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University Physics w/ Modern Physics - Hugh Young, Roger Freedman

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5.1 Summary

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Problem 5.92

Block B, with mass 5 kg, rests on block A, with mass 8 kg, which in turn is on a horizontal tabletop. There's no friction between block A and the tabletop, but the coefficient of coefficient of static friction between locks A and B is 0.75. A light string attached to block A passes over a frictionless, massless pulley, and block C is suspended from the other end of the string. What's the the largest mass that block C can have so that blocks A and B still slide together when the system is released from rest?

Solution: The largest mass that block C can have satisfies the following:

$$\begin{cases} m_C g - T &= m_C a \\ (m_A + m_B)a &= T \\ \mu_s m_B g &= (m_A + m_B)a \end{cases} \tag{5.1}$$

$\Longrightarrow m_C = 39\,\text{kg}$.

Problem 5.104

A 4 kg block is attached to a vertical rod by means of two strings. When the system rotates about the axis of the rod, the strings are extended and the tension in the upper string is 80 N.

a. What's the tension in the lower cord?

b. How many rpm does the system make?

c. Find the rpm at which the lower cord just goes slack.

d. Explain what happens if he number of rpm is less than that in part (c).

Solution:

a. We have $\sin\theta = 0.8$ and $T_1 \sin\theta - T_2 \sin\theta - mg = 0 \iff T_2 = 32\,\mathrm{N}$.
b. $r = \sqrt{l^2 - 0.25d^2} = 0.6\,\mathrm{m}$.
   $(T_1 + T_2)\sin\theta = \dfrac{mv^2}{r} \iff v = 3.55\,\mathrm{m\,s^{-1}} \iff \omega \approx 44.9\,\mathrm{rev/min}$.
c. If the lower cord goes slack (meaning $T_2 \longrightarrow 0$), then $T_1 \cos\theta = mg \iff T_1 \approx 49\,\mathrm{N}$. Then, $mv^2R^{-1} = T_1 \sin\theta \implies v \approx 2.35\ \mathrm{m/s} \implies \omega = \dfrac{v}{2\pi r} \approx 29.92$ rpm.
d. If the rpm count is less than the result the result obtained in (c), the lower cord becomes slacker still, reducing R further.

Problem 5.112 (Moving Wedge)

A wedge with mass M rests on a frictionless, horizontal tabletop. A block with mass m is placed on the wedge. There is no friction between the block and the wedge. The system is released from rest.
a. Calculate the acceleration of the block.
b. Do your answers in part (a) reduce to the correct results when M becomes very large?
c. To a stationary observer, what's the shape of the trajectory of the block?

Solution:

a. Let the accelerations of the wedge and the block be A and a, respectively and let the normal force of the block on the wedge be n. Let the $+x$ direction be horizontally upward and $+y$ be vertically upward.

   The force equations for the wedge and block are

$$\begin{cases} MA &= -n\sin\alpha \\ ma_x &= n\sin\alpha \\ ma_y &= n\cos\alpha - mg \end{cases} \tag{5.2}$$

   Next, we'll derive the kinematics equations from an inertial observer on the wedge: to this observer, the block is descending at an angle α but its horizontal acceleration is $a_x - A$. Thus we have $\dfrac{a_y}{a_x - A} = \tan\alpha$. Observing that $a_x = -\dfrac{M}{m}A$, and then using replacing that in the kinematic constraints to eliminate n and then a_y, we obtain

$$\begin{aligned} A &= -\frac{mg}{(M+m)\tan\alpha + M\cot\alpha} \\ a_x &= \frac{Mg}{(M+m)\tan\alpha + M\cot\alpha} \\ a_y &= -\frac{(M+m)g\tan\alpha}{(M+m)\tan\alpha + M\cot\alpha} \end{aligned} \tag{5.3}$$

b. When $M \gg m$, $A \longrightarrow 0$, $a - y \to -\dfrac{g\tan\alpha}{\tan\alpha + \cot\alpha} = -g\sin^2\alpha$ and $a_x \longrightarrow \dfrac{g}{\tan\alpha + \cot\alpha} = g\sin\alpha\cos\alpha$.
c. To a stationary observer, the block is falling down a straight line with slope $-\dfrac{m+M}{m}\tan\alpha$.

Problem 5.113

A wedge with M rests on a frictionless, horizontal tabletop. A block with mass m is placed on the wedge, and a horizontal force $\vec{F}$ is applied to the wedge. What must the magnitude of $\vec{F}$ be if the block is to remain at a constant height above the tabletop?

Solution: From Problem , we obtain $a_y = 0 \Longleftrightarrow mg = n\cos\alpha$ and $F = ma_x = n\sin\alpha \Longrightarrow a_x = g\tan\alpha$. For the block to remain at constant height, both the block and the wedge must experience the same horizontal acceleration, so $F = (M+m)a_x = (M+m)g\tan\alpha$.

Problem 5.114 (Double Atwood's Machine)

In Fig. P5.114 masses m_1 and m_2 are connected by a light string A over a light, frictionless pulley B. The ale of pulley B is connected by a light string C over a light frictionless pulley D to a mass m_3. Pulley D is suspended from the ceiling by an attachment to its cable. The system is released from rest. In terms of m_1, m_2, m_3 and g, what are

a. the acceleration of block m_3;
b. the acceleration of pulley B;
c. the acceleration of block m_1;
d. the acceleration of block m_2;
e. the tension in string A;
f. the tension in string C?
g. What do your expressions give for the special case of $m_1 = m_2$ and $m_3 = m_1 + m_2$? Is this reasonable?

Solution: Take the positive direction to be downward.

a. First, the can be not force acting on pulley B, so $T_C = 2T_A$.

   Second, since the length of the strings holding the masses are constant, we must have

$$\begin{aligned} X_3 + X_B &= \text{const} \\ \Longrightarrow a_3 + a_B &= 0 \quad \text{(taking double derivatives)} \\ \Longleftrightarrow a_3 &= -a_B \end{aligned} \tag{5.4}$$

   and similarly,

$$\begin{aligned} (X_1 - X_B) + (X_2 - X_B) &= \text{const} \\ \Longrightarrow a_1 - a_B + a_2 - a_n &= 0 \\ \Longleftrightarrow a_1 + a_2 + 2a_3 &= 0 \end{aligned} \tag{5.5}$$

   Combining these kinematic relationships with the Newtonian equations for m_1, m_2, m_3 we obtain

$$\begin{cases} m_3 g - T_C &= m_3 a \\ m_1 g - T_A &= m_1 a \\ m_2 g - T_A &= m_2 a \\ a_1 + a_2 + 2a_3 &= 0 \\ T_C &= 2T_A \end{cases} \tag{5.6}$$

   Solving for a_3, we get

$$a_3 = \frac{-4m_1m_2 + m_2m_3 + m_1m_3}{4m_1m_2 + m_2m_3 + m_1m_3} g. \tag{5.7}$$

b. $a_B = -a_3 = -\dfrac{-4m_1m_2 + m_2m_3 + m_1m_3}{4m_1m_2 + m_2m_3 + m_1m_3} g.$

c. $a_1 = \dfrac{4m_1m_2 - 3m_1m_3 + m_2m_3}{4m_1m_2 + m_2m_3 + m_1m_3} g.$

d. $a_2 = \dfrac{4m_1m_2 - 3m_1m_3 + m_2m_3}{4m_1m_2 + m_2m_3 + m_1m_3} g.$

e. $T_A = \dfrac{4m_1m_2m_3}{4m_1m_2 + m_2m_3 + m_1m_3}, T_C = 2T_A.$

f. For $\begin{cases} m_1 = m_2 \\ m_3 = m_1 + m_2 \end{cases}$, $a_1 = a_2 = a_3 = 0, T_A = mg, T_C = mg$, which is to be expected.

Problem 5.115

A ball is held at rest at position A in Fig. P5.115 by two light strings. The horizontal string is cut, and the ball starts swinging as a pendulum. Position B is the farthest to the right that the ball can go as it swings back and forth. What is the ratio of the tension in the supporting string at B to its value at A before the string was cut?

Solution: Before the horizontal string was cut, the string at position A cancels out the weight of the ball: $T_A \cos\beta = w \iff T_A = \dfrac{w}{\cos\beta}$. When it's at position B, the ball has instantaneous speed 0, there's no radial acceleration, and the tension in the rope is equal to the radial component of the weight, so $T_B = w\cos\beta \implies \dfrac{T_A}{T_B} = \cos^2\beta$.

6 Work and Kinetic Energy

6.1 Summary

6.2 Exercises

49. The work done by the variable force $F(x)$ on the box $W(x) = \displaystyle\int_{x_1}^{x_2} F(x)\,\mathrm{d}x = \int_0^{14} 18 - 0.53x\,\mathrm{d}x = 200.06\,\mathrm{J}$, but $W = \dfrac{mv^2}{2} \iff v = 8.17\,\mathrm{m\,s^{-2}}$.
50. The power done on the crate by the worker is $P(t) = \vec{F}\cdot\vec{v} = 15.12t^2 \implies P(5) = 378\,\mathrm{J}$.
57.
    (a) We have $P = \dfrac{nmgh}{60} = 4.41n \iff n = P/4.41$, so when $P = 0.5\,\mathrm{hp} = 384\,\mathrm{W}$, $n = 87$ crates

    (b) $P = 100\,\mathrm{W}$, $n = 22.68 \approx 23$ crates
62. The work applied on the cow $F(x) = \displaystyle\int_{x_1}^{x_2} F(x)\,\mathrm{d}x = \int_0^{6.9} -20 - 3x\,\mathrm{d}x = -209.415\,\mathrm{J}$.
80. Newton's 2nd law of motion gives us

$$F_p = ma = F\cos 30° - mg\sin 30° \tag{6.1}$$

$\iff F_p = 103.12\,\mathrm{N} \iff W = F_p \cdot s = 3.17\,\mathrm{m\,s^{-1}}$

By the work-energy theorem, we get

$$\frac{1}{2}m\left(v_2^2 - v_1^2\right) = W \iff v_2 \approx 3.17\,\mathrm{m\,s^{-1}} s = \frac{2\left(v_2^2 - v_1^2\right)}{g} = 1.28\,\mathrm{m}. \tag{6.2}$$

81. The work-energy theorem gives us

$$0,15m\left(v_2^2 - v_1^2\right) = 0,25mgs \iff s = \frac{2\left(v_2^2 - v_1^2\right)}{g} = 1{,}28\,\mathrm{m} \tag{6.3}$$

87. We have $280t + (86400 - t) \cdot 100 = 1{,}1 \cdot 10^7\,\mathrm{s} \iff t \approx 13\,111\,\mathrm{s} \approx 3,64\,h$.
75. By the work-energy theorem, we must have

$$W_{\text{tot}} = K_2 - K_1 = 0 = \frac{kx^2}{2} - \mu_k mgd \tag{6.4}$$

$\implies d = \frac{kx^2}{2\mu_k mg} = 1{,}1\,\mathrm{m}$.

86. We have $1000n \cdot 170 \cdot 9,8 = 2.17 \cdot 10^9\,\mathrm{W} \implies n = 1\,302.52\,\mathrm{m}^3$

Problem 93 (A Spring with Mass)

We usually ignore the kinetic energy of the moving coils of a spring, but let's try to get a reasonable approximation to this. Consider a spring of mass M, equilibrium length L_0 and force constant k. The work done to stretch or compress the spring by a distance L is $\frac{1}{2}kX^2$, where $X = L - L_0$. Consider a spring, as described above, that has one end fixed and the other end moving with speed v. Assume that the speed of point along the length of the spring varies linearly with distance l from the fixed end. Assume also that the mass M of the spring is distributed uniformly along the length of the spring.

(a) Calculate the kinetic energy of the spring in terms of M and v.
(b) In a spring gun, a spring of mass 0, 243 kg and force constant $k =$ 3200 N/m is compressed 2, 5 cm from its unstretched length.

Solution:

a. Let u be the speed function of each piece of the spring. Then, $u(l) = \frac{vl}{L}$

$$\implies K = \int \mathrm{d}K = \int_0^L \frac{M}{L}\frac{v^2 l^2}{L^2}\,\mathrm{d}l = \frac{Mv^3}{3}. \tag{6.5}$$

b. $0,5mv^2 = 0,5kx^2 \implies v \approx 6,14$ m/s.
c. $\frac{mv^2}{2} + \frac{Mv^2}{6} = \frac{kx^2}{2} \implies v \approx 3.86$ m/s.
d. $K_{\text{ball}} \approx 0{,}396$ J; $K_{\text{spring}} \approx 0{,}604$ J.

Problem 6.94

An airplane in flight is subject to an air resistance force proportional to the square of its speed v. But there is an additional resistive force because the airplane has wings. Air flowing over the wings is pushed down and slightly forward, so from Newton's third law the air exerts a force on the wings and airplane that is up and slightly backward. The upward force is the lift force that keeps the airplane aloft, and the backward force is called *induced drag.* At flying speeds, induced drag is inversely proportional to v^2, so the total air resistance force can be expressed by

$$F_{\text{air}} = av^2 + B/v^2,$$

where α and β are positive constants that depending on the shape and size of the airplane and the density of the air. For a Cessna 150, a small single-engine airplane, the $\alpha = 0{,}3\,\mathrm{N\,s^2\,m^{-2}}$ and $\beta = 3.5 \cdot 10^5\,\mathrm{N\,m^2\,s^{-2}}$. In steady flight, the engine must provide a forward force that exactly balances the air resistance force.

(a) Calculate the speed in km/h at which this airplane will have the maximum *range* (that is, ravel the greatest distance) for a given amount of fuel.
(b) Calculate the speed (in km/h) for which the airplane will have the maximum *endurance* (that is, remain in the air the longest time).

Solution:

a. In both cases a given amount of fuel represents the a given amount of work W_0 that engine does in moving the plane forward against the drag.

We have, since $W_0 = RF$, $R = vT$,

$$W_0 = T\left(av^3 + \frac{\beta}{v}\right) \tag{6.6}$$

.

$\frac{\mathrm{d}W_0}{\mathrm{d}v} = 0 \Longrightarrow \frac{\mathrm{d}R}{\mathrm{d}v}F + \frac{\mathrm{d}F}{\mathrm{d}v}R = 0$. Because the range is maximum,

$$\frac{\mathrm{d}R}{\mathrm{d}v} = 0 \Longrightarrow \frac{\mathrm{d}F}{\mathrm{d}v} = 0 \Longleftrightarrow v = \left(\frac{\beta}{a}\right)^{0{,}25} \approx 32.9\ \mathrm{m/s} = 118\ \mathrm{km/h}. \tag{6.7}$$

a. The maximum time corresponds with $\frac{\mathrm{d}}{\mathrm{d}v}Fv = 0 \Longleftrightarrow v = \left(\frac{\beta}{a}\right)^{0{,}25} =$ 25 m/s = 90 km/h.

7 Potential Energy and Energy Conservation

7.1 Summary

- Gravitational potential energy: $W_{\text{grav}} = mgy$.
- An object is *elastic* if it returns to its original form after being deformed.
- Elastic potential energy: $W_{\text{el}} = \frac{1}{2}kx^2$.
- When total mechanical energy is conserved, we have
$$\Delta K + \Delta U = 0.$$
- When total mechanical energy is *not* conserved, we have
$$\Delta K + \Delta U = W_{\text{other}}.$$
- (Non-)conservative forces are forces for which the work-energy theorem is (ir)reversible and are independent of the path taken to do the work (for conservative forces). The work done by conservative forces can be represented by a potential energy function, while nonconservative forces can't be described by such graphs.
- For conservative forces,
$$\Delta K + \Delta U + \Delta U_{\text{int}} = 0.$$
- For conservative forces, the force vector is the negative gradient vector of its energy:
$$\vec{\boldsymbol{F}} = -\boldsymbol{\nabla} U$$

7.2 Exercises

10.

a. The gravitational potential energy of the girl is $W_{\text{grav}} = mgh\cos(1 - \cos\theta) \approx 138.44\ \text{J}$.

b. Her velocity is $v = \sqrt{\frac{2W}{m}} \approx 3.33\ \text{m s}^{-1}$.

c. The tension in the string is radial while the displace is tangental to the circular motion; thus there is no work done on the child.

28.

a.
$$\begin{aligned} W &= \int_1^2 \vec{\boldsymbol{F}}\,\mathrm{d}\vec{\boldsymbol{l}} \\ &= \int_1^2 -\alpha x^2 \hat{\boldsymbol{\imath}}\,\mathrm{d}\hat{\boldsymbol{\jmath}} \\ &= 0. \end{aligned}$$

b.
$$W = \int_1^2 \vec{\boldsymbol{F}}\,\mathrm{d}\vec{\boldsymbol{l}}$$
$$= \int_1^2 -ax^2\hat{\boldsymbol{i}}\,\mathrm{d}\hat{\boldsymbol{i}}$$
$$= -\int_{0.1}^{0.3} ax^2\,\mathrm{d}x$$

$= -0.104$ J.

c.
$$W = \int_1^2 \vec{\boldsymbol{F}}\,\mathrm{d}\vec{\boldsymbol{l}}$$
$$= \int_1^2 -ax^2\hat{\boldsymbol{i}}\,\mathrm{d}\hat{\boldsymbol{i}}$$
$$= -\int_{0.3}^{0.1} ax^2\,\mathrm{d}x$$

$= 0.104$ J.

d. The sum work done along the straight paths $0.1\,\mathrm{m} \longrightarrow 0.3\,\mathrm{m}$ and $0.3\,\mathrm{m} \longrightarrow 0.1\,\mathrm{m}$ are zero, which means that the total work is done when the end and start are the same point, so $\vec{\boldsymbol{F}}$ is conservative. $W_{x_1 \longrightarrow x_2} = \dfrac{a\left(x_2^3 - x_1^3\right)}{3}$. The potential energy equation is $W_{x_1 \longrightarrow x_2} = U_1 - U_2$; comparing the two equations, we get $U = \dfrac{ax^3}{3}$.

31. The force producing U_x is $F_x(x) = -\dfrac{\mathrm{d}U}{\mathrm{d}x} = -4ax^3 = -2.52x^3$ N $\Longleftrightarrow$ $F(-0.8)1.29024$ N $\Longrightarrow$
This force has magnitude 1.29024 N and points away from the origin.

32. The force that one H atom exerts over another is $F(x) = -\dfrac{\mathrm{d}U}{\mathrm{d}x} = -\dfrac{6C_6}{x^7}F(x) < 0 \Longrightarrow$ This force is attractive.

34. $\vec{\boldsymbol{F}}(x, y) = -\boldsymbol{\nabla}U(x, y) = \dfrac{2a}{x^3}\hat{\boldsymbol{i}} + \dfrac{2a}{y^3}\hat{\boldsymbol{j}}$.

23.

a. The mass achieves its greatest speed when all of the elastic potential energy is converted to kinetic energy; in other words,

$$\frac{mv_{\max}^2}{2} = 11.5\text{ J}$$

$\Longrightarrow v_{\max} = 3.03\,\mathrm{m\,s^{-1}}$.

b. The acceleration of the block is $a = -\dfrac{kx}{m} \Longleftrightarrow a_{\max} = -\dfrac{kx_{\min}}{m} \approx 95.92\,\mathrm{m\,s^{-1}}$.

c. We have

$$K_2 - K_1 + U_2 - U_1 = W_{f_k}$$

$$\Longrightarrow v = \sqrt{\frac{2\left(U_1 - U_2 + W_{f_k}\right)}{m}} =\approx 0.335\,\mathrm{m\,s^{-1}}$$

69.

a. Applying $W_{\text{other}} = \Delta K + \Delta U$, we get $F_{x_B} = 0.5mv^2 + 1.25 = 5\text{ J} \Longrightarrow v \approx 3.87\,\text{m}\,\text{s}^{-1}$.
b. The distance compressed, x_c, is $0.5kx_c^2 = K_B + U_B = 5\text{ J} \Longleftrightarrow x_c = 0.5\,\text{m} \Longrightarrow$ The distance from the block to the wall is 0.6 – 0.5 = 0.1 m.

73. By the work-energy theorem,

$$\begin{aligned} W_{\text{other}} &= K_2 - K_1 + U_2 - U_1 \\ \Longleftrightarrow U_1 &= K_2 - K_1 - W_{\text{other}} \\ \Longleftrightarrow U_1 &= \frac{mv^2}{2} + mgd\sin\theta + \mu_k mgd\cos\theta \\ &\approx 119.04\text{J} \end{aligned}$$

74.

a.
$$\begin{aligned} W &= \int_1^2 \vec{\boldsymbol{F}}\,\mathrm{d}\vec{\boldsymbol{l}} \\ &= -\int_0^3 2.5y^3\,\mathrm{d}y \\ &= -50.625\text{J}. \end{aligned}$$

b. As $\vec{\boldsymbol{F}}$ is purely a $\hat{\boldsymbol{j}}$-force, the tool does no work as it travels along the x-axis, thus the work is $W_x = 0$.

On the y-axis, we have $x = 3$ m, so the work done is

$$\begin{aligned} W_y &= \int_1^2 \vec{\boldsymbol{F}}\,\mathrm{d}\vec{\boldsymbol{l}} \\ &= -\int_0^3 7.5y^2\,\mathrm{d}y \\ &= -67.5\text{J}. \end{aligned}$$

c. The work done by $\vec{\boldsymbol{F}}$ is different for the two paths in (a) and (b), so $\vec{\boldsymbol{F}}$ is nonconservative force.

80. a.
$$\begin{aligned} &\frac{\alpha}{x_0^2}\left[\left(\frac{x_0}{x}\right)^2 - \frac{x_0}{x}\right] \\ &= \frac{\alpha}{\alpha^2/\beta^2}\left(\frac{\alpha^2/\beta^2}{x^2} - \frac{\alpha/\beta}{x}\right) \\ &= \frac{\alpha}{x^2} - \frac{\beta}{x}. \end{aligned}$$

b. $v = \sqrt{-\frac{2}{m}U} = \sqrt{\frac{2\alpha}{mx_0^2}\left[\left(\frac{x_0}{x}\right) - \left(\frac{x_0}{x}\right)^2\right]}$. The proton moves in the positive direction, speeding up until it reaches a maximum speed, then slows down, although it never stops.

c. v reaches its max when the kinetic energy is maximum (and thus minimum for the potential energy), thus $\frac{\mathrm{d}U}{\mathrm{d}x} = 0 \iff x = 2x_0$. Then $\max v = \frac{\beta}{\sqrt{2ma}}$.

d. v reaches ít maximum at a point where $\frac{\mathrm{d}U}{\mathrm{d}x} = -F_x = 0$.

e. $x_1 = 3x_0$; $U(x_1) = -\frac{2a}{9x_0}$

$v(x) = \sqrt{\frac{2}{m}(U(x_1) - U(x))} = \sqrt{\frac{2a}{mx_0^2}\left[\frac{x_0}{x} - \left(\frac{x_0}{x}\right)^2 - \frac{2}{9}\right]}$.

The particle is still confined to the region where $U(x) < U(x_1)$. The maximum speed still occurs at $x = 2x_0$, but now the particle will oscillate between x_1 and some minimum value.

f. Note that $U(x) - U(x_1)$ can be written as

$$\frac{a}{x_0^2}\left[\frac{x_0}{x} - \frac{1}{3}\right]\left[\frac{x_0}{x} - \frac{2}{3}\right],$$

which is zero at $\left[\begin{array}{l} x = 3x_0 = x_1 \\ x = 1.5x_0 \end{array}\right.$. Thus when the proton is released from x_0, it will move forward indefinitely, and when it's released from x_1, it will oscillate between x_1 and $1.5x_0$.

58. Applying the work-energy theorem, we get

$$mv_0^2 + mgL\sin\alpha = \frac{mv^2}{2} = mgd\sin\beta + \mu_r mgd\cos\beta$$

$$\iff d = \frac{v_0^2/2g + L\sin\alpha}{\sin\beta + \mu_r\cos\beta},$$

where d is the distance the truck moves up the ramp before stopping.

59. a. We have $k\frac{0.28^2}{2} = \frac{mv^2}{2} + \mu_k mgx \iff v \approx 0.747\mathrm{J}$

b. Let x_0 be the initial compressed length of the spring and x be the length of the compressed spring when v reaches its maximum.

We then have $0.5kx_0^2 = 0.5mv^2 + \mu_k mg(x_0 - x) + 0.5kx^2$.

The maximum speed occurs when $m\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = 0 \implies x = \frac{mg\mu_k}{k}$.

Putting this value of x into the first equation and plugging the number, we get

$$\max v \approx 0.931 \text{ m/s}$$

67. a. $mgy_a = 0.5mv^2 - 0.5ky_a^2 \implies v = \sqrt{2gt_a - ky_a^2m^{-1}} \approx 0.48$ m.

b. v reaches its maximum when y is such that $v'(y) = 0 \iff y \approx 0.033$ m $\implies \max v \approx 0.5658$ m/s.

8 Momentum, Impulse and Collisions

8.1 Summary

- The momentum of a particle is the product of its mass and velocity and is a vector quantity: $\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}}$.
- According to Newton's 2nd law, the rate of change of a particle's momentum is equal to the net force acting on it:

$$\sum \vec{\boldsymbol{F}} = \frac{\mathrm{d}\vec{\boldsymbol{p}}}{\mathrm{d}t}. \tag{8.2}$$

- The impulse of a particle is the change in momentum of that particle:

$$\vec{\boldsymbol{J}} = \Delta\vec{p} = \vec{p}_2 - \vec{p}_1 = \sum \vec{\boldsymbol{F}}\Delta t = \int_{t_1}^{t_2} \vec{\boldsymbol{F}}\,\mathrm{d}t\,.$$

- For any system, forces that the particles of that system exert on each other are internal forces; any other force acted on the system by outside objects are called external forces.
- Systems on which no external forces act are isolated.
- Collisions are strong interactions between bodies in relatively short time frames.
- Elastic collisions are ones where the total kinetic energy of the system remains constant; in inelastic collisions, the TKE is less than before the collision, and if the bodies stick together and move as one after the collision, then it is completely inelastic.
- In any collision in which external forces can be ignored, momentum is conserved and the total momentum before equals the total momentum after; in elastic collisions only, the total kinetic energy before equals the total the total kinetic energy after.
- The center of mass of a body/collection of particles is the point which moves as if all the mass were concentrated at that point and it were acted on by a net force equal to the sum of all the external forces on the system.

8.2 Exercises

84. Let v_3 be the speed of the sphere and dart unit at the top of the loop. Applying Newton's 2nd law (with the $+x$ direction pointing downward), we get

$$T + mg = \frac{mv_3^2}{R}.$$

For the sphere to make a complete revolution, $T = 0 \Longrightarrow v_3 = \sqrt{gR}$.

Since the energy of the system is conserved, we have

$$0.5m_2v_2^2 = m_2(2gR) + 0.5v_3^2 \Longleftrightarrow v_2 = \sqrt{5gR}.$$

Applying the conservation of momentum to the dart, we have

$$v_1 = 5v_2 = 5\sqrt{5gR} \approx 58.57 \text{ m/s}.$$

97.

a. Let m_1, v_1, m_2, v_2 denote the mass and velocity of the heavier and lighter fragments, respectively.

   Since both total energy and momentum are conserved, we have as follows:

$$\begin{cases} m_1 v_1 &= m_2 v_2 \\ m_1 v_1^2 + m_2 v_2^2 &= 1720 \end{cases} \tag{8.3}$$

   Solving for v_1, v_2, we have $v_1 \approx 14.31$ m/s and $v_2 \approx 71.55$ m/s.

b. $0.5gt^2 = 80 \text{ m} \implies t \approx 4.04$ s. The distance between the two fragments is then $d = (v_1 + v_2)t \approx 346.87$ m.

98.

103. a. Let the positive direction be downward.

   We have $x = \dfrac{at^2}{2}$. Then,

$$\frac{gt^2}{2} = 3\frac{at^2}{2} \iff a = \frac{g}{3}.$$

b. $d = 14.7$ m.

c. $m = kd\big|_{t=3\text{ s}} = 29.8$g.

104.

a. We have $x_{\text{cm}} = \dfrac{1}{M}\displaystyle\int_0^L \rho A x \,\mathrm{d}x = \dfrac{\rho A}{M} \cdot \dfrac{L^2}{2} = \dfrac{L}{2}$.

b. $\displaystyle\int x \,\mathrm{d}m = \int_0^L A\alpha x^2 \,\mathrm{d}x = \frac{A\alpha L^3}{3}$, and $M = \int \mathrm{d}m = \displaystyle\int_0^L A\alpha x \,\mathrm{d}x = \frac{\alpha A L^2}{2}$.

