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Linear Algebra Done Right - Sheldon Axler

Contents

1: Vector Spaces

1A: $\mathbb{R}_n$ and $\mathbb{C}_n$

1A.1: Summary

Defintion 1.1 (Complex numbers)

A **complex number** is an ordered pair (a, b), where $a, b \in \mathbb{R}$, written as $a + bi$.

The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + b \mid a, b \in \mathbb{R}\}.$$

Addition and multiplication on $\mathbb{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i;$$

here $a, b, c, d \in \mathbb{R}$.

Proposition 1.3 (Properties of complex arithmetic)

- $\alpha + \beta = \beta + \alpha, \alpha\beta = \beta\alpha \quad \forall\ \alpha, \beta \in \mathbb{C}$;
- $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)\ \forall\ \alpha, \beta, \gamma \in \mathbb{C}$.
- $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda\ \forall\ \lambda \in \mathbb{C}$;
- For every $\alpha \in \mathbb{C}, \exists \beta \in \mathbb{C} : \alpha + \beta = 0$ (β unique);
- For every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, $\exists \beta \in \mathbb{C}$ such that $\alpha\beta = 1$.
- $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta\ \forall\ \lambda, \alpha, \beta \in \mathbb{C}$.

Defintion 1.5 ($-\alpha$, subtraction, $1/\alpha$, division)

Let $\alpha, \beta \in \mathbb{C}$.

- Let $-\alpha$ denote the additive inverse of α. Thus $-\alpha$ is the unique complex number such that

$$\alpha + (-\alpha) = 0.$$

- **Subtraction** on $\mathbb{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha).$$

- For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that

$$\alpha(1/\alpha) = 1.$$

- **Division** on $\mathbb{C}$ is defined by

$$\beta/\alpha = \beta(1/\alpha).$$

Covention 1.1 (Notation: $\mathbb{F}$)

Throughout this book, $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

Defintion 1.8 (List, length)

Suppose n is a nonnegative integer. A **list** of **length** n is an unordered collection of n elements (which might be numbers, other lists or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, ..., x_m).$$

Two lists are equal if and only if they have the same length and the same elements in the same order.

Defintion 1.11 (Coordinate, $\mathbb{F}^n$)

$\mathbb{F}^n$ is the set of all lists of length n of the elements of $\mathbb{F}$. For $(x_1, ..., x_n) \in \mathbb{F}^n$ and $j \in \{1, 2, ..., n\}$, we sat that x_j is the j^{th} **coordinate** of $(x_1, ..., x_n)$.

1B: Definition of Vector Space

1B.1: Summary

Defintion 1.19 (Addition, scalar multiplication)

- An **addition** on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A **scalar multiplication** on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.

From these definitions, we can derive the definition of a vector space:

Defintion 1.20 (Vector Space)

A **vector space** is a set V along with an addition and scalar multiplication on V such that the following propeties hold:
- $u + v = v + u \ \forall \ u, v, \in V$ (commutativity);
- $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V, a, b \in \mathbb{F}$ (associavity);
- $\exists 0 \in V : v + 0 = v \ \forall \ v \in V$ (additive identity);
- For every $v \in V, \exists w \in V : v + w = 0$ (additive inverse);
- $1v = v \ \forall \ v \in V$ (multiplicative identity);
- $a(u + v) = au + av, (a + b)v = av + bv \ \forall \ a, b \in \mathbb{F}, u, v \in V$ (distrubutive properties).

Defintion 1.21 (Vector, point)

Elements of a vector space are called ***vectors*** or ***points.***

Defintion 1.22 (Real vector space, complex vector space)

- A vector space over $\mathbb{R}$ is called a **real vector space.**
- A vector space over $\mathbb{C}$ is called a **complex vector space.**

Covention 1.24 ($\mathbb{F}^S$)

- If S is a set, then $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$.
- For $f, g \in \mathbb{F}^S$, the **sum** $f + g \in \mathbb{F}^S$ is the function defined by

$$(f+g)(x) = f(x) + g(x) \quad \forall\, x \in S.$$

- For $\lambda \in \mathbb{F}$, $f \in \mathbb{F}^S$, the **product** $\lambda f \in \mathbb{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x) \quad \forall\, x \in S.$$

Proposition 1.26 (Unique additive identity)

A vector space has a unique additive identity.

Proof. Suppose 0 and 0′ are both additive identities of some vector space V. Then

$$0 = 0 + 0' = 0' + 0 = 0'.$$

The first equality is true since because 0 is an additive identity, the second is true because of commutativity, and the third is true because 0′ is an additive identity. The result follows. □

Proposition 1.27 (Unique additive inverse)

Every element in a vector space has a unique additive inverse.

Proof. Suppose w and w' are two additive inverses of $v \in V$. Then

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w',$$

as desired. □

Covention 1.28 ($-v, w, -v$)

$0v = 0 \ \forall\, v \in V$.

Proof. $0v = (0+0)v = 0v + 0v \iff 0v = 0$. □

Covention 1.29 (V)

$a0 = 0 \; \forall \; a \in \mathbb{F}$.

