpha f(X) + \beta f(Y)$$

$\implies f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is not a linear map.

b. If, however, $X, Y \in \mathbb{F}^2 \wedge \alpha, \beta \in \mathbb{F}$, we get

$$f(\alpha X + \beta Y) = \alpha^2 f(X) + \beta^2 f(Y) = \alpha f(X) + \beta f(Y)$$

since $\alpha, \beta \in \mathbb{F}$. It's also easy to prove that f is an injective map, thus f is linear $\mathbb{F}^2$.

$\square$

Problem 1.2.4

Consider the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(x, y) = (x + 2y, x + y - 1).$$

Is the map f linear?

Problem 1.2.5

Consider the matrix $A = \begin{pmatrix} 1 & -2 & 2 & 0 \\ 2 & 0 & 4 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix}$. Describe the image of the vector $v = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$ through the linear map attached to A.

Problem 1.2.6

Give an example of a map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ which is *not* linear and for which

$$f(av) = af(v)$$

$\forall\ a \in \mathbb{R}$ and $v \in \mathbb{R}^2$.

1.3 Matrix Multiplication

1.3.1 Summary

1.3.2 Problems for Practice

Problem 1.3.5

Detemerine all matrices $A \in M_2(\mathbb{R})$ commuting with the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Problem 1.3.7 (Matrix representation of $\mathbb{C}$)

Let G be the set of the matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a, b \in \mathbb{R}$.

a. Prove that the sum and product of two elements of G is in G.
b. Consider the map $f : G \longrightarrow \mathbb{C}$ defined by

$$f\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) = a + ib.$$

Prove that f is a bijective map satisfying $f(A+B) = f(A) + f(B)$ and $f(AB) = f(A)f(B)\ \forall\ A, B \in G$.
c. Use this to compute the nth power of the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Solution.

a. Suppose $A, B \in G$ and $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, B = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$.

$A + B = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} \Longrightarrow A + B \in G.$

$AB = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix} \Longrightarrow AB \in G.$

b. $f(A) + f(B) = f(A+B)$ and $f(A)f(B) = f(AB)$ are easily proven; we'll prove that f is bijective.

We'll reuse A and B from (a). Suppose that $f(A) = f(B) \Longrightarrow \begin{cases} a = c \\ b = d \end{cases}$. Thus $A = B$ and f in injective.

Next, we observe that $\forall\ a, b \in \mathbb{R}, \exists\ X = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in G : f(X) = a + bi$. So f is surjective.

c. From (b), we get

$$\begin{aligned} f(A^n) = [f(A)]^n = (a+bi)^n &= \left(a^2+b^2\right)\left(\frac{a}{a^2+b^2}\right) + \frac{b}{a^2+b^2} \\ &= \left(a^2+b^2\right)^n (\cos\theta + i\sin\theta)^n = \left(a^2+b^2\right)^n (\cos n\theta + i\sin n\theta). \end{aligned} \quad (1.1)$$

Then, we obtain

$$A^n = \left(a^2+b^2\right)^n \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix},$$

where $\theta = \arccos \dfrac{a}{a^2+b^2}$.

□

Problem 1.3.8

For any real number x let

$$A(x) = \begin{bmatrix} 1-x & 0 & x \\ 0 & 1 & 0 \\ x & 0 & 1-x \end{bmatrix}.$$

a. Prove that for all $a, b \in \mathbb{R}$, we have

$$A(a)A(b) = A(a+b-2ab).$$

b. Given $x \in \mathbb{R}$, compute $A(x)^n$.

Problem 1.3.9

Compute A^{20}, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution. The power of a diagonal matrix is the power of its diagonal entries, so

$$A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 3^{20} \end{bmatrix} \quad (1.2)$$

. □

Problem 1.3.10

a. Give a detailed proof, by induction on k, for the *binomial formula:* if $A, B \in M_n(\mathbb{F})$ *commute* then

$$(A+B)^k = \sum_{j=0}^{k} \binom{k}{j} A^{k-j} B^j.$$

b. Give a counterexample to the binomial formula if we drop the hypothesis that A and B commute.

Solution.

a. There's nothing to prove the base case $k = 0$, so assume that the binomial formula is true $\forall\ k = n \in \mathbb{N}$.

For $k = n + 1$,

$$\begin{aligned}(A+B)^{n+1} &= (A+B)\cdot(A+B)^n \\ &= \sum_{j=0}^{n} \binom{k}{j} A^{n-j} B^j (A+B) \\ &= A^{n+1} + (n+1)A^n B + \ldots + (n+1)AB^n + B^{n+1} \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} A^{n+1-j} B^j.\end{aligned}$$

This closes the induction.

b. The claim isn't necessarily true: let $A = A = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $(A+B)^2 = \begin{bmatrix} 36 & 48 \\ 96 & 132 \end{bmatrix}$ and $\sum_{j=0}^{2} \binom{2}{j} A^{2-j} B^j = \begin{bmatrix} 32 & 44 \\ 96 & 136 \end{bmatrix}$, so in this case, $(A+B)^2 \neq \sum_{j=0}^{2} \binom{j}{2} A^{2-j} B^j$.

$\square$

Problem 1.3.11

a. Let

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Prove that $B^3 = O_3$.

b. Let $a \in \mathbb{R}$ Using part (a) and the binomial formula, compute A^n where

$$A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & -a \\ a & a & 1 \end{bmatrix}.$$

Problem 1.3.12

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

a. Prove that $(A - I_3)^3 = O_3$.
b. Compute $A^n \ \forall\, n \in \mathbb{N}^*$, by using part (a) and the binomial formula.

Problem 1.3.11

a. Prove that matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

satisfies $(A - I_3)^3 = O_3$.
b. Compute $A^n \ \forall\, n \in \mathbb{N}^*$.

Problem 1.3.16

A matrix $A \in M_n(\mathbb{R})$ is called a *permutation matrix* if each row and column of A has an entry equal to 1 and all other entries equal to 0. Prove that the product of two permutation matrices is a permutation matrix.

Problem 1.3.17

Consider the permutation σ of $1, 2, \ldots, n$ that is a bijective map

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

We define the associated permutation matrix P_σ as follows: the (i, j)-entry of P_σ is equal to 1 is $i = \sigma(j)$ and 0 otherwise.

a. Prove that any permutation matrix is of the form P_σ for unique permutation σ.
b. Deduce that there are $n!$ permutation matrices.
c. Prove that

$$P_{\sigma_1} \cdot P_{\sigma_2} = P_{\sigma_1 \circ \sigma_2}.$$

for all permutations σ_1, σ_2.
d. Given a matrix $B \in M_n(\mathbb{F})$, describe the matrices $P_\sigma B$ and BP_σ in terms of B and of the permutation σ.

1.4 Block Matrices

1.4.1 Summary

1.4.2 Problems for Practice

If $A = [a_{ij}] \in M_{m_1,n_1}(\mathbb{F})$ and $B \in M_{m_2,n_2}(\mathbb{F})$ are matrices, the *Kronecker product* or *tensor product* of A and B is the matrix $A \otimes B \in M_{m_1m_2,n_1n_2}(\mathbb{F})$ defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1,n_1}B \\ a_{21}B & a_{22}B & \dots & a_{2,n_1}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m_1,1}B & a_{m_1,2}B & \cdots & a_{m_1,n_1}B \end{bmatrix}.$$

Problem 1.4.1

Compute the Kronecker product of the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Solution: $A \otimes B = \begin{bmatrix} 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$

Problem 1.4.2

Do we always have $A \otimes B = B \otimes A$?

Problem 1.4.3

Check that $I_m \otimes I_n = I_{mn}$.

1.5 Invertible Matrices

1.5.1 Summary

1.5.2 Problems for Practice

Problem 1.5.3

Is the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in M_3(\mathbb{F}_2)$$

invertible? If so, compute its inverse.

Problem 1.5.5

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in M_5(\mathbb{R}).$$

Prove that A is invertible and compute its inverse.

Problem 1.5.6

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

a. Compute the inverse of A by solving, for each $b \in \mathbb{R}^4$, the system $AX = b$.
b. Prove that $A^2 = 3I_4 + 2A$. Deduce a new way of computing A^{-1}.

Problem 1.5.7

Let A be the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 5 & -2 & 3 \\ -1 & 0 & -1 \end{bmatrix}.$$

a. Check that $A^3 = O_3$.
b. Compute $(I_3 + A)^{-1}$.