   Multiplying the two results together, we obtain

$$x_{\text{cm}} = \frac{1}{M}\int x \,\mathrm{d}m = \frac{2L}{3}.$$

105. First notice that $V = \dfrac{\pi a^2}{2}$ and $M = \rho V$.

To find x_{cm}, divide the plate into thin strips parallel to the y-axis and vice versa.

$x_{\text{cm}} = M^{-1}\displaystyle\int x \,\mathrm{d}m = \frac{\rho t}{M}\int_{-a}^{a} x\sqrt{a^2 - x^2}\,\mathrm{d}x = 0.$

Similarly, we get $y_{\text{cm}} = \displaystyle\int_0^a \frac{2\rho t}{M} y\sqrt{a^2 - y^2}\,\mathrm{d}y = \frac{4a}{3\pi}$.

17. Applying conservation of momentum,

$$P_b - P_r + P_g = 0.$$

$P_g = 0.8528$ kg · m/s. The gases travel along the direction of the bullet.

9 Rotation of Rigid Bodies

9.1 Summary

- Rotational kinematics:

$$\omega_z = \lim_{\Delta t \to 0} \frac{d\theta}{dt} = \frac{d\theta}{dt} \tag{9.1}$$

$$\alpha_z = \lim_{\Delta t \to 0} \frac{d\omega}{dt} = \frac{d\omega}{dt} \tag{9.2}$$

- If α_z = const:

$$\theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2 \tag{9.3}$$

$$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t \tag{9.4}$$

$$\omega_z = \omega_{0z} + \alpha_z t \tag{9.5}$$

$$\omega_z^2 - \omega_{0z}^2 = 2\alpha_z(\theta - \theta_0) \tag{9.6}$$

- Relationship between linear and angular kinematics:

$$v = r\omega \tag{9.7}$$

$$a_{\text{tan}} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{9.8}$$

$$a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r \tag{9.9}$$

- The *moment of inertia* I of a body about a given axis is its rotational inertial; the great the value of I, the harder it is to change the state of a body's rotation.

$$I = \sum_{i=1}^{n} m_r r_i^2. \tag{9.10}$$

- Rotational kinetic energy of a body:

$$K = \frac{1}{2}I\omega^2. \tag{9.11}$$

- Parallel-axis theorem:

$$I_p = I_{\text{cm}} + Md^2, \tag{9.12}$$

where I_p is the moment of inertia of the object at the point of consideration, I_{cm} is its moment of inertia at the center of mass, and d the distance from that point to its center of mass.

- For an object with continuous mass, its moment of inertia is

$$I = \int r^2\,\mathrm{d}m\,. \tag{9.13}$$

9.2 Exercises

91. We have

$$\begin{aligned} I &= \int r^2\,\mathrm{d}m \\ &= \frac{3MR^2}{2h^5}\int_0^h z^4\,\mathrm{d}z \\ &= \frac{3}{10}MR^2 \end{aligned}$$

92. a. $r(\theta) = \int_0^\theta r_0 + \beta\theta\,\mathrm{d}\theta = r_0\theta + \dfrac{\beta\theta^2}{2}$.

b. $s = vt = r_0\theta + \dfrac{\beta\theta^2}{2} = 0 \Longrightarrow \theta(t) = \dfrac{-r_0 + \sqrt{r_0^2 + 2vt\beta}}{\beta}$.

c. $\omega(t) = \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dfrac{v}{\sqrt{r_0^2 + 2v\beta t}}$.

$\alpha(t) = \dfrac{\mathrm{d}\omega}{\mathrm{d}t} = -\dfrac{\beta v^2}{\left(r_0^2 + 2v\beta t\right)^{1.5}}$.

55. a. $M = \int_0^L \gamma x\,\mathrm{d}x = 0.5\gamma L^2$.

b. $I_b = \int x^2\,\mathrm{d}m = \int_0^L \gamma x^3\,\mathrm{d}x = \dfrac{\gamma L^4}{4} = \dfrac{ML^2}{2}$.

c. $I_c = \int (L-x)^2\,\mathrm{d}m = \int_0^L \gamma(L-x)^2 x\,\mathrm{d}x = \dfrac{ML^2}{6}$.

52. a. $I_a = \dfrac{1}{3}\cdot\dfrac{M}{2}\cdot\left(\dfrac{L}{2}\right)^2 + \dfrac{1}{3}\cdot\dfrac{M}{2}\cdot\left(\dfrac{L}{2}\right)^2 = \dfrac{ML^2}{12}$.

b. $I_b = 2\left(\dfrac{1}{12}\dfrac{ML^2}{2\cdot 4} + \dfrac{ML^2}{2\cdot 16}\right) = \dfrac{ML^2}{12}$

c. $I = 4\left(\dfrac{1}{12}\cdot\dfrac{M}{4}a^2 + \dfrac{M}{4}\cdot\dfrac{a^2}{4}\right) = \dfrac{Ma^2}{3}$.

45. The velocity of the block after descending 3 m is $v_y = \dfrac{2(y - y_0)}{t} = 33$ m/s. $\omega = v_y R^{-1} \approx 10.71$ rad / s. Conservation of mechanical energy gives us

$$mgy = \frac{mv^2}{2} + \frac{I\omega^2}{2}$$

$I \approx 1.824\ \mathrm{kg} \cdot m^2 \Longrightarrow M = \dfrac{2I}{R^2} \approx 46.5$ kg. Using the work-energy theorem theorem, we obtain

$$mgR = \frac{1}{2}\left(mv^2 + \frac{1}{2}m\omega^2R^2\right)$$

$\Longrightarrow \omega = \sqrt{\dfrac{4g}{3R}}$.

10 Dynamics of Rotational Motion

10.1 Summary

- Torque is the quantitative measure to cause or change a body's rotational motion of a force: $\vec{\tau} = \hat{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$.
- 3 ways of calculating torque:

$$\tau = Fl = Fr\sin\theta = F_{\tan}r, \tag{10.1}$$

  where
  - l is the lever/moment arm;
  - θ is $\widehat{\left(\hat{\boldsymbol{r}}, \vec{\boldsymbol{F}}\right)}$;
  - $F_{\tan} = F\sin\theta$.
- Analogy of Newton's second law: The net torque acting on a solid object about an axis is equal toi the product of its moment of inertia times angular acceleration about that axis.

$$\sum \tau_z = I\alpha_z. \tag{10.2}$$

- If a rigid object experiences both translational and rotational motion, the total kinetic energy of that object is

$$K = \frac{1}{2}Mv_{\mathrm{cm}}^2 + \frac{1}{2}Ic_{\mathrm{cm}}^2. \tag{10.3}$$

- Work done by a torque: $W = \displaystyle\int_{\theta_1}^{\theta_2} \tau_z\,\mathrm{d}\theta$. If $\tau_z = \mathrm{const}$, then $W = \tau_z\Delta\theta$.
- The total work done on a rotating rigid body (work done by the net torque) is

$$W_{\mathrm{tot}} = \int_{\omega_1}^{\omega_2} I\omega_z\,\mathrm{d}\omega_z = \frac{1}{2}I\Delta\omega^2. \tag{10.4}$$

Power due to a torque: $P = \dfrac{\mathrm{d}W}{\mathrm{d}t} = \tau_z\dfrac{\mathrm{d}\theta_z}{\mathrm{d}t} = \tau_z\omega_z$.

- Angular momentum:

$$\boxed{\vec{\boldsymbol{L}} = \hat{\boldsymbol{r}} = \vec{\boldsymbol{p}} = \hat{\boldsymbol{r}} \times m\vec{\boldsymbol{v}}. \tag{10.5}}$$

- The rate of change angular momentum of a particle equals the torque of the net force acting on it: $\frac{\mathrm{d}\vec{\boldsymbol{L}}}{\mathrm{d}t} = \tau$.
- For a solid object: $\vec{\boldsymbol{L}} = I\vec{\omega}$.
- For system of particle,

$$\boxed{\sum \vec{\tau} = \frac{\mathrm{d}\vec{\boldsymbol{L}}}{\mathrm{d}t}. \qquad (10.6)}$$

- The law of conservation of angular momentum says that the total angular momentum of a system is preserved if the net torque acting on it is 0.

10.2 Exercises

3. The net torque acting on the plate is $\sum \tau_z = \sum_{i=1}^{3} \tau_z = -18 \sin 45° \approx 2.5$ Nm.

16. a. The tensions of the wire satisfy

$$\begin{cases} T & = m_1 a \\ m_2 g - T_2 & = m_2 a \\ \tau & = 0.6MR^2\alpha = (T_2 - T_1)R \end{cases}$$

$\Longrightarrow a = \frac{m_2}{m_2 + m_1 + 0.5M} g \approx 2.72$ m/ s². Then $T_1 = m_1 a = 32.6$ N, and $T_2 = m_2(g - a) = 35.4$ N.

b. $a = 2.72$ m/ s²

c. $F_x = T_1 = 32.6$ N and $F_y = T_2 + Mg = 55$ N.

40. a. Yes, momentum is conserved as the moment arm of tension is 0.

b. $L_1 = L_2 \Longrightarrow I\omega_1 = I\omega_2$. Then

$$\begin{aligned} \omega &= \left(\frac{r_1}{r_2}\right)^2 \omega_2 \\ &= 11.4 \text{ rad / s.} \end{aligned}$$

c. $\Delta K = 0.5m\left(r_2^2\omega_2^2 - r_1^2\omega_1^2\right) = 0.027\mathrm{J}$

d. $W = \Delta K = 0.027\mathrm{J}$

54. a. The rotational inertia is $I = 0.5MR^2$. The net torque is $I\alpha_z = 0.5MR^2\frac{\alpha - \alpha_0}{t}$, and it is equal to

$$FR - \tau_f - \mu_k nR,$$

where FR is the torque of the applied force, τ_f is the torque due to friction $\Longrightarrow F = R^{-1}\left(\tau + \tau_f + \mu_k nR\right) = 67.6$ N.

b. A constant angular speed means that $\tau = 0 \Longrightarrow F = R^{-1}\left(\tau_f + \mu_k nR\right) = 62.9$ N.

c. $\tau_f = \frac{\mathrm{d}\vec{\boldsymbol{L}}}{\mathrm{d}t} \Longrightarrow t = \frac{L}{\tau_f} = 3.27$ s.

59. Let $+y$ be downward for A and $+y$ be upward for B. We have, for the blocks,

$$m_A g + T_a = m_A a$$
$$-m_B g + T_B = m_B a$$

For the pulley, $(T_A - T_B)R = I\alpha = IaR^{-1}$.

$\Longrightarrow (m_A - m_B)g = \left(m_A + m_B + IR^{-2}\right)a \Longleftrightarrow a = \dfrac{m_A - m_B}{m_A + m_B + IR^{-2}}g = 0.921\ \mathrm{m/\ s^2}$.

We have $a = \left(\dfrac{W_1 - W_2}{W_1 + W_2 + 0.5W}\right)g = 0.2083g$. Then, $T_1 = W_1\left(1 - \frac{a}{g}\right) = 90.62\ \ \mathrm{N}$ and $T_2 = W_2(1 + ag_1) = 96.96\ \ \mathrm{N} \Longrightarrow F_{\text{pulley}} = T_1 + T_2 + W_{\text{pulley}} = 270\ \ \mathrm{N}$.

75. The total kinetic energy, which is equal to the potential energy, is

$$K = Mgy = \frac{1}{2}\left(Mv^2 + \frac{4MR^2}{2} \cdot \frac{v^2}{4R^2} + \frac{MR^2}{2} \cdot \frac{v^2}{R^2} + Mv^2\right) = \frac{3}{2}Mv^2$$

$$\Longrightarrow v^2 = \frac{2}{3}gy = 2ay \Longrightarrow a = \frac{g}{3}.$$

11 Equilibrium and Elasticity

11.1 Summary

- For an object to be in equilibrium, it must not exhibit any rotational or translational motion:

$$\begin{gathered}\sum \tau = 0 \quad \text{about any point} \\ \sum \vec{\boldsymbol{F}} = 0\end{gathered} \tag{11.1}$$

- The torque due to the weight of a body can be found by assuming that tis entire weight is concentrated at a singular point called the *center of gravity*, which occupies the same spot as the center of mass if $\vec{g}$ has the same value at all points.
- *Hooke's law* states that stress (force/area) is proportional to strain (deformation). The ratio of stress to strain is called the *elastic modulus.*

$$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus}$$

- If a material returns to its original form after removing the stress acting on it, it is *elastic;* otherwise, it is *plastic.*
- Tensile stress appears when an object is stretched or compressed at their ends, and its modulus is equal to

$$Y = \frac{F_\perp}{A} \cdot \frac{l_0}{\Delta l}.$$

- Bulk stress appears when an object is compressed by pressure from all sides, and its modulus is equal to

$$B = -\frac{(\Delta p) \cdot V_0}{\Delta V}$$

- Compressibility represents the fractional decrease of volume per uni increase of pressure and is the inverse of bulk modulus: $k = B^{-1}$.
- Shear stress appears when an object experiences tangental forces to opposite sides of its surface, and its modulus is equal to

$$S = \frac{F_{\parallel}}{A} \cdot \frac{h}{x}.$$

- The proportional limit is the maximum amount of stress a material can take before Hooke's law no longer applies to that material.
- The elastic limit is the maximum stress for which elasticity can be preserved.
- The breaking stress/ultimate strength/tensile strength is the stress at which the material breaks.

11.2 Exercises

1. We'll set the point of origin at the center of gravity of the bar and the (+) direction to the right. Then,

$$x_{\text{cm}} = \frac{-0.5Lm_1 + 0.5Lm_2 + M \cdot 0}{M + m_1 + m_2} \approx 4.8 \text{ cm},$$

so the fulcrum should be placed $0.5L + x_{\text{cm}} = 29.8$ cm away from the left end.

2. Letting the point of origin be the object's center of gravity and the (+) direction be left, we get $x_{\text{cm}} = \dfrac{mD}{m + M} = 9.53$ cm, which means this mass should have its center of gravity located 9.53 cm to the left.

11. a. The heaviest beam that the cable can support satisfies

$$\begin{aligned} \sum \tau &= 0 \\ \frac{Tl \sin\theta}{3} - \frac{wl}{2} &= 0 \\ \iff w &\approx 533 \ \text{ N}. \end{aligned}$$

b.

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= 0 \\ -T\cos\theta + F_x &= 0 \\ T\sin\theta + F_y - w &= 0 \\ \implies F_x &= 600 \ \text{ N} \\ \implies F_y = w - T\sin\theta &= -267 \ \text{ N}. \end{aligned}$$

$\implies \vec{F}_y$ points downward.

c. The center of gravity of the sandbox lies at its center. Setting the origin at the fulcrum and letting the positive direction to the right, we obtain

$$\frac{m_1x_1 + m_2x_2}{m_1 + m_2} = 0$$

$$\iff m_1 = -m_2\frac{x_2}{x_1} = 20 \text{ kg}.$$

16. a. Denote the length of the stick as l.

    Taking torques about the right end of the stick, we have $fl = wl/2 \iff f = w/2$. Taking torques about where the cord is attached to the wall, we get $l(n\tan\theta) = wl/2 \iff n\tan\theta = w/2$. Then, $\frac{f}{n} = \tan\theta \le \mu_s = 0.4$, so

    $$\theta < \arctan(0.4) \approx 22°.$$

    b. Taking torques about the same points as (a), we get $fl = 0.5wl + w(l - x)$ and $nl\tan\theta = 0.5wl + wx$. Then in terms of μ_s,

    $$\begin{aligned}\mu_s > \frac{f}{n} &= \frac{0.5l + l - x}{0.5l + x}\tan\theta \\ &= \frac{3l - 2x}{l + 2x}\tan\theta \\ \iff x > \frac{l}{2}\frac{3\tan\theta - \mu_s}{\mu_s + \tan\theta} &= 0.302 \text{ m}\end{aligned}$$

    c. Setting $x = 0.1$ m and $l = 1$ m gives $\mu_s > 0.625$.

17. a. Choose the coordinates shown in Figure 4.

    Thus

    $$\begin{aligned}F_B &= 2w = 1.47 \text{ N} \\ \sin\theta 0.5 &\implies \theta = 30° \\ \sum \tau_p &= 0 \\ \implies -wR + 2RF_C\cos\theta &= 0 \\ \iff F_C &\approx 0.424 \text{ N} \\ F_C = F_A &= 0.424\ N.\end{aligned}$$

    b. $n = \frac{F_C}{\sin\theta} = 0.848 \text{ N}.$

18. Set the origin at the edge of the table and the (+x) direction be to the right. Suppose, too, that each block has mass m. Let the overhang be l.

    a. To obtain maximum overhang, the topmost block must be placed such that its center is at the end of the second block. Then the length x that the bottom block hangs satisfies

    $$\begin{aligned}0.5mL + m(x - L) &= 0 \\ \implies x &= L/4 \\ \implies l &= 3L/4\end{aligned}$$

b. A similar argument applies to $n = 3$:

$$\begin{aligned} 2mx + m(x - 0.5L) &= 0 \\ \iff x &= \frac{L}{6} \\ \implies l &= \frac{11L}{12}. \end{aligned}$$

Again for $n = 3$:

$$\begin{aligned} 3mx + m(x - 0.5L) &= 0 \\ \iff x &= \frac{L}{8} \\ \implies l &= \frac{25L}{24}. \end{aligned}$$

c. With just $n = 4$, we can set a stack of blocks such that the topmost block is not directly over the table.

19. a. We first notice that the strain on the wire is affected by the weight it's attached to: $W = k|\Delta l|$. Then

$$\begin{aligned} Y &= \frac{W}{A} \cdot \frac{l_0}{\Delta l} \\ \iff Y &= \frac{k}{A} \cdot l_0 \\ \iff k &= \frac{AY}{l_0}. \end{aligned}$$

b. $Y = 11 \cdot \cdot 10^{10}$ Pa $\implies k = 1.49 \cdot \cdot 10^{8}$ N/ m.

12 Fluid Mechanics

12.1 Summary

- A *fluid* refers to any substance that can change flow and the shape of the shape of the volume it takes.
- Density is defined as the mass per unit volume:

$$\rho = \frac{m}{V}. \tag{12.1}$$

- The *specific gravity (relative density)* of a material is the ratio of its density to that of water at 4°C.
- *Pressure* is the amount of normal force per unit area:

$$p = \frac{\mathrm{d}F_\perp}{\mathrm{d}A}. \tag{12.2}$$

- *Atmospheric pressure* p_a is the pressure of the earth's atmosphere, and its value is

$$\boxed{(p_a) = 1 \text{ atm} = 101325 \text{ Pa} \approx 1.013 \text{ bar}.} \tag{12.3}$$

- The relationship between elevation and pressure:

$$\mathrm{d}p = -\rho g \,\mathrm{d}y\,. \tag{12.4}$$

  - In particular, if the fluid is of uniform density, then the pressure at depth h relative to the pressure of the fluid at its surface is

$$p = p_0 + \rho g h. \tag{12.5}$$

- *Pascal's law* states that the pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.
  - One application of Pascal's law is the hydraulic lift: If two pistons have differing cross-sectional areas, the applied pressure on both of them is the same if we place a weight on one of them, while the other experiences a force proportional to its area compared to the first piston:

$$F_2 = \frac{A_2}{A_1} F_1. \tag{12.6}$$

- *Gauge pressure* is the excess pressure above the atmospheric pressure, while *total pressure* refers to the sum of of gauge and atmospheric pressure.
- Since liquids are generally incompressible (i.e. their densities don't change), an important consequence is the *continuity equation*, which says that the volume flow rate of a fluid is the same at every point in its flow tube:

$$A_1 v_1 = A_2 v_2. \tag{12.7}$$

  The product Av is called the *volume flow rate*, the rate of volume crossing a section of the tube:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = Av. \tag{12.8}$$

  The *mass flow rate* is the rate of mass flowing through a section and is equal to $\rho \cdot \frac{\mathrm{d}V}{\mathrm{d}t}$.
  - The continuity equation can be generalized to compressible liquids:

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2.$$

- Archimedes's principle says that when a body is partially or completely submerged in a fluid, it experiences a buoyancy force equal to the weight of the displaced volume by the body from the fluid.
- Bernoulli's equation tells us that a change in either the pressure, the elevation or flow speed of a fluid results in a change in the remaining factors:

$$\boxed{p_1 + \rho g y_1 + \frac{\rho v_1^2}{2} = p_2 + \rho g y_2 + \frac{\rho v_2^2}{2}} \tag{12.9}$$

- Viscosity is the internal friction within a fluid. The more viscous a liquid is, the “thicker” it flows.
- Once the flow speed of a fluid exceeds a critical threshold, its flow is longer laminar (flowing in sheets) and instead becomes extremely complex and chaotic, and changes continuously with time. This type is chaotic flow is call *turbulent*.

12.2 Exercises

113. a. Pressure is force divided by area, so

$$p_0 - p = F\left(\frac{\pi D^2}{4}\right)^{-1}$$
$$\iff F = \frac{(p_0 - p)\pi D^2}{4}.$$

b. For $p = 2.5 \cdot\cdot 10^{-2}$ atm and $R = 5$ cm, we obtain $F = 776$ N.

114.

a. $p - p_0 = \rho g y = 1.1 \cdot\cdot 10^8$ Pa

b. On the surface, 1 m^3 of seawater has density 1.03 kg / m^3. At the bottom of the Mariana Trench, the change of volume due to compressibility of water is $\Delta = -\frac{\Delta p \cdot V_0}{B} = -0.05$ m^3 $\Longrightarrow V = V_0 + \Delta V =$ 0.95 m^3; the density then is

$$\rho = \frac{m}{V} = 1.08 \cdot\cdot 10^3 \text{ kg / m}^3;$$

this equates to a 5% increase in density.

115. a. The force at the bottom of the pool is simply the weight of the water: $F_b = \rho g V = 588000$ N.

b. Consider the force acting on a thin strip of one end of the pool: $\mathrm{d}F = p dA = \rho g y \cdot L\, \mathrm{d}y$. Then,

$$F = \int \mathrm{d}F$$
$$= \int_0^h \rho g y L\, \mathrm{d}y = 0.5 \rho g l y^2$$
$$= 176400 \text{ N}.$$

116. Let the torque on the upper and lower halves of the dam be τ_u and τ_h, respectively. Then the total torque is $\tau = \tau_l - \tau_h$. For the upper half of the gate, we have $\mathrm{d}F = p\,\mathrm{d}A = W\rho g y\,\mathrm{d}y$. The moment arm is $H/2 - y$, so

$$\tau = \int_0^{H/2} W\rho g y(H/2 - y)\,\mathrm{d}y$$
$$= 6553 \text{ N}.$$

For the lower half, $\mathrm{d}F = p\,\mathrm{d}A = W\rho g(H/2 + y)\,\mathrm{d}y$. The moment arm is y, so

$$\tau = \int_0^{H/2} W\rho g y(H/2 + y)\,\mathrm{d}y$$
$$= 32670 \ \text{N}$$
$$\Longrightarrow \tau = \tau_l - \tau_u = 26117 \ \text{N}$$

117. $p_1 + \rho g y_1 + \frac{\rho v_1^2}{2} = p_2 + \rho g y_2 + \frac{\rho v_2^2}{2} \Longleftrightarrow v_2 = \sqrt{2(p_1 - p_2) + 2gy_1} \approx 28.4$ m/s. ($A_1 v_1 = A_2 v_2$, but $A_1 \gg A_2$, so $v_1 \gg v_2$ and $0.5\rho v^2$ can be neglective.)
118. a.

$$\rho_g V_g - \rho_L V_s g = 0$$
$$\Longleftrightarrow V_s = \frac{\rho_B}{\rho_L} V$$
$$\Longleftrightarrow V_f = \left(1 - \frac{\rho_B}{\rho_L}\right) V$$

b. $p - p_0 = \rho_w g d + \rho_L g(L - D)$. Applying Newton's second law, we get

$$F_\perp = mg$$
$$\Longrightarrow A(\rho_W g d + \rho_l g(L - D)) = \rho_B L A g$$
$$\Longleftrightarrow \rho_W d + \rho_L (L - D) = \rho_B L$$
$$\Longleftrightarrow d = \left(\frac{\rho_B - \rho_L}{\rho_W - \rho_L}\right) L$$

c. $d = 4.6$ cm.

119. Bernoulli's equation gives us

$$p_1 + \rho g y_1 + \frac{\rho v_1^2}{2} = p_2 + \rho g y_2 + \frac{\rho v_2^2}{2}$$

$A_2 v_2 = 2.4 \cdot \cdot 10^{-4}$ m^3/ s $\Longleftrightarrow v = 1.6$ m/s. We have $A_1 \gg A_2 \Longrightarrow v_1 \ll v_2$; neglect the $\rho v^2/2$ part. If the height of the water in the bucket is x, then

$$\rho g x + p_a = p_a + \frac{\rho v_2^2}{2} \Longrightarrow x = 0.131 \text{ m} = 13.1 \text{ cm}.$$

13 Gravitation

13.1 Summary

- Newton's law of gravitation between two masses:

$$\boxed{\vec{\boldsymbol{F}}_g = \frac{G m_1 m_2}{r^3} \hat{\boldsymbol{r}}} \tag{13.1}$$

.

- Given the mass of a planet m_p, the weight of an object of mass m on that planet is

$$\boxed{w = F_p = \frac{Gmm_p}{R_p^2}} \tag{13.2}$$

and its acceleration due to gravity on the surface is

$$\boxed{g_p = \frac{Gm_E}{R_p^2}.} \tag{13.3}$$

At a point r from the surface, the weight is then

$$\boxed{w = F_p = \frac{Gmm_p}{r^2}.} \tag{13.4}$$

- Gravitational potential energy:

$$\boxed{U_g = -\frac{Gmm_E}{r}.} \tag{13.5}$$

- If a satellite is to maintain a circular orbit around the earth, its speed must be $v = \sqrt{\frac{Gm_E}{r}}$, and its period is $T = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}}$.
- The three Kepler's laws:
  - Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.
  - A line from the sun to a given planet sweeps out equal areas in equal times.
  - The periods of the planets are proportional to the 3/2 powers of the major axis lengths of their orbits.
- If a non-rotating spherical mass distribution with total mass M has a radius less than its Schwarzchild radius R_s, it is a black hole, and the Schwarzchild radius is calculated by

$$R_s = \frac{2GM}{c^2}. \tag{13.6}$$

13.2 Exercises

81. Let $+x$ be upward. From Example 13.35, the gravitation force caused by a thin ring is

$$F_x = -\frac{GMmx}{(x^2 + a^2)^{3/2}}.$$

Now, for this disk, we split it into infinitesimally thin concentric rings of radius dr. Then, we have

$$
\begin{aligned}
\mathrm{d}F_x &= -\frac{2GMmx}{a^2(x^2+r^2)^{3/2}}\,\mathrm{d}r \\
\Longrightarrow F_x &= \int_0^a \mathrm{d}F_x \\
&= -\frac{2GMm}{a^2}\left(1-\frac{x}{\sqrt{x^2+a^2}}\right).
\end{aligned}
$$

Part II - Waves/Acoustics

14 Periodic Motion

14.1 Summary

- Simple harmonic motion: $x = A\cos(\omega t + \varphi)$.
- The phase angle φ can be found if we're given the initial displacement and velocity x_0 and v_{0x}:

$$= \arctan\left(-\frac{v_{0x}}{\omega x_0}\right). \tag{14.1}$$

  Similarly, the amplitude A can also be found:

$$A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}}. \tag{14.2}$$

- In SHM, total mechanical forces are conserved:

$$E = K + U = \frac{1}{2}kA^2 = \text{const.} \tag{14.3}$$

- At a given displacement x, the velocity can be solved:

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}. \tag{14.4}$$

- Suppose we hang a spring with force constant k and suspend from a body with mas m. Let $(+x)$ direction be upward; the force exerted on the mass will be

$$F_{\text{net}} = k(\Delta l - x) - mg = -kx. \tag{14.5}$$

  It works the same way if the spring is compressed by a block of mass m above it.
- Angular SHM occurs in the balance springs of mechanical watches, and given the balance spring's moment of inertia I about its axis and torsion constant κ, we can derive its angular frequency ω:

$$\omega = \sqrt{\frac{\kappa}{I}}. \tag{14.6}$$

- A *physical pendulum* is any pendulum that uses an extended body. Its angular frequency is given by

$$\omega = \sqrt{\frac{mgd}{I}}, \tag{14.7}$$

  where m is the object's mass, d the distance from its rotation axis, and I its moment of inertia.
- Oftentimes, forces like friction cause oscillations to die out. The decrease in amplitude caused by dissipative forces is called *damping*, and the corresponding movement is called a *damped oscillation*. If the force is sufficiently small, then the motion is

$$x = Ae^{-(b/2m)t}\cos(\omega' t + \varphi). \tag{14.8}$$

  The angular frequency of such a motion is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \tag{14.9}$$

- If $\omega' = 0 \iff b = 2\sqrt{km}$, then the system is *critically damped* - it no longer oscillates and returns to the equilibrium position without oscillating when it's displaced. If $\omega' > 2\sqrt{k/m}$, then the system is *overdamped*. In this case, the system returns to equilibrium slower than with critical damping and the position is determined by

$$x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t}, \tag{14.10}$$

  where C_1, C_2 depend on the constants and a_1, a_2 are determined by m, k, and b.
- If a periodic force is applied to keep a damped harmonic oscillator moving, the resulting motion is called a *forced oscillation*. Its amplitude is given by

$$A = \frac{F_{\max}}{\sqrt{\left(k - m\omega_d^2\right)^2 + b^2\omega_d^2}}. \tag{14.11}$$

14.2 Exercises

54. We have, $T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{mr^2 + mr^2}{mgd}} = 2\pi\sqrt{\frac{2r}{g}} \Leftrightarrow r = \frac{T^2 g}{8\pi^2} = 0.496\,\mathrm{m}$
57. Pendulum A is a simple pendulum: Its period is $T_A = 2\pi\sqrt{l/g}$; as for pendulum B, it's a solid pendulum, and its period is given by

$$
\begin{aligned}
T &= 2\pi\sqrt{\frac{I}{mgd}} \\
&= 2\pi\sqrt{\frac{\left(\frac{2}{5}\right)M\left(\frac{L^2}{4}\right) + ML^2}{MgL}} \\
&= 2\pi\sqrt{1.1l/g} \approx 1.05T_A.
\end{aligned}
\tag{14.12}
$$

It takes longer for pendulum B to complete a swing.