Proof. $a \cdot 0 = a(0 + 0) = a0 + a0 \iff a0 = 0$. □

Proposition 1.30 (0 times a vector)

$(-1)v = -v \; \forall \; v \in V$.

Proof. $(-1)v + v = (-1)v + 1v = (-1 + 1)v = 0v = 0$. □

1B.2: Exercises 1.B

Problem 1B.1

Prove that $-(-v) = v$ for all $v \in V$.

Proof.

$$\begin{aligned} -(-v) &= (-1)(-1)v \quad \text{(Prop. 1.31)} \\ &= [(-1)(-1)]v \\ &= v, \end{aligned}$$

as desired. □

Problem 1B.1

Suppose $a \in \mathbb{F}, v \in V, av = 0$. Prove that $a = 0 \lor v = 0$.

Proof. If $a = 0$, we're done.

If $a \neq 0$, then there exists the inverse of a, a^{-1}. Hence

$$v = 1v = \left(aa^{-1}\right)v = a^{-1}(av) = a^{-1} \cdot v = 0$$

□

Problem 1B.1

Show that in the definition of a vector space (1.19), the additive inverse condition can be replaeced with the condition that

$$0v = 0 \quad \forall\, v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the RHS is the additive identity of V. (The phrase “a condition can be replaced” in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.)

Proof. $0v = [1 + (-1)]v = v + (-1)v = 0$. Letting $w = (-1)v$, we get the original additive inverse condition. □

Problem 1B.1

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & (t < 0) \\ 0 & (t = 0) \\ \infty & (t > 0) \end{cases} \qquad t(-\infty) = \begin{cases} \infty & (t < 0) \\ 0 & (t = 0) \\ -\infty & (t > 0) \end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty, \quad \infty + (-\infty) = 0.$$

Is $R \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$? Explain.

Proof. This is not a vector space in $\mathbb{R}$; using the definitions of addition and multiplication above, we should obtain

$$\infty = (2 - 1)\infty = 2\infty - \infty = \infty - \infty = 0.$$

However, that would mean that for any $t \in \mathbb{R}$,

$$t = 0 + t = \infty + t = \infty = 0$$

which violates the uniqueness of the zero vector. □

1C: Subspaces

1C.1: Summary

Defintion 1.33 (Subspaces)

A subset U of V is called a **subspace** of V if U is also a vector space using the same addition and scalar multiplication as on V.

Example 1 (Subspaces)

$\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{F}\}$ is a subspace of $\mathbb{F}^3$.

Proposition 1.34 (Conditions of a subspace)

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

- $0 \in U$;
- $u, w \in U \Longrightarrow u + w \in U$;
- $a \in \mathbb{F}, u \in U \Longrightarrow au \in U$.

Proof. If U is a subspace of V then by definition it satisfies the three conditions.

Now assume that U is a subset satisfying all 3 conditions mentioned. Condition #1 ensures that the additive identity of V is in U. Conditions #2 and #3 ensure that addition and scalar multiplication exist in U.

Since $u \in U \Longrightarrow (-1)u = -u \in U$. Thus there always exists an additive identity for every vector in U.

The rest of the definitions for a vector space, such as associavity and commutativity are satisfied in U because they hold on V. Thus U is a vector space and a subspace of V. $\square$

Defintion 1.36 (Sum of subspaces)

Suppose $U_1, U_2, \dots, U_n$ are subsets of V. The **sum** of $U_1, U_2, \dots, U_n$, denoted $U_1 + \dots + U_m$, is the set of all possible sums of elements

$$U_1, U_2, \dots, U_n.$$

More precisely,

$$U_1 + \dots + U_n = \{u_1 + \dots + u_m : u_i \in U_i (i = 1, 2, \dots, n)\}.$$

Proposition 1.40 (Sum of subspaces is the smallest containing subspace)

Suppose $U_1, U_2, ..., U_n$ are subspaces of V. Then $U_1 + ... + U_n$ is the smallest subspace of V containing $U_1, U_2, ..., U_n$.

Proof. □

Defintion 1.41 (Direct sum)

Suppose $U_1, U_2, ..., U_n$ are subspaces of V.

- The sum $U_1 + U_2 + ... + U_m$ is called **direct sum** if each element of $U_1 + U_2 + ... + U_m$ can be written in only one way as a sum $u_1 + u_2 + ... + u_n$ where each $u_j \in U_j$.
- If $U_1 + U_2 + ... + U_m$ is a direct sum, then $U_1 \oplus U_2 \oplus ... \oplus U_n$ with the $\oplus$ notation serving as indication that this is a direct sum.

Proposition 1.45 (Condition for a direct sum)

Suppose $U_1, U_2, ..., U_m$ are subspaces of V. Then $U_1 + U_2 + ... + U_m$ is a direct sum if and only if the only write $0 = u_1 + u_2 + ... + u_m$, where $u_j \in U_j (j = 1, 2, ..., m)$ and $u_j = 0$.