Problem 1.5.10

Let $A, B \in M_{\mathbb{R}}$ such that

$$A + B = I_n \text{ and } A^2 + B^2 = O_n.$$

Prove that A and B are invertible and that $\left(A^{-1} + B^{-1}\right)^n = 2^n I_n \forall\ n \in \mathbb{N}^*$.

Proof. We have

$$(A + B)^2 = A^2 + AB + BA + B^2 = AB + BA = I_n.$$

On the other hand,

$$A(A + B) = AI_n = (A + B)A = A^2 + AB = A^2 + BA.$$

Combining these two equations, we obtain $AB = BA \Longrightarrow AB + BA = 2AB = 2BA = I_n \Longrightarrow A$ has inverse and $A^{-1} = 2B$.

$\left(A^{-1} + B^{-1}\right)^n = (A + B)^n\left(A^{-1} + B^{-1}\right)^n = \left(2I_n + BA^{-1} + AB^{-1}\right)^n = \left(2I_n + A^2 + B^2\right)^n = 2^n I_n$, as desired. □

Problem 1.5.11

Let $A \in M_n(\mathbb{R})$ be in invertible be an invertible matrix such that

$$A^{-1} = I_n - A.$$

Prove that $A^6 - I_n = O_n$.

Proof. So,

$$\begin{aligned} A^6 - I_n &= (A - I_n)(A + I_n)\left(A^2 + A + I_n\right)\left(A^2 - A + I_n\right) \\ &= B\left(A^2 - A + I_n\right) = BO_n = O_n, \end{aligned}$$

where $B = (A - I_n)\left(A^2 + A + I_n\right)(A + I_n)$, as desired. □

Problem 1.5.14

Suppose that an upper-triangular matrix $A \in M_n(\mathbb{C})$ is invertible. Prove that A^{-1} is also upper-triangular.

Problem 1.5.15

Let $a, b, c \in \mathbb{R}_+$, not all of them equal and consider the matrix

$$A = \begin{bmatrix} a & 0 & b & 0 & c & 0 \\ 0 & a & 0 & c & 0 & b \\ c & 0 & a & 0 & b & 0 \\ 0 & b & 0 & a & 0 & c \\ b & 0 & c & 0 & a & 0 \\ 0 & c & 0 & b & 0 & a \end{bmatrix}.$$

Prove that A is invertible.

1.6 The Transpose of a Matrix

1.6.1 Summary

1.6.2 Problems for Practice

Problem 1.6.2

Let $\theta \in \mathbb{R}$ and let

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

a. Prove that A is orthogonal.
b. Find all values of θ for which A is symmetric.
c. Find all values of θ for which A is skew-symmetric.

Problem 1.6.3

Which matrices $A \in M_n(\mathbb{F}_2)$ are the sum of a symmetric matrix and of a skew-symmetric matrix?

Problem 1.6.4

Write the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

and the sum of a symmetric matrix and of a skew-symmetric matrix with real entries.

Problem 1.6.5

All matrices in the following statements are sqaure matrices of the same sizes. Prove that

a. The product of two symmetric matrices is a symmetric matrix if and only if the two matrices commute.
b. The product of two antisymmetric matrices is a symmetric matrix if and only if the two matrices commute.
c. The product of a symmetric and a skew-symmetric matrix is a symmetric matrix if and only if the two matrices commute.

Problem 1.6.7

Consider the map $\varphi : M_3(\mathbb{R}) \longrightarrow M_3(\mathbb{R})$ defined by

$$\varphi(A) = A^{\mathsf{T}} + 2A.$$

Prove that φ is linear, that is,

$$\varphi(cA + B) = c\varphi(A) + \varphi(B)$$

$\forall\ A, B \in M_3(\mathbb{R}),\ c \in \mathbb{R}$.

Problem 1.6.8

Let $A \in M_n(\mathbb{R})$ be a matrix such that $A \cdot A^{\mathsf{T}} = O_n$. Prove that $A = O_n$.

Problem 1.6.9

Find all skew-symmetric matrices $A \in M_m(\mathbb{R})$ such that $A^2 = O_n$.

Proof. If a skew-symmetric matrix $A = [a_{ij}] \in M_n(\mathbb{R}) : A^2 = O_n$ then its diagonal entries must be zero, or

$$\sum_{i=1}^{n} a_{1i}^2 = 0$$
$$\sum_{i=1}^{n} a_{2i}^2 = 0$$
$$\vdots$$
$$\sum_{i=1}^{n} a_{ni}^2 = 0$$
$$\Longrightarrow a_{ij} = 0\ \forall\ i, j \leq n \Longleftrightarrow A = 0_n.$$

$\square$

Problem 1.6.10

Let $a_1, \dots, a_k \in M_n(\mathbb{R})$ be matrices such that

$$\sum_{i=1}^{k} A_i \cdot A_i^{\mathsf{T}} = O_n$$

Prove that $A_1 = \dots = A_k = O_n$.

Problem 1.6.11

a. Let $A \in M_3(\mathbb{R})$ be a skew-symmetric matrix. Prove that there exists a non-zero vector $X \in \mathbb{R}^3 : AX = 0$.
b. Does the result in part (a) remain true if we replace $M_3(\mathbb{R})$ with $M_2(\mathbb{R})$?

Proof.

a. Suppose that $A \in M_3(\mathbb{R}) = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$. We then have four cases to $AX = 0$:
   - $a = b = c = 0$: Any nonzero vector $X \in \mathbb{R}^3$ satisfies the problem.
   - One of a, b, c is 0. Then the value of the varible with the zero coefficient can be any real number, while the remaining two are 0.
   - One of a, b, c is 0. Then the value of the two varibles with the zero coefficient can be any real number (but they can't be both 0), while the remaining one is 0.
   - $a, b, c \neq 0$. Then $x = \begin{bmatrix} c \\ -b \\ a \end{bmatrix}$.

b. The result of (a) holds true only for $a = 0$; if $a \neq 0$, then $X = 0$.

□

Problem 1.6.12

Describe all upper-triangular matrices $A \in M_m(\mathbb{R})$ such that

$$A \cdot A^{\mathsf{T}} = A^{\mathsf{T}} \cdot A.$$

2 Square Matrices of Order 2

2.1 The Trace and Determinant Maps

2.1.1 Summary

Defintion 2.1

Consider a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(\mathbb{C})$. We define

- the *trace* of A is

$$\mathrm{Tr}(A) = a_{11} + a_{22}.$$

- the *determinant* of A as

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

We also write

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

for $\det A$.

Proposition 2.2

For all matrices $A, B \in M_2(\mathbb{C}), z \in \mathbb{C}$,
(1) $\mathrm{Tr}(A + zB) = \mathrm{Tr}(A) + z\,\mathrm{Tr}(B)$;
(2) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$;
(3) $\mathrm{Tr}\left(A^\intercal\right) = \mathrm{Tr}(A)$.

Proposition 2.4

For all matrices $A, B \in M_2(\mathbb{C})$ and all $\alpha \in \mathbb{C}$ we have
(1) $\det(AB) = \det A \cdot \det B$;
(2) $\det A^\intercal = \det A$;
(3) $\det(\alpha A) = \alpha^2 \det A$.

2.1.2 Problems for Practice

Problem 2.1.4

Prove that $\forall\ A \in M_2(\mathbb{C})$, we have

$$\det A = \frac{(\mathrm{Tr}(A))^2 - \mathrm{Tr}\left(A^2\right)}{2}.$$

Problem 2.1.5

Prove that $\forall\ A, B \in M_2(\mathbb{C})$ we have

$$\det(A + B) = \det A + \det B + \mathrm{Tr}(A) \cdot \mathrm{Tr}(B) - \mathrm{Tr}(AB).$$

Problem 2.1.6

Let $f : M_2(\mathbb{C}) \longrightarrow \mathbb{C}$ be a map with the property that $\forall\ A, B \in M_2(\mathbb{C})$ and $z \in \mathbb{C}$, we have

$$f(A + zB) = f(A) + zf(B) \quad \text{and} \quad f(AB) = f(BA).$$

a. Consider the matrices

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and define $x_{ij} = f(E_{ij})$. Check that $E_{12}E_{21} = E_{11}$ and $E_{21}E_{12} = E_{22}$, and deduce that $x_{11} = x_{22}$.

b. Check that $E_{11}E_{12} = E_{12}$ and $E_{12}E_{11} = O_2$, and deduce that $x_{12} = 0$. Using a similar argument, prove that $x_{21} = 0$.

c. Conclude that $\exists\ c \in \mathbb{C}$ such that

$$f(A) = c \cdot \mathrm{Tr}(A)$$

$\forall$ #mts A.