60. $A_2 = Ae^{(-b/2m)t} = Ae^{(-b/2m)5} = 0.1\,\text{m} \Rightarrow b = 0.022\,\text{kg}\,\text{s}^{-2}$.
61. a. The frequency of the rat's oscillation is $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = 2.47$
    b. The motion will be critically dampened at $b = 2\sqrt{km} = 1.73\,\text{kg}\,\text{s}^{-1}$.
62. a. The times when the mas isn't moving is when it reaches its amplitudes, or, $v = 0 \iff t = 0, t = 1\,\text{s}, t = 2\,\text{s}, t = 3\,\text{s}, t = 4\,\text{s}$.
    b. The original energy of the system is $E_0 = kA_0^2/2 = 0.551\,25$ J.
    c. $\Delta E = E_4 - E_1 = \frac{k(A_4^2 - A_1^2)}{2} = -0.303\,75$ J. The mechanical energy lost is transformed into the the work done by forces such as friction.
64. $\varphi = \arccos(-1.5/6) = 104.48° \Longrightarrow t = T \cdot \frac{\varphi}{360°} = 0.09\,\text{s}$.
68. The amplitude oscillation A is such that the maximum restoring force does not exceed the static friction of block m.

    Then, we have

$$
\begin{aligned}
\mu_s mg = ma &= \frac{kA}{m + M} \\
\Leftrightarrow A &= \frac{\mu_s(m + M)g}{k}
\end{aligned}
\tag{14.13}
$$

71. Let the stretched length of the spring be L. The bounce frequency is double that of the pendulum, which means $\frac{4g}{L} = \frac{k}{m} \Longleftrightarrow L = \frac{4W}{k} = 2.67\,\text{m} \Longrightarrow$ The natural length of the spring is

$$
l = L - \Delta x = L - \frac{w}{k} = 2\,\text{m}. \tag{14.14}
$$

72. a. Letting the (+) direction be downward, we have, according to Newton's 2nd law,

$$
\begin{aligned}
-\rho g x_1 A + Mg &= 0 \\
\Longleftrightarrow x_1 &= \frac{M}{\rho A}.
\end{aligned}
\tag{14.15}
$$

    b. Similar to part (a), we have

$$-\rho g x_2 A + Mg + F = 0$$
$$\iff x_2 = \frac{M}{\rho A} + \frac{F}{\rho g A}. \quad (14.16)$$

With $\vec{F}$ pressing down on the object, the object is plunged an extra distance $\Delta x = \frac{F}{\rho g A}$.

c. The restoring force in this situation is the buoyancy force itself. Here, $k = \rho g A$.

   The period of the mass is then $T = 2\pi\omega^{-1} = 2\pi\sqrt{\frac{M}{k}} = 2\pi\left(\frac{M}{\rho g A}\right)^{0.5}$.

73. The center of mass of the square is its geometric center, so its distance is $d = L/\sqrt{2}$.

    Each of the bars has inertia $I_{\text{cm}} = \frac{1}{12}mL^2$. Applying the parallel-axis theorem wrt the center of the square, we obtain the moment of inertia of each stick: $\frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{mL^3}{3}$, and thus the m.o.i. of the square is $\frac{4mL^2}{3}$. Applying the parallel-axis theorem again wrt the axis at the hook, we obtain $I = \frac{4}{3}mL^2 + \frac{4mL^2}{2} = \frac{10mL^2}{3}$.

    Finally, the frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{mgd}{I}} = \sqrt{\frac{6}{5\sqrt{2}}}\left(\frac{1}{2\pi}\sqrt{\frac{g}{L}}\right). \quad (14.17)$$

83. Linear momentum is conserved, so $m_B v_B = mv \implies v \approx 2.24\,\text{m}\,\text{s}^{-1}$.

    Mechanical energy is also conserved, so $\frac{1}{2}mv^2 = \frac{1}{2}kA^2 \implies k \approx 223\,\text{N}\,\text{m}^{-1}$, and the period of oscillation is $T = 2\pi\sqrt{\frac{m}{k}} = 0.421\,\text{s}$.

86. The angular frequency of the bell is $\omega_b = \sqrt{\frac{mgd}{I}} = 11.11$ rad/s. The length L of the rod is such that $\omega_r = \omega_b \implies L = 0.88\,\text{m}$.

The torque of the rod about the pivot is $\tau = -\left(k\frac{L}{2}\theta\right)\frac{L}{2} = I\alpha = \frac{ML^2}{12}\theta''(t)$

$\implies \alpha = -\frac{3k}{M}\theta = -\frac{\kappa}{I}\theta \implies$ The rod moves in angular SHM. The period is $T = 2\pi\omega^{-1} = 2\pi\sqrt{\frac{M}{3k}}$.

88. The L-bar has moment $I = 2(1/3)ML^2 = (2/3)ML^2$ about the pivot, and the center of gravitated is situated $d = \frac{L}{2\sqrt{2}}$ below the pivot.

$F = \dfrac{1}{2\pi}\sqrt{\dfrac{mgd}{I}} = \dfrac{1}{2\pi}\sqrt{\dfrac{6g}{4\sqrt{2}L}} = \dfrac{1}{4\pi}\sqrt{\dfrac{6g}{\sqrt{2}L}}.$

93. a. Let $\Delta x_1, \Delta x_2$ be positive if the springs are stretched and negative if expanded. $\Delta x = \Delta x_1 = \Delta x_2$ and $\sum F_x = -(k_1 + k_2)\Delta x$, so $k_{\text{eff}} = k_1 + k_2$.
    b. Despite the orientation of the springs, the fact that one spring compresses while the other expands (which means that $\Delta x = \Delta x_1 - \Delta x_2$), and that the spring forces in the same direction means that the result obtained from (a) still holds: $k_{\text{eff}} = k_1 + k_2$.
    c. For massless springs, the tension acting on each one are the same: $F_1 = F_2 = F$. Then, $\Delta x_1 = -\dfrac{F}{k_1}, \Delta x_2 = -\dfrac{F}{k_2}$. $\Delta x = -\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right)F \implies k_{\text{eff}} = \dfrac{k_1 k_2}{k_1 + k_2}$.
    d. Cutting the spring in half doubles its force constant, and so $f_1/f_2 = \omega_1/\omega_2 = 1/\sqrt{2}$.
94. a. We divide the spring into small pieces of length dl whose individual mass is d$m = M\,$dl. Each piece also has linear velocity $v_x = vl$. The kinetic energy of the entire spring is then

$$\begin{aligned} K &= \int \mathrm{d}K \\ &= \int_0^L \frac{Mv^2 l^2}{2}\,\mathrm{d}l \\ &= \frac{Mv^2}{6}. \end{aligned} \tag{14.18}$$

    b. Recall that the conservation of energy equation for a massless spring is

$$E = K + U = 0.5\left(kx^2 + mv^2\right) = 0.5kA^2. \tag{14.19}$$

    Taking the time derivative, we get

$$\begin{aligned} kvx + mva &= 0 \\ \iff a &= -\frac{k}{m}x, \end{aligned} \tag{14.20}$$

    which fits Eq.(14.8). The frequency is then $\omega = \sqrt{\dfrac{k}{m}}$.
    c. Applying the same procedure in (b) to our massive spring, we get

$$\begin{aligned} E = K + U = Mv^2/6 + kx^2/2 &= 0.5kA^2 \\ \implies Mva/3 + kvx &= 0 \\ \iff a = \frac{-k}{3M}x &= \omega'^2 x \end{aligned} \tag{14.21}$$

    Then the frequency of the oscillation is $\omega = \sqrt{\dfrac{k}{3M}} = \sqrt{\dfrac{k}{M'}}$. Thus the effective mass in terms of M is $M' = \dfrac{M}{3}$.

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Part II - Waves/Acoustics

15 Mechanical Waves

15.1 Summary

- Waves occur when a system is disturbed from equilibrium and when the disturbance propagates from one region of the system to another.
- Mechanical waves are waves that propagate within a physical *medium*.
- As the wave travels through the material, the particles comprising the medium experience displacements, which can be either of these two types: *longitudinal*, where the particles move parallel to the direction of the wave, and *transversal*, where they move perpendicular to the wave. Such waves that cause these kinds of motions are called *longitudinal* and *transversal* waves, respectively.
- The *wavelength* of a periodic wave is the shortest distance between any point to the next on the next repetition of the waveform: $\lambda = vT = v/f$.
- The *wave function* of a sinusoidal wave propagating in the $+x$-direction is
$$y(x,t) = A\cos(kx - \omega t), \tag{15.1}$$
where $k = \dfrac{2\pi}{\lambda}$ is a quantity called the *wave number* and ω is the angular frequency.
- The *wave equation* relates a traveling wave to its phase velocity and curvature at a particular point:
$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}. \tag{15.2}$$
Furthermore, it leads to an important consequence: Any disturbance that satisfies the wave equation is a wave (not necessarily harmonic) propagating along the $+x$ direction with speed v.
- The speed v at which the waves spread is dependent on two factors: the tension F on the string and its mass per unit length μ, which varies based on the material of the string:
$$v = \sqrt{F/\mu}. \tag{15.3}$$
- Inverse square-law: $\dfrac{I_1}{I_2} = \dfrac{r_2^2}{r_1^2}$.
- Average power of power of a sinusoidal wave: $P_{\text{av}} = \frac{1}{2}\sqrt{\mu F}\omega^2 A^2$.
- *Interference* occurs when two or more waves pass through the same region at the same time.
- The *principle of superposition* tells us that a wave reflects when it reaches a boundary of its medium. At any point where two or more waves overlap, the total displacement is the sum of the displacements of the individual wave.

- *Standing waves* are waves created by SHM motions on a string that is fixed on one or both ends and whose wave patterns don't move in either direction along the string, in contrast to **traveling waves**.
  - There are two types of points worth noting when studying standing waves: the first are *nodes*, which don't move at all, and the second are *antinodes*, which oscillate with the greatest amplitude and are always $\lambda/2$ away from each node.
- For a string that's fixed on one end at $x = 0$, its standing wave function is

$$y(x, t) = (A_{\mathrm{SW}} \sin kx) \sin \omega t, \tag{15.4}$$

  where $A_{\mathrm{SW}} = 2A$ is the standing-wave amplitude, double that of either the original traveling waves.
- The positions of the nodes and antinodes for a one-fixed string are as follows:

$$x_n = n\frac{\lambda}{2} \quad (n = 0, 1, 2, \ldots) \tag{15.5}$$

  and

$$x_{an} = x_n + \frac{\lambda}{2}. \tag{15.6}$$

- For a standing wave to exist on a 2-fixed string (such as a guitar string), the length of the string must be equal to an integer number of half-wavelengths:

$$L = n\frac{\lambda}{2}. \tag{15.7}$$

  Otherwise, there can't be a steady wave pattern of with nodes & antinodes and a standing wave cannot exist.
- The possible values of the wavelength, λ_n, is equal to

$$\lambda_n = \frac{2L}{n}. \tag{15.8}$$

- Corresponding to each term in λ_n is a frequency f_n related to its wavelength by $f_n = v/\lambda_n$. The smallest of them, the *fundamental frequency*, corresponds to the longest wavelength $\lambda_1 = 2L$: $f_1 = v/2L$. Similarly, we can express all frequencies as

$$f_n = n\frac{v}{2L} = \frac{n}{2L}\sqrt{\frac{F}{\mu}}. \tag{15.9}$$

- Note: $v = \dfrac{\omega}{k}$.

15.2 Exercises

3. The speed of these waves is $v = \lambda T^{-1} = 800$ km/h $= 222.22\,\mathrm{m\,s^{-1}}$.

6. $\lambda = 1.06\,\mathrm{m}$

a. The speed of the waves is $v = \lambda T^{-1} = 1.92\,\mathrm{s}$.
b. The amplitude of each wave is $A = 0.53\,\mathrm{m}$.

8. a. The amplitude of the wave is $A = 6.5\,\mathrm{mm}$.
   b. Its wavelength is $\lambda = 28\,\mathrm{cm}$.
   c. Its frequency is $f = T^{-1} = 27.8\,\mathrm{Hz}$.
   d. Its speed of propagation is $v = \lambda f = 7.784\,\mathrm{m\,s^{-1}}$.
   e. The direction of propagation is to the $+x$-direction, owing to the minus sign.

12. a. We'll rewrite Eq. 15.3: $y(x,t) = A\cos\left[\omega\left(\frac{x}{v} - t\right)\right]$. Then, we can rewrite $\omega = 2\pi/T = (2\pi/\lambda)v$ and $\omega/v = 2\pi f/\lambda f = 2\pi/\lambda$, giving us

$$y(x,t) = A\cos\left[\frac{2\pi}{\lambda}(x - vt)\right]. \tag{15.10}$$

   b. The transverse velocity v_y is simply the derivative of y wrt t:

$$\begin{aligned}\frac{\partial y(x,t)}{\partial t} &= v_y(x,t) \\ &= Akv\sin(kx - kvt)\end{aligned} \tag{15.11}$$

   The speed is greatest when $\sin(kx - kvt) = 1$, at which point $v_y = Akv$. $v_y = v \iff A = k^{-1}$, $< v \iff A < k^{-1}$ and $> v \iff A > k^{-1}$.

18. a. $v = \omega k^{-1} = 28.08\,\mathrm{m\,s^{-1}}$. It takes $t = 0.053\,\mathrm{s}$ to travel the entire length of the string.
   b. $\mu = mL^{-1} = 0.000\,85\,\mathrm{kg\,m^{-1}} \implies W = v^2\mu = 0.67\,\mathrm{N}$.
   c. $\lambda = 2\pi k^{-1} = 0.037\,\mathrm{m} \implies n = \frac{L}{\lambda} = 41.1$.
   d. The equation for waves traveling down the string is

$$y'(x,t) = (8.5\,\mathrm{mm})\cos(172\ \mathrm{rad/m}\ x + 4830\ \mathrm{rad/s}\ t). \tag{15.12}$$

22. a. $\mu = m/L = 0.003\,75\,\mathrm{kg\,m^{-1}}$, $\omega = 2\pi f = 753.98$ rad/s. Then the average power carried by the wave is $P_{\mathrm{av}} = 0.5\sqrt{\mu F}\omega^2 A^2 = 0.223\,\mathrm{W}$.
   b. The average power will decrease by a factor of 4 if the amplitude is halved.

24. If the wavelength is doubled, then the frequency, and thus the angular frequency will also be halved, and average power will be reduced to 25% $\implies P_2 = 0.1\,\mathrm{W}$.

44. a. The frequency of the sound produced by the string and the string are equal: $f = v/\lambda = 449.7\,\mathrm{Hz}$. Since the string is vibrating in its 2nd overtone, its wavelength is then $\lambda_s = \frac{2L}{3} = 0.5\,\mathrm{m} \implies v_s = \lambda_s f = 224.85\,\mathrm{m\,s^{-1}} \implies F = v_s^2\mu = 589.84\,\mathrm{N}$.
   b. The fundamental frequency of the string is $f_1 = f_3/3 = 149.9\,\mathrm{Hz}$.

45. a. $\lambda_1 = 2L = 1.27\,\text{m} \Longrightarrow v = \lambda f = 311.15\,\text{m}\,\text{s}^{-1}$.
    b. The new transverse speed would be $\sqrt{1.01}$ times that of the original, so the new fundamental frequency is $f_2 = \sqrt{1.01} f_1 = 246.22\,\text{Hz}$.
    c. The frequency of the air and guitar string are the same: $f = 245\,\text{Hz}$ and its velocity is $\lambda_a = v_a/f = 1.4\,\text{m}$.
63. a. $F = v^2\mu = 0.675\,\text{N}$ and $\omega = 2\pi\frac{v}{\lambda} = 942.5$ rad/s. Then, the amplitude of the wave is $A = \sqrt{\frac{2P}{\sqrt{\mu F}}\frac{1}{\omega}} = 7{,}07\,\text{cm}$.
    b. $P \sim v^3 \Longrightarrow$ Average power will be increased by a factor of 8 $\Longrightarrow A_2 = 400\,\text{W}$.
66. a. The wave speed is $v = \sqrt{\frac{F}{\mu}} = 491.93\,\text{m}\,\text{s}^{-1}$. $\lambda = 2L = 4.4\,\text{m}$ and the maximum transverse speed is $2A\pi v\lambda^{-1} \Longrightarrow A \approx 1.28\,\text{cm}$.
    b. The maximum transverse acceleration of the wave is $a_{\text{max}} = Ak^2v^2 = 6\,316\,\text{m}\,\text{s}^{-2}$.
71. For when the rock is submerged (or isn't), the frequency is calculated as follows:

$$\begin{aligned} f_1 &= \frac{1}{2L}\sqrt{\frac{\rho_1 gV}{\mu}} \\ f_2 &= \frac{1}{2L}\sqrt{\frac{(\rho_1 - \rho_2)gV}{\mu}} \end{aligned} \tag{15.13}$$

$\Longrightarrow \frac{f_1}{f_2} = \sqrt{\frac{\rho_1}{\rho_1 - \rho_2}} = \frac{42}{28} = 1.5 \Longrightarrow \rho_2 = 1\,778\,\text{kg}\,\text{m}^{-3}$.

16 Sound and Hearing

16.1 Summary

- *Sound* can be defined as longitudinal waves traveling through a medium, be it air, liquid, or solid.
- The typical human ear can hear sounds in the frequency range from 20 - 20000 Hz; this is called the *audible range*.
- The pressure caused by a sinusoidal sound wave can be written as $p(x,t) = BkA\sin(kx - \omega t)$, where $BkA = p_{\text{max}}$ is known as the *pressure amplitude*.
- Three main factors decide how we perceive sound: *loudness* (amplitude), *frequency* (how high the notes are), *timbre* (harmonic content).
- In a fluid, the speed of sound is

$$v = \sqrt{\frac{B}{\rho}}, \tag{16.1}$$

  where B is the bulk modulus of the fluid, and ρ its density.

- In a solid, the speed of sound is

$$v = \sqrt{\frac{Y}{\rho}}, \tag{16.2}$$

  where Y is the Young's modulus of the solid, and ρ its density.
- In an ideal gas, the speed of sound is

$$v = \sqrt{\frac{\gamma RT}{M}}, \tag{16.3}$$

  where γ is the ratio of heat capacities, R is the gas constant, T is the absolute temperature in Kelvin, and M is the molar mass of the gas.
- The intensity of a sound wave at a given point and time (x, t) is the average value of the product of its pressure and velocity: $I = \frac{1}{2}B\omega kA^2$

$$\Longrightarrow I = \frac{1}{2}\sqrt{\rho B}\omega^2 A^2. \tag{16.4}$$

- The *sound intensity level* is a logarithmic scale used to measure loudness to human ears: $\beta = (10\,\mathrm{dB})\log\frac{I}{I_0}$, where β is the intensity level (measured in decibels) and I_0 is the reference intensity, $10^{-12}\,\mathrm{W\,m^{-2}}$.
- A pressure node is always a displacement antinode and a pressure antinode is always a displacement node.
- An *open pipe* is open on both ends; in contrast, a *stopped pipe* is only open at one end.
- Open pipes create standing waves with n displacement nodes and $n + 1$ antinodes, including one on either end of the pipe, where n is the number of the nth harmonic. Its frequency is determined by

$$f_n = \frac{nv}{2L}, \tag{16.5}$$

  where $n = 1, 2, 3, \ldots.$
- Stopped pipes generate standing waves with an antinode at the open end and a node on the closed end; only odd harmonics are possible. The normal-mode frequencies are given by

$$f_n = \frac{nv}{4L}, \tag{16.6}$$

  where $n = 1, 3, 5, \ldots$
- Suppose there are two sound sources S_1, S_2 (speakers) and a point P. The interference of the waves at P is **constructive** when the distance traveled by them differ by a whole number of wavelengths and **destructive** when that difference is a half-integer number:

$$\begin{aligned} |d_2 - d_1| &= n\lambda \quad \text{(constructive interference)} \\ |d_2 - d_1| &= (2n+1)\frac{\lambda}{2} \quad \text{(destructive interference)} \end{aligned} \tag{16.7}$$

- *Beats* are variations of loudness caused by the amplitude variations of two waves which have the same amplitude but slightly different frequency, and the frequency with which it varies is called the *beat frequency*.

$$f_{\text{beat}} = f_a - f_b. \tag{16.8}$$

- *The Doppler effect* occurs when a sound source and the listener are moving relative to each other. For a moving listener and a stationary source, the frequency to the listener is

$$f_L = \left(1 + \frac{v_L}{v}\right) f_S. \tag{16.9}$$

  When both the listener and the source are moving, however, the frequency relative to the listener is then

$$f_L = \frac{v + v_L}{v + v_S} f_S. \tag{16.10}$$

- The angle of the shockwaves produced by a supersonic sound source is determined by $\sin\alpha = \dfrac{v}{v_S}$.

16.2 Exercises.

3. We have $B = 1.42 \cdot 10^5$ Pa, $A = 2 \cdot 10^{-5}$ m and $p_{\max} = BkA = BA\dfrac{2\pi}{\lambda} = BA\dfrac{2\pi f}{v}$.

   a. $p_{\max} = 7.78$ Pa.

   b. $p_{\max} = 77.8$ Pa.

   c. $p_{\max} = 778$ Pa.

10.

$$\begin{aligned} v &= \sqrt{\frac{\gamma RT}{M}} \\ \implies v^2 &= \frac{\gamma RT}{M} \\ \implies 2v\,\mathrm{d}v &= \gamma RM^{-1}\,\mathrm{d}T \\ \implies \frac{\mathrm{d}v}{v} &= \frac{1}{2}\frac{\mathrm{d}T}{T} \end{aligned} \tag{16.11}$$

15. $\log\dfrac{I_2}{I_1} = \dfrac{60 - 20}{10} = 4 = \log\dfrac{r_1^2}{r_2^2} \implies r_2 = 15$ cm.

17. a. $f_1 = v/2L = 524$ Hz $\implies L = 0.328$ m.

   b. $\lambda_2 = 4L = 12.19$ m. The new fundamental frequency is then $f_{1_2} = 0.5 f_{1_1} = 262$ Hz.

36. The wavelength is $\lambda = v/f = 1.67\,\text{Hz}$. $\Delta l = L - 2x \iff x = \dfrac{L - \Delta l}{2}$. For points of constructive interference, $\Delta l = \lambda/2$ and for destructive interference, $\Delta l = (n + 0.5)\lambda$.
   a. $n = 0 \implies \Delta l = 0.835\,\text{m}$; $x = 0.58\,\text{m}$. $n = -1 \implies \Delta l = -0.835\,\text{m}$; $x = 1.42\,\text{m}$. No other values of n place P between the speakers.
   b. $n = 0 \implies \Delta l = 0.835\,\text{m}$; $x = 0.58\,\text{m}$. $n = -1 \implies \Delta l = -0.835\,\text{m}$; $x = 1.42\,\text{m}$. No other values of n place P between the speakers.
   c. Treating the speakers as points are a poor approximation, and the sound travels to the points after bouncing off of walls or other nearby objects.

45. Let the (+x) direction point to the source. $f_S = 1\,200\,\text{Hz}$, $f_L = 1\,240\,\text{Hz}$, $v_S = -25\,\text{m}\,\text{s}^{-1}$, $v_L = 0$. Then, $v = 775\,\text{m}\,\text{s}^{-1}$. The speed of sound on Arrakis is $775\,\text{m}\,\text{s}^{-1}$.

55. a. We first treat the car as the source and then as the observer.
   First, $f_L = \left(\dfrac{v + v_L}{v + v_S}\right) f_S = \left(\dfrac{v + v_c}{v}\right) f_S$. Then, treating the stationary car as the observer gives us

$$1\,250\,\text{Hz} = \left(\frac{v}{v - v_c}\right)\left(\frac{v + v_c}{v}\right) 1\,200\,\text{Hz} \qquad (16.12)$$
$$\implies v_c = 7.02\,\text{m}\,\text{s}^{-1}.$$

   a. We repeat the calculations in (a), only with $v_p = 20\,\text{m}\,\text{s}^{-1}$.
   Waves reflected by the car: $f_L = \dfrac{v + v_c}{v - v_p} f_S = 1\,300\,\text{Hz}$
   Waves received by the police car: $f_L = \dfrac{v + v_P}{v - v_c} f_S = 1\,404\,\text{Hz}$.

56. $\sqrt{\dfrac{c + v}{c - v}} = 1.1 \implies v = 0.095 \times 10^{-3}\, c = 2.85 \cdot 10^{7}\,\text{m}\,\text{s}^{-1}$.

57. a. $\alpha = \arcsin \dfrac{v}{v_s} \approx 36°$.
   b.
   c. $t = \dfrac{s}{v} = \dfrac{h}{\tan 36° v} = 2.94\,\text{s}$.

64. a. $f = \dfrac{v}{4L} = 349\,\text{Hz} \implies 24.64\,\text{cm}$.
   b. $\dfrac{f_1}{f_2} = \dfrac{349}{370} = \sqrt{\dfrac{T_1}{T_2}} \implies T_2 = 56.34℃$.

65. We have $\Delta d = d_2 - d_1 = \sqrt{x^2 + 4} - x$ and $\lambda = 0.438\,\text{m}$. For points of constructive interference, $\Delta d = \sqrt{x^2 + 4} - x = \beta\lambda$ ($\beta \in \mathbb{Z}_+$) and for destructive interference, $\Delta d = \sqrt{x^2 + 4} - x = \beta\dfrac{\lambda}{2}$, where β is an odd multiple of 0.5. In either case, $x = \dfrac{h^2}{2\beta\lambda} - \dfrac{\beta}{2}\lambda$.

a. Destructive interference occurs at distances 9.01 m, 2.71 m, 1.27 m, 0.53 m, 0.026 m.
b. Constructive interference occurs at 4.34 m, 1.84 m, 0.86 m, 0.26 m.

62. Applying equilibrium conditions gives us $T_A = 128.75\,\mathrm{N}$, $T_B = 221.25\,\mathrm{N}$. Then $f_A = 88.33\,\mathrm{Hz}$, $f_B = 115.8\,\mathrm{Hz} \implies f_{\text{beat}} = 27.47\,\mathrm{Hz}$.

76. a. Here, k' is the change of force per length change, so the force change per fractional length change is $k'L$. The force applied at one end is $F = k'L\frac{v_y}{v}$, and the longitudinal impulse by that force over a time t is Ft. The change in linear momentum is $\frac{mvt}{Lv_y}$. Equating the impulse and linear momentum equations, we get $v^2 = L^2k/m$.