Proposition 1.46 (direct sum of two subspaces)

Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

1C.2: Exercises 1.C

Problem 1C1

For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$:

a. $A = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;
b. $B = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;
c. $C = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\}$;
d. $D = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$;

Solution.

a. Subset A is a subspace of $\mathbb{F}^3$ because it has a zero vector $(0, 0, 0)$, and if $u, v \in A$, it can easily be proven that $u + v \in A$ and for $a \in F$, $au \in A$.

b. B is not a subspace of $\mathbb{F}^3$ because there exists no zero vector in B.
c. C is also not a subspace beacuse if $u = (0, 1, 1) \in C, w = (1, 0, 0) \in C \implies u + w = (1, 1, 1) \notin C$.
d. D is a subspace because it has an additive identity $0 = (0, 0, 0)$ and given $u = (5x_3, x_2, x_3), w = (5y_1, y_2, y_3)$, $u + w \in D$ and $\lambda u \in U \quad \forall\, \lambda \in \mathbb{F}$.

□

Problem 1C1

Is $\mathbb{R}^2$ a subspace of the complex vector space $\mathbb{C}^2$?

Solution. No, $\mathbb{R}^2$ is not a subspace of $\mathbb{C}^2$ because even though $\mathbb{R}^2$ has an additive identity and is closed under addition, scalar multiplication doesn't hold if $\lambda \in \mathbb{C}$. □

Problem 1C1

a. Is $A = \{(a, b, c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?
b. Is $B = \{(a, b, c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Solution.
a. Yes. In $\mathbb{R}$, $a^3 = b^3 \implies a = b$, hence this subset is a subspace of $\mathbb{R}^3$.
b. No. Suppose that

$$u = \left(1, \frac{-1 + \sqrt{3}i}{2}, 0\right) \in B$$
$$w = \left(1, \frac{-1 - \sqrt{3}i}{2}, 0\right) \in B.$$

Then $x + y = (2, -1, 0) \notin B$. Thus B is not closed under addition and therefore not a subspace of $\mathbb{C}^3$.

□

Problem 1C.1

Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Proof. Let $u, v \in U_1 \cap U_2 \iff u, v \in U_1 \wedge u, v \in U_2 \implies u + \lambda v \ \ (\lambda \in \mathbb{F}) \in U_1 \wedge u + \lambda v = U_2 \iff u + \lambda v \in U_1 \cap U_2$. Thus $U_1 \cap U_2 \in V$. $\square$

Problem 1C.1

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. $\square$

Problem 1C.1

Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces is contained within the other.

Proof. $\square$

Problem 1C.1

Suppose U is a subspace of V. What is $U + U$?

Solution. Because U is a subspace of V, it's closed under addition. Therefore, for $x, y \in U$, we have $x + y \in U \implies U + U \subset U$. If $x \in U$, then $x = x + 0 \in U + U$, hence $U \subset U + U$. It follows that $U + U = U$. $\square$

Remark ()

Two sets are equal if and only if they are the subset of each other, i.e. $A = B \iff A \subset B \wedge B \subset A$.

Problem 1C.1

If U and W are subspaces of V, is $U + W = W + U$?

Solution. Yes.

U and W are both subspaces of U, so they're closed under addition. Then, for $x \in U, y \in W, x + y = y + x \in W + U \implies U + W \subset W + U$ and $W + U \subset U + W \implies U + W = W + U$. $\square$

Problem 1C.1

If U_1, U_2, U_3 are subspaces of V, is $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$?

Solution. Yes. For $x_1 \in U_1, x_2 \in U_2, x_3 \in U_3$,

$(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3) \in U_1 + (U_2 + U_3) \Longrightarrow (U_1 + U_2) + U_3 \subset U_1 + (U_2 + U_3)$. Using a similar argument, $U_1 + (U_2 + U_3) \subset (U_1 + U_2) + U_3$, so $U_1 + (U_2 + U_3) = (U_1 + U_2) + U_3$. □

Problem 1C.1

Prove or give a counterexample: If U_1, U_2, W are subspaces of W such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Solution. This is not necessarily true. For instance: if $U_1 = \{a \in \mathbb{R}\}$, $U_2 = \{bi \in \mathbb{C}\}$, $V = \{c + di \in \mathbb{C}\}$, it's obvious that $U_1 \neq U_2$, even though $U_1 + W = U_2 + W$. □

Problem 1C.1

Suppose

$$U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Proof. Let $W = \{(0, z - x, y, t - y) \in \mathbb{F}^4 : x, y, z, t \in \mathbb{F}^4\}$. Then $U \cap W = \{0\}$ and $U + W = (x, y, z, t)$ where $x, y, z, t \in \mathbb{F}^4$. Thus $U \oplus W = \mathbb{F}^4$. □

Problem 1C.1

Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^5 = U \oplus W$.

Proof. Let

$$W = \{(0, 0, z - x - y, t - x + y, u - 2x) \in \mathbb{F}^5 : x, y, z, t, u \in \mathbb{F}^5\}$$

. Then $U \cap W = \{0\}$ and $U + W = (x, y, z, t, u)$ where $x, y, z, t \in \mathbb{F}^5$. Thus $U \oplus W = \mathbb{F}^5$. □

Problem 1C.1

Prove or give a counterexample: If U_1, U_2, W are subspaces of W such that

$$U_1 \oplus W = V = U_2 \oplus W,$$

then $U_1 = U_2$.