2.2 The Characteristic Polynomial and the Cayley-Hamilton Theorem

2.2.1 Summary

The *characteristic polynomial* of A is the polynomial denoted

$$\det(XI_2 - A) = X^2 - \mathrm{Tr}(A)X + \det A.$$

Note that AB and BA have the same characteristic polynomial for all $A, B \in M_2(\mathbb{C})$. In particular, if P is in invertible, then A and PAP^{-1} have the same characteristic polynomial.

For any $z \in \mathbb{C}$, if we evaluate the characteristic polynomial of A at z, we know the value of $\det(zI_2 - A)$.

Worked Example 2.6

For any $A, B \in M_2(\mathbb{C})$, there is a complex number u such that

$$\det(A + zB) = \det A + uz + \det B \cdot z^2$$

$\forall\ z \in \mathbb{C}$ If A, B have integer/rational/real entries, so does u.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then,

$$\det(A+zB) = \det\left(\begin{bmatrix} a+ze & b+zf \\ c+zg & d+zh \end{bmatrix}\right)$$
$$= ad - bc + z^2(eh - fg) + (ah + de - bg - cf)z$$
$$= \det A + uz + \det B \cdot z^2,$$

where $u = ah + de - bg - cf$. □

Worked Example 2.7

Let $U, V \in M_2(\mathbb{R})$. Using the polynomial $\det(U + XV)$, prove that

$$\det(U+V) + \det(U-V) = 2\det U + 2\det V.$$

Proof. We have, from Worked Example ,

$$\det(U+V) + \det(U-V)$$
$$= \det U + m + \det V + \det U - m + \det V$$
$$= 2\det U + 2\det V$$

for some $m \in \mathbb{R}$. □

Worked Example 2.8

Let $A, B \in M_2(\mathbb{R})$. Using the previous problem, prove that

$$\det\left(A^2 + B^2\right) + \det(AB + BA) \geq 0.$$

Proof. From the previous problem, we can derive

$$\det\left(A^2+B^2\right) + \det(AB+BA) = \det\left[\frac{(A+B)^2 + (A-B)^2}{2}\right] + \det\left[\frac{(A+B)^2 - (A-B)^2}{2}\right]$$
$$= \det\left[(A+B)^2\right] + \det\left[(A-B)^2\right]$$
$$= [\det(A+B)]^2 + [\det(A-B)]^2 \geq 0$$

as desired. □

Worked Example 2.9

Let $A, B \in M_2(\mathbb{R})$. Using the polynomial

$$f(X) = \det[I_2 + AB + X(BA - AB)],$$

prove that

$$\det\left(I_2 + \frac{2AB + 3BA}{5}\right) = \det\left(I_2 + \frac{3AB + 2BA}{5}\right).$$

Proof. We'll need to prove the more general result $f(X) = f(1 - X)$; the statement of the problem asks us to verify this for $X = \frac{2}{5}$.

Our claim can be proved by observing that

$$\begin{aligned} f(X) &= \det[I_2 + AB + X(BA - AB)] \\ &= 1 + u + \det[(1 - X)AB + XBA] \\ &= 1 + u + \det[AB - X(AB - BA)] \\ &= 1 + u + \det AB - Xv + X^2 \det(AB - BA) \\ &= 1 + u + \det BA - Xv + X^2 \det(BA - AB) \\ &= 1 + u + \det[(1 - X)BA + XAB] \\ &= \det[I_2 + BA + X(AB - BA)] \\ &= f(1 - X) \text{ for some } u, v \in \mathbb{R}, \end{aligned}$$

as desired. Applying this result to $X = \frac{2}{5}$ gives us the result we want. □

Defintion 2.10 (Eigenvalues)

The *eigenvalues* of a matrix $A \in M_2(\mathbb{C})$ are the roots of its characteristic polynomial in other words, they are the complex solutions λ_1, λ_2 of the equation

$$\det(tI_2 - A) = t^2 - \operatorname{Tr}(A)t + \det A = 0.$$

Theorem 2.11 (Cayley-Hamilton theorem)

For any $A \in M_2(\mathbb{C})$, we have

$$A^2 - \operatorname{Tr}(A) \cdot A + (\det A) \cdot I_2 = O_2.$$

Proof. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\begin{aligned} &A^2 - \mathrm{Tr}(A)\cdot A + (\det A)\cdot I_2 = O_2 \\ &= \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+ad \\ ac+cd & ad+d^2 \end{bmatrix} - \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= O_2. \end{aligned}$$

□

Remark 2.12

a. In other words, the matrix A is a solution of the characteristic polynomial

$$\det(tI_2 - A) = t^2 - \mathrm{Tr}(A)t + \det A = 0.$$

b. Expressed in terms of the of the eigenvalues λ_1, λ_2 of A, the Cayley-Hamilton theorem can written either as

$$A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 \cdot I_2 = O_2 \tag{2.1}$$

or equivalently

$$(A - \lambda_1 I_2)(A - \lambda_2 I_2) = O_2. \tag{2.2}$$

Worked Example 2.13

Let $A \in M_2(\mathbb{C})$ have eigenvalues λ_1 and λ_2. Prove that $\forall\ n \geq 1$ we have

$$\mathrm{Tr}(A^n) = \lambda_1^n + \lambda_2^n.$$

Deduce that λ_1^n and λ_2^n are eigenvalues of A^n.

Proof. Let $x_n = \mathrm{Tr}(A^n)$. Multiplying (2.1) by A^n and taking its trace, we get

$$x_{n+2} - (\lambda_1 + \lambda_2)x_{n+1} + \lambda_1\lambda_2 x^n = 0.$$

□

Since $x_0 = 2$ and $x_1 = \lambda_1 + \lambda_2 = \mathrm{Tr}(A)$, a simple induction shows that $\mathrm{Tr}(A^n) = \lambda_1^n + \lambda_2^n$.

Worked Example 2.14

Let $A \in M_2(\mathbb{C})$ be a matrix with $\mathrm{Tr}(A) \neq 0$. Prove that a matrix $B \in M_2 \in (\mathbb{C})$ commutes with A if and only if B commutes with A^2.

Worked Example 2.15

Prove that for any matrices $A, B \in M_2(\mathbb{R})$ there is a real α such that $(AB - BA)^2 = \alpha I_2$.

Worked Example 2.16

Let $X \in M_2(\mathbb{R})$ be a matrix such that $\det\left(X^2 + I_2\right) = 0$. Prove that $X^2 + I_2 = O_2$.

Worked Example 2.17

A matrix $A \in M_2(\mathbb{C})$ in invertible if and only if $\det A \neq 0$. If this is the case, then

$$A^{-1} = \frac{1}{\det A}[\mathrm{Tr}(A) \cdot I_2 - A].$$

Worked Example 2.19

Let $A, B \in M_2(\mathbb{C})$ be two matrices such that $AB = I_2$. Then A is invertible and $B = A^{-1}$. In particular, we have $BA = I_2$.

Theorem 2.20

If $A \in M_2(\mathbb{C})$ and $z \in \mathbb{C}$, then the following assertions are equivalent:

a. z is an eigenvalue of A;

b. $\det(zI_2 - A) = 0$;

c. There is a nonzero vector $v \in \mathbb{C}^2$ such that $Av = zv$.

Worked Example 2.21

Let $A \in M_2(\mathbb{C})$ have two distinct λ_1, λ_2. Prove that we can find an invertible matrix $P \in \mathrm{GL}_2(\mathbb{C})$ such that

$$A = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}P^{-1}.$$

Worked Example 2.22

Solve in $M_2(\mathbb{C})$ the following equations:

a. $A^2 = O_2$;

b. $A^2 = I_2$;

c. $A^2 = A$.

Worked Example 2.23

Let $A \in M_2(\mathbb{C})$ be a matrix. Prove that the following statements are equivalent:

a. $\operatorname{Tr}(A) = \det A = 0$.
b. $A^2 = O_2$.
c. $\operatorname{Tr}(A) = \operatorname{Tr}\left(A^2\right) = 0$.
d. $\exists\, n \geq 2 : A^n = O_2$.

Worked Example 2.24

Find all matrices $X \in M_2(\mathbb{R}) : X^3 = I_2$.