Part III - Thermodynamics

17 Temperature

17.1 Summary

- *Thermal equilibrium* in a system is achieved when no interaction between its elements can cause change to the system.
- 0^{th} law of thermodynamics: Given three systems A, B, C, if C is in thermal equilibrium with both A and B, then A and B are also in thermal equilibrium to each other.
- Two systems are in thermal equilibrium iff they have the same temperature.
- $T_F = \frac{9}{5}T_C + 32°$; $T_C = \frac{5}{9}(T_F - 32°)$.:
- The definition of the Kelvin scale: The ratio of two temperatures in kelvins equals the ratio of the corresponding pressures in a constant-volume gas thermometer.

$$T_2/T_1 = p_2/p_1. \tag{17.1}$$

  To complete the definition of T, the Kelvin temperature of a single state is used: the triple point of water, a state where ice, liquid water, and water vapor can all coexist. It occurs at a temp. of 0.01℃ and a water-vapor pressure of 610 Pa.
- When an object experiences temperature change, it goes through thermal expansion. If said object is a rod, the expansion is *linear*, and the new length is expressed as

$$L = L_0 + \Delta L = L_0(1 + \alpha\Delta T), \tag{17.2}$$

where α is the **coefficient** of linear expansion, L_0 is the original length, and ΔT is the change in temperature. If the object is a solid object, then V, V_0 replace L, L_0 and *volume expansion* happens:

$$\Delta V = \beta V_0 \Delta T, \tag{17.3}$$

where β is the coefficient of volume expansion. **Note**: $\beta = 3\alpha$.

- Should one clamp down the ends of a rod to prevent thermal expansion, *thermal stresses* develop as a result of the rod not being able to contract or expand. The thermal stress is given by

$$F/A = -Y\alpha\Delta T, \tag{17.4}$$

  where F is the force needed to keep the length constant, Y is its Young's modulus.
- *Heat flow (or heat transfer)* is the process of exchanging energy between two objects due to the difference of temperature between them.
- A *calorie* is the amount of heat required to raise the temperature of 1 g of water from 14.5℃ to 15.5℃ and 1 cal = 4.186 J.
- Heat needed to change temperature of a certain mass: $\mathrm{d}Q = mc\,\mathrm{d}T$. c is the *specific heat* of the material in question, which represents how much energy it takes to heat up 1 gram of that substance by 1℃.
- Sometimes it's more convenient to calculate the heat with the number of moles n that it has. Letting $m = nM$, where M is the molar mass of the object, we obtain $Q = nC\Delta T$, where $C = Mc$ is called the *molar heat capacity* can be expressed as $C = \frac{1}{n}\frac{\mathrm{d}Q}{\mathrm{d}T} = Mc$.
- Phases are specific states of matter, i.e. steam and ice are phases of water.
- In general, given the latent heat L needed to change phase for mass m of a certain material, the heat transfer is

$$Q = \pm mL. \tag{17.5}$$

  Q is positive when heat is entering and negative when it's leaving.
- Heat transfers through three means: conduction, convection and radiation.
  - *Conduction* occurs within a body or between two bodies.

    In conduction (say, within a rod), a quantity of heat $\mathrm{d}Q$ is transferred in a time $\mathrm{d}t$, so the rate of heat flow is $\frac{\mathrm{d}Q}{\mathrm{d}t}$. This rate is called the heat current, denoted by H. Given the thermal conductivity of the material k and the temperatures of the ends of the rod along with its length L, we get

$$H = \frac{\mathrm{d}Q}{\mathrm{d}t} = kA\frac{T_1 - T_2}{L}. \tag{17.6}$$

The quantity $(T_1 - T_2)/L$ is called the magnitude of the temperature gradient.

- *Convection* is the transfer of heat from one area to another due to the movement of fluids. If the fluid is moved with a pump or blower, it's a *forced convection*; if the movement is caused by the differences in density due to thermal expansion, the *convection* is *free*.
- *Radiation* is heat transferred by means of electromagnetic waves. The rate of radiation is $\propto T^4, A$. It also depends on the emissivity e of the body in question, where $0 \leq e \leq 1$.

  The heat current in radiation is

  $$H = Ae\sigma T^4. \tag{17.7}$$

  This relationship is called the **Stefan-Boltzmann law** and σ is called the **Stefan-Boltzmann constant**; its current numerical value is

  $$\sigma = 5.670\,373\,21 \cdot 10^{-8}\,\mathrm{W\,m^{-2}\,K^4}. \tag{17.8}$$

  – In most cases that we'll be considering, objects don't radiate their heat in a vacuum; its surroundings at temperature T_s are also radiating, whose heat the object absorbs. Then the Stefan-Boltzmann equation becomes $H_{\text{net}} = Ae\sigma\left(T^4 - T_s^4\right)$, where T is the absolute temperature of the body in question.

17.2 Exercises

5. a. The temperature change is also -10℃.
   b. In °F, the temperature change is $1.8\Delta T_C = -18$℉.
16. The increase in volume of the sphere is $\Delta V = \beta V_0 \Delta T \approx 313.61\,\mathrm{m^3}$.
17. We have $\Delta V_m - \Delta V_g = V_0\Delta T\left(\beta_m - \beta_g\right) = 8.95\,\mathrm{cm^3} \Longleftrightarrow \beta_g \approx 1.73\,\mathrm{K^{-1}}$. The glass has coefficient of volume expansion of $1.73\,\mathrm{K^{-1}}$.
22. The force needed to prevent the rod from contracting is $F = -YA\alpha\Delta T \approx 39\,810\,\mathrm{N}$.
23. The amount of heat to be added to raise the temperature to 85℃ is $Q = \Delta T(m_a c_a + m_w c_w) = 512\,580\,\mathrm{J}$.
34. Let the mass of the water that's originally in the beaker and that of the amount needed be m_1 and m_2, respc. Then,

    $$\begin{aligned} c_w(m_1\Delta T_1 + m_2\Delta T_2) &= 0 \\ \Longleftrightarrow m_2 &= -\frac{\Delta T_1}{\Delta T_2}m_1 = 1\,950\,\mathrm{g}. \end{aligned} \tag{17.9}$$

35. a. The specific heat of the metal satisfies

$$m_w c_w \Delta T_1 + m_m c_m \Delta T_2 = 0 \iff c_m = 214.87\,\mathrm{J\,kg^{-1}\,K}. \tag{17.10}$$

b. An equal weight of water will be more useful for storing heat as its specific heat is higher, therefore storing more heat.
c. If the amount of heat absorbed by Styrofoam is not negligible, the true specific heat c_m will be larger.

67. Suppose the pot's walls are thin so that the temperature at the walls is nearly the same as the water it contains. $R = \sqrt{0.75V\pi} = 0.056\,36\,\mathrm{m}$, $A = 0.04\,\mathrm{m}^2$. Then the heat current is $H = Ae\sigma\left(T^4 - T_s^4\right) \approx 15\,\mathrm{W}$.

68. The amount of power needed to maintain the tungsten sphere at $3\,000\,\mathrm{K}$ is the amount of heat loss to the environment: $H = Ae\sigma\left(T^4 - T_s^4\right) \approx 8.015 \cdot 10^{10}\,\mathrm{W}$.

85. $\mathrm{d}Q = nC\,\mathrm{d}T = nk\dfrac{T^3}{\theta^3}\,\mathrm{d}T \implies Q = \displaystyle\int_{T_1}^{T_2} nk\frac{T^3}{\theta^3}\,\mathrm{d}T = \frac{nk}{4\theta^3}\left(T_2^4 - T_1^4\right)$

a. Applying the above equation, we have $Q = 83.61\,\mathrm{W}$.
b. $C_{\text{av}} = \dfrac{1}{n}\dfrac{\Delta Q}{\Delta T} = 1.858\ \mathrm{J/mol} \cdot K$
c. $C = k\dfrac{T^3}{\theta^3} = 5.596\ \mathrm{J/mol} \cdot K$.

88. $Q = \displaystyle\int_{300}^{500} 3\left(29.5 + 8.20 \cdot 10^{-3}T\right)\mathrm{d}T = 6\,556\,\mathrm{W}$.

79. a. If the length of the rod is allowed to change by an amount ΔT, then the total length change is the sum of the change due to both tension and heat:

$$\begin{aligned} \frac{FL_0}{AY} + \alpha L_0 \Delta T &= \Delta L \\ \implies \frac{F}{A} &= Y\left(\frac{\Delta L}{L_0} - \alpha\Delta T\right). \end{aligned} \tag{17.11}$$

b. Since the brass bar is heavy and the wires are fine, we can assume that the stress by the wires on the bar due to thermal expansion is insignificant. This means that the change of length ΔL is not zero, but is in fact the amount of $\alpha_b L_0 \Delta T$:

$$F/A = Y_s(\alpha_b - \alpha_s)\Delta T = 1.2 \cdot 10^8\,\mathrm{Pa}. \tag{17.12}$$

103. First, we notice that the rod has the same heat current over all of its sections. $k_B = 109\,\mathrm{W\,m^{-1}\,K}$, $k_C = 385\,\mathrm{W\,m^{-1}\,K}$, $k_A = 205\,\mathrm{W\,m^{-1}\,K}$. Then,
$H = k_B\dfrac{100 - T_1}{12} = k_C\dfrac{T_1 - T_2}{18} = k_A\dfrac{T_2}{24} \implies T_1 \approx 59.809℃, T_2 \approx 42.740℃ \implies \dfrac{\mathrm{d}Q}{\mathrm{d}t} = 8.4\,\mathrm{W}$.

71. $l = 10.5\,\mathrm{m}$, $R = 0.175\,\mathrm{m}$. The temperature increase ΔT is such that $\alpha_s L_0 \Delta T + \alpha_b R_0 \Delta T = 2 \cdot 10^{-3}\,\mathrm{m} \implies T \approx 35.44℃$.

73. a. We have 0 deg M = −39℃, 100 deg M = 357℃ $\Longrightarrow$ 100 deg M = 396℃ $\Longrightarrow$ 1 deg M = 3.96℃.

    0 deg M = −39℃, so $T_M = \frac{1}{3.96}(T_c + 39)$.

    a. A 10 deg M change corresponds to $3.96 \cdot 10 = 39.6$℃.

78. The maximum temperature is 4℃ + $\frac{106 - 103.4}{106 \cdot 9.5 \cdot 10^{-4}} \approx 32$℃. To have the maximum amount of fuel, she should have filled the tanks right before flying.

53. $0.25 \cdot 4190(40 - 75) + m(2100 \cdot 20 + 334000 + 4190 \cdot 40) = 0 \Longleftrightarrow m \approx 0.0674$ kg

Remark

The ice first melts into water, absorbs a large amount of heat to change phase, and then heats up to 40℃.

84. a. Suppose that the normal melting point of iron is above 745℃, so the iron initially is solid.

    $Q_s = 0$. The heat released by the iron slug when it's cooled to 100℃ is $Q_i = 3.03 \cdot 10^4$ J; the heat absorbed by the water to get to that temperature is $Q_w = 2.85 \cdot 10^4$ J, which leaves $1.81 \cdot 10^3$ J to convert some mass m of the water to steam. Then the mass of steam is $m = QL^{-1} = 0.801$ g.

    b. The final temperature of the water is 100℃.

    The final mass of the iron is still the same: 100 g, while the remaining water has mass 84.2 g.

18 Thermal Properties of Matter

18.1 Summary

- An *ideal gas* is one where the *ideal-gas law* (aka the Clapeyron-Mendeleev equation) is obeyed:

  $$\boxed{pV = nRT}$$

  where R is the **gas constant**, which is equal to 8.314 J mol^{-1} K (accurate to 4 decimal places).

- Most gasses, however, aren't ideal. To approximate their behavior more correctly, the *van der Waals* equation is used, which takes into account intermolecular forces within the gas:

  $$\left(p + \frac{an^2}{V^2}\right)(V - nb) = nRT \tag{18.1}$$

The variables a and b depend on the gas in question. b approximately represents the volume of a mole of gas, and a depends on the attractive intermolecular forces in the gas.

- The *kinetic-molecular model* of an ideal gas assumes 4 axioms:
  1. A container with volume V contains a very large number of identical molecules N, each with mass m.
  2. The size of each molecule is miniscule relative to the size of the container and the average distance between them.
  3. The particles in such a gas is in constant motion.
  4. The walls of the container are indefinitely massive, rigid and don't move.
- The pressure exerted by molecules of an ideal gas on the sides of its containers is dependent on the number of molecules, the mass of each molecule and their speed.

$$\boxed{p = F/A = \frac{Nmv_x^2}{V}} \tag{18.2}$$

- Average translational kinetic energy in a gas:

$$\boxed{K_{\text{tr}} = \frac{3}{2}nRT} \tag{18.3}$$

To derive the kinetic energy of a single molecule, we divide K_{tr} by the number of molecules N:

$$\begin{aligned} \frac{K_{\text{tr}}}{N} &= \frac{3}{2}\frac{n}{N}RT \\ &= \frac{3}{2}\frac{R}{N_A}T \\ &= \frac{3}{2}kT, \end{aligned} \tag{18.4}$$

where $k = \dfrac{R}{N_A}$ is the **Boltzmann constant**, whose value is

$$k = 1.381 \cdot 10^{-23}\ \text{J/molecule} \cdot K. \tag{18.5}$$

- Root-mean-square speed of a molecule:

$$\boxed{v_{\text{rms}} = \sqrt{v_{\text{av}}^2} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}},} \tag{18.6}$$

where M is the molar mass of a gas and m the mass of one molecule. root

- The average distance traveled between collisions of a molecule is called the *mean free path*:

$$\boxed{\lambda = vt_{\text{mean}} = \frac{V}{4\sqrt{2}\pi r^2 N}} \quad (18.7)$$

- Ideal gas law on a "per-molecule" basis: $pV = nkT$, where k is the Boltzmann constant and n the number of moles.
- The principle *equipartition of energy* states that each velocity component, linear or angular, has, on average, an associated kinetic energy of $\frac{1}{2}kT$. The number of velocity components needed to fully describe the motion of a molecule or atom is called its *degree of freedom*.
  - A **monatomic** gas has 3 translational components in all three axes, so its molar heat capacity (i.e. its average kinetic energy) is $\frac{3}{2}R$.
  - A **diatomic** gas has two more rotational axes, so its m.h.c. is $\frac{5}{2}R$.
- Molar heat capacity of a solid: $C_V = 3R$.
- Average speed of molecules in a gas given its temperature T: $v_{\text{av}} = \int_0^\infty v f(v)\,\mathrm{d}v$.
- The rms velocity is simply the square root of the average of v: $v_{\text{rms}} = \sqrt{\int_0^\infty v^2 f(v)\,\mathrm{d}v}$.
- The Maxwell-Boltzmann distribution: $f(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}$.

  Letting the kinetic energy of a molecule be $\epsilon = \frac{mv^2}{2}$, we obtain

$$f(\epsilon) = \frac{8\pi}{m}\left(\frac{m}{2\pi kT}\right)^{3/2} \epsilon e^{-\epsilon/KT} \quad (18.8)$$

18.2 Exercises

1. $V = 20\,\text{L}$, $m = 4.86 \cdot 10^{-4}\,\text{kg}$, $M = 4\,\text{g}\,\text{mol}^{-1}$, $T = 18℃$
   a. $n = \frac{4.86 \cdot 10^{-4}}{4 \cdot 10^{-3}} = 0.121\,5\,\text{mol}$
   b. $p = nRTV^{-1} = 14\,698\,\text{Pa} = 0.145\,\text{atm}$
2. $V_1 = 3.2\,\text{L}$, $p_1 = 0.18\,\text{atm}$, $T_1 = 314\,\text{K}$, $V_2 = 6.4\,\text{L}$, $p_2 = 0.36\,\text{atm}$.
   a. $T_2 = 4T_1 = 983℃$.
   b. $n = p_1V_1RT_1^{-1} \approx 0.223\,\text{mol}$
7. $T_2 = T_1\frac{p_2V_2}{p_1V_1} \approx= 748\,\text{K} = 475℃$

8. a. $n_1 = \dfrac{pV}{rT} \approx 11.7\,\text{mol} \Longrightarrow m = 374.4\,\text{g}$.

   b. $n_2 = \dfrac{p_2V}{r_2T_2} \approx 8.59\,\text{mol} \Longrightarrow \Delta m = m_2 - m_1 = M(n_2 - n_1) = 99.52\,\text{g}$.

> **Remark**
>
> Remember that we need to use **absolute pressure** when using $pV = nRT$.

9. $V_1 = 0.75\,\text{m}^3$, $V_2 = 0.41\,\text{m}^3$, $p_1 = 7.5 \cdot 10^3\,\text{Pa}$, $T_1 = 300\,\text{K}$, $T_2 = 430\,\text{K}$.

   $p_2 = \dfrac{V_1T_2}{V_1T_2} = 1.97 \cdot 10^4\,\text{Pa}$.

24. a. $N/V = 8 \cdot 10^7, T = 7\,500\,\text{K}$

   $\Longrightarrow p = \dfrac{nRT}{V} = \dfrac{N}{V}\dfrac{R}{N_A}T = 8.28 \cdot 10^{-12}\,\text{Pa}$.

   a. The depiction of turbulence in sci-fi movies isn't accurate because the pressure in nebulae is too small to truly affect the ship.

25. $p = 1$ atm, $T = 273\,\text{K}$, $N = 7 \cdot 10^9$.

   $V = \dfrac{NkT}{p} = 2.613 \cdot 10^{-16}\,\text{m}^3 \Longrightarrow V = L^{1/3} \approx 6.4 \cdot 10^{-6}\,\text{m}$

34. $p = 3.545\,5\,\text{Pa}, r = 2 \cdot 10^{-10}\,\text{m}, T = 300\,\text{K}$

   $\Longrightarrow \lambda = \dfrac{V}{4\sqrt{2}\pi r^2 N} = \dfrac{kT}{4\sqrt{2}\pi r^2 p} \approx 164\,381\,\text{m}$

35. $T_N = ?, M_H = 2\,\text{g}\,\text{mol}^{-1}, M_N = 28\,\text{g}\,\text{mol}^{-1}, T_H = 293\,\text{K}$

   $v_{\text{rms}_H} = \sqrt{\dfrac{3RT_H}{M_H}} \approx 1\,912\,\text{m}\,\text{s}^{-1} \Longrightarrow T_N = \dfrac{v^2M_N}{3R} \approx 4\,103.95\,\text{K} = 3\,800\,℃$.

38. $\Delta T_1 = 2.5℃, Q = 300\,\text{J}$.

   $C_1 = 1.5R, C_2 = 2.5R, Q_1 = Q_2 \Longrightarrow \Delta T_2 = (C_2/C_1)\Delta T_1 \approx 4.17℃$.

42. If 94.7 % of all molecules have a speed less than v, then

$$v_{\text{rms}} = 0.625v \Longrightarrow T = 0.390625\frac{Mv^2}{3R}. \tag{18.9}$$

   Applying it to $v = 1\,500\,\text{m}\,\text{s}^{-1}, 1\,000\,\text{m}\,\text{s}^{-1}, 500\,\text{m}\,\text{s}^{-1}$, we obtain $T = 987\,\text{K}, 438\,\text{K}, 110\,\text{K}$.

45. 1 mol of liquid water has mass 18 g $\Longrightarrow$ Its volume @ 20℃ is $V = mp^{-1} \approx 1.804\,\text{m}^3$, and the ratio of that volume to 1 mol of water at the critical point point is $V/V_{\text{cp}} \approx 0.322$.

50. $m_1 = 3\,\text{kg}, m_2 = 12\,\text{kg}, h_1 = 1\,\text{m}$.

Since temperature is constant and no gas leaks out of the tank, $p_1V_1 = p_2V_2$.
$\Longrightarrow \dfrac{p_1}{p_2} = \dfrac{V_2}{V_1} = \dfrac{h_2}{h_1} = \dfrac{m_1}{m_2} = \dfrac{1}{4} \Longrightarrow h_2 = 1\,\mathrm{m}$.

51. $T = \text{const} = 295\,\mathrm{K}$, $p_1 = 1.3 \cdot 10^6\,\mathrm{Pa}$, $p_2 = 3.4 \cdot 10^5\,\mathrm{Pa}$, $V = \text{const} = \pi r^2 h = 0.011\,\mathrm{m^3}$, $M = 44.1\,\mathrm{g\,mol^{-1}}$.
$n_1 = \dfrac{p_1V}{RT}, n_2 = \dfrac{p_2V}{RT} \Longrightarrow n_2 = \dfrac{p_2}{p_1}n_1 \Longrightarrow \Delta m = \dfrac{p_1 - p_2}{p_1}Mn_1 = \dfrac{V(p_1 - p_2)}{RT}M \approx 0.19\,\mathrm{kg}$

75. $$\int_0^\infty v^2 f(v)\,\mathrm{d}v = \int_0^\infty v^2 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}\,\mathrm{d}v$$
$$= 4\pi\left(\frac{m}{2\pi kT}\right)^{1.5} \int_0^\infty v^4 e^{-mv^2/2kT}\,\mathrm{d}v$$
$$= 4\pi\left(\frac{m}{2\pi kT}\right)^{1.5} \frac{12k^2T^2}{8m^2}\frac{2\pi kT}{m} = 3kT/m = (v_{\mathrm{av}}^2).$$

78. $$\int_0^\infty v f(v)\,\mathrm{d}v = \int_0^\infty v 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}\,\mathrm{d}v$$
$$= 4\pi\left(\frac{m}{2\pi kT}\right)^{1.5} \int_0^\infty v^3 e^{-mv^2/2kT}\,\mathrm{d}v$$
$$= 2\pi\left(\frac{m}{2\pi kT}\right)^{1.5} \frac{1}{m^2/4k^2T^2} = \sqrt{\frac{8kT}{\pi m}} = v_{\mathrm{av}}.$$

Problem 18.49 ( )

A hot-air balloon stays aloft because hot air at atmospheric pressure is less dense than cooler air at the same pressure. If the volume of the balloon is $500\,\mathrm{m^3}$ and the surrounding air is at 15℃, what must the temperature of the air in the balloon be for it to lift a total load of 290 kg (in addition to the mass of the hot air)? The density of air at 15℃ and atmospheric pressure is $1.23\,\mathrm{kg\,m^{-3}}$.

Solution: The buoyancy force must be able to support the weight of the hot air inside of the balloon and and the weight of the load: $F_B/g = 290\,\mathrm{kg} + \rho_h V = \rho_a V$

$\Longrightarrow \rho_h = 0.65\,\mathrm{kg\,m^{-3}}$. Applying the ideal gas equation for T_h and T_a while noticing that pressure, molar count and volume stay constant, we obtain $T_h\rho_h = T_a\rho_a \Longrightarrow T_h = 545\,\mathrm{K} = 272$℃.

Problem 18.79 ( )

A vertical cylinder of radius r contains an ideal gas and is fitted with a piston of mass m that's free to move. The piston and the walls of the cylinder are frictionless, and the entire cylinder is placed in an const-T bath. The outside air pressure is p_0. In cylinder, (a)

Problem 18.84 (Earth's Atmosphere -)

In the *troposphere*, the part of the atmosphere that extend from the Earth's surface to an altitude of about 11 km, the temperature is not uniform but decreases with increasing elevation.

a. Show that if the temperature variation is approximated by the linear relationship

$$T = T_0 - \alpha y \tag{18.10}$$

where T_0 is the temperature at the Earth's surface and T the temperature at height y, the pressure p at height y is

$$\ln\left(\frac{p}{p_0}\right) = \frac{Mg}{R\alpha}\ln\left(\frac{T_0 - \alpha y}{T_0}\right), \tag{18.11}$$

where p_0 is the pressure at the Earth's surface and M is the molar mass for air. The coefficient α is called the lapse rate of temperature. It varies with atmospheric conditions, but an average value is about 0.6℃/100 m.

b. Show that above result reduces to the result of Ex. 18.4 (Sect. 18.1) in the limit of $\alpha \to 0$.

c. With $\alpha = 0.6$℃/100 m, calculate p for $y = 8\,863$ m and compare your answer to the result of Ex. 18.4. Take $T_0 = 288$ K and $p_0 = 1$ atm.

Solution:

a. First, we'll realize that

$$\begin{aligned} \rho &= \frac{m}{V} \\ &= \frac{pm}{nRT} = \frac{pMn}{nRT} \\ &= \frac{pM}{RT} \end{aligned} \tag{18.12}$$

and $\mathrm{d}p = -\rho g\,\mathrm{d}y$. Thus

$$
\begin{aligned}
\frac{\mathrm{d}p}{p} &= \frac{Mg}{R(-T_0 + \alpha y)}\,\mathrm{d}y \\
\Longrightarrow \int_{p_0}^{p} \frac{1}{p}\,\mathrm{d}p &= \int_0^y \frac{Mg}{R(\alpha y - T_0)}\,\mathrm{d}y \\
\Longleftrightarrow \ln\left(\frac{p}{p_0}\right) &= \frac{Mg}{R\alpha}\ln\left(\frac{T_0 - \alpha y}{T_0}\right)
\end{aligned}
\tag{18.13}
$$

b. As $\alpha \to 0$, $\ln\left(1 - \frac{\alpha y}{T_0}\right) \to -\frac{\alpha y}{T_0} \Longrightarrow p = p_0 e^{-Mgy/RT_0}$.

c. $p \approx 0.315$ atm.

19 The First Law of Thermodynamics

19.1 Summary

- *Thermodynamic systems* are collections of objects that can be regarded as one unit and are capable of exchanging energy with its surroundings; processes that change the state of the thermodynamic system are called *thermodynamic processes*.
- Signing conventions of the heat Q added to a system and the work W done by it:
  - $Q > 0$ when there's heat flow *into* a system and $Q < 0$ when heat flows out of it.
  - $W > 0$ when work is done by the system and $W < 0$ when there is work done on it by its surroundings.
- Work done in a volume change:

$$\boxed{\mathrm{d}W = p\,\mathrm{d}V\,. \tag{19.1}}$$

- Heat and work in thermodynamic processes depend not only on the initial and final states but also on the path.
- *Internal energy* of a system is defined as the sum total of the kinetic energies of its particles and the potential energies of interaction between those particles.
- *The first law of thermodynamics* states that any amount of heat Q entering a system remains partially as the change of internal energy ΔU of that system; the remainder goes to doing some work W:

$$\boxed{\Delta U = Q - W \Longleftrightarrow Q = \Delta U + W \tag{19.2}}$$

One can derive the first law for infinitesimal processes:

$$\boxed{\mathrm{d}U = \mathrm{d}Q - p\,\mathrm{d}V} \tag{19.3}$$

- Two cases of this law are worth mentioning. The first one is a *cyclic process* that returns to its initial state:

$$\boxed{U_2 = U_1 \Longrightarrow Q = W \text{(cyclic process)}} \tag{19.4}$$

- The second is an *isolated system*, where it neither exchanges heat nor do any work.

$$\boxed{W = Q = 0 \Longrightarrow \Delta Q = W \text{(isolated system)}} \tag{19.5}$$

- There are four common kinds of thermodynamic processes worth taking into consideration.
  - An **adiabatic** process is one where there's no heat transfer into or out of a system:

$$Q = 0 \quad \text{(adiabatic)} \tag{19.6}$$

  - An **isochoric** process has constant volume, which does no work on its surroundings: $W = 0$. Thus

$$Q = \Delta U \quad \text{(isochoric)} \tag{19.7}$$

  - An *isobaric* process is a constant-pressure process. In such processes,

$$W = \int_{V_1}^{V_2} p\,\mathrm{d}V \quad \text{(isobaric)} \tag{19.8}$$

  - An **isothermic** process has constant temperature.
- The internal energy U of an ideal gas depends only on its temperature T, not on its pressure or volume.
- The relationship between isobaric and isochoric heat capacities: Molar heat capacities at constant pressure will always be greater than that of constant volume because additional work W is done when the gas expands in the former case.

$$\boxed{C_p = C_V + R} \tag{19.9}$$

- The *ratio of specific heats*:

$$\boxed{\gamma = \frac{C_p}{C_V}} \tag{19.10}$$

- For an ideal gas, the internal energy change in *any* process is given by $\Delta U = nC_V\Delta T$ *irrespective of there being any volume change.*
- For an ideal gas in a adiabatic process, the relationship between the initial and final states (T_1, V_1) and (T_2, V_2) is given by

$$\boxed{TV^{\gamma-1} = \text{const}} \tag{19.11}$$

or, by letting $T = \dfrac{nRT}{p}$, we obtain

$$\boxed{pV^{\gamma} = \text{const.}} \tag{19.12}$$

- The work of an ideal gas in any adiabatic process is given by

$$\begin{aligned} W &= nC_V(T_1 - T_2) = \frac{C_V}{R}(p_1V_1 - p_2V_2) \\ &= \frac{1}{\gamma - 1}(p_1V_1 - p_2V_2) = -\Delta U. \end{aligned} \tag{19.13}$$

- Heat absorbed or released by an ideal gas: $Q = nC_p\Delta T$.

19.2 Exercises

1.

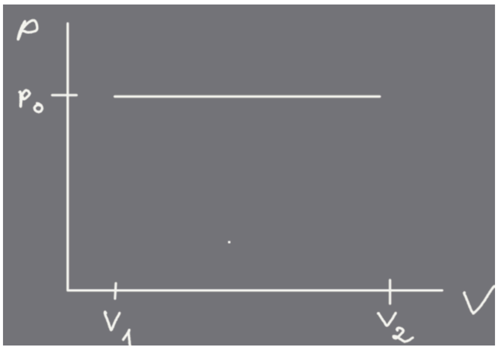
   a. 