Problem 1C.1

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called **even** if

$$f(-x) = f(x) \quad \forall\ x \in \mathbb{R}.$$

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called **odd** if

$$f(-x) = -f(x) \quad \forall\ x \in \mathbb{R}.$$

Let U_c denote the set of real-valued functions on $\mathbb{R}$ and U_0 the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_c \oplus U_o$.

Proof. We must find $U_e \cap U_o$. Suppose f is one such function: then $f(x) = f(-x) = -f(x) \implies f(x) = 0$. Hence $U_o \cap U_e = \{0\}$.

Next, we notice that $f(x) = f_e(x) + f_o(x)$, where

$$f_o(x) = \frac{f(x) - f(-x)}{2}$$

and

$$f_e(x) = \frac{f(x) + f(-x)}{2}.$$

$f_o(-x) = -f_o(x)$; $f_e(-x) = f_e(x) \implies f_o \in U_o, f_e \in U_e$. Thus $U_e \oplus U_o = \mathbb{R}^{\mathbb{R}}$ by Proposition 1.45. □

2: Finite-dimensional Vector Spaces

2A: Span and Linear Independence

2A.1: Summary

Defintion 2.2 (Linear combinations)

A *linear combination* of a list $v_1, v_2, \ldots, v_m$ of vectors in V is a vector of the form

$$\sum_{i=1}^{m} a_i v_i,$$

where $a_1, a_2, \ldots, a_m$.

Defintion 2.4 (Span)

The set of all linear combinations of a list of vectors $v_1, v_2, \ldots, v_m$ in V is called the *span* of $v_1, v_2, \ldots, v_m$, denoted by

$$\operatorname{span}(v_1, v_2, \ldots, v_m).$$

In other words,

$$\operatorname{span}(v_1, v_2, \ldots, v_m) = \left\{ \sum_{i=1}^{m} a_i v_i ; v_1, v_2, \ldots, v_m \in \mathbb{F} \right\}.$$

Lemma 2.6 (Span is the smallest containing subspace)

The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Proof. Suppose $v_1, \ldots, v_m$ is a list of vectors in V.

We'll prove that $U = \operatorname{span}(v_1, \ldots, v_m)$ is a subspace in V. First, the additive identity exists:

$$0 = 0(v_1 + \ldots + v_m).$$

Then, addition and scalar multiple are closed under U:

$$\sum_{i=1}^{m} a_i v_i + \lambda \sum_{i=1}^{m} b_i v_i = \sum_{i=1}^{m} (a_i + \lambda b_i) v_i.$$

Thus U is a subspace of V.

Every v_k is a linear combination of $v_1, \ldots, v_m$. Thus each $v_k \in U$. Since addition and scalar multiple are closed under subspaces, every subspace containing v_k contains U, which means $\operatorname{span}(v_1, \ldots, v_m)$ is the smallest subspace containing all the vectors $v_1, \ldots, v_m$. $\square$

Defintion 2.7 (Spans)

If span $(v_1, v_2, \ldots, v_m) = V$, we say that the list $v_1, v_2, \ldots, v_m$ spans V.

Defintion 2.9 (Finite-dimensional vector space)

A vector space is called *finite-dimenional* if some list of vectors in it spans the space.

Defintion 2.10 (Polynomials)

- A function $p : \mathbb{F} \longrightarrow \mathbb{F}$ is called a *polynomial* w/ coefficients in $\mathbb{F}$ if there exists $a_0, \ldots, a_m \in \mathbb{F}$ such that

$$p(z) = \sum_{i=0}^{m} a_i z^i \qquad \forall\, z \in \mathbb{F}.$$

- $\mathscr{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$.

Defintion 2.11 (Degree of a polynomial, $\deg p$)

- A polynomial $p \in \mathscr{P}(\mathbb{F})$ is said to have *degree m* if $\exists a_0, a_1, \ldots, a_m \in \mathbb{F}(a_m \neq 0)$ such that $\forall\, z \in \mathbb{F}$, we have

$$p(z) = \sum_{i=0}^{m} a_i z^i.$$

- The polynomial that is identically 0 is said to have degree $-\infty$.
- The degree of a polynomial p is denoted by $\deg p$.

Covention 2.12 ($\mathscr{P}_m(\mathbb{F})$)

For $m \geq 0 \in \mathbb{Z}$, $\mathscr{P}_m(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree of at most m.

Defintion 2.13 (finite-dimensional vector space)

A vector space is called *infinite-dimensional* if it is not finite-dimensional.

Defintion 2.15 (Linear independence)

- A list $v_1, \ldots, v_n$ of vectors in V is called *linearly independent* if

$$\sum_{i=1}^{m} a_m v_m = 0$$

  implies $a_1 = \ldots = a_n = 0$.
- The empty list is also declared to be linearly independent.