2.2.2 Problems for Practice

Problem 2.2.1

Let $A, B \in M_2(\mathbb{R})$ be commuting matrices. Prove that

$$\det\left(A^2 + B^2\right) \geq 0$$

Proof.

$$\begin{aligned}\det\left(A^2 + B^2\right) &= \det(A + iB) \cdot \det(A - iB) \\ &= (\det A - \det B + ui)(\det A - \det B - ui) \\ &= (\det A - \det B)^2 + u^2 \text{ for some } u \in \mathbb{R}\end{aligned}$$

□

The last equation clearly shows that $\det\left(A^2 + B^2\right) \geq 0$.

Problem 2.2.2

Let $A, B \in M_2(\mathbb{R})$ such that $AB = BA$ and $\det\left(A^2 + B^2\right) = 0$. Prove that $\det A = \det B$.

Hint: use the hint of the previous problem and consider the polynomial $\det(A + XB)$.

Proof. Since $\det(A + iB) = \overline{\det(A - iB)}$ and $\det\left(A^2 + B^2\right) = 0$, this means that $\det(A + iB) = \det(A - iB) = 0 \implies \det A + iu - \det B = \det A - iu - \det B = 0 \implies \det A - \det B = 0 \implies \det A = \det B$. □

Problem 2.2.3

Let $A, B, C \in M_2(\mathbb{R})$ be pairwise commuting matrices and let

$$f(X) = \det\left(A^2 + B^2 + C^2 + x[AB + BA + CA]\right).$$

a. Prove that $f(2) \geq 0$.
b. Prove that $f(-1) \geq 0$.
c. Deduce that

$$\det\left(A^2 + B^2 + C^2\right) + 2\det(AB + BC + CA) \geq 0.$$

Solution.
a.

□

Problem 2.2.4

Let $A, B \in M_2(\mathbb{C})$ be matrices with $\mathrm{Tr}(AB) = 0$. Prove that $(AB)^2 = (BA)^2$.

Problem 2.2.5

Let $A \in M_2(\mathbb{Q})$ with the property that

$$\det\left(A^2 - 2I_2\right) = 0.$$

Prove that $A^2 = 2I_2$ and $\det A = -2$.

Problem 2.2.6

Let $x \in \mathbb{R}_+$ and let $A \in M_2(\mathbb{R})$ such that $\det\left(A^2 + xI_2\right) = 0$. Prove that

$$\det\left(A^2 + A + xI_2\right) = x.$$

Problem 2.2.7

Let $A, B \in M_2(\mathbb{R})$ be such that $\det(AB - BA) \leq 0$. Conisder the polynomial

$$f(X) = \det(I_2 + [1 - X]AB + XBA).$$

a. Prove that $f(0) = f(1)$.
b. Deduce that

$$\det(I_2 + AB) \leq \det\left(I_2 + \frac{1}{2}[AB + BA]\right).$$

Problem 2.2.8

Let $n \geq 3$ be an integer. Let $X \in M_2(\mathbb{R})$ be such that

$$X^n + X^{n-2} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

a. Prove that $\det X = 0$.
b. Let $t = \mathrm{Tr}(X)$. Prove that

$$t^n + t^{n-2} = 2.$$

c. Find all possible matrices X satisfying the original equation.

2.3 The Powers of a Square matrix of Order 2

2.3.1 Summary

Theorem 2.25

Let $A \in M_2(\mathbb{C})$ and let λ_1, λ_2 be its eigenvalues.
a. If $\lambda_1 \neq \lambda_2$, then $\forall\ n \geq 1$ we have $A^n = \lambda_1^n B + \lambda_2^n C$, where

$$B = \frac{1}{\lambda_1 - \lambda_2}(A - \lambda_2 I_2) \text{ and } C = \frac{1}{\lambda_2 - \lambda_1}(A - \lambda_1 I_2).$$

b. If $\lambda_1 = \lambda_2$, then $\forall\ n \geq 1$ we have $A^n = \lambda_1^n I_2 + n\lambda_1^{n-1}(A - \lambda_1 I_2)$.

Worked Example 2.26

Compute A^n, where $A = \begin{bmatrix} 1 & 3 \\ -3 & -5 \end{bmatrix}$.

Corollary 2.27

For any matrix $A \in M_2(\mathbb{C})$ there are sequences $(x_n)_{n\geq 0}, (y_n)_{n\geq 0}$ of complex numbers such that

$$A^n = x_n A + y_n I_2.$$

Worked Example 2.28

Let m, n be positive integers and let $A, B \in M_2(\mathbb{C})$ be two matrices such that $A^m B^n = B^n A^m$. If A^m and B^n are not scalar, prove that $AB = BA$.

Worked Example 2.29

Let $t \in \mathbb{R}$ and let

$$A_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Compute A_t^n for $n \geq 1$.

2.3.2 Problems for Practice

Problem 2.3.1

Consider the matrix

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$

a. Let n be a positive integer. Prove the existence of a unique pair of intergers $(x_n, y_n) \in \mathbb{Z}$ such that

$$A^n = x_n A + y_n I_2.$$

b. Compute $\lim\limits_{x \to \infty} \frac{x_n}{y_n}$.

Problem 2.3.2

Given a positive integer n, compute the nth power of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Problem 2.3.3

Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}^*$. Compute the nth power of the matrix $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

Problem 2.3.4

Let $x \in \mathbb{R}$ and let

$$A = \begin{bmatrix} \cos x + \sin x & 2\sin x \\ -\sin x & \cos x - \sin x \end{bmatrix}.$$

Compute A^n for all positive integers n.

2.4 Application to Linear Recurrences

2.4.1 Summary

Let a, b, c, d, x_0, y_0 be complex numbers and consider the sequences x_n, y_n defined by

$$\begin{cases} x_{n+1} = ax_n + by_n \\ y_{n+1} = cx_n + dy_n \end{cases}, n \geq 0.$$

Noting that $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \forall\ n \geq 0$, we yield, via induction,

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

For second-order linear recurrences with constant coefficients

$$x_{n+1} = ax_n + bx_{n+1} \quad n \geq 1,$$

we can reduce it back to the first case by letting $y_n = x_{n-1}$ for $n \geq 1$ and $y_0 = b^{-1}(x_1 - ax_0)$ if $b \neq 0$. Indeed, we then get the following system

$$\begin{cases} x_{n+1} = ax_n + by_n \\ y_{n+1} = x_n \end{cases}, \quad n \geq 0.$$

Finding x_n is now a matter of computting the powers of the matrix

$$A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is $\lambda^2 - a\lambda - b = 0$. If λ_1, λ_2 are the roots of this equation, then Theorem 2.25 allows us to yield the following:

- If $\lambda_1 \neq \lambda_2$, we can find constants u, v such that

  $$x_n = u\lambda_1^n + v\lambda_2^n \quad \forall\ n.$$

  They can be easily computed by solving for u, v in

  $$\begin{cases} x_0 = u + v \\ x_1 = u\lambda_1 + v\lambda_2 \end{cases}.$$

- If $\lambda_1 = \lambda_2$, we can find u, v such that $\forall\ n \geq 0$,

  $$x_n = (un + v)\lambda_1^n$$

  and u, v can be found by solving

  $$\begin{cases} x_0 = v \\ x_1 = (u + v)\lambda_1 \end{cases}.$$

2.4.2 Problems for Practice

Problem 2.4.1

Find the general term of the sequence $(x_n)_{n\geq 0}$ defined by $x_1 = 1, x_2 = 0$ and for $n \geq 1$,

$$x_{n+2} = 4x_{n+1} - x_n.$$

Solution. Letting $y_{\{n+1\}} = x_n$, we obtain

$$\begin{cases} x_{n+1} = 4x_n - x_{n-1} \\ y_{n+1} = x_n \end{cases}$$

Calculating the eigenvalues of the matrix $A = \begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix}$, we get

$$\lambda_1 = 2 + \sqrt{3}, \lambda_2 = 2 - \sqrt{3}.$$

Thus there are constants u, v such that $x_n = u\lambda_1^n + v\lambda_2^n$ and

$$\begin{cases} x_0 = -4 = u + v \\ x_1 = 1 \quad = u\lambda_1 + v\lambda_2 \end{cases}$$

$$\Longrightarrow u = \frac{3\sqrt{3} - 4}{2} - 2, v = \frac{-3\sqrt{3} - 4}{2}$$

$$\Longrightarrow x_n = \frac{\left(3\sqrt{3} - 4\right)\left(2 + \sqrt{3}\right)^n}{2} - \frac{\left(3\sqrt{3} + 4\right)\left(2 - \sqrt{3}\right)^n}{2}$$

□

Problem 2.4.2

Consider the sequence $(x_n)_{n\geq 0}$ defined by $x_0 = 1, x_1 = 2$ and for $n \geq 1$,

$$x_{n+2} = x_{n+1} - x_n.$$

Is this series periodical? If so, find its minimal period.