   Figure 1: Problem 19.1

   b. $W = \displaystyle\int_{V_1}^{V_2} p\,\mathrm{d}V = \int_{T_1}^{T_2} nR\,\mathrm{d}T = 1\,330.24\ \mathrm{J}$

5.

   a. $n = 0.305\,\mathrm{mol}$, $T = 295\,\mathrm{K}$, $W = -392\ \mathrm{J}$, $p_2 = 1.76$ atm.
   $\dfrac{W}{nRT} = \ln\left(\dfrac{p_1}{p_2}\right) \Longleftrightarrow p_1 = p_2 e^{W/nRT} \Longrightarrow p_2 = 1.04$ atm.

   b.

6.

   a.

b. The first process is isochoric, so it does no work: $W_{ab} = 0$.

The second process is isobaric, so the total work is $W = W_{ab} + W_{bc} = p(V_c - V_b) = -40\,000$ J.

8.

a. $T_2 = 0.25T_1$, $p = 1.5$ atm, $V = 5 \cdot 10^{-3}$ m^3

$V_2 = 0.25V_1 = 0.125$ L.

b. $W = p(V_2 - V_1) \approx -57$ J. The work is negative, so the environment is doing work on the gas.

c. For an ideal gas, $Q = nC_p\Delta T$, T decreases so the heat leaves the gas from a to b.

d. $\Delta U = nC_V\Delta T$, so the internal energy of the gas decreases from a to b.

10. $n = 5$ mol, $C_V = 1.5R$, $C_p = 2.5R$, $Q = 1\,500$ J, $W = 2\,100$ J.

$$\Delta U = Q - W = -600\text{ J} \Longrightarrow \Delta T = \frac{\Delta U}{nC_V} - 80℃ \Longrightarrow T_2 = 47℃.$$

11. $n = 0.017\,5$ mol, $V_a = V_B = 2$ L, $V_c = 6$ L, $p_a = 0.2$ atm, $p_b = 0.5$ atm, $p_c = 0.3$ atm

a. Applying the ideal gas equation to all three points in the process abc, we obtain $T_a \approx 279$ K, $T_b \approx 696$ K, $T_c \approx 836$ K

The lowest T occurs when pV has its smallest values, which is at point a: $T_a = \frac{p_aV_a}{nR} = 278$ K.

a. Process ab is isochoric, so there's no work done here: $W_{ab} = 0$.

The work done by bc is positive since the gas expands and its magnitude is simply the area underneath the line bc: $W = W_{bc} = 162.08$ J.

b. $\Delta U = Q - W = 52.92$ J.

26. 5 mol, $C_V = 1.5R$, $C_p = 2.5R$, $p_1 = 2.5 \cdot 10^3$ Pa, $V_1 = 2.1$ m^3, $W = 1\,480$ J.

$p_1V_1^\gamma = p_2V_2^\gamma \Longleftrightarrow p_2 = p_1\left(\frac{V_1}{V_2}\right)^\gamma$. We only need to find V_2.

Since $pV^\gamma = \text{const}$, let that constant be K. Then

$$p = \frac{K}{V^\gamma} \Longrightarrow W = \int_{V_1}^{V_2} p\,\mathrm{d}V = \int_{V_1}^{V_2} \frac{K}{V^\gamma}\,\mathrm{d}V = \frac{K}{1-\gamma}\left(V_2^{1-\gamma} - V_1^{1-\gamma}\right)$$

$K = p_1V_1^\gamma \approx 8\,609$ N m^3 $\Longrightarrow V_2 = 2.869\,7$ m^3 $\Longrightarrow p_2 = 1\,490$ Pa.

33. $V_a = 0.07$ m^3, $V_b = 0.11$ m^3, $p_a = 1 \cdot 10^5$ Pa, $p_b = 1.4 \cdot 10^5$ Pa.

a. $pV = nRT$. In this process, both pressure and volume increase, which means temperature increases as well.

b. The work W done by the gas is the area below the process: $W = 0.5(100000 + 140000)(0.11 - 0.07) = 4\,800$ J.

45. $n = 2.5$ mol, $C_V = 20.76$ J mol^{-1} K, $C_p = 29.07$ J mol^{-1} K

a. $\Delta T = \dfrac{Q}{nC_V} = 262$ K, $T_a = 293$ K $\Longrightarrow T_b = 555$ K. When V doubles, so does T, and $T_c = 2T_b = 1\,110$ K $= 837$℃.

b. Process ab is isochoric, which means there is no work done, so the total work is done by the isobaric process bc: $W = p\Delta V = nR\Delta T = 11\,535.675$ J

c. For process bc, $Q = nC_p\Delta T = 40\,334.625$ J

d. $\Delta U = nC_V\Delta T = 2.5 \cdot 20.76(1110 - 293) \approx 42.4$ kJ

51. $C_V = 20.8$ J mol^{-1} K, $p_1 = 1.01 \cdot 10^5$ Pa, $p_2 = 3.8 \cdot 10^5$ Pa,
$C_V = 20.8$ J mol^{-1} K, $C_p = 29.141$ J mol^{-1} K $\Longrightarrow \gamma = 1.4$, $h = 0.25$ m, $n = 20$ mol

a. $p_1/p_2 = (h_1/h_2)^\gamma \Longrightarrow h_2 = 0.082$ m $\Longrightarrow \Delta h = 0.168$ m.

b. $T_2 = T_1(h_1/h_2)^{\gamma-1} = 468.7$ K $= 195$℃

c. $W = nC_V(T_1 - T_2) = -70\,179.2$ J

20 The Second Law of Thermodynamics

20.1 Summary

- All thermodynamic processes occurring in nature are *irreversible;* that is, they happen in one direction and not the other.
- Reversible processes are *equilibrium processes* with the system always in thermal equilibrium.
- *Heat engines* are devices that partly convert heat into work using a quantity of *working substance* which undergoes thermal processes, expansion and compression.
- The *thermal efficiency* is the ratio of work converted from heat input to original heat input: $e = \dfrac{W}{Q_H} = 1 + \dfrac{Q_C}{Q_H} = 1 - \left|\dfrac{Q_C}{Q_H}\right|$
- Most gas engines utilize an *Otto cycle.* The efficiency of such an engine is given by

$$\boxed{e = 1 - \frac{1}{r^{\gamma-1}}, \qquad (20.1)}$$

where r is the compression ratio of the engine and γ is the ratio of heat capacities.

- *Refrigerators* are heat engines working in reverse; they extract heat Q_C from a colder place and transfer it and discards heat $|Q_H|$ a at warmer place. The economical effectiveness of a refrigerator is measured by its coefficient of performance K, which is the ratio of heat extracted to the amount of work:

$$Q = \left|\frac{Q_C}{W}\right| = \frac{|Q_C|}{|Q_H - Q_C|}. \tag{20.2}$$

- **The second law of thermodynamics** states that it's impossible for any system to undergo a process in which it absorbs heat from a reservoir at a single temperature and converts the heat completely into mechanical work with the system ending in the same state in which it began. Another statement of this law is that it's impossible to for any process to have, as its sole result, the transfer of heat from a cooler to a hotter body. In other words: there can be no perfectly efficient heat engine or refrigerator.
- The Carnot engine consists of two reversible isothermal cycle and two reversible adiabatic cycles and the following steps:
  1. The gas expands isothermally at temperature T_H, absorbing heat Q_H.
  2. It expands adiabatically until its temperature drops to T_C.
  3. It is compressed isothermal at T_C, rejecting $|Q_C|$.
  4. It is compressed adiabatically back to its initial state at temperature T_H.
- Heat transfer in a Carnot engine: $\dfrac{Q_C}{Q_H} = -\dfrac{T_C}{T_H}$
- The efficiency of a Carnot engine depends only on the temperatures of the two heat reservoirs:

$$e_{\text{Carnot}} = 1 - \frac{T_C}{T_H} = \frac{T_H - T_C}{T_H} \tag{20.3}$$

- No engine can be more efficient than a Carnot engine operating between the same temperatures, due to its reversibility; thus all Carnot engines operating between the same two temperatures have the same efficiency irrespective of the nature of the working substance.
- *Entropy* is a quantitative measure of randomness whose symbol is S. The infinitesimal change dS during an infinitesimal reversible process change at absolute temperature dT is $\mathrm{d}S = \dfrac{\mathrm{d}Q}{T}$.
- Entropy change in a reversible process:

$$\Delta S = \int_1^2 \mathrm{d}Q\,/T \tag{20.4}$$

- Expression for entropy in microscopic terms: $S = k \ln w$.

20.2 Exercises

7. $r = 8.8, \gamma = 1.4$
   a. The ideal efficiency of the Mercedes engine is $e_1 = 58.1\%$.
   b. The Dodge Viper has an ideal efficiency of $e_2 = 59.53\%$. $e_2/e_1 = 1.019$, so the Dodge has a marginal efficiency increase of only 1.9%.

21 Electric Charge and Electric Field

21.1 Summary

21.2 Exercises

22 Gauss's Law

22.1 Summary

- The *electric flux* is the amount of flow of an electric field through a surface, and defined by

$$\varphi_e = \int \vec{\boldsymbol{E}}\,\mathrm{d}\vec{\boldsymbol{S}} = \int E\,\mathrm{d}S\cos\theta \tag{22.1}$$

22.2 Exercises

23 Electric Potential

23.1 Summary

- Electric potential energy is the work an electric field dows on a charged particle:

$$W_{a\longrightarrow b} = U_a - U_b. \tag{23.1}$$

- Electric potential is potential energy per unit charge:

$$\boxed{V = \int \frac{1}{4\pi\varepsilon_0 r}\,\mathrm{d}q \quad \text{(due to a charge distribution)}} \tag{23.2}$$

  or

$$\boxed{V_b - V_a = -\int_a^b \vec{\boldsymbol{E}}\,\mathrm{d}\vec{\boldsymbol{l}}.} \tag{23.3}$$

23.2 Exercises

24 Capacitance and Dielectrics

24.1 Summary

24.2 Exercises

25 Current, Resistance and Electromotive Force

25.1 Summary

- A *current* is any motion of charge from one region to another.
- Current through a cross-sectional area is defined by the net charge flowing through said area per unit of time: $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$.
- Vector current density: $\vec{J} = nq\vec{v}_d$, where $\vec{v}_d$ is the drift velocity vector, n is the concentration of moving charged particles.
- $I = \frac{\mathrm{d}Q}{\mathrm{d}t} = n|q|v_d A$.
- Ohm's law says that the *resistivity of a material* is $\rho = \frac{E}{J}$.
- For a conductor of resistivity ρ, the current density $\vec{J}$ at a point where the electric field is $\vec{E}$ is given by

$$\boxed{\vec{E} = \rho\vec{J}} \tag{25.1}$$

- A material that obeys Ohm's law *reasonably* well is called an *ohmnic* material; for such materials, ρ = const that does not depend on E. A *nonohmnic* material behaves otherwise, and J depends on E in a more complicated manner.
- Over a small temperature range (up to about 100℃), the resistivity of a metal can be approximated by

$$\rho(T) = \rho_0[1 + \alpha(T - T_0)] \tag{25.2}$$

- Resistance of a conductor: $R = \frac{\rho L}{A}$.
- The *electromotive force* is the influence that causes currents to flow against its natural tendency by making them flow from lower to higher potential.
- Real sources of emf have at least some internal resistance, so the effective potential difference is

$$\boxed{V_{ab} = \mathcal{E} - Ir,} \tag{25.3}$$

  where r is the internal resistance.
- Power delivered to a resistor:

$$P = V_{ab}I = I^2R = \frac{V_{ab}^2}{R} \tag{25.4}$$

- Resistivity of a metal: $\rho = \frac{m}{ne^2\tau}$

25.2 Exercises

Problem 26.59 (Truncated cone resistor)

A material of resistivity ρ is formed into a solid, truncated cone of height h and radii r_1 and r_2 at either end. (a) Calculate the resistance of the cone between the two flat end faces. (b) Show that your result agrees with Eq. (25.10) when $r_1 = r_2$.

Solution:

a. Divide the cone into infinitesimally thin disks of thickness dx. We have

$$\begin{aligned} \mathrm{d}R &= \frac{\rho\,\mathrm{d}x}{\pi(r_2 + x\beta)^2} \quad \left(\beta = \frac{r_1 - r_2}{h}\right) \\ \Longrightarrow R &= \int_0^h \frac{\rho}{\pi(r_2 + \beta x)^2}\,\mathrm{d}x \\ &= \frac{\rho}{\pi\beta}\left(\frac{1}{r_2} - \frac{1}{r_2 + h\beta}\right) = \frac{\rho h}{\pi r_1 r_2} \end{aligned} \tag{25.5}$$

b. $r_1 = r_2 = r \Longrightarrow R = \dfrac{\rho h}{\pi r^2}$

Problem 25.60 (Spherical conductor)

The region between two concentric conducting spheres with radii a and b is filled with a conducting material with resistivity ρ.

a. Show that the resistance between the spheres is given by

$$R = \frac{\rho}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right). \tag{25.6}$$

b. Derive an expression for the current of density as a function of radius, in therms of the potential difference V_{ab} between the spheres. (c) Show that the result in (a) reduces to Eq. (25.10) when the separation $L = b - a$ between the spheres is small.

Solution:

a.

$$\begin{aligned} \mathrm{d}R &= \frac{\rho\,\mathrm{d}r}{4\pi r^2} \\ \implies R &= \int_a^b \frac{\rho}{4\pi r^2}\,\mathrm{d}r \\ &= \frac{\rho}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{\rho(b-a)}{4\pi ab} \end{aligned} \tag{25.7}$$

b. $I = \frac{V_{ab}}{R}$, $J = \frac{I}{A} = \frac{V_{ab}ab}{\rho(b-a)r^2}$.

c. When $L = b - a \to 0$, $A \longrightarrow 4\pi r^2 \implies R \longrightarrow \frac{\rho L}{4\pi r^2} = \frac{\rho L}{A}$

Problem 25.77

The resistivity of a semi conductor can be modified by adding different amounts of impurities. A rod of semi-conducting material of length L and cross-sectional area A lies along the x axis between $x = 0$ and $x = L$. The material obeys Ohm's law, and its resistivity varies along the rod according to $\rho(x) = \rho_0 e^{-x/L}$. The end of the rod at $x = 0$ is at a potential V_0 greater than the end at $x = L$.

a. Find the total resistance of the rod and the current in the rod.
b. Find the electric-field magnitude $E(x)$ as a function of x.
c. Find the electric potential $V(x)$ in the rod as a function of x.

Solution:

a.

$$\begin{aligned} \mathrm{d}R &= \frac{\rho_0 e^{-x/L}\,\mathrm{d}x}{A} \\ \implies R &= \int_0^L \rho_0 \frac{e^{-x/L}}{A}\,\mathrm{d}x \\ &= \frac{L\rho_0}{A}(1 - e^{-1}) \end{aligned} \tag{25.8}$$

b. $I = \frac{V_0}{R} = \frac{V_0 A}{L\rho_0(1-e^{-1})} \implies J = \frac{V_0}{L\rho_0(1-e^{-1})}$

$\implies E(x) = \rho(x)J = \frac{V_0 e^{-x/L}}{L(1-e^{-1})}$

c. $V = V_0 - IR(x)$

$$= \left(\frac{V_0 A}{\rho_0 L[1 - e^{-1}]}\right)\left(\frac{\rho_0 L}{A}\right)(1 - e^{-x/L}) = V_0 \frac{e^{-x/L} - e^{-1}}{1 - e^{-1}}$$

Problem 25.78

An external resistor with resistance R is connected to a battery that has emf $\mathscr{E}$ and internal resistance r. Let P be the electrical power output of the source. By conversion on energy, P = power consumed by R. What is the value of P in the limit that R is

a. very small;

b. very large?

c. Show that the power output of the battery is at the maximum when $R = r$. What's this maximum power P in terms of $\mathscr{E}$ and r?

d. A battery has $\mathscr{E} = 64\,\text{V}$, $r = 4\,\Omega$. What's the power output of this battery when it is connected to a resistor R for $R = 2\,\Omega, 4\,\Omega, 6\,\Omega$? Are your results consistent with the general result that you derived in part (b)?

Solution:

a. $\mathscr{E} = (R + r)I \Longrightarrow I = \dfrac{R + r}{\mathscr{E}}$

$P = I^2 R = \left(\dfrac{\mathscr{E}}{R + r}\right)^2 R$. When $R \longrightarrow 0$, $P \longrightarrow 0$.

b. When $R \longrightarrow \infty$, $P \longrightarrow 0$.

c. The maximum power is reached $P'(R) = 0 \Longleftrightarrow R = r$. At that point, $P_{\max} = \dfrac{\mathscr{E}^2}{4r}$.

26 DC Circuits

26.1 Summary

- Effective resistance of a circuit
  - in parallel: $\dfrac{1}{R} = \sum_{i=1}^{n} \dfrac{1}{R_i}$
  - in series: $R = \sum_{i=1}^{n} R_i$.

- Kirchhoff's rules for circuits:
  - At any junction, $\sum I = 0$.
  - At any loop, $\sum V = 0$.
- R-C circuits:
  - Capacitor charing:

$$\boxed{\begin{aligned} q &= C\mathcal{E}\left(1 - e^{-t/RC}\right) \\ i &= \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{E}{R} e^{-t/RC} \end{aligned}} \tag{26.1}$$

  - Capacitor discharging:

$$\boxed{\begin{aligned} q &= Q_0 e^{-t/RC} \\ i &= \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{Q_0}{RC} e^{-t/RC} \end{aligned}} \tag{26.2}$$

26.2 Exercises

Problem 26.66 ( )

In the circuit shown in Fig. P26.66 all the resistors are rated at maximum power of 2 W. What's the maximum emf $\mathcal{E}$ that be battery can have without burning up any of the resistors?

Solution: The current running through the 40 Ω resistor is the same one running through the emf; the currents running through every other resistors must be less than or equal to that value. Let $P_{40} = 2\,\mathrm{W}$; $I^2 R = 2\,\mathrm{W} \Longleftrightarrow I = 0.22\,\mathrm{A}$. The equivalent resistance of the circuit is 126 Ω. Thus $\mathcal{E} = R_{\mathrm{eq}} I = 27.72\,\mathrm{V}$

Problem 26.74 ( )

A uniform wire of resistance R is cut into three equal lengths. One of these is formed into a circle and connected between the other two. What is the resistance between the opposite ends of a and b?

Solution: $R_{\mathrm{eq}} = R/3 + R/3 + \left(\frac{6}{R} + \frac{6}{R}\right)^{-1} = 3R/4.$

Problem 26.1 (The Wheatstone Bridge -)

The circuit shown in Fig P26.74, called a Wheatstone bridge, is used to determine the value of an unknown resistor X by comparison with three resistors M, N and P whose resistances can be varied. For each setting, the resistance of each resistor is precisely known. With switches S_1 and S_2 closed, these resistors are varied until the current in the galvanometer G is zero; the bridge is then said to be *balanced.*

a. Show that under this condition the unknown resistance is given by $X = MP/N$ (This method permits very high precision in comparing resistors.)

b. If galvanometer G shows zero deflection when $M = 850\,\Omega$, $N = 15\,\Omega$, $P = 33.48\,\Omega$, what's the unknown resistance X?

Solution:

a. Applying Kirchhoff's loop and rule laws to loops *abca*, *bcdb* and junctions b, c, we get

$$\begin{cases} I_N - 0 - I_M = 0 \\ I_P - I_X - 0 = 0 \\ V_P - V_N = 0 \\ V_M - V_X = 0 \end{cases} \Longrightarrow \begin{cases} I_N = I_M \\ I_P = I_X \\ R_P I_P = R_N I_N \\ R_M I_M = R_X V_X \end{cases} \tag{26.3}$$

$\Longrightarrow I_P/I_N = P/N = M/X \Longleftrightarrow X = MP/N.$

b. $X = 1\,897\,\Omega.$

Problem 26.84 (An Infinite Network -)

As shown in Fig. P26.83, a network of resistors of resistances R_1 and R_2 extends to infinity toward the right. Prove that the total resistance R_T of the infinite network is equal to

$$R_T = R_1 + \sqrt{R_1^2 + 2R_1R_2}. \tag{26.4}$$

Solution: Noticing that the resistance of the network to the right of junctions c and d is also R_T, we have $R_T = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_T}\right)^{-1} + R_1$. Solving for R_T, we obtain

$$R_T = R_1 + \sqrt{R_1^2 + 2R_1R_2}. \tag{26.5}$$

Problem 26.84 (A Resistor Cube -)

Suppose a resistor R lies along each edge with a cube (12 resistors in all) with connections at the corners. Find the equivalent resistance between two diagonally opposite corners of the cube.

Solution: At a and b there are 3 branches, so each has current $I/3$ running through them. At the next junction there are two equivalent branches, so current flowing each branch is $I/6$. Thus $V = \frac{5}{6}IR$. However, $V = R_{\text{eq}}I$, so $R_{\text{eq}} = 5R/6$.

27 Magnetic Field and Magnetic Forces

27.1 Summary

- If a point charged is placed in a magnetic field $\vec{\boldsymbol{B}}$, then the *Lorentz force* acting on the charge by the field is: $\vec{\boldsymbol{F}}_L = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$; $F = |q|vB\sin\varphi$.
- If both a magnetic and electric field are present, the force acting on the point charge is then

$$\vec{\boldsymbol{F}} = q\left(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\right). \tag{27.1}$$

  Note that only the electric field is responsible for accelerating the point charge; the Lorentz force serves to deflect and bend the trajectory of the charged particle.

- Magnetic flux through a surface: $\varphi_B = \displaystyle\int \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{A}$.
- *Gauss's law for magnetism:*

$$\boxed{\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{A} = 0} \tag{27.2}$$

- Radius of a circular in a magnetic field:

$$\boxed{R = \frac{mv}{|q|}} \tag{27.3}$$

  From here, we can derive the angular velocity:

$$\omega = \frac{v}{R} = \frac{|q|B}{m}. \tag{27.4}$$

- If a wire carrying current I is placed within a magnetic field $\vec{\boldsymbol{B}}$, the force acting on it is given by

$$\boxed{\begin{aligned} d\vec{\boldsymbol{F}} &= I\, d\vec{l} \times \vec{\boldsymbol{B}} \\ F &= IlB \sin\varphi \end{aligned}} \tag{27.5}$$

- The magnetic torque acting on a looped wire is

$$\boxed{\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}},} \tag{27.6}$$

  where $\mu = IA$ is called the **magnetic (dipole) moment** of the loop.
- Potential energy for a magnetic dipole:

$$U = -\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{B}} = -\mu B \cos\varphi. \tag{27.7}$$

27.2 Exercises

15.

   a. $\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$, and since the point charge is negative, $\vec{\boldsymbol{B}}$ points into the page.

      $B = \dfrac{mv}{|q|R} = 1.6 \cdot 10^{-4}\,\mathrm{T}$.

   b. $t = \pi r v_0^{-1} =$

16. We're repeating Ex. 27.15 for $m_p = 1.67 \cdot 10^{-27}\,\mathrm{kg} \wedge q_p = 1.6 \cdot 10^{-19} C$.

   a. This time, $\vec{\boldsymbol{B}}$ points out of the page, and $B = 0.294\,337\,5\,\mathrm{T}$ and $t = 5.57 \cdot 10^{-9}\,\mathrm{s}$.

25. $0.5\pi r = 0.011\,8\,\mathrm{m} \Longrightarrow r = 0.075\,1\,\mathrm{m} \Longrightarrow B = \dfrac{mv}{|q|r} = 1.67 \cdot 10^{-3}\,\mathrm{T}$ $R = \dfrac{mv}{|q|B}$

35. Let the three segments in the wire be a, b, c. Segment a has length x, segment b is 30 cm long, and segment c has length $0.6 - x$ m. Thus the combined force acting on a and c is $F_{ac} = F_a + F_c = 4.5 \cdot 0.6 \cdot 0.24 = 0.648\,\mathrm{N}$, whose direction points to the bottom of the page. $F_b = 0.324\,\mathrm{N} \Longrightarrow F \approx 0.724\,\mathrm{N}$ and $\varphi = \arctan \dfrac{F_{ac}}{F_b} \approx 63.43°$.

36. $B = 0.55\,\mathrm{T}, r = 2.5 \cdot 10^{-3}\,\mathrm{m}, I = 10.8\,\mathrm{A}$

    $F = IlB \sin\varphi = 0.297\,\mathrm{N}$.

40.

   a. $A = 4 \cdot 10^{-3}\,\mathrm{m}^2, B = 0.19\,\mathrm{T}, I = 6.2\,\mathrm{A}$.

      $\tau = \mu B \sin\varphi = 4.712 \cdot 10^{-3}\,\mathrm{N\,m}$.

   b. $\mu = 0.024\,8\,\mathrm{N\,m}^2$.

   c. The perimeter of this rectangular loop is 26 cm. For maximum torque with the same current and magnetic field, the area of this loop must be the greatest, which implies that this would be a circular loop whose perimeter is also 26 cm.

$$R_{\text{loop}} \approx 4.13 \cdot 10^{-2}\,\text{m} \Longrightarrow \max\tau \approx 6.617 \cdot 10^{-3}\,\text{N m}.$$

42.

61. For the force on wire induced by the magnetic field to counteract its gravity, the current I must flow from the left of the wire to its right.

    Then,

$$IBL\cos\theta = Mg\sin\theta$$
$$I = \frac{Mg\tan\theta}{BL}. \tag{27.8}$$

Problem 27.68 (A Magnetic Railgun)

A conducting bar with mass m and length L slides over the horizontal rails that are connected to a voltage source. The voltage source maintains a constant current I in the rails and bar, and a constant, uniform, vertical magnetic field $\vec{\boldsymbol{B}}$ fills the region between the rails.

a. Find the magnitude and direction of the net force on the conducting bar.
b. If the bar has mass m, find the distance d that the bar must move along the rails from rest to attain speed v.
c. It has been suggested that railguns based on this principle could accelerate payloads into orbit or beyond. Find that distance the bar must travel if it's to reach the escape velocity of the Earth ($11.2\,\text{km}\,\text{s}^{-1}$). Let $B = 0.8\,\text{T}$, $I = 2 \cdot 10^3\,\text{A}$, $m = 25\,\text{kg}$, $L = 0.5\,\text{m}$. For simplicity, assume the net force on the object is equal to the magnetic force, as in parts (a) and (b), even though gravity plays an important role in an actual launch in space.

Solution:

a. The bar is supported by the rails, so the sum of vertical forces acting on it is zero: $\vec{\boldsymbol{F}}_y = \mathbf{0}$. Thus the only force acting on the bar is the force caused by the magnetic field: $F = IBL$. Because the current I travels from the left side of the bar to its right, the resulting $\vec{\boldsymbol{F}}$ points away from the U-bend.
b. $v^2 = 2ad = 2IBLm^{-1}d \Longleftrightarrow d = \dfrac{mv^2}{2IBL}.$
c. Plugging in the numbers in (c) into the result we obtained in (b), we get $d = 1\,960\,\text{km}$.

Problem 27.70

A uniform bar of length L carries a current I in the direction of point a to point b (Fig. P27.70.). The bar is in a uniform magnetic field that's directed into the page. Consider the torque in about an axis perpendicular to the bar at point a that is due to the force that magnetic field exerts on the bar.

a. Suppose that an infinitesimal section of the bar has length $\mathrm{d}x$ and is located a distance x from point a. Calculate the torque $\mathrm{d}\tau$ about a due to the magnetic force on this infinitesimal section.
b. Use $\tau = \int_a^b \mathrm{d}\tau$ to calculate the total torque τ on the bar.
c. Show that τ is the same as though all of the magnetic force acted at the midpoint of the bar.

Solution:

a. Here, $\varphi = 1$ because the bar is perpendicular to the plane; $\mathrm{d}F = IB\,\mathrm{d}x$. Then, the torque about a is then $\tau_z = x\,\mathrm{d}F \sin\varphi = IBx\,\mathrm{d}x$.
b.
$$\begin{aligned}\tau &= \int_0^L \mathrm{d}\tau_z \\ &= \int_0^L IBx\,\mathrm{d}x \\ &= \frac{IBL^2}{2}\end{aligned} \tag{27.9}$$
c. $\frac{IBL^2}{2} = IBL\frac{L}{2} = FL/2$, so τ is the same as though all of the magnetic force acted at the midpoint of the bar.

Problem 27.72

A uniform bar has mass 0.012 kg and is 0.3 m long. It pivots without friction about an axis perpendicular to the bar at point a. The gravitational force on the bar acts in the $-y$-direction. The bar is in a uniform magnetic field that is directed into the page and has magnitude $B = 0.15\,\mathrm{T}$.

a. What must be the current I in the bar for the bar to be in rotational equilibrium when it is at an angle $\varphi = 30°$ above the horizontal?
b. For the bar to be in rotational equilibrium, in what direction should I be?