Defintion 2.17 (Linear dependence)

A list of vectors in V is called *linearly dependent* if it's not linearly independent in other words, in a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exit $a_1, \ldots, a_m \in \mathbb{F}$, not all 0, such that

$$\sum_{i=1}^{m} a_i v_i = 0.$$

Lemma 2.19 (Linear dependence lemma)

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then $\exists k \in \{1, 2, \ldots, m\}$ such that

$$v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}).$$

Furthermore, if k satisfies the condition above and the kth term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

Proof. Because $v_1, \ldots, v_m$ is linearly dependent, there exist $a_1, \ldots, a_m$, not all of them zero, such that

$$\sum_{i=1}^{m} a_i v_i = 0.$$

Let k be the greatest among $\{1, \ldots, m\}$ such that $a_k \neq 0$. Then we have

$$v_k = -\frac{1}{a_k}\left(\sum_{i=1}^{k-1} a_i v_i\right)$$

$\Longrightarrow v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$, as desired.

Now, suppose k is any element in $\{1, \ldots, m\}$ such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$. Then $\exists\ b_1, \ldots, b_{k-1}$ such that

$$v_k = \sum_{i=1}^{k-1} b_i v_i.$$

Let $u \in \text{span}(v_1, \ldots, v_m)$. Then $\exists c_1, \ldots, c_m$:

$$u = \sum_{i=1}^{m} c_i v_i.$$

Replacing v_k in the expression for u with our result, we find that indeed u can be obtained by removing the kth term from $v_1, \ldots, v_m$. Thus removing the kth term does not change the span of the list. □

Lemma 2.22 (Length of linearly independent list ≤ length of spanning list)

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof. Suppose $u_1, \ldots, u_m$ is linearly independent and $v_1, \ldots, v_n$ spans V. We will prove that $m \leq n$.

Step 1 Let B be the list $v_1, \ldots, v_n$ whose span is V. Adding u_1 to this list makes a linearly dependent list, since u_1 can be written as a linear combination of $v_1, \ldots, v_n$. This means that

$$u_1, v_1, \ldots, v_n$$

is linearly dependent.

By the linear dependence lemma, one of the vectors in the list above is a linear combination of the rest of the vectors. Since $u_1 \neq 0$ because $u_1, \ldots, u_m$ is linearly independent, it is not in the span of the previous vectors in the list above. Thus the linear dependence lemma tells us that we can remove one of w's and the new list B consisting of u_1 and the remaining w's span V.

Step k ($k = 2, \ldots, m$**)**

The list B from step $k - 1$ spans V. In particular, u_k is in the span of the list B. Thus the list of length $n + 1$ obtained by adjoining u_k to B immediately after $u_1, \ldots, u_{k-1}$, is linearly dependent. By the linear dependence lemma, one of the vectors in the list is in the span of the previous ones, and because $u_1, \ldots, u_k$ is linearly independent, this vector can not be one of the u's. Hence there remains at least one w in this step. Again, the linear dependence lemma implies that we can remove from the new list a v that is a linear combination of the previous vectors on the list, so that the new list B of length n consists of $u_1, \ldots, u_k$ and the remaining v's span V.

After the mth step, all the u's have been added and we stop the process. At each step we replace one of v with a u. The linear dependence lemma shows that there is some v to replace; thus $m \leq n$. □

Remark

The above proof relies on the fact the we can use the linear dependence lemma to replace v with u one vector at a time.

Lemma 2.25 (Finite-dimensional)

Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof. Suppose U is a subspace of a finite-dimensional vector space V. We shall follow these steps.

Step 1

If $U = \{0\}$, then U is finite-dimensional and we're done. If not, choose a nonzero vector $u_1 \in U$.

Step k

If $U = \operatorname{span}(u_1, \ldots, u_{k-1})$, U is finite-dimensional and we're done. If not, choose a vector $u_k \in U$ such that

$$u_k \notin \operatorname{span}(u_1, \ldots, u_{k-1}).$$

After each step, we've constructed a list of vectors with every not being a linear combination of the previous ones, so by the linear dependence lemma, this new list can't be longer than any spanning set of V, thanks to Lemma 2.22. Thus the process will eventually stop, which implies that U is finite-dimensional. □

2A.2: Exercises 2.A

Problem 2A.1

Find a list of four distinct vectors in $\mathbb{F}^3$ whose span equals

$$V = \{x, y, z\} \in \mathbb{F}^3 : x + y + z = 0.$$

Solution. Such a subspace would mean that $z = -(x + y)$, so a possible spanning set of V could be $\{(1, 0, 0), (0, 1, 0), (0, 0, -2), (0, 0, 0)\}$. □

Problem 2A.1

Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Solution. Noticing that

$$\begin{aligned} v_1 &= (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4 \\ v_2 &= (v_2 - v_3) + (v_3 - v_4) + v_4 \\ v_3 &= (v_3 - v_4) + v_4, \end{aligned}$$

it becomes clear that $V = \operatorname{span}(v_1, v_2, v_3, v_4) = \operatorname{span}[(v_1 - v_2), (v_2 - v_3), (v_3 - v_4), v_4]$. □

Problem 2A.1

a. Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.
b. Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

Proof.

a. If the vector in the list is not 0, then $av = 0 \implies a = 0$ by Problem 2A.2.