Solution. Setting $y_n = x_{n+1}$, we get

$$\begin{cases} x_{n+1} = x_n - x_{n-1} \\ y_{n+1} = x_n \end{cases}.$$

Solving for the eigenvalues of the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, we get $\lambda_1 = \frac{1 + \sqrt{3}i}{2}, \lambda_2 = \frac{1 - \sqrt{3}i}{2}$. This means that there exists constants, u, v such that

$$x_n = u\lambda_1^n + v\lambda_2^n.$$

. Solving the system $\begin{cases}1 = u + v\\ 2 = u\lambda_1 + v\lambda_2\end{cases}$, we obtain $u = \frac{1-\sqrt{3}i}{2}, v = \frac{1+\sqrt{3}i}{2}$. Thus

$$x_n = \frac{\left(1-\sqrt{3}i\right)\left(1+\sqrt{3}i\right)^n}{2^{n+1}} + \frac{\left(1+\sqrt{3}i\right)\left(1-\sqrt{3}i\right)^n}{2^{n+1}}.$$

□

Problem 2.3.3

Find the general terms of the sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ satisfying $x_0 = y_0 = 1, x_1 = 1, y_1 = 2$ and

$$x_{n+1} = \frac{2x_n + 3y_n}{5}, y_{n+1} = \frac{2y_n + 3x_n}{5}.$$

Problem 2.4.4

A sequence $(x_n)_{n\geq 0}$ satisfies $x_0 = 2, x_1 = 3$ and $\forall\ n \geq 1$,

$$x_{n+1} = \sqrt{x_{n-1}x_n}.$$

Find the general term of this sequence.

Solution. Taking the logarithm base 2 of this recurrence relation gives us

$$\lg x_{n+1} = \frac{1}{2}\lg x_n + \frac{1}{2}\lg x_{n-1}.$$

Similarly to the above questions, if we take $y_n = \lg x_{n-1}$ and $z_n \lg x_n$, we get

$$\begin{cases}z_{n+1} = 0.5z_n + 0.5z_{n-1}\\ y_{n+1} = z_n\end{cases}$$

□

The eigenvalues of the matrix $A = \begin{bmatrix}0.5 & 0.5\\ 1 & 0\end{bmatrix}$ are $\lambda_1 = \frac{1+\sqrt{7}i}{4}, \lambda_2 = \frac{1-\sqrt{7}i}{4}$. Then, we must have

$$x_n = u\lambda_1^n + v\lambda_2^n.$$

Solving for $\begin{cases}2 = u + v\\ 3 = \lambda_1 u + \lambda_2 v\end{cases}$, we obtain $u = 1 - \frac{5\sqrt{7}}{7}i, v = 1 + \frac{5\sqrt{7}}{7}i$.

Problem 2.4.5

Consider a map $f : (0, \infty) \longrightarrow (0, \infty)$ such that

$$f(f(x)) = 6x - f(x)$$

$\forall\ x > 0$. Let $x > 0$ and define a sequence $(z_n)_{n \geq 0}$ by $z_0 = x$ and $z_{n+1} = f(z_n)$ for $n \geq 0$.

a. Prove that

$$z_{n+2} + z_{n+1} - 6z_n = 0 \forall\ n \geq 0.$$

b. Deduce the existence of $a, b \in \mathbb{R}$ such that

$$z_n = a \cdot 2^n + b \cdot (-3)^n\ \forall\ n \geq 0.$$

c. Using the fact that $z_n > 0\ \forall\ n$, prove that $b = 0$ and conclude that $f(x) = 2x\ x > 0$.

2.5 Solving the Equation $X^n = A$

2.5.1 Summary

2.5.2 Problems for Practice

Problem 2.5.1

Let $n > 1 \in \mathbb{N}^*$. Prove that the equation

$$X^n = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has no solutions in $M_2(\mathbb{C})$.

Problem 2.5.2

Solve in $M_2(\mathbb{C})$ the binomial equation

$$X^4 = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}.$$

Problem 2.5.1

Let $n > 1 \in \mathbb{N}^*$. Prove that the equation

$$X^n = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$$

has no solutions in $M_2(\mathbb{Q})$.

Problem 2.5.4

Find all matrices $X \in M_2(\mathbb{R})$ such that

$$X^3 = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}.$$

Problem 2.5.5

Find all matrices $A, B \in M_2(\mathbb{C})$ such that

$$AB = O_2 \quad \text{and} \quad A^5 + B^5 = O_2.$$

Problem 2.5.6

Solve in $M_2(\mathbb{R})$ the binomial equation

$$X^n = \begin{bmatrix} 7 & -5 \\ -15 & 12 \end{bmatrix}.$$

Problem 2.5.7

Solve in $M_2(\mathbb{R})$ the binomial equation

$$X^n = \begin{bmatrix} -6 & -2 \\ 21 & 7 \end{bmatrix}.$$

2.6 Application to Pell's Equations

2.6.1 Summary

2.6.2 Problems for Practice

Problem 2.6.1

A triangular number is a number of the form $1 + 2 + \dots + n$ for some positive integer n. Find all triangular numbers which are perfect squares.

Problem 2.6.2

Find all positive intgers n such that $n + 1$ and $3n + 1$ are simultaneously perfect squares.

Problem 2.6.3

Find all integers a, b such that $a^2 + b^2 = 1 + 4ab$.

Problem 2.6.4

The difference of two consecutive cubes equals n^2 for some positive integer n. Prove that $2n - 1$ is a perfect square.

Problem 2.6.5

Find all triangles whose side lengths are consecutive integers and whose area is also an integer.

3 Matrices and Linear Equations

3.1 Linear Systems: The Basic Vocabulary

3.1.1 Summary

A *linear equation in the variables* $x_1, \ldots, x_n$ is an equation of the form

$$\sum_{i=1}^{n} a_i x_i = b,$$

where $a_1, \ldots, a_n, b \in \mathbb{F}$ are given scalars and n is a given positive integer. The unknowns $x_1, \ldots, x_n$ are supposed to be elements in $\mathbb{F}$. A *linear system in the variables* $x_1, \ldots, x_n$ is a family of linear equations. It can also be written as

$$AX = b \tag{1}$$

where $A = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. If $C_1, \ldots, C_n$ are the columns of A, (1) is equivalent to

$$\sum_{i=1}^{n} x_i C_i = b. \tag{2}$$

Defintion 3.1

a. The linear system is called *homogenous* if $b_1 = \ldots = b_m = 0$.
b. The homogeneous linear system associated with the system (1) is the system $AX = 0$.

Proposition 3.2 (Superposition principle)

Let $A \in M_{m,n}(F)$ and $b \in \mathbb{F}^m$. Let $\mathcal{S} \subset \mathbb{F}^n$ be the set of solutions of the homogeneous linear system $AX = 0$. If the system $AX = b$ has a solution X_0, then the set of solutions of this systems is $X_0 + \mathcal{S}$.

Defintion 3.13

A linear system is called *consistent* if it has at least one solution and is *inconsistent* otherwise.

Defintion 3.4

a. Two linear systems are *equivalent* if they have exactly the same set of solutions.
b. If A, B are matrices of the same size and are equivalent, we write $A \sim B$.

3.1.2 Problems for Practice

Problem 3.1.3

Let $a, b \in \mathbb{R}$, not both equal to 0.

a. Prove that the system

$$\begin{cases} ax_1 + bx_2 &= 0 \\ -bx_1 + ax_2 &= 0 \end{cases}$$

has only the trivial solution.

b. Prove that $\forall\ c, d \in \mathbb{R}$, the system

$$\begin{cases} ax_1 + bx_2 &= c \\ -bx_1 + ax_2 &= d \end{cases}$$

has a unique solution and find this solution in terms of a, b, c, d.

Solution:

a. Becauase the system is homogeneous, it has the trivial solution. Supposing that this system has at least one non-trivial solution.