Solution:

a. The bar is in rotational equilibrium when

$$\begin{aligned} \tau &= 0 \\ \iff IBL - 0.5mgL\cos 30^\circ &= 0 \\ \iff I &= \frac{0.5\cos 30^\circ mg}{B} = 6 \cdot 10^{-3}\,\text{A}. \end{aligned} \tag{27.10}$$

b. By the right hand rule, I is directed from $a \longrightarrow b$.

Problem 27.82 (A Cycloidal Path)

A particle with mass m and positive charge q starts from rest at the origin shown in Fig P27.82. There's a uniform electric field $\vec{\boldsymbol{E}}$ in the $+y$-direction and a uniform magnetic field $\vec{\boldsymbol{B}}$ directed out of the page. It is shown in more advanced books that the path is a *cycloid* whose radius of curvature at the top points is twice the y-coordinate at that level.

a. Explain why the path has this general shape and why it is repetitive.
b. Prove that the speed at any point is equal to $\sqrt{2qEy/m}$.
c. Applying Newton's second law at the top point and taking as given that the radius of curvature here equals $2y$, prove that speed at this point is $\frac{2E}{B}$.

Solution:

a. The maximum speed occurs at the top of the cycloidal path, and hence the radius of curvature is greatest there. Past that point, the particle gets slowed down by the magnetic field, and at $y = 0$, it's stopped completely. At this point, the electric field provides acceleration the $+y$-direction, leading to repetitive motion.

b. Conservation of energy tells us that the kinetic energy is equal to the work done by the electric field $\vec{\boldsymbol{E}}$ on the particle: $\frac{mv^2}{2} = qEy \iff v = \sqrt{\frac{2qEy}{m}}$.

c. At the highest point,

$$\begin{aligned} F_y = qE - qvB &= -\frac{mv^2}{R} \\ &= -\frac{m}{2y}\frac{2qEy}{m} \\ &= -qE \\ \iff qvB &= 2qE \\ \iff v &= \frac{2E}{B} \end{aligned} \tag{27.11}$$

28 Sources of Magnetic Field

28.1 Summary

- Magnetic field due to a point charge with constant velocity:

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{r}}}{r^3}, \tag{28.1}$$

  where μ_0 is called the *magnetic constant* and

$$\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{T\,m\,A^{-1}}. \tag{28.2}$$

- The *Biot-Savart law*:

$$\boxed{\mathrm{d}\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{\boldsymbol{l}} \times \hat{\boldsymbol{r}}}{r^3}.} \tag{28.3}$$

- The magnetic field on the axis of a circular current-carrying loop is

$$B_x = \frac{\mu_0 I r^2}{2(x^2 + r^2)^{3/2}}, \tag{28.4}$$

  where r is the radius of the loop and and x is the distance from a point on the x axis to the center of the loop.

  If there are N loops, the magnitude of the field at the center is then

$$B = \frac{\mu_0 N I}{2r}.$$

- The magnitude of the force caused by two parallel conductors is

$$F = I'LB = \frac{\mu_0 I I' L}{2\pi r}, \tag{28.5}$$

where I, I' are the currents of the first and second conductors respectively, while r is the distance between them. Thus the magnetic force per unit length is

$$\boxed{\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r}.} \tag{28.6}$$

Note that two parallel conductors carrying the same charge direction attract one another, while two carrying opposite-direction charges repel.

- *Ampère's law* dictates that the line integral of a magnetic field around a closed path is equal to the net current enclosed by that path times the magnetic constant:

$$\boxed{\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I} \tag{28.7}$$

- The magnitude of a magnetic field caused by a straight, current-carrying wire at a point is

$$B = \frac{\mu_0 I}{4\pi} \frac{2a}{x\sqrt{x^2 + a^2}}, \tag{28.8}$$

where a is the length of the wire and x the distance from the point to the wire. As $a \longrightarrow 0$, Equation (28.8) reduces to

$$B = \frac{\mu_0 I}{2\pi x}.$$

28.2 Exercises

1. $v = \left(8 \cdot 10^6\,\mathrm{m\,s^{-1}}\right)\hat{\boldsymbol{j}}$.
   a. $\vec{\boldsymbol{r}} = 0.5\,\mathrm{m}\hat{\boldsymbol{i}}, r = 0.5\,\mathrm{m}$

      $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{r}} = vr\hat{\boldsymbol{j}} \times \hat{\boldsymbol{i}} = -vr\hat{\boldsymbol{k}}$

      $\Longrightarrow \vec{\boldsymbol{B}} = -\frac{\mu_0}{4\pi}\frac{qv}{r^2}\hat{\boldsymbol{k}} = -1.92 \cdot 10^{-5}\,\mathrm{T}\hat{\boldsymbol{k}}.$
   b. $\vec{\boldsymbol{r}} = 0.5\,\mathrm{m}\hat{\boldsymbol{j}}, r = 0{,}5\,\mathrm{m}.$

      $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{r}} = \boldsymbol{0} \Longrightarrow \vec{\boldsymbol{B}} = \boldsymbol{0}.$
   c. $\vec{\boldsymbol{r}} = 0.5\,\mathrm{m}\hat{\boldsymbol{k}}, r = 0.5\,\mathrm{m}$

      $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{r}} = vr\hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}} = vr\hat{\boldsymbol{i}}$

      $\Longrightarrow \vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{qv}{r^2}\hat{\boldsymbol{i}} = 1.92 \cdot 10^{5}\,\mathrm{T}\hat{\boldsymbol{i}}.$

d. $\vec{\boldsymbol{r}} = (0{,}5\,\mathrm{m})\hat{\boldsymbol{j}} + 0.5\,\mathrm{m}\hat{\boldsymbol{k}}, r \approx 0.707\,1\,\mathrm{m}$.

$\vec{\boldsymbol{v}} \times \vec{\boldsymbol{r}} = 4 \cdot 10^6\,\mathrm{m^2\,s^{-1}}\hat{\boldsymbol{i}}$

$\Longrightarrow \vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{r}}}{r^3} = 9.6 \cdot 10^{-6}\,\mathrm{T}\,\hat{\boldsymbol{i}}$.

24. Equation (28.8) gives us

$$B = \frac{\mu_0 I}{4\pi}\frac{2a}{x\sqrt{x^2 + a^2}};$$

For the long sides, $a = 4.75$ cm, $x = 2.1$ cm, so $B_1 = 2B_l = 1.7421 \cdot\cdot 10^{-5} I$ T.

For the short sides, $a = 2.1$ cm, $x = 4.75$ cm, $B_2 = 2B_s = 3.405 \cdot\cdot 10^{-6} I$ T

Combining the two, we have $B = B_1 + B_2 = 2.0826 \cdot\cdot 10^{-5} I$ T $= 5.5 \cdot\cdot 10^{-5}$ T $\Longleftrightarrow I = 2.64$ A.

28. For the central wire, it's evenly placed between the two outermost wires and the current running through it is opposite to the other two wires; thus the repulsive forces between the center wire and the outer wires cancel out and the net force per unit length for it is 0.

For the outermost wires, the force per unit length on it is $F/L = \frac{\mu_0 I}{2\pi d} - \frac{\mu_0 I}{4\pi d} = \frac{\mu_0 I}{4\pi d}$ (positive direction facing away from the inner wire). The net force is repulsive.

34. Let $+z$ point into the page.

$\mathrm{d}B = \frac{\mu_0}{4\pi}\frac{I \cdot R \cdot R}{R^3}\hat{\boldsymbol{r}} \times \vec{e}_r\,\mathrm{d}\theta = \frac{\mu_0}{4R} I\,\mathrm{d}\theta\,\vec{e}_z \Longrightarrow B = \int_0^\pi \frac{\mu_0}{4R} I\,\mathrm{d}\theta = \frac{\mu_0}{4R} I$

35. Again, we'll let $+z$ point into the page.

Then, from Problem 34, $\vec{\boldsymbol{B}}_1 = \frac{\mu_0}{4R} I_1 \vec{e}_z$ and $\vec{\boldsymbol{B}}_2 = -\frac{\mu_0}{4R} I_2 \vec{e}_z \Longrightarrow \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_1 + \vec{\boldsymbol{B}}_2 = \frac{\mu_0}{4R}(I_1 - I_2)$.

If $I_1 = I_2$, then $\vec{\boldsymbol{B}} = \vec{\boldsymbol{0}}$.

36. $r = 2.4$ cm, $N = 800$, $B_a = 0.077$ T, $B_b = 0.0385$ T.

a. $B_a = \frac{\mu_0 N I}{2r} \Longleftrightarrow I_a = \frac{2Br}{\mu_0 N} \approx 3.68$ A.

b. $B_b = \frac{\mu_0 N I r^2}{2(x^2 + r^2)^{3/2}} = \frac{1}{2} B_a \Longrightarrow x = \sqrt{2^{2/3} - 1}\,r \approx 1.83$ cm.

38. $B_b = \frac{\mu_0 N I r^2}{2(x^2 + r^2)^{3/2}} = 6.39 \cdot 10^{-4}\,\mathrm{T}, r = x = 6\,\mathrm{cm}, I = 2.5\,\mathrm{A} \Longrightarrow N \approx 69$ turns.

39. $B = 0.15$ T, $N = 400$, $L = .55$ m.

The magnetic field of a solenoid is $B = \mu_0 n I = \mu_0 \frac{N}{L} I \Longrightarrow I \approx 16.41$ A

40. a. $B_a = \dfrac{\mu_0 I}{2\pi r}$.
    b. $B_b = 0$.

46. $B = \dfrac{\mu_0 N I}{L} \approx 0.0402$ T.

Problem 28.57 ( )

Two long, parallel transmission lines, 40 cm apart, carry 25 A and 75 A currents. Find all locations where the net magnetic field of the two wires is 0 if the currents are in

a. the same direction and
b. the opposite direction.

Problem 28.65 ( )

Two long, parallel wires hang by 4 cm-long cords from a common axis. The wires have a mass per unit length of 0.0125 kg / m and carry the same current in opposite directions. What is the current of in each wire if the cords hang at an angle of 6° with the vertical?

Solution: Denote the length of the cords as l and that of the wires L; let the distance between the wires be r. Then, $r = 2l\sin\theta$, where $\theta = 6°$, $l = 4$ cm.

For any one of the wires, we can derive the following equations:

$$\begin{cases} T\cos\theta = mg = \lambda L g \\ T\sin\theta = \dfrac{\mu_0 I^2 L}{2\pi r} = \dfrac{\mu_0 I^2 L}{4\pi l \sin\theta} \end{cases}$$

$$\iff \frac{\lambda L g}{\cos\theta} = \frac{\mu_0 I^2 L}{4\pi l \sin^2\theta}$$

$$\iff I = \sqrt{\frac{4\pi \cdot \tan\theta \sin\theta l \lambda g}{\mu_0}} \approx 23.2 \text{ A}.$$

Problem 28.66 ( )

The wire semicircles shown in Fig P28.66 have radii a and b. Calculate the net magnetic field (magnitude and direction) that the current in the wires produces at point P.

Solution: Let $+z$ face away from the page.

From Problem 28.34, we have $\vec{\boldsymbol{B}}_a = \frac{\mu_0 I}{4a}\vec{e}_z, \vec{\boldsymbol{B}}_b = -\frac{\mu_0 I}{4b} \Longrightarrow B = \frac{\mu_0 I}{4}\left(\frac{1}{a} - \frac{1}{b}\right)\vec{e}_z$.

Problem 28.68 ( )

Calculate the magnetic field at a point P due to a current $I = 12$ A in the wire shown in Fig P28.68. Segment BC is an arc of a circle with radius 30 cm, and point P is the center of curvature in the arc. Segment DA is an arc of a circle with radius 20 cm, and point P is at its center of curvature. Segments CD and AB are straight lines of length 10 cm each.

Solution: Let $+z$ point into the page. Using similar calculations in Problem 28.34, this time with $\theta = \frac{2\pi}{3}$, we obtain

$$B = \frac{\mu_0 I}{6R}$$

Then, the magnetic field caused by BC and DA at P is $\vec{\boldsymbol{B}}_{BC} = -\frac{\mu_0 I}{6R_a}\vec{e}_z, \vec{\boldsymbol{B}}_{DA} = \frac{\mu_0 I}{6R_b}\vec{e}_z \Longrightarrow \vec{\boldsymbol{B}} \approx 4.19\mu\text{T}$.

Problem 28.70 ( )

The wire shown in Fig P28.70 is infinitely long and carries a current I. Calculate the net magnetic field that this current produces at P.

Solution: The horizontal wire doesn't cause a magnetic field at P because P lies on its axis of symmetry; the vertical wire, however, does; its magnetic field is half that of an infinitely long current-carrying conductor, and is equal to

$$\frac{\mu_0 I}{4\pi a}$$

where $\vec{\boldsymbol{B}}$ points out of the page.

Problem 28.71 ( )

A long, straight cylinder, oriented with its axis in the z-direction, carries a current whose current density is $\boldsymbol{J}$. The current density, although symmetric about the cylinder axis, is not constant but varies according to the relationship

$$\begin{aligned} \vec{\boldsymbol{J}} &= \frac{2I_0}{\pi a^2}\left[1 - \left(\frac{r}{a}^2\right)\right] && \text{for } r \le a \\ &= \mathbf{0} && \text{for } r \ge a \end{aligned}$$

where a is the radius of the cynlinder, r the radial distance from the cynlinder axis, and I_0 a constant measured in A.

a. Show that I_0 is the total current passing through the entire cross section of the wire.
b. Using Ampère's law, derive an expression for the magnitude of the magnetic field $\vec{\boldsymbol{B}}$ in the region $r \ge a$.
c. Obtain an expression for the current I contained in a circular cross section of radius $r \le a$ and centered at the cylinder axis.
d. Using Ampère's law, derive an expression for the magnitude of the magnetic field $\vec{\boldsymbol{B}}$ in the region $r \le a$. How do your results in (b) and (d) compare for $r = a$?

Solution:

a. $\mathrm{d}I = J \cdot 2\pi r\,\mathrm{d}r = \frac{4I_0 r}{a^2}\left[1 - \frac{r^2}{a^2}\right]\mathrm{d}r \Longrightarrow I = \int_0^a \frac{4I_0 r}{a^2} - \frac{4I_0 r^3}{a^4}\,\mathrm{d}r = 2I_0 - I_0 = I_0.$
b. We have $\vec{\boldsymbol{B}} = B\vec{e}_\theta$. Letting the Ampèrian loop be a circle of radius r whose center lies on the wire's axis, we get

$$B = \frac{\mu_0 I_0}{2\pi r}.$$

c. $I_2 = \frac{2I_0 r^2}{a^2} - \frac{I_0 r^4}{a^4}.$
d. Using the same method as in (b), we get

$$\frac{\mu_0 I}{2\pi}\left(\frac{2I_0 r}{a^2} - \frac{I_0 r^3}{a^4}\right)$$

Problem 28.73 (An Infinite Current Sheet -)

Long, straight conductors with square cross sections and each carrying current I are laid side by side to form an infinite current sheet. The conductors lie on the xy-plane, are parallel to the y-axis, and carry current in the $+y$-direction. There are n conductors per unit length measured along the x-axis.

a. What's the magnetic field a distance a below the current sheet?
b. What's the magnetic field a distance a above the current sheet?

Solution: By the right-hand rule, the magnetic field caused by the field is counterclockwise, so for points above the sheet, $\vec{\boldsymbol{B}} = -B\vec{e}_x$, and for points below, $\vec{\boldsymbol{B}} = B\vec{e}_x$. Let our loop be a rectangle with length L and width $2a$. Then, by Ampère's law,

$$\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I$$
$$\Longrightarrow B \cdot 2L = \mu_0 nIL$$
$$\Longleftrightarrow B = \frac{\mu_0 nI}{2}$$

Problem 28.80 ( )

A wide, long, insulating belt has a uniform positive charge per unit area σ on its upper surface. Rollers at each end move the belt to the right at a constant speed v. Calculate the magnitude and direction of the magnetic field produced by the moving belt at a point just above its surface.

Solution: We'll let our Ampèrian loop be the same as that in Problem 28.73. Then, we have

$$2BL = \frac{\mu_0 \sigma v}{2} \cdot L$$
$$\Longleftrightarrow B = \frac{\mu_0 \sigma v}{4}.$$

37. $L = 80$ mH, $C = 1.25$ nF, $i_{\text{max}} = 0.75$ A.

    a. $Q = i_{\text{max}}\sqrt{LC} = 7.5 \cdot 10^{-6}$ C.
    b. $f = \dfrac{1}{2\pi\sqrt{LC}} \approx 15.92$ kHz.

c. $q = 1.25\cos(10^5 t) \Longrightarrow q\big|_{t=2.5\text{ ms}} \approx -0.428\text{ nF}$
$\Longrightarrow i^2 = \omega^2(Q^2 - q^2)$

29 Electromagentic Induction

29.1 Summary

- The *magnetic flux* over a surface is defined as

$$\varphi_{\boldsymbol{B}} = \int \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = \int B\,\mathrm{d}A\cos\theta. \tag{29.1}$$

- *Faraday's law* states that the induced emf of a closed loop is equal to the negative of the change of the magentic flux through it.

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_{\boldsymbol{B}}}{\mathrm{d}t} \tag{29.3}$$

- To find the direction of an induced emf, we'll do the following steps:
  1. Define a positive direction for $\vec{\boldsymbol{A}}$.
  2. From the directions of the $\vec{\boldsymbol{A}}, \vec{\boldsymbol{B}}$, determine $\Phi_{\boldsymbol{B}}$ and $\frac{\mathrm{d}\Phi_{\boldsymbol{B}}}{\mathrm{d}t}$.
  3. Use the right-hand rule to determine the direction of the emf. If the induced emf is *positive*, it is in the same direction as your curled fingers and *negative* otherwise.
- *Lenz's law* dictates that any induced current opposes the change that causes it.
- If a coil has N identical turns and the flux varies at the same rate through each turn, the emf in such a coil is then

$$\mathcal{E} = -N\frac{\mathrm{d}\Phi_{\boldsymbol{B}}}{\mathrm{d}t}. \tag{29.4}$$

- The induced emf caused by a slidewire generator is

$$\mathcal{E} = -BLv.$$

- *Motional emf* in general is expressed by

$$\mathcal{E} = \oint\left(\vec{v} \times \vec{\boldsymbol{B}}\right) \cdot \mathrm{d}\vec{l}. \tag{29.7}$$

  If the conductor has length L moves perpendicularly through a uniform magnetic field speed v, then the emf becomes

$$\mathcal{E} = BLv. \tag{29.6}$$

- If the angle θ doesn't change, then the magnetic flux must change in order to generate an induced emf. In that case, there exists an *induced electric field* in the conductor and *Faraday's law for a stationary intergration path* says that the work done by $\vec{\boldsymbol{E}}$ per unit charge is

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \tag{29.10}$$

- In real life, electrical equipment often have masses of metal moving in magnetic fields or in alternating magnetic fields, and the induced currents circulating within such components are called *eddy currents.*
- The *displacement current* through an area is

  $$i_D = \varepsilon\frac{\mathrm{d}\Phi_E}{\mathrm{d}t} \tag{29.14}$$

  and the *displacement current density* is

  $$j_D = \varepsilon\frac{\mathrm{d}E}{\mathrm{d}t}. \tag{29.16}$$

  This current has physical significance: when a round capacitor with plates of radius R is charged, there exists a magnetic field while the capacitor is charging, due to Ampère's law, and this field has magnitude

  $$B = \frac{\mu_0}{2\pi}\frac{r}{R^2}. \tag{29.17}$$

  When $r > R$, $\vec{\boldsymbol{B}}$ is the same as if the wire was continuous.
- Maxwell's equations of electro magnetism:
  1. Gauss's law for $\vec{\boldsymbol{E}}$:

     $$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = \frac{Q_{\text{encl}}}{\varepsilon_0} \tag{29.10}$$

  2. Gauss's law for $\vec{\boldsymbol{B}}$:

     $$\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = 0 \tag{29.11}$$

  3. Faraday's law for a stationary integration path:

     $$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \tag{29.12}$$

  4. General form of Ampère's law for a stationary integration path:

     $$\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0(i_C + i_D) = \mu_0\left(i_c + \varepsilon_0\frac{\mathrm{d}\Phi_E}{\mathrm{d}t}\right) \tag{29.13}$$

- The minimum external magnetic field to eliminate superconductivity at a temperature below the critical temperature T_c of a given material is called the *critical field,* denoted by B_c.

29.2 Exercises

1.
   a. $|\mathcal{E}| = \left|\frac{d\Phi_B}{dt}\right| = A\left|\frac{dB}{dt}\right| = 0.0171\text{V}$
   b. $I = 0.0285$ A

2. $t = 0.04$ s, $A = 12\text{cm}^2$, $N = 200$ turns, $B = 6 \cdot 10^{-5}$ T, $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$.
   a. The total magnetic flux in the coil before it's rotated is

   $$N\Phi_{B_1} = NBA\cos\theta_1 = 1.44 \cdot 10^{-5}\text{ Wb}.$$

   After it is rotated, $\theta_2 = \frac{\pi}{2}$, so $N\Phi_{B_2} = 0$.
   b. The average emf induced by the coil is

   $$\mathcal{E} = -\frac{\Delta\Phi_B}{\Delta t} = 3.6 \cdot 10^{-4}\text{V}$$

13. The total magnetic flux of the coil is $\Phi_B = BS\cos(\omega t) \Longrightarrow \mathcal{E} = -N\frac{d\Phi_B}{dt} = \omega NBS\sin(\omega t) \Longrightarrow \mathcal{E}_{\text{max}} = \omega NBS = 24\text{ mV} \Longleftrightarrow \omega \approx 10.42\text{ rad / s}$

18. $I(t) = I_0 e^{-bt}$ decreases as time decreases, which means that B also decreases. By Lenz's law, the field generated by the loop must counteract that decrease in magnetic field strength, so the induced current in the coil must be *counterclockwise* by the right-hand rule.

22. $r = 0.048$ m, $R = 0.16\ \Omega$, $\frac{dB}{dt} = -0.68$ T/ s.
   a. By Lenz's law, the direction of the induced current has to be counterclockwise in order to counter the decrease in field strength.
   b. $|\mathcal{E}| = S\left|\frac{dB}{dt}\right|$; $P = \frac{|\mathcal{E}|^2}{R} \approx 1.51 \cdot 10^{-4}$ W.

24. $I = \frac{\mathcal{E}}{R} = \frac{BLv}{R}$; $F = ILB = v\frac{(BL)^2}{R} = 6.48 \cdot 10^{-3}$ N.

   Since Φ_B is decreasing, the induced current in the wire is clockwise, and $\vec{F} = I\vec{l} \times \vec{B}$ indicates $\vec{F}$ is to the left.

25.
   a. $\mathcal{E} = vBL = 0.675\text{V}$.
   b. The positive charges move to b, so it has higher potential
   c. The electric field has direction from $b \longrightarrow a$, and its magnitude is $E = \frac{\mathcal{E}}{L} = 2.25\text{V/ m}$.
   d. b has an excess of positive charges.
   e. The potential difference across the rod is 0
      i. if it moves parallel to ab should it have no appreciable thickness
      ii. if it moves out of the page because then it would move parallel to the magnetic field.

30. $L = 0.65$ m, $v = 5$ m/s, $B = 0.75$ T, $R = 25\ \Omega$; magnetic field points into the page
    a. $|\mathcal{E}| = BLv = 2.4375\text{V}$.
    b.
        i. $\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}(q > 0)$, so I is clockwise.
        ii.
        iii. Lenz's law dictates that since Φ_B increases, the current I is such that the induced magnetic field resists the change, which means that I is clockwise.
    c. $I = \dfrac{\mathcal{E}}{R} = 0.0975$ A.
33. The counterclockwise rotation of the current indicates a decreasing Φ_B, which means the bar is moving the left. $I = \dfrac{vBL}{R} \Longleftrightarrow v = 35$ m/s.
35. $L = 75$ cm, $v = 3$ m/s, $B = 1.25$ T, $R = 12.5\ \Omega$; magnetic field points out of the page
    a. The current runs counterclockwise. $I = \dfrac{vBL}{R} = 0.225$ A
    b. In the field, $\Phi_B = \text{const}$, so $\mathcal{E} = 0$ and thus $I = 0$.
    c. The current runs clockwise. $I = -\dfrac{vBL}{R} = -0.225$ A.
38. $|E| = \dfrac{1}{2\pi r}\left|\dfrac{\mathrm{d}\Phi_B}{\mathrm{d}t}\right| = \dfrac{\mu_0 nr}{2}\left|\dfrac{\mathrm{d}I}{\mathrm{d}t}\right|$
    a. $|E| \approx 1.02 \cdot 10^{-4}\text{V}$
    b. $|E| \approx 2.04 \cdot 10^{-3}\text{V}$
39. $\left|\dfrac{\mathrm{d}i}{\mathrm{d}t}\right| = \dfrac{2Er}{R^2\mu_0 n} = 9.2$ A/ s.
43. $\varepsilon = 4.7\varepsilon_0$, $d = 2.5 \cdot 10^{-3}$ m, $A = 3 \cdot 10^{-4}$ m^2, $v = 120\text{V}$, $i_C = 6$ mA.
    a. $q = \dfrac{\varepsilon A}{d}v \approx 5.99 \cdot 10^{-10}$ C
    b. $\dfrac{\mathrm{d}q}{\mathrm{d}t} = i_C = 6$ mA.
    c. $j_D = \varepsilon\dfrac{\mathrm{d}E}{\mathrm{d}t} = \dfrac{i_C}{A} = j_C$, so $j_D = j_C = 6 \cdot 10^{-3}$ A.
46. $B = 2.9$ T, $L = 0.04$ m, $m = 0.024\text{g}$, $R = 5 \cdot 10^{-3}\ \Omega$, $F_{\text{ext}} = 0.18$ N, magnetic field pointing into the page
    a. We notice that by Lenz's law, the current I flowing in the loop must be counterclockwise to counteract the decrease in Φ_B.

        Then, $F_B = ILB = v\dfrac{(BL)^2}{R}$ and $F = F_{\text{ext}} - F_B = 0.99264$ N $= ma \Longrightarrow a =$ 4.136 m/s^2
    b. Terminal speed is reached when $F_{\text{ext}} = F_B$, thus $a = 0$, and $v = \dfrac{F_B R}{(BL)^2} \approx$ 0.0669 m/s.

c. The acceleration of the loop once it's completely out of the magnetic field is $a = F_{\text{ext}}/m = 7.5\ \text{m/s}^2$.

56.

a. The induced current is counterclockwise, so it resists the battery's voltage. $I = \dfrac{\mathcal{E} - vLB}{R}$, and $F = m\dfrac{\mathrm{d}v}{\mathrm{d}t} = ILB = \dfrac{\mathcal{E}BL - v(LB)^2}{R} \Longrightarrow v = \dfrac{\mathcal{E}}{BL}\left(1 - e^{-\frac{B^2L^2t}{mR}}\right) = 14\left(1 - e^{-\frac{t}{6}}\right)$

b. $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{7}{3}e^{-\frac{t}{6}}$, so $a|_{t=0} \approx 2.33\ \text{m/s}^2$.

c. $a|_{v=2\ \text{m/s}} = 2\ \text{m/s}^2$.

d. The bar's terminal speed is $v_t = \lim\limits_{t\to\infty} v(t) = 14\ \text{m/s}$.

Problem 29.61 ( )

A rectangular loop with width L and a slide wire with mass m are as shown in Fig P29.61. A uniform magnetic field $\vec{B}$ is directed perpendicular into the plane of the figure. The slide wire is given an initial speed v_0 and is then released. There is no friction between the slide wire and the loop, and the resistance of the the loop is negligible in comparison to the resistance R of the slide wire.

a. Obtain an expression for F, the magnitude of the force exerted on the wire while it is moving at speed v.