   If the list is linearly independent, then $v \neq 0$. Otherwise, we have $1v = v = 0$, a contradiction.
b. If $v_1, v_2 \in V$ are linearly independent, then neither of them is a scalar multiplier of the other. Otherwise, we can assume without loss of generality that $v_1 = cv_2 \implies 1v_1 + cv_2 = 0 \implies v_1, v_2$ are not linearly dependent, a contradiction.

   Conversely, if $v_1, v_2 \in V$ are linearly dependent, then $\exists a, b : av_1 + bv_2 = 0$ and a, b are not both zero. Without loss of generality, we can assume that $a \neq 0$, then $av_1 + bv_2 \implies \frac{-b}{a} v_2 = v_1$, which is also a contradiction.

□

Problem 2A.1

Find a number such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbb{R}^3$.

Solution. t is such that $a(3, 1, 4) + b(2, -3, 5) = (3a + 2b, a - 3b, 4a + 5b) = (5, 9, t)$. Solving for a, b, we get $a = 3, y = -2 \implies t = 2$. $\square$

Problem 2A.1

a. Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list $1 + i, 1 - i$ is linearly independent.
b. Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the list $1 + i, 1 - i$ is linearly dependent.

Proof.

a. Suppose that $\mathbb{C}$ is indeed a vector space over $\mathbb{R}$. Then for $a, b \in \mathbb{R}$,

$$a(1 + i) + b(1 - i) = 0 \iff a = b = 0.$$

Thus $1 + i, 1 - i$ is linearly independent.
b. If, however, $\mathbb{C}$ is a vector space over $\mathbb{C}$, then the above would be linearly dependent, as $-i(1 + i) = -i + 1 = 1 - i$.

$\square$

Problem 2A.1

Suppose v_1, v_2, v_3, v_4 are linearly independent in V. Prove that the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is also linearly independent.

Proof. If we take the linear combination of the new list to be 0, we get

$$a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0.$$

Letting $a_1 = b_1, a_2 - a_1 = b_2, a_3 - a_2 = b_3, a_4 - a_3 = b_4$, we then must have $b_1 = b_2 = b_3 = b_4 = 0 \iff a_1 = a_2 = a_3 = a_4 = 0$, closing the proof. $\square$

Problem 2A.1

Prove or give a counterexample: If $v_1, \dots, v_m$ is a linearly independent list of vectors in V and $\lambda \in F$, $\lambda \neq 0$, then $\lambda v_1, \dots, \lambda v_m$ is also linearly independent.

Solution. $\sum_{i=1}^{m} a_i(\lambda v_i) = \lambda \sum_{i=1}^{m} a_i v_i = 0$. Since $v_1, \dots, v_m$ is linearly independent, it means that $a_1 = \dots = a_m = 0$, as desired. □

Problem 2A.1

Prove or give a counterexample: If $v_1, \dots, v_m$ and $w_1, \dots, w_m$ are linearly independent lists of vectors in V, then the list $v_1 + w_1, \dots, v_m + w_m$ is linearly independent.

Solution. This is not true: If each $w_k = -v_k (k = 1, \dots, m)$, then it means that $v_k + w_k = 0$, and so the list $v_1 + w_1, \dots, v_m + w_m$ becomes $0, \dots, 0$, which is linearly dependent. □

Problem 2A.1

Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \dots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \dots, v_m)$.

Proof. $v_1 + w, \dots, v_m + w$ being linearly dependent implies that there exist $a_1, \dots, a_n$, not all of them zero, such that

$$\sum_{i=1}^{m} a_i(v_i + w) = 0.$$

If $a_1 + \dots + a_n = 0 \Longrightarrow \sum_{i=1}^{m} a_i v_i = 0 \Longleftrightarrow a_i \equiv 0$. Hence $\sum_{i=1}^{m} a_i \neq 0$. Then

$$w = \frac{\sum_{i=1}^{m} a_i v_i}{\sum_{i=1}^{m} a_i} \in \operatorname{span}(v_1, \dots, v_m).$$

□

Problem 2A.1

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, \ldots, v_m, w \text{ is linearly independent} \iff w \notin \operatorname{span}(v_1, \ldots, v_m).$$

Proof. The statement of the problem is tantamount to proving that $v_1, \ldots, v_m, w$ is linearly dependent if and only if $w \in \operatorname{span}(v_1, \ldots, v_m)$.

If $v_1, \ldots, v_m, w$ is linearly dependent, $\exists a_1, \ldots, a_m, a_w \in \mathbb{F}$, not all 0, such that

$$a_w w + \sum_{i=1}^{m} a_i v_i = 0.$$

If $a_w = 0$, then we can immediately conclude that $a_i \equiv 0$. Hence $a_w \neq 0$, and

$$w = -\frac{1}{a_w} \sum_{i=1}^{m} a_i v_i \in \operatorname{span}(v_1, \ldots, v_m).$$

Conversely, if $w \in \operatorname{span}(v_1, \ldots, v_m)$, then w is a linear combination of $v_1, \ldots, v_m$, which makes the list $v_1, \ldots, v_m, w$ linearly dependent. □

2B: Bases

2B.1: Summary

Defintion 2.26 (Basis)

A *basis* of V refers to any linearly independent spanning set of V.