If one a, b is equal to 0 (without loss of generality, let $b = 0$), then $ax_1 = ax_2 = 0 \Longrightarrow x_1 = x_2 = 0$. If $a, b \neq 0$, then we have $x_2/x_1 = \dfrac{b}{a} = -\dfrac{a}{b}$, a contradiction. In either case, each result gives a contradiction. Thus $\begin{cases} ax_1 + bx_2 &= 0 \\ -bx_1 + ax_2 &= 0 \end{cases}$ only has the trivial solution.

b. The system can be written as $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$. $\det A > 0 \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{ac-bd}{a^2+b^2} \\ \frac{ad+bc}{a^2+b^2} \end{bmatrix}$.

Problem 3.1.4

Let $A \in M_2(\mathbb{C})$ be a matrix and consider the homogeneous system $AX = 0$. Prove that the following statements are equivalent:

a. This system has only the trivial solution.
b. A is invertible.

Solution. Since a is invertible, we get $X = A^{-1}AX = 0$, so the homogeneous system only has the trivial solution. Now assume that A only has the trivial solution but is also not invertible. Writing $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we must have $ad - bc = 0$. If $a = 0 \implies bc = 0$. If $c = 0$ then $X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a non-trivial solution to $AX = 0$. If $c \neq 0 \implies b = 0 \implies X = \begin{bmatrix} -d \\ c \end{bmatrix}$.

If $a \neq 0 \implies d = \dfrac{bc}{a} \implies A = \begin{bmatrix} -b \\ a \end{bmatrix}$. As such, for each case we have a non-trivial solution to $AX = 0$, a contradiction to our assupmtion. Hence A is invertible. □

Problem 3.1.5

Let A, B be $n \times n$ matrices such that the system $ABX = 0$ has only the trivial solution. Show that the system $BX = 0$ also has only the trivial solution.

Problem 3.1.6

Let C, D be $n \times n$ such that the system $CDX = b$ is consistent for every choice of $b \in \mathbb{R}^n$. Show that the system $CY = b$ is consistent for every choice of $b \in \mathbb{R}^n$.

3.2 The Reduced Row-Echelon Form and Its Relevance to Linear Systems

3.2.1 Summary

3.2.2 Problems for Practice

Problem 3.2.3

Determine the fundamental solutions of the homogeneous linear system of equations $AX = 0$, where

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -2 & 4 & 0 & 2 \\ -1 & 2 & 1 & 2 \end{bmatrix}.$$

Solution. $A = \begin{bmatrix} 2a+b \\ -a \\ -b \\ b \end{bmatrix}$, where $a, b \in \mathbb{R}$. □

Problem 3.2.4

a. Write the solutions of the homogeneous system of equations $AX = 0$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ -1 & 1 & 4 \end{bmatrix}.$$

b. Find a solution to the system for which the sum of the first two coordinates is 1.

Solution.

a. $X = \begin{bmatrix} -3a \\ -7a \\ a \end{bmatrix}$.

b. $-10a = 1 \implies a = -0.1 \implies X = \begin{bmatrix} 0.3 \\ 0.7 \\ -0.1 \end{bmatrix}$.

□

Problem 3.2.7

Let $n > 2 \in \mathbb{Z}$. Solve in $\mathbb{R}$ the linear system

$$x_2 = \frac{x_1 + x_3}{2}, x_3 = \frac{x_2 + x_4}{2}, \dots, x_{n-1} = \frac{x_{n-2} + x_n}{2}.$$

Solution. We'll prove the following:

Claim

If $X = \begin{bmatrix} a_1 \\ a_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$, then

$$x_n = (n-1)a_2 - (n-2)a_1 \quad \text{for } n \ge 3.$$

Indeed, $x_3 = 2a_2 - a_1$, so assume the claim is true $\forall\ n \ge 3$. For $n+1$, $x_n = \frac{x_{n-1} + x_{n+1}}{2} \implies x_{n+1} = 2x_n - x_{n-1} = 2[(n-1)a_2 - (n-2)a_1] - (n-2)a_2 + (n-3)a_1 = na_2 - (n-1)a_1$. Thus we've proved our claim via induction, and

$$A = \begin{bmatrix} a_1 \\ a_2 \\ 2a_2 - a_1 \\ \vdots \\ (n-1)a_2 - (n-2)a_1 \end{bmatrix} \quad (a_1, a_2 \in \mathbb{R}).$$

$\square$

3.3 Solving the system $AX = b$

3.3.1 Sumamry

3.3.2 Problems for Practice

Problem 3.2.6

Find all possible values of h and k such that the system with augmented matrix

$$\begin{bmatrix} 1 & 2 & \text{bar} & h \\ 2 & k & \text{bar} & 12 \end{bmatrix}$$

has
a. a unique solution.
b. infinitely many solutions.
c. no solution.

Problem 3.3.8

Let $a, b \in \mathbb{R}$. Solve in real numbers the system

$$\begin{cases} x + y = a \\ y + z = b \\ z + t = a \\ t + x = b \end{cases}.$$

3.4 Computing the Inverse of a Matrix

3.4.1 Summary

3.4.2 Problems for Practice

Problem 3.4.2

For which $x \in \mathbb{R}$ is the matrix

$$A = \begin{bmatrix} 1 & x & 1 \\ 0 & 1 & x \\ 1 & 0 & 1 \end{bmatrix}$$

invertible? For any such number x, compute the inverse of A.

Problem 3.3.3

Let $x, y, z \in \mathbb{R}$. Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 3.4.4

Determine the inverse of the matrix

$$A = \begin{bmatrix} n & 1 & 1 & \cdots & 1 \\ 1 & n & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n \end{bmatrix} \in M_n(\mathbb{R}).$$

Problem 3.4.5

Let $a \in \mathbb{R}$. Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a & 1 & 0 & \cdots & 0 & 0 \\ a^2 & a & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a^{n-2} & a^{n-3} & a^{n-4} & \cdots & 1 & 0 \\ a^{n-1} & a^{n-2} & a^{n-3} & \cdots & a & 1 \end{bmatrix} \in M_m(\mathbb{R}).$$

4 Vector Spaces and Subspaces

4.1 Vector Spaces: Definintion, Basic Properties and Examples

4.1.1 Summary

4.1.2 Problems for Practice

Problem 4.1.1

Consider the set $V = \mathbb{R}^2$ endowed with addition rule defined by

$$(x, y) + (x', y') = (x + x', y + y')$$

and with a multiplication rule by elements λ of $\mathbb{R}$ as follows:

$$\lambda \cdot (x, y) = (2x, 0).$$

Is V endowed with these operations a vector space over $\mathbb{R}$?

Solution. No V isn't a $\mathbb{R}$-vector space as $\forall\ v = (x, y) \in V, 1(x, y) = (x, 0) \neq (x, y) = v$. □

Problem 4.1.2

Define an operation $+_*$ on $(0, \infty)$ by

$$a +_* = ab$$

for $a, b \in (0, \infty)$ and an external multiplication by real numbers as follows:

$$a \cdot_* b = b^a$$

for $a \in \mathbb{R}, b \in (0, \infty)$. Does $(0, \infty)$ endowed with this new addition and scalar multiplication become a vector space over $\mathbb{R}$?

Problem 4.1.3 (Complexification of a real vector space)

Let V be a vector space over $\mathbb{R}$. Let $V_{\mathbb{C}}$ be the Cartesian product $V \times V$ endowed with the following addition rule:

$$(x, y) + (x', y') = (x + x', y + y').$$

Also, for each $z = a + bi \in \mathbb{C}$, consider the "multiplication by z rule"

$$z \cdot (x, y) = (ax - by, ay + bx)$$

on $V_{\mathbb{C}}$. Prove that $V_{\mathbb{C}}$ endowed with these operations becomes a $\mathbb{C}$-vector space (this space is called the *complexification* of the vector space V).

4.2 Subspaces

4.2.1 Summary

Defintion 4.5

Let V be an $\mathbb{F}$-vector space. A *subspace* of V is a nonempty subset W of V which is stable under the operations of addition and scalar multiple: $v + w \in W$ and $cv \in W$ $\forall$ $v, w \in W$ and $c \in \mathbb{F}$.

Remark 4.7

a. Note that a subspace must contain the zero vector.
b. If W is a subspace of V, then W becomes an $\mathbb{F}$-vector space itself, by limiting the operations in V to W.
c. A nonempty subset W is a subspace if and only if $av + bw \in W, v, w \in W, a, b, \in \mathbb{F}$.
d. If $(W_i)_{i \in I}$ is a family of subspaces of V, then

$$W \coloneqq \bigcap_{i \in I} W_i.$$

is also a subset of V.