   (Additionally, obtain an expression for F as a function of time.)

b. Show that the distance x that the wire moves before coming to rest is

$$x = \frac{mv_0R}{L^2B^2}.$$

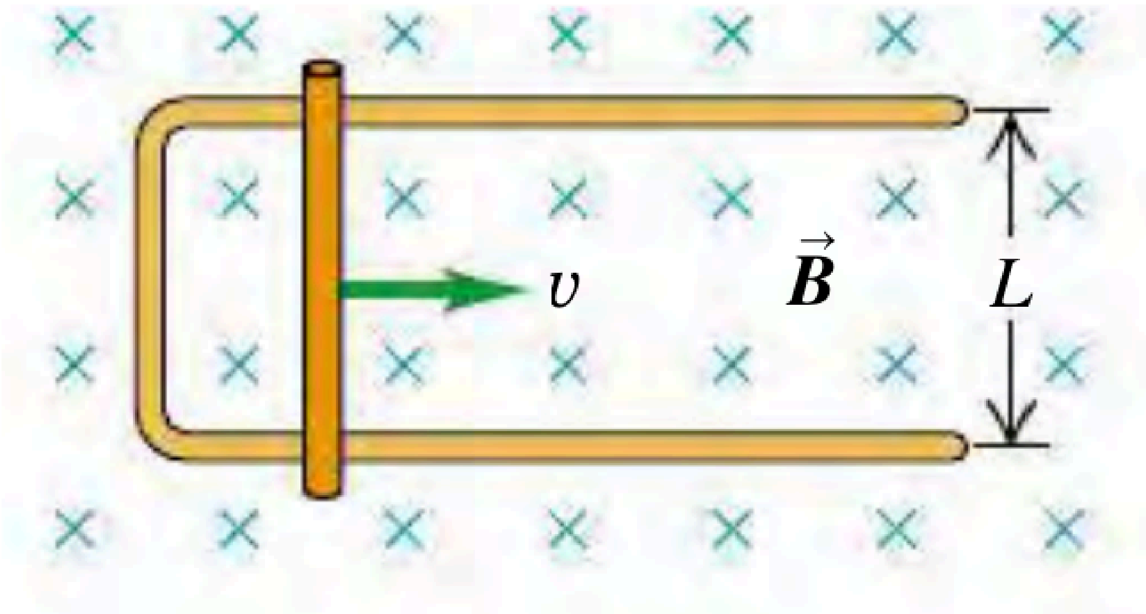


Figure 2: Problem 29.61.

Solution:

a. As usual, we have

$$\mathcal{E} = vBL \Longrightarrow I = \frac{vBL}{R}$$

$$\Longrightarrow F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{(BL)^2}{R}$$

$$\Longrightarrow v = v_0 e^{-\frac{B^2L^2}{Rm}t}$$

$$\Longrightarrow F = ma = -\frac{mv_0B^2L^2}{R}e^{-\frac{B^2L^2}{mR}t}$$

b. $x = \int_{v_0}^{0} v\,\mathrm{d}x = \frac{mv_0R}{B^2L^2}$.

65. $i_D = \varepsilon\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = 84\cdot 10^{-8}t^2 = 21\mu\mathrm{A} \Longleftrightarrow t = 5$ s.

Problem 29.69 ( )

A metal bar with length L, mass m and resistance R is placed oon frictionless metal rails that are inclined at an angle φ above the horizontal. The rails have negligible resistance. A uniform magnetic field of magnitude B is directed downward as shown in Fig P29.69. The bar is released from rest and slides down the rails.

a. Is the direction of the current induced in the bar from $a \longrightarrow b$ or $b \longrightarrow a$?
b. What's the terminal speed of the bar?
c. What is the induced current in the bar when the terminal speed has been reached?
d. After the terminal speed has been reached, at what rate is electrical energy being converted to thermal energy in the resistance of the bar?
e. After the terminal speed has been reached, at what rate is work being done on the bar by gravity? Compare your answer to that in part (d).

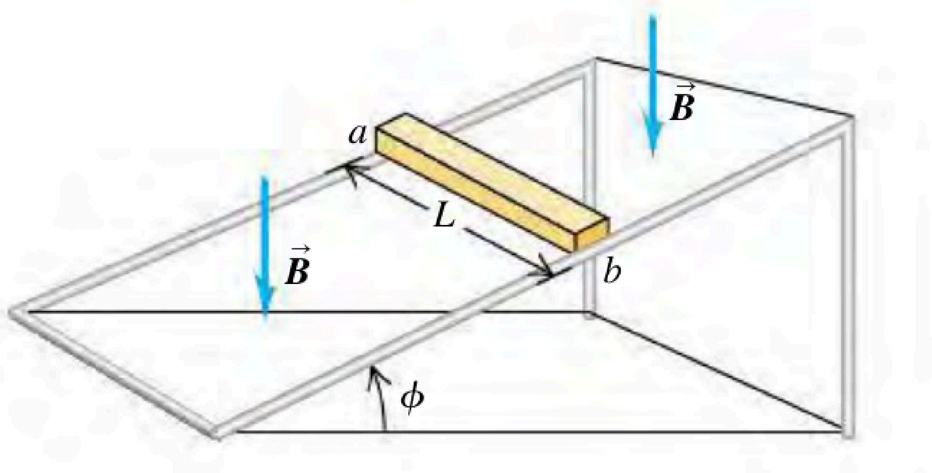


Figure 3: Problem 29.69.

Solution:

a. The current induced by the wire is from $a \longrightarrow b$, because by Lenz's law, the direction of I is such that the magnetic field generated resists the decrease in magnetic flux Φ_B.

$$F_B = iBL = \frac{\varepsilon}{R}BL = \frac{BL}{R}\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$
$$= \frac{BL}{R}(B\cos\varphi)\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{LB^2}{R}(vL\cos\varphi) = \frac{vL^2B^2}{R}\cos\varphi$$

b. The bar hits terminal velocity when the force of the magnetic field on the bar cancels out the force of gravity on it:

$$\frac{vL^2B^2}{R}\cos\varphi = mg\tan\varphi \Longleftrightarrow v = \frac{Rmg\tan\varphi}{L^2B^2\cos\varphi}.$$

c. The induced electric current in the bar is then

$$i = \frac{\mathcal{E}}{R} = \frac{1}{R}\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = \frac{B\cos\varphi}{R}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{vBL\cos\varphi}{R} = \frac{mg\tan\varphi}{BL}$$

d. $P = i^2R = R\left(\frac{mg\tan\varphi}{BL}\right)^2$

e. $P_g = mgv\cos(90° - \varphi) = mgv\sin\varphi = R\left(\frac{mg\tan\varphi}{BL}\right)^2$, exactly the same as in (d); it's a reasonable result because of conservation of energy.

30 Inductance

30.1 Summary

- When a changing current i_1 in one circuit causes a change in the magnetic flux of the second, an induced emf $\mathcal{E}_2$ appears in the second circuit, and vice versa. If the circuits are coils of wire with N_1 and N_2 turns, the *mutual inductance M* can be expressed in terms of the average flux Φ_{B2} through each turn of coil 2 caused by i_1, and vice versa. The unit of mutual inductance is the henry, H :

$$M = \frac{N_2\Phi_{B2}}{i_1} = \frac{N_1\Phi_{B1}}{i_2} \tag{30.5}$$

- *Mutually induced emfs:*

$$\mathcal{E}_2 = -M\frac{\mathrm{d}i_1}{\mathrm{d}t} \text{ and } \mathcal{E}_1 = -M\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{30.4}$$

- Any current carrying a variable current has an emf induced in it by the variation in its own magentic field; such an emf is called a *self-induced emf.* It can occur in any circuit, though the effect is magnified when there's a coil, called an *inductor,* present. The self-inductance L is defined as

$$L = \frac{N\Phi_B}{i} \tag{30.6}$$

  and the self-induced emf is

$$\mathcal{E} = -L\frac{\mathrm{d}i}{\mathrm{d}t} \tag{30.7}$$

- Inductors are essential components in electronics.
  - In DC circuits, they help regulate current despite any fluctuations in the applied emf.
  - In AC circuits, inductors oppose any rapid changes in the current.
- Energy stored in an inductor:

$$U = \int_0^I i\,\mathrm{d}i = \frac{1}{2}Li^2 \tag{30.9}$$

- Magnetic energy density in a vacuum:

$$u = \frac{B^2}{2\mu_0} \tag{30.10}$$

  In a material with permeability $\mu = K_m\mu_0$,

$$u = \frac{B^2}{2\mu} \tag{30.11}$$

- In a *RL* circuit, the current flowing through the entire circuit is

$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \tag{30.14}$$

  and the time constant in such a circuit is

$$\tau = \frac{L}{R}. \tag{30.16}$$

  - The current in an *RL* circuit decays according to

$$i = I_0 e^{\frac{-Rt}{L}} \tag{30.18}$$

- The angular frequency of oscillation of an *LC* circuit is given by

$$\omega = \sqrt{\frac{1}{LC}} \tag{30.22}$$

- Conservation of energy in an LC circuit:

$$\frac{1}{2}Li^2 + \frac{q^2}{2C} = \frac{Q^2}{2C}. \tag{30.25}$$

- In an LC circuit, $i = \pm\omega\sqrt{Q^2 - q^2}$
- When $R^2 < \frac{4L}{C}$,

$$q = Ae^{-\frac{Rt}{2L}}\cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}t + \varphi\right) \tag{30.28}$$

- $$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

30.2 Exercises

1. a. $|\mathcal{E}_2| = M\frac{\mathrm{d}i_1}{\mathrm{d}t} = 0.26975\mathrm{V}.$

   b. $|\mathcal{E}_1| = M\frac{\mathrm{d}i_2}{\mathrm{d}t} = 0.26975\mathrm{V}.$

4. $N_1 = 25, N_2 = 300, L_2 = 0.25\text{ m}, r_2 = 0.01\text{ m}, I_2 = 0.12\text{ A}, \frac{\mathrm{d}i}{\mathrm{d}t} = 1.75 \cdot 10^3\text{ A/ s}$

   a. $\Phi_{B2} = \mu_0 N_2 I A_2 \approx 5.68 \approx (-8)\text{ Wb}$

   b. $M = \frac{\mu_0 A N_1 N_2}{L} \approx 1.18 \cdot 10^{-5}\text{ H}$

   c. $|\mathcal{E}_1| = -M\frac{\mathrm{d}i_2}{\mathrm{d}t} \approx -0.02065\mathrm{V}.$

5. a. $M = \frac{N_2\Phi_{B2}}{i_1} \approx 1.96\text{ H}.$

   b. $\Phi_{B1} = \frac{Mi_2}{N_1} = 7.112 \cdot 10^{-3}\text{ Wb} + -.$

6. a. $M = \frac{\mu_0 N_1 N_2 A}{2\pi r}$

   b. $M = 2.4 \cdot 10^{-5}\text{ H}.$

7.

   a. $L = \frac{N\Phi_B}{i} = \frac{\mu_0 N^2 A}{2\pi r} \Longleftrightarrow N \approx 1936\text{ turns}$

   b. $\frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{\mathcal{E}}{L} = -800\text{ A/ s}$

11.

   a. $\mathcal{E} = 4.68 \cdot 10^{-3}\mathrm{V}.$

   b. a is at a higher potential.

13.
   a. $N = \sqrt{\dfrac{2\pi r L}{\mu_0 A}} = 1000.$

   a. $R \approx 2.09\ \Omega$

15. a. $L = \dfrac{N\Phi_B}{i} = \dfrac{N \cdot \mu_0 N A i}{l i} = \dfrac{\mu_0 N^2 A}{l}.$

    b. Plugging in the numbers, we get $L \approx 1.11 \cdot 10^{-7}$ H.

17. $U_L = \dfrac{1}{2} L I^2 \Longrightarrow N = \dfrac{2}{I}\sqrt{\dfrac{\pi r L}{\mu_0 A}} \approx 2850.$

19.
    a. $B = \mu_0 n I \approx 0.161$ T
    b. $u = \dfrac{B^2}{2\mu_0} \approx 10.3 \cdot 10^{-4} J/\ \mathrm{m}^3.$
    c. $U = uV = 0.12875\mathrm{J}.$
    d. $L = \dfrac{2U}{I^2} \approx 4.02 \cdot 10^{-5}$ H.

21. $U = \dfrac{B^2}{2\mu_0} V \approx 9167\mathrm{J}.$

23.
    a. $\left.\dfrac{\mathrm{d}i}{\mathrm{d}t}\right|_{t=0} = \dfrac{\mathcal{E}}{L} = 2.4\ \mathrm{A/\ s}$
    b. $\mathcal{E} - iR - L\dfrac{\mathrm{d}i}{\mathrm{d}t} = 0 \Longrightarrow \left.\dfrac{\mathrm{d}i}{\mathrm{d}t}\right|_{i=0.5\ \mathrm{A}} = 0.8\ \mathrm{A/\ s}$
    c. $i\big|_{t=0.25\ \mathrm{s}} = 0.41$ A.
    d. $i\big|_{t=\infty} = \dfrac{\mathcal{E}}{R} = 0.75$ A.

29. $\dfrac{\mathcal{E}}{L} e^{-\frac{Rt}{L}} = 7.2\ \mathrm{A/\ s} \Longleftrightarrow t \approx 0.015\ \mathrm{s} \Longrightarrow i = \dfrac{\mathcal{E}}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \approx 0.32\ \mathrm{A} \Longrightarrow v_{\mathrm{ab}} = 15.36\mathrm{V}.$

30. a. $P = 0$ because the inductor prevents a sudden buildup of current.

    a. $U_L = \dfrac{1}{2} L I^2 = 0.703125\mathrm{J}$

       $p_R = i^2 R = 4.5$ W

       The energy supplied by the battery is also 4.5 W.

31. a. $Q = \sqrt{LC} I = 4.43 \cdot 10^{-7}$ C.

    b. $q\big|_{i=0.5\ \mathrm{mA}} \approx 3.58 \cdot 10^{-7}$ C

34. $C = 18\mu\mathrm{F}, \mathcal{E} = 22.5\mathrm{V}, L = 12$ mH.

    a. $Q = Cv \Longrightarrow i_{\max} = \dfrac{Q}{\sqrt{LC}} \approx 0.87$ A

b. $T = 2\pi\sqrt{LC} \approx 2.92$ ms. The first time the capacitor discharges is at $0.25T = 0.73$ ms; the second time, $0.75T = 2.19$ ms.

36. $C = 4.18$ pF

a. $f = \dfrac{1}{2\pi\sqrt{LC}} \Longleftrightarrow L \approx 2.37$ mH.

b. $C_{\text{max}} = \dfrac{1}{(2\pi f)^2 L} \approx 3.67 \cdot 10^{-11}$ C = 36.7 pF.

37. $L = 80$ mH, $C = 1.25$ nF, $i_{\text{max}} = 0.75$ A.

a. $Q = i_{\text{max}}\sqrt{LC} = 7.5 \cdot 10^{-6}$ C.

b. $f = \dfrac{1}{2\pi\sqrt{LC}} \approx 15.92$ kHz.

c. $q = 1.25\cos(10^5 t) \Longrightarrow q|_{t=2.5\text{ ms}} \approx -0.428$ nF

$\Longrightarrow i^2 = \omega^2(Q^2 - q^2)$

39. $L = 0.45$ H, $C = 2.5 \cdot 10^{-5}$ F, $R = 320\ \Omega$.

a. The angular frequency of the circuit is $\omega = \sqrt{\dfrac{1}{LC}} \approx 298.14$ rad / s

b. $\omega' = \sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}} = 0.95\sqrt{\dfrac{1}{LC}} \Longrightarrow R \approx 83.79\ \Omega$.

40.

a. $t = 0 \Longrightarrow q|_{t=0\text{ s}} = A = 2.8 \cdot 10^{-4}$ C; $\varphi = k2\pi$.

b. $\omega = \sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}} \approx 1847$ rad /s

c. $T = \dfrac{2\pi}{\omega'} \approx 14.2$ ms.

d. $q' \approx 9.75 \cdot 10^{-7}$ C.

44. $N = 400$, $i = 680\cos\left(\dfrac{\pi t}{0.025}\right)$ mA, $L = 7.5$ mH

a. $\mathcal{E}_{\text{max}} = L \cdot \dfrac{680 \cdot 10^{-3}\pi}{0.025} = 0.641$V.

b. $\max \Phi_B = \dfrac{Li_{\text{max}}}{N} = 1.275 \cdot 10^{-5}$ T/m²

c. $\mathcal{E}|_{t=0.018\text{ s}} \approx -0.494$V.

46. a. $\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 i = B \cdot 2\pi r$

$\Longrightarrow \vec{\boldsymbol{B}} = \dfrac{\mu_0 i}{2\pi r}\vec{e}_{\theta}$.

b. $\mathrm{d}\Phi_B = l\dfrac{\mu_0 i\,\mathrm{d}r}{2\pi r}$.

c. $\Phi_B = \int_a^b l\frac{\mu_0 i\,\mathrm{d}r}{2\pi r}\,\mathrm{d}r = l\frac{\mu_0 i}{2\pi}\ln\left(\frac{b}{a}\right)$.

d. $\mathrm{d}L = \frac{\mathrm{d}\Phi_B}{i} \Longrightarrow L = l\frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$.

e. $U = \frac{1}{2}Li^2 = \frac{\mu_0 l i^2}{4\pi}\ln\left(\frac{b}{a}\right)$.

47. a. Same as in Problem 30.46: $\vec{\boldsymbol{B}} = \frac{\mu_0 i}{2\pi r}\vec{e}_\theta$.

b. $\mathrm{d}U = u\,\mathrm{d}V = \frac{\mu_0 i^2}{8\pi^2 r^2}\cdot 2\pi r l = \frac{\mu_0 l i^2}{4\pi r}\,\mathrm{d}r$

c. $U = \int_a^b \mathrm{d}U = \frac{\mu_0 l i^2}{4\pi}\ln\left(\frac{b}{a}\right)$

d. $L = \frac{2U}{i^2} = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right)$, exactly as in part (d) of Problem 30.46.

48. a. $i = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \Longrightarrow P_R = i^2 R = \frac{\mathcal{E}^2}{R}\left(1 - 2e^{-\frac{Rt}{L}} + e^{-\frac{2Rt}{L}}\right)$.

P_R is at a maximum when $t \longrightarrow \infty$, at which point $P_R = \frac{\mathcal{E}^2}{R}$

b. $P_L = -Li\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{\mathcal{E}^2}{R}\left(1 - e^{-\frac{Rt}{L}}\right)e^{-\frac{Rt}{L}}$.

c. $P_L\big|_{t\longrightarrow\infty} = \frac{\mathcal{E}^2}{R}; P_L\big|_{t\longrightarrow\infty} = 0$

d. $\frac{\mathrm{d}P_L}{\mathrm{d}t} = 0 \Longrightarrow t = -\frac{L}{R}\ln\left(\frac{1}{2}\right) = \frac{L}{R}\ln(2)$.

e. $P_{\mathcal{E}} = \mathcal{E}\cdot i = \frac{\mathcal{E}^2}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \Longrightarrow \max P_{\mathcal{E}} = \frac{\mathcal{E}^2}{R} \Longleftrightarrow t \longrightarrow \infty$

50.

31 AC Currents

31.1 Summary

- $i = I\cos\omega t$
- $I_{\mathrm{rms}} = \frac{I}{\sqrt{2}}$
- $V_{\mathrm{rms}} = \frac{V}{\sqrt{2}}$
- $Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left[\omega L - \left(\frac{1}{\omega C}\right)\right]^2}$
- $\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R} = \frac{Z_L - Z_C}{R}$
- $P_{\mathrm{av}} = \frac{1}{2}VI\cos\varphi$

31.2 Exercises

49.

a. $I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$

b. $P = \frac{1}{2}I^2 R = \frac{0.5VR^2}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$

c. In either case $\omega \longrightarrow 0 \vee \omega \longrightarrow \infty,\ Z \longrightarrow \infty \Longrightarrow I, P \longrightarrow 0$.

32 Electromagnetic Waves

32.1 Summary

- Electromagnetic wave in vaccum:

$$E = cB \tag{32.4}$$

$$B = \mu_0 \varepsilon_0 c E \tag{32.8}$$

- Speed of electromagnetic waves in vacuum:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \tag{32.9}$$

- Key properties of electromagnetic waves:
  1. The wave is *transverse:* both $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are perpendicular to the direction of propagation of the wave. The electric and magnetic fields are also perpendicular to each other. The direction of propagation is the direction of the vector product $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$.
  2. $E = cB$.
  3. The wave travels in vaccum with an unchanging speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.
  4. Unlike mechanical waves, which need the particles of a medium such as air to transmit a wave, electromagnetic waves require no medium.

- Speed of electromagnetic waves in a dielectric:

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{KK_m}} \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{c}{\sqrt{KK_m}}. \tag{32.21}$$

- If an electromagnetic wave is propagating in the $+x$-direction, then

$$\begin{aligned}\vec{\boldsymbol{E}}(x,t) &= E_{\max}\cos(kx-\omega t)\vec{e}_y \\ \vec{\boldsymbol{B}}(x,t) &= B_{\max}\cos(kx-\omega t)\vec{e}_z\end{aligned} \tag{32.17}$$

If it’s propagating in the $-x$-direction, on the other hand, then

$$\begin{aligned}\vec{\boldsymbol{E}}(x,t) &= E_{\max}\cos(kx+\omega t)\vec{e}_y \\ \vec{\boldsymbol{B}}(x,t) &= -B_{\max}\cos(kx+\omega t)\vec{e}_z\end{aligned} \tag{32.19}$$

- Energy density of an electromagnetic wave:

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0} = \varepsilon_0 E^2 = \frac{B^2}{\mu_0} \tag{32.25}$$

- The *Poynting vector* respresents the direction of the energy flow rate of an electromagnetic wave and is defined as

$$\vec{\boldsymbol{S}} = \frac{1}{\mu_0}\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} \tag{32.28}$$

whose value is

$$S = \frac{EB}{\mu_0}. \tag{32.27}$$

- Intensity of a sinusoidal electromagnetic wave in vacuum:

$$I = S_{\text{av}} = \frac{E_{\max}B_{\max}}{2\mu_0} = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}E_{\max}^2 = \frac{1}{2}\varepsilon_0 c E_{\max}^2 = \frac{E_{\max}^2}{2\mu_0 c} \tag{32.29}$$

- Flow rate of electromagnetic momentum:

$$\frac{1}{A}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{S}{C} = \frac{EB}{\mu_0 c} \tag{32.31}$$

- If a perfect reflecting surface is placed at $x = 0$, the incident and reflected waves form a standing wave. Nodal planes for $\vec{\boldsymbol{E}}$ occur at $kx = 0, \pi, 2\pi, \ldots$, and nodal planes for $\vec{\boldsymbol{B}}$ at $kx = \pi/2, 3\pi/2, 5\pi/2$. At each point, the sinusoidal variations of $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ with time are 90° out of phase.
- Radiation pressure:

$$p_{\text{rad}} = \frac{S_{\text{av}}}{c} = \frac{I}{c} \quad \text{(wave totally absorbed)} \tag{32.32}$$

$$p_{\text{rad}} = \frac{2S_{\text{av}}}{c} = \frac{2I}{c} \quad \text{(wave totally absorbed)} \tag{32.33}$$

32.2 Exercises

Problem 32.28 ( )

A laser has diameter 1.2 mm. What is the amplitude of the electric field of the electromagnetic radiation in this beam if it exerts a force of $3.8 \cdot 10^{-9}$ N on a totally reflecting surface?

Solution: $A \approx 1.131 \cdot 10^{-6}\ \mathrm{m}^2 \implies p = \dfrac{E^2}{\mu_0 c^2} = \dfrac{F}{A} \iff E = \sqrt{\dfrac{F\mu_0}{A}}c \approx 1.9 \cdot 10^{-4}\ \mathrm{V/\ m}$.

34. $\dfrac{\partial E_y(x,t)}{\partial x} = -\dfrac{\partial B_z(x,t)}{\partial t} \implies \varphi = 0 \wedge kE = \omega B$.
$-\dfrac{\partial B_z(x,t)}{\partial x} = \mu_0\varepsilon_0 \dfrac{\partial E_y(x,t)}{\partial t} \implies k = \mu_0\varepsilon_0\omega$
$\implies E_{\max} = cB_{\max}$.

Problem 32.51 ( )

Electromagnetic radiation is emitted by acclerating charges. The rate at which energy is emitted from an accelerating charge that has charge q and acceleration a is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{q^2a^2}{6\pi\varepsilon_0 c^3}$$

where c is the speed of light.

a. Verify that this equation is dimensionally correct.
b. If a proton with a kinetic energy of 6 MeV is traveling in a particle accelerator in a circular orbit of radius 0.75 m, what fraction of its energy does it radiate per second?
c. Consider an electron orbiting with the same speed and radius. What fraction of its energy does it radiate per second?

Solution:

a.
b. $a = \dfrac{0.5mv^2}{0.5mR} \approx 1.53 \cdot 10^{15}\ \mathrm{m/s^2} \implies \dfrac{\mathrm{d}E}{\mathrm{d}t} \approx 1.3 \cdot 10^{-23}\mathrm{J/s}$
$\implies \dfrac{\frac{\mathrm{d}E}{\mathrm{d}t}}{E} \approx 1.39 \cdot 10^{-11}$.
c. Applying the same calculations for the electron, we get

$$\frac{\frac{dE}{dt}}{E} \approx 2.54 \cdot 10^{-8}.$$

Part V - Optics

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Part VI - Modern Physics

37 Relativity

37.1 Summary

- To formulate his theory of relativity, Einstein put forward two hypotheses:
  1. The laws of physics are the same in ever inertial frame of reference.
  2. The speed of light in vacuum is the same in all inertial frames of reference and is independent of the motion of the source.
- For speeds $v \ll c$, we can use the *Galilean coordinate and velocity transformations* to describe the position and velocity of a moving and stationary object, whose inertial frames of reference are S' and S', respectively:

$$x = x' + ut, \quad y = y', \quad z = z' \tag{37.1}$$

$$\frac{dx}{dt} = \frac{dx'}{dt} + u \tag{37.2}$$

- Proper time is the time interval between two events that occur at the same point.
- The *time dilation* of a moving frame of reference relative to a statoinary frame is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{37.6}$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$ is the *Lorentz factor;* thus the above equation can be written as

$$\Delta t = \gamma \Delta t_0, \tag{37.7}$$

with Δt_0 being the proper time between two events, measured in rest frame.

- *Length contraction:*

$$l = l_0 \sqrt{1 - \frac{u^2}{c^2}} = \frac{l_0}{\gamma} \tag{37.16}$$

where l_0 is the proper length of the object.

- For more general cases (say, when v is a significant percentage of c), we can use the *Lorentnz coordinate transformations* to describe the movement of a frame S' relative to S:

$$\begin{aligned} x' &= \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \frac{t - \frac{ux}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma\left(t - \frac{ux}{c^2}\right) \end{aligned} \tag{37.21}$$

Conversely, we have

$$\begin{aligned} x &= \frac{x' + ut'}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma(x' + ut') \\ y' &= y \\ z' &= z \\ t &= \frac{t' - \frac{ux'}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma\left(t' + \frac{ux'}{c^2}\right) \end{aligned}$$

- In Lorentz velocity transformation,
  - The velocity of S' in terms of velocity in S is

$$v'_x = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \tag{37.22}$$

- The velocity of S in terms of velocity in S' is

$$v_x = \frac{v'_x + u}{1 + \frac{uv'_x}{c^2}} \tag{37.22}$$

- Electromagnetic waves also experience the Doppler effect when the source is moving, and when it's approaching the observer,

$$f = \sqrt{\frac{c + u}{c - u}} f_0 \tag{37.25}$$

When it's moving away from the observer,

$$f = \sqrt{\frac{c - u}{c + u}} f_0 \tag{37.26}$$

- The *rest mass m* of a particle is its mass when measured at rest relative to us; a *material particle* is one that has a nonzero rest mass.
- The *relativistic momentum* $\vec{\boldsymbol{p}}$ is

$$\vec{\boldsymbol{p}} = \frac{m\vec{\boldsymbol{v}}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m\vec{\boldsymbol{v}} \tag{37.27}$$

- Relativistic kinetic energy:

$$K = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2 = (\gamma - 1)mc^2 \tag{37.36}$$

- The total energy of a material particle is

$$E = K + mc^2 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma mc^2 \tag{37.38}$$

- Total energy relates to rest energy and momentum as follows:

$$E^2 = \left(mc^2\right)^2 + (pc)^2 \tag{37.39}$$

37.2 Exercises

5. $\Delta t_0 = 26$ ns, $\Delta t = 420$ ns

a. $v = \sqrt{1 - \frac{26^2}{420^2}}c \approx 0.998c.$

b. $d = c\Delta t \approx 7.76$ m.

6. $v = 0.8c$

   a. $\Delta t = 0.5$ s; $\Delta t_0 = \Delta t/\gamma = 0.33$ s

   b. $d = v\Delta t_0 = 800000$ m.

   c. $s = 1.2 \cdot 10^8$ m.

7. $\Delta t = 365$ days, $v = 4.8 \cdot 10^6$ m/s.

   $\Delta t_0 = \Delta t\gamma = 364.95$ days. The shorter elapsed time is shown on the clock in the spacecraft.

Problem 37.11 (Why Are We Bombarded by Muons?)