Theorem 2.28 (Criterion for basis)

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be uniquely written in the form

$$v = \sum_{i=1}^{n} a_i v_i$$

where $v_1, \ldots, v_n \in \mathbb{F}$.

Lemma 2.30 (Every spanning list contains a basis)

Every spanning set in a vector space can be reduced to a basis of the vector space.

Lemma 2.31 (Basis of finite-dimensional vector space)

Every finite-dimensional vector space has a basis.

Theorem 2.32 (Every linearly independent list extends to a basis)

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Theorem 2.33 (Ever subspace of V is part of a direct sum equal to V)

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

2B.2: Exercises 2B

Problem 2B.1

Find all vector spaces that have exactly one basis.

Solution. The only satisfactory vector space is $\{0\}$; if there exists a nonzero vector in v in a basis, we can get a new basis by changing v to $2v$. □

Problem 2B.1

a. Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \land x_3 = 7x_4\}.$$

Find a basis of U.

b. Extend the basis in (a) to a basis in $\mathbb{R}^5$.

c. Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Solution.

a. Such a basis for V is $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$.

b. A basis for $\mathbb{R}^5$ could be $(3, 1, 0, 0, 0), (1, 0, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)$.

c. From B, we obtain $W = \operatorname{span}\{(1, 0, 0, 0, 0), (0, 0, 0, 0, 1)\}$.

□

Problem 2B.1

Suppose V is finite-dimensional and U, V are subspace of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup V$.

Problem 2B.1

Prove or give a counterexample: If p_0, p_1, p_2, p_3 is in a list in $\mathscr{P}_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then p_0, p_1, p_2, p_3 is not a basis of $\mathscr{P}_3(\mathbb{F})$.

Solution. This statement is false: If $p_3 = z^3 + z^2 + z + 1, p_2 = z^3, p_1 = z, p = 1$, then it is clear that any linear combination $p = az^3 + bz^2 + cz + d(a, b, c, d \in \mathbb{F})$ is possible by letting writing p as

$$p = (a - b)p_2 + bp_3 + (c - b)p_2 + (d - b)p_1.$$

□

Problem 2B.1

Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Proof. First, we need to show that the new list is linearly independent. Letting $a(v_1 + v_2) + b(v_2 + v_3) + c(v_3 + v_4) + dv_4 = 0$, we have

$$av_1 + (a + b)v_2 + (b + c)v_3 + dv_4 = 0$$

$$\iff \begin{cases} a = 0 \\ a + b = 0 \\ b + c = 0 \\ d = 0 \end{cases} \iff a = b = c = d = 0 \quad (v_1, \ldots, v_4 \text{ is a basis of } V).$$

Thus our new list of vectors is linearly independent.

Since the first list is a basis of V, every $v \in V$ can be written as

$$
\begin{aligned}
v &= av_1 + bv_2 + cv_3 + dv_4 \qquad \forall\, a, b, c, d \in \mathbb{F} \\
&= a(v_1 + v_2 - v_2 - v_3 + v_3 + v_4 - v_4) \\
&\quad + b(v_2 + v_3 - v_3 - v_4 + v_4) + c(v_3 + v_4 - v_4) + dv_4 \\
&= a(v_1 + v_2) + (b - a)(v_2 + v_3) + (c - b + a)(v_3 + v_4) + (d - c + b - a)v_4 \\
&= e(v_1 + v_2) + f(v_2 + v_3) + g(v_3 + v_4) + hv_4 \quad (e, f, g, h \in \mathbb{F}).
\end{aligned}
$$

Therefore span $(v_1, \ldots, v_4) =$ span $(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4) = V$ and therefore $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is a basis in V. □

Problem 2B.1

Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$
w_k = \sum_{i=1}^{k} v_i.
$$

Show that $v_1, \ldots, v_m$ is a basis of V if and only if $w_1, \ldots, w_m$ is a basis of V.

Proof. ($\Longrightarrow$) $v_1, \ldots, v_m$ being a basis of V means that $\sum_{i=1}^{m} a_i v_i = 0$ forces $a_1 = \ldots = a_n = 0$. By setting the $\sum_{i=1}^{m} a_i w_i = 0$, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{m} a_i w_i = 0 \\
\Longleftrightarrow &\left(\sum_{i=1}^{n} a_i\right) v_1 + \left(\sum_{i=1}^{m-1} a_i\right) v_2 + \ldots + a_n v_m = 0 \\
\Longleftrightarrow &\begin{cases} a_n = 0 \\ a_{n-1} = 0 \\ \vdots \\ a_1 = 0 \end{cases}.
\end{aligned}
$$

Thus $w_1, \ldots, w_m$ is linearly independent. Since each vector w_k is a linear combination of v vectors, the w list also spans V, which means $w_1, \ldots, w_m$ is a basis of V.