4.2.2 Problems for Practice

Problem 4.2.1

Show that none of the following sets of vectors is a subspaces pf $\mathbb{R}^3$:
a. The set U of vectors $x = (x_1, x_2, x_3)$ such that $\sum_{i=1}^{3} x_i^2 = 1$.
b. The set V of vectors in R^3 all of whose coordinates are integers.
c. Thet set W of vectors $\mathbb{R}^3$ that have at least one coordinate equal to 0.

Problem 4.2.2

Determine if U is a subspace of $M_2(\mathbb{R})$, where
a. U is the set of 2×2 matrices such that the sum of the entries in the first column is 0.
b. U is the set of 2×2 matrices such that the product of the product of the entries in the first column is 0.

Problem 4.2.6

Let V be the set of twice differerentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for all x we have

$$f''(x) + x^2 f'(x) - 3f(x) = 0.$$

Is V a subspace of the space of all maps $f : \mathbb{R} \longrightarrow \mathbb{R}$?

Problem 4.2.7

Let V be the set of differerentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\forall\ x$ we have

$$f'(x) - f(x)^2 = x$$

Is V a subspace of the space of all maps $f : \mathbb{R} \longrightarrow \mathbb{R}$?

Problem 4.2.8

a. Is the set of bounded sequences of real numbers a vector space of the space of all sequences of real numbers?
b. Answer the question if instead of bounded sequences we consider monotonic sequences.

Problem 4.2.9

Let V be the set of all sequences $(x_n)_{n\geq 0}$ of real numbers such that

$$x_{n+2} + nx_{n+1} - (n-1)x_n = 0 \qquad n \geq 0.$$

Prove that V is a subspace of the space of all sequence of real numbers.

Problem 4.2.10

Let V be the space of real-valued maps on $\mathbb{R}$ and let W be the $\subset$ of V consisting of maps f such that $f(0) + f(1) = 0$.
a. Check that W is a subspace of V.
b. Find a subspace S of V such that $V = W \oplus S$.

Problem 4.2.12

Let V be the space of convergent sequences of real numbers. Let W be the $\subset$ of V consisting of sequences converging to 0 and let Z be the $\subset$ of V consisting of constant sequences. Prove or disprove that W, Z are subspaces of V and $W \oplus Z = V$.

Problem 4.2.13 (Quotient space)

Let V be a vector space over $\mathbb{F}$ and let $W \subset V$ be a subspace. For a vector $v \in V$, let $[v] = \{v + w : w \in W\}$. Note that $[v_1] = [v_2]$ if $v_1 - v_2 \in W$. Define the *quotient space* V/W to be $\{[v] : v \in V\}$. Define an addition and scalar multiplication on V/W by $[u] + [v] = [u + v]$ and $a[v] = [av]$. Prove that the addition and multiplication above are well-defined and V/W equipped with these operations is vector space.

4.3 Linear Combinations and Span

4.3.1 Summary

4.3.2 Problems for Practice

Problem 4.3.1

Show that the vector $\begin{pmatrix}1 & 1 & 1\end{pmatrix}$ can't be expressed as a linear combination of

$$a_1 = \begin{pmatrix}1 & -1 & 0\end{pmatrix}, \quad a_2 = \begin{pmatrix}1 & 0 & -1\end{pmatrix} \quad \text{and } a_3 = \begin{pmatrix}0 & 1 & -1\end{pmatrix}.$$

Problem 4.3.3

Let W be the $\subset$ of $\mathbb{R}^n$ consisting of vectors whose sum of coordinates equals 0. Let $Z = \operatorname{Span}(1, \ldots, 1)$ in $\mathbb{R}^n$. Prove or disprove that $W \oplus Z = \mathbb{R}^n$.

Problem 4.3.4

Let $P = \operatorname{Span}\left[\begin{pmatrix}1 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1 & -1\end{pmatrix}\right]$ in $\mathbb{R}^3$, and let $D = \operatorname{Span}(0, 1, -1)$. Is it true that $P \oplus D = \mathbb{R}^3$?

Problem 4.3.6

Let V be the vector space of real-valued on $\mathbb{R}$ and let f_n (respectively g_n) be the map $x \mapsto nx$ (respectively $\cos^n x$). Prove or disprove that

$$\operatorname{Span}(\{f_n \text{ bar } n \geq 0\}) = \operatorname{Span}(\{g_n \text{ bar } n \geq 0\}).$$

4.4 Linear Independence

4.4.1 Summary

Defintion 4.4.1 (Linear (in)dependence)

a. Vectors $v_1, \ldots, v_n$ in some vector space are *linearly dependent* if

$$\sum_{i=1}^{n} c_i v_i = 0$$

forces at least one of $c_1, \ldots, c_n$ to be nonzero.

b. Vectors $v_1, \ldots, v_n$ in some vector space v are *linearly independent* if

$$\sum_{i=1}^{n} c_i v_i = 0$$

forces $c_1 = \ldots = c_n = 0$.

Remark 4.4.2

a. A subfamily of a linearly independent family is linearly independent.
b. If two vectors in a family of vectors are equal, then this family is automatically linearly dependent.

Remark 4.4.3

Given some vectors

4.4.2 Problems for Practice

Problem 4.4.1

Are the vectors

$$v_1 = (1 \;\; 2 \;\; 1), \quad v_2 = (-3 \;\; 4 \;\; 5), \quad v_3 = (0 \;\; 2 \;\; -3)$$

linearly independent in $\mathbb{R}^3$?

Problem 4.4.2

Consider the vectors

$$v_1 = (1 \;\; 2 \;\; 1 \;\; 3), \quad v_2 = (1 \;\; -1 \;\; 1 \;\; -1), \quad v_3 = (3 \;\; 0 \;\; 3 \;\; 1)$$

.

a. Prove that v_1, v_2, v_3 are linearly dependent.
b. Express one of these vectors as a linear combination of two other vectors.

Problem 4.4.3

Let V be the vector space of polynomials with real coefficients whose degree does not exceed 3. Are the following vectors

$$1 + 3X + x^2, \quad X^3 - 3X + 1, \quad 3X^3 - X^2 - X - 1$$

linearly independent in V?

Problem 4.4.4

Let V be the space of all real-valued on $\mathbb{R}$.

a. If $a_1 < \ldots < a_n \in \mathbb{R}$, compute

$$\lim_{x\to\infty} \sum_{i=1}^{n} e^{(a_i - a_n)}.$$

b. Prove that the family of maps $(x \mapsto e^{ax})_{a\in\mathbb{R}}$ is linearly independent in V.

Problem 4.4.5

Let V be the space of all maps $\varphi : [0,\infty) \longrightarrow \mathbb{R}$. For each $a \in (0,\infty)$ consider the map $f_a \in V$ defined by

$$f_a(x) = \frac{1}{x+a}.$$

a. Let $a_1 < \ldots < a_n$ be positive real numbers and suppose that $\alpha_1, \ldots, \alpha_n$ are real numbers such that

$$\sum_{i=1}^{n} \alpha_i f_{a_i}(x) = 0 \quad \forall\, x \geq 0.$$

Prove that $\forall\, x \in \mathbb{R}$, we have

$$\sum_{i=1}^{n} \alpha_i \cdot \prod_{j\neq i} (x + a_j) = 0.$$

By making suitable choices of x, deduce that $\alpha_1 = \ldots = \alpha_n = 0$.

b. Prove that the family that $(f_a)_{a>0}$ is linearly independent in V.

Problem 4.4.6

Consider the $V = \mathbb{R}$, seen as a vector space over $F = \mathbb{Q}$.

a. Prove the $1, \sqrt{2}, \sqrt{3}$ is a linearly independent set in V.
b. Prove that the set of numbers $\ln p$, where p runs over the prime numbers, is linearly independent in V.

Problem 4.4.7

a. If $m, n \neq 0 \in \mathbb{Z}$, compute

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,\mathrm{d}x\,.$$

b. Deduce that the maps $x \mapsto \cos nx$ with $n \neq 0 \in \mathbb{Z}$, form a linearly independent set in the space of all real-valued maps on $\mathbb{R}$.
c. Let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^n$. Is it always the case that $v_1, v_1 + v_2, \ldots, v_1 + v_2 + \ldots + v_n$ are linearly independent?

4.5 Dimension Theory

Problem 1

Given an irrep V of a finite group G, it is known that $\dim(V) \mid |G|$.

Using this, if $Z(G)$ denotes the center of the group, prove that $\dim(V) \mid |G/Z(G)|$.