Muons are unstable subatomic particles that decay to electrons with a mean lifetime of 2.2μs. They are produced when cosmic rays bombard the upper atmosphere about 10 km the earth's surface and they travel to the speed of light. The problem we want to address is why we see any of them at the earth's surface.

a. What is the greatest distance a muon could travel during its 2.2μs lifetime?
b. According to your anser in (a), it'd seem that muons could never make it to the ground. But the 2.2μs lifetime is measured in the frame of the muon and muons are moving very fast, ≈ $0.999c$, what's the mean lifetime of a muon as measured by an observer at rest on the erath? How far would the muon travel in this time? Does this result explain why we find muons in cosmic rays?
c. From the muon's POV, it still lives for only 2.2μs, so how does it make it to the ground? What is the thickness of the 10 km of atmosphere through which the muon must travel, as measured by the muon? Is it now clear how the muon is able to reach the ground?

Solution: $\Delta t_0 = 2.2\mu\text{s}$,

a. $d_{\text{max}} = c\Delta t = 660$ m.

b. $\Delta t = \frac{\Delta t_0}{\gamma} \approx 4.9 \cdot 10^{-5}$ s.

   The distance said muon would travel relative to a terrestial observer is $d \approx 14.7$ km.

c. For the muon, its average lifetime would still be the same, so for it to reach the ground, the atmosphere from its point of view is relativistically contracted: $\Delta l = \Delta l_0/\gamma \Longrightarrow \Delta l_0 = 447$ m.

15. Simply use

$$v_x = \frac{v'_x + u}{1 + \dfrac{v'_x \cdot u}{c^2}}$$

    a. $v \approx 0.806c$
    b. $v \approx 0.974c$
    c. $v \approx 0.997c$

17. a. The velocity of the cruiser is directed *toward* from the pursuit ship relative to the pursuit ship because the distance between them is decreasing.
    b. $u = 0.8c, v_x = 0.6c \Longrightarrow v'_x = -0.385c$.

18. $u = 0.4c, v' = 0.7c$.
    a. $v = \dfrac{v' + u}{1 + \dfrac{uv'}{c^2}} \approx 0.8594c$
    b. $t = d/v \approx 31$ s.

19. $u = 0.65c, v = -0.95c$

$$|v'| = \frac{v + u}{1 + \dfrac{uv}{c^2}} \approx 0.784c.$$

20. $v' = 0.938c, u = -0.938c$

$$|v| = \frac{v' - u}{1 - \dfrac{uv'}{c^2}} = 0.9998c.$$

21. $v' = 0.89c$

$$v'_x = \frac{2v}{1 + \dfrac{v^2}{c^2}} \Longleftrightarrow v \approx 0.611c.$$

25. Per the Doppler effect, we can conclude that the source is approaching ~~provoking black clouds in isolation~~ and its speed is

$$u = \frac{(f/f_0)^2 - 1}{(f/f_0)^2 + 1}c \approx 0.21951c$$

26. a. $F = ma = 0.145$ N

    b. $F = \gamma^3 ma = 1.75$ N

c. $F = \gamma^3 ma = 51.65$ N

d. $F = 0.145$ N, $F = 0.333$ N, $F = 1.02$ N

28. $v = \sqrt{1 - \frac{1}{\gamma^2}} c$

a. $v = 0.14c$.

b. $v = 0.4166c$.

c. $v = 0.866c$.

31. $K = (\gamma - 1)mc^2 =$

a. $\gamma = 2 \iff v = 0.866c$.

b. $\gamma = 6 \iff v \approx 0.986c$.

36. a. $(\gamma - 1)mc^2 = 7.5 \cdot 10^5$ eV $\iff v \approx 0.731c$

b. When computed from the principles of classical mechanics, we have $v = \sqrt{\frac{2k}{m}} \approx 5.14 \cdot 10^8$ m/s.

Problem 37.45 (   )

The starships of the Solar Federation are marked with the symbol of the fedderation, a circle, while starships of the Denebian Empire are marked with the empire's symbol, an ellipse whose major axis is 1.4 times longer than its minor axis ($a = 1.4b$). How fast, relative to an observer, does an empire ship have to travel for its marking to be confused with the marking of a fed ship?

Solution: For an imperial ship to be confused with a federation ship, we must have

$$l = b = \frac{l_0}{\gamma} = 1.4\frac{b}{\gamma} \implies \gamma = 1.4 \iff v \approx 0.7c.$$

46. Only the edge that's parallel to the x-axis experiences relativistic length contraction, so the volume of the cube is then

$$V = a^3\sqrt{1 - \frac{v^2}{c^2}}.$$

51. $u = 250$ m/s, $\Delta t = 4$ h.

The elapsed time on the airliner will be shorter, and the difference in the readings is $\Delta t - \Delta t_0 = \Delta t\left(1 - 1 + \sqrt{1 - u^2c^2}\right) \sim \Delta t \cdot \dfrac{u^2}{2c^2} = 5$ ns.

54. $p = 2.52 \cdot 10^{-19}$ kg $\cdot$ m/s; $v = 1.35 \cdot 10^8$ m/s

    $\gamma \approx 0.89 \Longrightarrow m = \dfrac{p}{\gamma v} \approx 1.67 \cdot 10^{-27}$ kg $\Longrightarrow$ The particle is a proton.

56. S is the lab frame and S' is the frame of the proton moving in the x-direction, $v'_x = -0.7c$.

    In S, each proton has speed ac; $u = ac, v_x = -ac$; $v_x = \dfrac{v_x + u}{1 + \dfrac{uv'_x}{c^2}} = -ac$.

    $$\begin{aligned} &\Longleftrightarrow 0.7a^2 - 2a + 0.7 = 0 \\ &\Longleftrightarrow a = 2.45 \vee a = 0.408. \end{aligned}$$

    Thus each proton has measured speed $0.408c$.

58. Setting $x = 0$, we have $x' = -\gamma ut$ and $t' = \gamma t$.

    $\Longrightarrow (t')^2 - \left(\dfrac{x'}{c}\right)^2 = t^2 \Longleftrightarrow x' = c\sqrt{t'^2 - t^2} \approx 3.53 \cdot 10^8$ m.

Problem 37.69 (Kaon Production -)

In high-energy physics, new particles can be created collisions of fast-moving projective with stationary particles. Some of the kinetic energy of the incident particle is used to create the mass of the new particle. A proton-proton collision can result in the creation of a negative kaon (K^-) and a positive kaon (K^+):

$$p + p \longrightarrow p + p + K^- + K^+.$$

a. Calculate the minimum kinetic energy of the incident proton that will allow this reaction to occur if the second (target) proton is intially at rest. The rest energy of each kaon is 493.7 MeV, and the the rest energy of each proton is 938.3 MeV.
b. How does this calculated minimum kinetic enegy compare with the total rest mass energy of the created kaons?
c. Suppose that instead the two protons are both in motion with velocities of equal kinetic energy of the two protons that will allow the reaction occur. How does this calculated minimum kinetic energy compare with the total rest mass energy of the created kaons?

---

a. It's useful here to to work in the frame in which the total momentum is zero. But note that the Lorentz transformation must be used to relate the velocities in the lab frame to those in the zero-total-momentum frame.
c. This example shows that when colliding beams of particles are used instead of a stationary target, the energy requirements for producing new particles are reduced substantially.

Solution:

a. $2(\gamma_{\text{cm}} - 1)m_p c^2 = 2m_K c^2 \Longrightarrow \gamma_{\text{cm}} = 1 + \dfrac{m_K}{m_p} \approx 1.526$. Then, the velocity of a proton in the center of the momentum frame is then

$$v_{\text{cm}} = c\sqrt{\frac{\gamma_{\text{cm}}^2 - 1}{\gamma_{\text{cm}}^2}} = 0.7554c.$$

Taking the lab frame to be the unprimned frame moving to the left, $u = v_{\text{cm}}$ and $v' = v_{\text{cm}}$.

$$v_{\text{lab}} = \frac{v' + u}{1 + \frac{v_{\text{cm}}^2}{c^2}} = 0.9619c$$

$$\implies \gamma_{\text{lab}} = \frac{1}{\sqrt{1 - \frac{v_{\text{lab}}^2}{c^2}}} = 3.658 \implies K_{\text{lab}} = (\gamma_{\text{lab}} - 1)m_p c^2 = 2494 \text{ MeV}.$$

b. $\dfrac{K}{m_K} \approx 2.526.$
c. $K_{\text{cm}} = 2m_K = 987.4$ MeV.

Problem 37.70 (Relativity and the Wave Equation)

a. Consider the Galilean transformation along the x-direction: $x' = x - vt$ and $t' = t$. In frame S the wave equation for electromagnetic waves in a vacuum is

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2} = 0$$

where E represents the electric field in the wave. Show that by using the Galilean transformation the wave equation in frame S' is found to be

$$\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 E(x',t')}{\partial x'^2} + \frac{2v}{c^2}\frac{\partial^2 E(x',t')}{\partial x' \partial t} - \frac{\partial^2 E(x',t')}{\partial t'^2} = 0.$$

This has a different form than the wave equation in S. Hence the Galilean transformation vioaltes the first relativity postulate that all physical laws have the same form in all inertial references.

b. Repeat the analysis of part (a), but use the Lorentz transformations, and show that in S' the wave equation has the form as in frame S. Explain why this shows that the speed of light in vacuum is c in both frames S and S'.

38 Photons: Light Waves Behaving as Particles

38.1 Summary

- The *photoelectric effect* is a phenomenon in which photons strike the surface of a material (namely, metals), which causes electrons to eject.
- The *stopping potential* is the minimum voltage applied to the anode to stop to the photoelectric effect, and the work by an electron moving in an electric field that has such a voltage is also the maximum kinetic energy of any electron:

$$K_{\text{max}} = \frac{1}{2}mv_{\text{max}}^2 = eV_0 \tag{38.1}$$

where V_0 is the stopping potential.

- The expermients testing if the photoelectric effect is consistent with the wave model of light produced very different results. To resolve this, Albert Einstein proposed that light propagates in the form of packets of energy called *photons.* Each photon has energy

$$E = hf = \frac{hc}{\lambda}. \tag{38.2}$$

Here, h is called the *Plank constant,* and its value is

$$h \approx 6.626 \cdot 10^{-34}\,\text{J} \cdot \text{s}.$$

- Einstein postulated that the each photon gets absorbed by one electron, and photoeletrons will be ejected only when $hf > \varphi$.
- The photoelectric effect:

$$eV_0 = hf - \varphi \tag{38.4}$$

- The momentum of a photon is

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}. \tag{38.5}$$

- The inverse of the photoelectric effect is also true: we can make a surface emit electromagnetic radiation by bombarding it with electrons.
- When a cathode is heated, it releases electrons. Their movement is halted by the anode, creating X rays in the process; this process is called *bremsstrahlung.*
- In bremsstrahlung, the energy of an emitted photon relates to the lost energy of an electron as folows:

$$eV_{\text{AC}} = hf_{\text{max}} = \frac{hc}{\lambda_{\text{min}}}. \tag{38.6}$$

- *Compton scattering* occurs when a photon collides with an electon. In this process, the photon is scattered, imparting some of its energy and momentum to the electron. The new wavelength is related by

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\varphi). \tag{38.7}$$

- Using the photon model, we can explain the phenomenon of *pair production,* where a gamma-ray of sufficiently short wavelength generates a electon-positron pair upon hitting a target. The inverse process, *electron-positron pair annihilation,* generates photons with total energy of at least $2m_e c^2 = 1.022\text{ MeV}$.
- The principle of *complementarity* by Niels Bohr states that it's impossible to observe or measure certain pairs of complementary properties, and the Heisenberg uncertainty principle is the particulariztion of that principle to the position and momentum of a particle:

$$\boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}} \qquad (38.17)$$

where

$$\hbar = \frac{h}{2\pi}.$$

The Heisenberg uncertainty principle also applies for energy and time:

$$\boxed{\Delta t \Delta E \geq \frac{\hbar}{2}.} \qquad (38.24)$$

38.2 Exercises

5. a. $E = 1.545\text{ eV}$
   b. $\lambda = \frac{hc}{E} \approx 804\text{ nm}$, infrared.
7. We have

$$\frac{mv_{\max}^2}{2} = \frac{hc}{\lambda} - \varphi$$

$$v = \sqrt{2\frac{hc\lambda^{-1} - \varphi}{m}} \approx 2.56 \cdot 10^5\text{ m/s}.$$

8. Visible light: 380 – 750 nm.
   $\Longrightarrow$ Minimum work function = longest wavelength
   $\Longleftrightarrow \varphi_{\min} \frac{hc}{\lambda_{\max}} \approx 1.66\text{ eV}$
9. $\lambda_1 = 400\text{ nm}, \lambda_2 = 300\text{ nm}\ \varphi = E - K_{\max} \approx 2\text{ eV}. \Longrightarrow K = \frac{hc}{\lambda_2} - \varphi \approx 2.13\text{ eV}.$
10. a. $V_0 = e^{-1}\left(\frac{hc}{\lambda} - \varphi\right) \approx 2.59\text{ eV}.$
    b. $K_{\max} = 2.59\text{ eV}.$
    c. $v = \sqrt{2K_{\max}m^{-1}} = 9.54 \cdot 10^5\text{ m/s}.$
14. a. $V_{\min} = \frac{hc}{e\lambda} \approx 8266\text{V}.$

b. $\lambda_{\min} = \dfrac{hc}{eV} = 0.413\text{Å}.$

15. Energy is conserved when the X ray hits the electron:

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + K_e$$
$$\Longrightarrow K_e \approx 1.81 \cdot 10^{-16}\,\text{J} = 1.13125\text{ eV}.$$

17. $\lambda' - l = \dfrac{h}{mc}(1 - \cos\varphi)$

$\lambda'_{\max} = \dfrac{2h}{mc} + \lambda = 0.715\text{Å}$.

18. $\dfrac{hc}{\lambda'} = \dfrac{hc}{\lambda} - K_e$

20. $0.1\lambda = \dfrac{2h}{mc} \Longleftrightarrow \lambda = 0.485\text{Å}.$

19. a. $\Delta\lambda = 4.39 \cdot 10^{-13}m.$

b. $\lambda' = 4.2939 \cdot 10^{-11}m.$

c. Energy gained by the electron:

$$hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = 0.2725\text{ eV}.$$

26. a. $n = \dfrac{0.1p}{hf} \approx 3.62 \cdot 10^{19}$ photons

b. $\dfrac{n}{4\pi r^2} = 10^{11}$ photons $/\text{cm}^2 \Longrightarrow n \approx 5367\text{ cm} = 53.67\text{ m}.$

29. a. $\lambda' = \dfrac{2h}{mc} + \lambda = 0.09485\text{ nm} \Longrightarrow p' = \dfrac{h}{\lambda'} = 6.99 \cdot 10^{-24}\text{ kg}\cdot\text{ m/s}.$

b. $K = hc\left(\dfrac{1}{\lambda} - \dfrac{1}{\lambda'}\right) \approx 705\text{ eV}.$

31. $\lambda' = \dfrac{h}{2mc} + \lambda \approx 0.1062\text{ nm}.$

$p_e = 6.31 \cdot 10^{-24}\angle 0 - 6.24 \cdot 10^{-24}\angle 60° \approx 6.28 \cdot 10^{-24}\text{ kg}\cdot\text{ m/s}, \varphi_e = -59.4°.$

36. a. $\lambda = \left(\dfrac{1}{\lambda'} + \dfrac{mc^2}{hc\sqrt{1 - v^2c^{-2}}}\right)^{-1}.$

b. $\varphi = \arccos\left(1 - \dfrac{mc\Delta\lambda}{h}\right).$

Problem 38.40 ( )

Consider Compton scattering of a photon by a *moving* electron. Before the collision the phton has wavelength λ and is moving in the $+x$-direction, and the electron is moving in the $-x$-direction with total energy E (including its rest energy mc^2). The photon and electron collide head-on. After the collision, both are moving in the $-x$-direction (that is, the proton has been scattered by 180°).

a. Derive an expression for the wavelength λ' of the scattered proton. Show that if $E \gg mc^2$, where m is the rest mass of the electron, your result reduces to

$$\lambda' = \frac{hc}{E}\left(1 + \frac{m^2c^4\lambda}{4hcE}\right).$$

b. A beam of infrared from a CO2 laser ($\lambda = 10.6\mu$m) collides head-on with a beam of electrons, each of total energy $E = 10$ GeV. Calculate the wavelength λ' of the scattered protons, assuming a 180° scattering angle.
c. What kind of scattered photons are these (infrared, microred, ultraviolet, etc.)? Can you think of an application of this effect?

Solution:

a. Momentum: $p - P = -p - P' \iff p' = P - (p + P')$

Energy:

$$pc + E = p'c + E' = p'c + \sqrt{(P'c)^2 + (mc^2)^2}$$

$$\Longrightarrow (pc - p'c + E)^2 = (P'c)^2 + \left(mc^2\right)^2 = (Pc)^2 + [(p - p')c]^2 - 2P(p + p')c^2 + \left(mc^2\right)^2$$

$$(pc - p'c)^2 + E^2 = E^2 + (pc + p'c)^2 - 2Pc^2(p + p') + 2Ec(p - p') - 4pp'c^2 + 2Ec(p - p')$$

$$+2\left(Pc^2\right)(p + p') = 0$$

$$\Longrightarrow p'\left(Pc^2 - 2pc^2 - Ec\right) = p\left(-Ec - Pc^2\right)$$

$$\Longrightarrow p' = p\frac{Ec + Pc^2}{2pc^2 + Ec - Pc^2} = p\frac{E+Pc}{2pc+(E-Pc)}$$

$$\lambda' = \lambda\left[\frac{2hc/\lambda + (E - Pc)}{E + Pc}\right] = \lambda\left(\frac{E - Pc}{E + Pc}\right) + \frac{2hc}{E + Pc}$$

$$\Longrightarrow \lambda' = \frac{\lambda(E - Pc) + 2hc}{E + Pc}.$$

If $E \gg mc^2$,

$$Pc = \sqrt{E^2 - (mc^2)^2} = E\sqrt{1 - \frac{m^2c^4}{E^2}} \approx E\left(1 - \frac{m^2c^4}{2E^2} + \dots\right)$$

$$\Longrightarrow E - Pc \approx \frac{1}{2}\frac{m^2c^4}{E} \Longrightarrow \lambda' \approx \frac{\lambda m^2c^4}{4E^2} + \frac{hc}{E} = \frac{hc}{E}\left(1 + \frac{m^2c^4\lambda}{4hcE}\right).$$

b. Plugging in the numbers, we get $\lambda' = 7.05 \cdot 10^{-15}$ m.

c. The scattered photons we produced using that infrared CO2 laser are gamma rays. It could be used in high-energy physics, nuclear physics, nuclear spectroscopy, etc… anywhere controlled gamma ray sources prove useful.

39 Particles Behaving as Waves

39.1 Summary

- De Brogile made the following important observation: that just as light, *matter* is also dualistic, meaning that matter exhibits both wave and particle behaviour. He proposed that a particle with rest mass m, moving with non-relativistic speed v, will have wavelength

$$\lambda = \frac{h}{p} = \frac{h}{mv}. \tag{39.1}$$

  λ is called the *de Brogile* wavelength of that particle.

  If the particle is moving at a relativistic speed, we replace mv with γmv.

- The de Brogile wavelength of an electron is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV}} \tag{39.2}$$

  where V is the accelerating voltage.

- According to classical physics, Rutherford's model of the atom (where electrons revolve around the nucleus) would fail since they'd eventually lose energy, quickly spiraling into the nucleus while radiating a continuous spectrum in so doing. But in fact, atoms are stable, and they only emit (and absorb light) at only specific frequencies. To resolve this conundrum, Niels Bohr proposed a new idea: Atoms can only have certain energy levels, and once excited, they can only have those specific amounts of energy, but *not* the intermediate amount between two such levels. When excited, they transition from a higher to a lower state, emitting a photon in the process. That photon has energy equal to the difference between the two states:

$$hf = \frac{hc}{\lambda} = E_i - E_f \tag{39.5}$$

- The fact that atoms are stable proves that for each atom, there exists a *ground state* in which its energy is at its lowest.
- Bohr also discovered that the angular momentum of an electron in a hydrogen atom (the simplest one) is quantized: the magntiude must be a multiple of $\hbar$.

$$L_n = mv_n r_n = n\frac{h}{2\pi} = n\hbar, \tag{39.6}$$

  where n is the *principal quantum number, assigned to each orbit.*
- The radius of the nth orbit in the Bohr model is

$$r_n = \varepsilon_0 \frac{n^2 h^2}{\pi m e^2} = n^2 a_0, \tag{39.8}$$

  where

$$a_0 = \varepsilon_0 \frac{h^2}{\pi m e^2} \approx 5.29 \cdot 10^{-11} \text{ m}$$

  is called the *Bohr radius*, corresponding to the radius of the ground state.
- The orbital speed of the nth orbit in the Bohr model is

$$v_n = \frac{1}{\varepsilon_0} \frac{e^2}{2nh}. \tag{39.9}$$

- Some wavelength series to remember:
  - **Lyman (n = 1): Ultraviolet**
  - **Balmer (n = 2): Visible light and infrared**
  - Paschen (n = 3): Infrared
  - Brackett (n = 4): Infrared
  - Pfund (n = 5): Infrared
- Total energy for nth orbit in the Bohr model:

$$E_n = -\frac{hcR}{n^2} = -\frac{1}{\varepsilon_0} \frac{me^4}{8n^2h^2}, \tag{39.15}$$

  where

$$R = \frac{me^4}{8\varepsilon_0^2 h^3 c}$$

  is the *Rydberg constant.*
- Unlike gases, liquids and solids emit a continuous spectrum.

- The Stefan-Boltzmann law for a blackbody:

$$\boxed{I = \sigma T^4. \qquad (39.19)}$$

- Wien displacement law for a blackbody:

$$\boxed{\lambda_m T = 2.9 \cdot 10^{-3}\mathrm{m} \cdot \mathrm{K} \qquad (38.21)}$$

- Planck radiation law:

$$\boxed{I(\lambda) = \frac{2\pi hc^2}{\lambda^5(e^{hc/(\lambda kT)} - 1)}. \qquad (39.24)}$$

39.2 Exercises

1. a. $\lambda = 1.55\text{Å}$
   b. $\lambda = 8.43 \cdot 10^{-14}$ m
2. a.
   b.
4. $\lambda = \dfrac{h}{\sqrt{2mE}} = 7.01 \cdot 10^{-15}$ m
8.

Problem 39.55 ( )

A sample of H atoms is irradiated with light of wavelength 85.5 nm, and electrons are observed leaving the gas.

a. If each hydrogen atom were initially in its ground level, what would be the maximum kinetic energy in eV of these photoelectons?
b. A few electrons a detected with energies as much as 10.2 eV greater than the maximum kinetic energy calculated in part (a). How can this be?

Solution:

a. $E_f = E_i + hc\lambda^{-1} = 0.9$ eV.
b. Some atoms are already in the $n = 2$ state, having initial energy $E_2 = -3.4$ eV. When struck by the given light source, these electrons will have maximum kinetic energy 11.1 eV, 10.2 eV greater than the maximum kinetic energy in part (a).

Problem 39.87 ( )

a. Show that in the Bohr model, the frequency of revolution of an electron in its circular orbit around a stationary hydrogen nucleus is

$$\frac{me^4}{4\varepsilon_0^2 n^3 h^3}.$$

b. In classical physics, the frequency of revolution of the is equal to the frequency of the radation it emits. Show that when n is very large, the frequency of radiation does indeed equal the radiated frequency calculated from Eq. 395 for a transition from $n_1 = n + 1$ to $n_2 = n$.

Solution:

a. $f = \frac{v_n}{2\pi r_n} = \frac{\varepsilon^2}{2\varepsilon_0 nh} \frac{me^2}{2\varepsilon_0 n^2 h^2} = \frac{me^4}{4\varepsilon_0^2 n^3 h^3}.$

b.

$$\begin{aligned} hf = E_{n+1} - E_n &= \frac{me^4}{8\varepsilon_0 h^2}\left[-\frac{1}{n^2} + \frac{1}{(n+1)^2}\right] \\ \iff f &= \frac{me^4}{8\varepsilon_0 h^3} \frac{1+2n}{n^2(n+1)^2} \\ &\sim \frac{me^4}{8\varepsilon_0 h^3} \cdot \frac{2n}{n^4} \\ &= \frac{me^4}{4\varepsilon_0 n^3 h^3} \quad (n \longrightarrow \infty) \end{aligned}$$

40 Quantum Mechanics I: Wave Functions

40.1 Summary

- Notation of wave functions in quantum mechanics:
  - $\Psi(x, y, z, t)$: wave function of a particle, including temporal element
  - $\psi(x, y, z)$: wave function of a particle, without temporal element
- A *free particle* is one that moves along a straight line without being acted on by any forces. In such a particle, the relationship between angular frequency and wave number is as follows:

$$\boxed{\hbar\omega = \frac{\hbar^2 k^2}{2m}. \qquad (40.8)}$$

- The sinusoidal wave function representing a free particle in 1D is

$$\Psi(x,t) = A[\cos(kx - \omega t) + i\sin(kx - \omega t)] = Ae^{i(kx-\omega t)} = Ae^{ikx}e^{-i\omega t} \qquad (40.16)$$

- The wave function can be interpreted as follows: $\Psi(x,t)$ is the distribution of a particle in space, and the square of its absolute (absolute because the function has real and imaginary parts), $|\Psi|^2$, tells us the probability of finding that particle around each point.

  For a particle moving along the x-axis only, the quantity $|\Psi(x,t)|^2\,\mathrm{d}x$ is the probability of finding it at time t within the region from x to $x + \mathrm{d}x$. For this to be true, the wave function must be normalized, meaning that $\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\,\mathrm{d}x = 1$.
- *Wave packets* are localized parts of a wave function in space where said portions have both characteristics of waves and particles, and can be described as

$$\Psi(x,t) = \int_{-\infty}^{+\infty} A(k)e^{i(kx-\omega t)}\,\mathrm{d}k\,. \qquad (40.19)$$

- General 1D Schrödinger equation:

$$\boxed{-\frac{\hbar}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + U(x)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}. \qquad (40.20)}$$

- Time-dependent wave function for a state of definite energy E:

$$\boxed{\Psi(x,t) = \psi(x)e^{\frac{-iEt}{\hbar}}. \qquad (40.21)}$$

  From the above equations, we obtain the *time-independent one-dimensional Schrödinger equation:*

$$\boxed{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + U(x)\psi(x) = E\psi(x). \qquad (40.23)}$$

- To study the behavior of particles, the *particle-in-a-box* model proves useful. In this scenario, a particle is bound in a region in which the potential energy is 0 within the walls and infinite everywhere else:

$$\boxed{U(x) = \begin{cases} 0 & x \in [0, L] \\ \infty & \text{otherwise} \end{cases}}$$

  Here are some important results for particles in an infinitely deep potential well:

  - The energy for a particle in a box is

$$\boxed{E_n = \frac{p_n^2}{2m} = \frac{n^2h^2}{8mL^2} = \frac{n^2\pi^2\hbar^2}{2mL^2}.} \tag{40.31}$$

- The stationary-state wave function in this case is

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}.} \tag{40.35}$$

- The time-dependent wave function is

$$\Psi_n(x,t) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} e^{-iE_n t/\hbar}. \tag{40.36}$$

- A potential well is a potential-energy function $U(x)$ that has a minimum, whose approximation is a *finite well,* one that has straight sides of finite height:

$$U(x) = \begin{cases} 0 & x \in [0, L] \\ U_0 & \text{otherwise} \end{cases}.$$

  We say a particle is in a *bound state* if its total mechanical energy E is less than U_0.

  Inside the well, the time-independent wave function is

$$\psi(x) = A\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + B\sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) \tag{40.38}$$

  while outside it's

$$\psi(x) = Ce^{\kappa x} + De^{-\kappa x} \tag{40.40}$$

  where

$$\kappa = \frac{\sqrt{2m(U_0 - E)}}{\hbar}.$$

- The ground-level energy of an infinitely deep well is

$$E_{1-\mathrm{IDW}} = \frac{\pi^2\hbar^2}{2mL^2}. \tag{40.41}$$

- In contrast to a potential well, a *potential barrier* is a potential-energy function with a maximum. Normally, Newtonian mechanics wouldn't allow a particle to cross the barrier, quantum-mechanical particles may appear on the other side thanks to *quantum tunneling.* The *tunneling probability* T is given by

$$T = Ge^{-2\kappa L} \tag{40.42}$$

  where

$$G = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right).$$

- Energy levels for a quantum harmonic oscillator:

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{k'}{m}} = \left(n + \frac{1}{2}\right)\hbar\omega. \qquad (40.46)}$$

40.2 Exercises