($\Longleftarrow$) Conversely, let's assume that $w_1, \ldots, w_m$ is a basis of V. Then every $v \in V$ can be written as

$$
\sum_{i=1}^{m} a_i w_i = \left(\sum_{i=1}^{m} a_i\right) v_1 + \ldots + a_m v_m \quad (a_1, \ldots, a_m \in \mathbb{F}).
$$

By Theorem 2.28, $v_1, \ldots, v_m$ is a basis in V. □

Problem 2B.1

Prove or give a counterexample: If $v_1, \ldots, v_4$ is a basis of V and U is a $\subset$ of V such that $v_1, v_2 \in U$ and $v_3 \notin U, v_4 \notin U$, then v_1, v_2 is a basis of U.

Solution. Let $V = \mathbb{R}^4$, $v_1 = (1, 0, 0, 0)$, $v_2 = (0, 1, 0, 0)$, $v_3 = (0, 0, 1, 1)$, $v_4 = (0, 0, 0, 1)$ and

$$U = \{(x, y, z, 0) \mid x, y, z \in \mathbb{R}\}. \tag{2.1}$$

$v_1, v_2 \in U; v_3, v_4 \notin U$, but v_1, v_2 isn't a basis of U since $(0, 0, 1, 0)$ can't be expressed by a linear combination of v_1 and v_2. □

2C: Dimension

2C.1: Summary

Theorem 2.34 (Basis length does not depend on basis)

Any two bases of a finite-dimensional vector space have the same length.

Defintion 2.35 (Dimension)

- The *dimension* of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of a finite-dimensional vector space V is denoted by $\dim V$.

Theorem 2.37 (Dimension of a subspace)

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Theorem 2.38 (Linearly independent list of the right length is a basis)

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V of length $\dim V$ is a basis of V.

Theorem 2.39 (Subspace of full dimension equals the whole space)

If V is finite-dimensional a U is a subspace such that $\dim U = \dim V$, then $U = V$.

Theorem 2.42 (Spanning list of the right length is a basis)

Suppose V is finite-dimensional. Then every spanning list of vectors in V of length $\dim V$ is a basis of V.

Theorem 2.43 (Dimension of a sum)

If V_1, V_2 of a subspace of a finite-dimensional vector space, then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

2C.2: Exercises 2C

Problem 2C.1

Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, all lines in $\mathbb{R}^2$ containing the origin, and $\mathbb{R}^2$.

Problem 2C.2

Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, all lines in $\mathbb{R}^3$ containing the origin, all planes in $\mathbb{R}^3$ containing the origin, and $\mathbb{R}^2$.

Problem 2C.3

a. Let $U = \{p \in P_4(\mathbb{F}) : p(6) = 0\}$. Find a basis of U.
b. Extend the basis in part (a) to basis in $\mathscr{P}_4(\mathbb{R})$.
c. Find a subspace W of $\mathscr{P}_4(\mathbb{F})$ such that $\mathscr{P}_4(\mathbb{F}) = U \oplus W$.

Problem 2C.6

a. Let $U = \{p \in P_4(\mathbb{F}) : p(6) = p(2) = p(5)\}$. Find a basis of U.
b. Extend the basis in part (a) to basis in $\mathscr{P}_4(\mathbb{R})$.
c. Find a subspace W of $\mathscr{P}_4(\mathbb{F})$ such that $\mathscr{P}_4(\mathbb{F}) = U \oplus W$.

Problem 2C.7

a. Let $U = \left\{p \in P_4(\mathbb{F}) : \int_{-1}^{1} p = 0\right\}$. Find a basis of U.
b. Extend the basis in part (a) to basis in $\mathscr{P}_4(\mathbb{R})$.
c. Find a subspace W of $\mathscr{P}_4(\mathbb{F})$ such that $\mathscr{P}_4(\mathbb{F}) = U \oplus W$.

Problem 2C.10

Suppose m is a positive integer and $p_0, \dots, p_m \in \mathscr{P}(\mathbb{F})$ are such that each p_k has degree k. Prove that $p_0, p_1, \dots, p_m$ is a basis of $\mathscr{P}_m(\mathbb{F})$.

The basis in this exercise leads to what are called *Bernstein polynomials*. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on $[0, 1]$.

Problem 2C.11

Suppose U, W are both 4-dimensional subspaces of $\mathbb{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

Problem 2C.12

Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3, \dim W = 5, U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Problem 2C.13

Suppose U and W are both 5-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

Problem 2C.14

Suppose V is a 10-dimensional vector space and V_1, V_2, V_3 are subspaces of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Problem 2C.16

Suppose V is a finite-dimensional and U is a subspace of V with $U \neq V$. Let $n = \dim V, m = \dim U$. Prove that there exists $n - m$ subspaces of V, each of dimension $n - 1$, whose intersect equals U.

Problem 2C.17

Suppose that $V_1, \dots, V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \dots + V_m$ is finite-dimensional and

$$\dim(V_1 + \dots + V_m) \leq \dim V_1 + \dots + \dim V_m.$$

Problem 2C.18

Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist 1-dimensional subspaces $V_1, \ldots, V_n$ of V such that

$$V = V_1 \oplus \ldots \oplus V_n.$$