5 Linear Transformations

5.1 Definitions and Objects Canonically Attached to a Linear Map

5.1.1 Summary

Defintion 5.1

Let V, W be vector spaces over F. A *linear map* (or *linear transformation* or *homomorphism*) between V and W is a map $T : V \longrightarrow W$ satisfying the following two properties:
1) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for vectors $v_1, v_2 \in V$;
2) $T(cv) = cT(v)\ \forall\ v \in V, c \in F$.

In practice, instead of checking separately that T respects addition and scalar multiplication, it may be advantageous to prove directly that

$$T(v_1 + cv_2) = T(v_1) + cT(v_2)\ \forall\ v_1, v_2 \in V, c \in \mathbb{F}.$$

Worked Example 5.2

If $T : V \longrightarrow W$ is a linear transformation, then $T(0) = 0 \wedge T(-v) = -T(v)\ \forall\ v \in V$.

Defintion 5.4

Let $T : V \longrightarrow V$ be a linear map on a vector space V. A subspace W of V is called *stable under T* or *T-stable* if $T(W) \subset W$.

Worked Example 5.5

Consider the map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ sending $(x_1, x_2) \longrightarrow (x_2, -x_1)$. Find all subspaces of $\mathbb{R}^2$ which are stable under T.

Solution. Let W be a subspace of $\mathbb{R}^2$ that's T-stable. Obviously $\mathbb{R}^2$ and $\{0\}$ are stable under W, so let's assume W is neither of those spaces. Then $\dim W$ is necessarily 1, that is, $W = \mathbb{R}v$ for $v = (x_1, x_2)$. Since W is T-stable, this means that $T(v) = c(v)$ for some $c \in \mathbb{R} \Longrightarrow (cx_1, cx_2) = (x_2, -x_1) \Longrightarrow x_1 = -cx_1^2 \Longleftrightarrow x_1 = x_2 = 0 (c \in \mathbb{R}) \Longleftrightarrow v = 0$, an obvious contradiction. Thus the only subspaces stable under T are $\mathbb{R}^2$ and $\{0\}$. □

Worked Example 5.7

Let V be a vector space over some field F and let $T : V \longrightarrow V$ be a linear transformation. Suppose that all lines in V are stable subspaces under T. Prove that there is a scalar $c \in F$ such that $T(x) = cx \forall\ 2x \in V$.

Proposition 5.8

Let V, W be vector spaces. The set $\operatorname{Hom}(V, W)$ of linear transformations between V and W is a subspace of $M(V, W)$.

Defintion 5.9

The *kernel* (or *null space*) of a linear transformation $T : V \longrightarrow W$ is

$$\ker T = \{v \in V, T(v) = 0\}.$$

The *image* (or *range*) $\operatorname{Im}(T)$ of T is the set

$$\operatorname{Im}(F) = \{T(v) \mid v \in V\} \subset W.$$

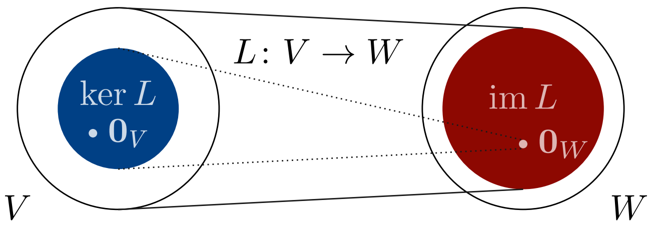

Figure 1: Image and kernel

Proposition 5.10

If $T : V \longrightarrow W$ is a linear transformation, then T is injective if and only if $\ker T = \{0\}$.

Proposition 5.14

If $T : V \longrightarrow W$ is a linear transformation, then $\ker T$ and $\mathrm{Im}(T)$ are subspaces of V, respectively W. Moreover, $\ker T$ is stable under T, and if $V = W$ then $\mathrm{Im}(T)$ is stable under T.

5.1.2 Problems for Practice

In the next problems $\mathbb{F}$ is a field.

Problem 5.1.2

Consider the map $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by

$$f(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 + x_4; 2x_1 + x_2 - x_3 + x_4; x_1 - x_2 + x_3 - x_4).$$

a. Prove that f is a linear map.
b. Give a basis for $\ker F$.

Solution.
a. f is a linear map because $\forall\ X = (x_1, \ldots, x_4), Y = (y_1, \ldots, y_4)$, we have

$$\begin{aligned} f(X+Y) = (&x_1 + y_1 + x_2 + y_2 + x_3 + y_3 + x_4 + y_4, \\ &2x_1 + 2y_1 + x_2 + y_2 - x_3 - y_3 + x_4 + y_4, \\ &x_1 + y_1 - x_2 - y_2 + x_3 + y_3 - x_4 - y_4) \\ &= f(X) + f(Y) \end{aligned}$$

and $T(cX) = cT(X)\ \forall\ c \in \mathbb{R}$.
b. $\ker f$ satisfies

$$f(X) = 0\ \forall\ X \in \mathbb{R}^4.$$

Writing down the coordinates of $f(X)$ in matrix form gives us

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

So a basis for $\ker f$ is $(0, 1, 0, -1)$. $\square$

Problem 5.1.3

Let V be the space of polynomials with real coefficients whose degree does not exceed 3, and let the map $f : V \longrightarrow \mathbb{R}^4$. be defined by

$$f(P) = [P(0), P(1), P(-1), P(2)].$$

a. Prove that f is an linear map.
b. Is f injective?

Solution.

a. $\forall\ P, Q \in \mathbb{R}_3[X]$, we have $f(P+Q) = f(P) + f(Q)$ and $f(cP) = cf(P)$ for some $c \in \mathbb{R}$ and F is a linear map by definition.
b. Let $P(x) = ax^3 + bx^2 + cx + d$. $\ker f$ satisfies

$$f(P) = (0, 0, 0, 0)$$

$\Longrightarrow a = b = c = 0 \Longrightarrow \ker f = \{0\}$. Thus f is injective.

□

Problem 5.1.4

Let $n \in \mathbb{N}^*$ and let V be the space of real polynomials whose degrees is not greater than n. Consider the map

$$f : V \longrightarrow V, \quad f[P(X)] = P(X) + (1 - X)P'(X),$$

where $P'(X)$ is the derivative of P.

a. Explain why f is a well-defined linear map.
b. Give a basis for $\ker f$.

Problem 5.1.5

Find all subspaces of $\mathbb{R}^2$ which are stable under the linear transformation

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad T(x, y) = (x + y, -x + 2y).$$

Problem 5.1.8

Let V be an vector space over F and let $T_1, \ldots, T_n : V \longrightarrow V$ be linear transformation. Prove that

$$\bigcap_{i=1}^{n} \ker T_i \subseteq \ker\left(\sum_{i=1}^{n} T_i\right).$$

Problem 5.1.11

For each of the following maps $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, check that T is linear and then check whether $\ker T$ and $\operatorname{Im}(T)$ are in direct sum.

a. $T(x, y, z) = (x - 2y + z, x - z, x - 2y + z)$;
b. $T(x, y, z) = (3x + 3y + 3z, 0, x + y + z)$.

Problem 5.1.12

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map such that $f(x + y) = f(x) + f(y)\ \forall\ x, y \in \mathbb{R}$. Prove that f is a linear map of $\mathbb{Q}$-vector spaces between $\mathbb{R}$ and itself.

Problem 5.1.13 (Quotient space)

Let V be a finite-dimensional vector space over F and let $W \subset V$ be a subspace. For a vector, let

$$[v] = \{v + w : W \in W\}.$$

Note that $[v_1] = [v_2]$ if $v_1 - v_2 \in W$. Define the quotient space V/W to be $\{[v] : v \in V\}$. Define and addition and scalar multiplication on V/W by $[u] + [v] = [u + v]$ and $a[v] = [av]$. We recall that the addition and multiplication above are well defined and V/W equipped w/ these operations is a vector space.

a. Show that the map $\pi : V \longrightarrow V/W$ defined by $\pi(v) = [v]$ is linear with kernel W.
b. Show that $\dim W + \dim(V/W) = \dim V$.
c. Show that $U \subset V$ is any subspace with $W \oplus U = V$. Show that $\pi|_U : U \longrightarrow V/W$ is an isomorphism.
d. Let $T : V \longrightarrow U$ be a linear map, let $W \subset \ker T$ be a subspace, and $\pi : V \longrightarrow V/W$ be the projection onto the quotient space. Show that there is a unique linear map $S : V/W \longrightarrow U$ such that $T = S \circ \pi$.