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Calculus: A Complete Course - Robert A. Adams, Christopher Essex

> Is that you, John Wayne? Is this me?
>
> — a silly goober

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1 Limits and Continuity

1.1 Examples of Velocity, Growth Rate, and Area

1.1.1 Summary

1.1.2 Exercises

1. The average velocity of the object near the time interval $[t, t+h]$ is

$$\frac{\Delta x}{\Delta t} = h + 2t.$$

1.2 Limits of Functions

1.2.1 Summary

Defintion 1 (Informal definition of limit)

If $f(x)$ defined $\forall x$ near a, except possibly at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a, we say that function f approaches the *limit* L as $x \longrightarrow a$ and we write

$$\lim_{x \longrightarrow a} f(x) = L. \tag{1.1}$$

1.2.2 Exercises

57. $\lim\limits_{x\to a^-} \dfrac{|x-a|}{x^2-a^2} = \lim\limits_{x\to a^-} -\dfrac{1}{x+a} = -\dfrac{1}{2a}.$
58. $\lim\limits_{x\to a^+} \dfrac{|x-a|}{x^2-a^2} = \lim\limits_{x\to a^+} \dfrac{1}{x+a} = \dfrac{1}{2a}.$
59. $\lim\limits_{x\to 2^-} \dfrac{x^2-4}{|x+2|} = \lim\limits_{x\to 2^-} x-2 = 0.$
60. $\lim\limits_{x\to 2^+} \dfrac{x^2-4}{|x+2|} = \lim\limits_{x\to 2^+} x-2 = 0.$

1.3 Limits at Infinity and Infinite Limits

1.3.1 Summary

Defintion 3 (Limits at ∞ and $-\infty$ [informal definition])

If the function f is defined on an interval (a, ∞) and we can ensure that $f(x)$ is as close was we want to the number L by taking x large enough, then we say that $f(x)$ *approaches the limit* L as $x \longrightarrow \infty$ and we write

$$\lim_{x \longrightarrow \infty} f(x) = L. \tag{1.2}$$

If the function f is defined on an interval $(-\infty, b)$ and we can ensure that $f(x)$ is as close was we want to the number M by taking x small enough, then we say that $f(x)$ *approaches the limit* M as $x \longrightarrow -\infty$ and we write

$$\lim_{x \longrightarrow -\infty} f(x) = M. \tag{1.3}$$

1.3.2 Exercises

27. $\lim\limits_{x\to\infty} \dfrac{x\sqrt{x+1}\left(1-\sqrt{2x+3}\right)}{7-6x+4x^2} = -\dfrac{1}{4}\sqrt{2}.$
28. $\lim\limits_{x\to\infty} \dfrac{x^2}{x+1} - \dfrac{x^2}{x-1} = \lim\limits_{x\to\infty} -\dfrac{2x^2}{x^2-1} = -2.$
29. $\lim\limits_{x\to-\infty} \sqrt{x^2+2x} - \sqrt{x^2-2x} = -2.$
30. $\lim\limits_{x\to\infty} \sqrt{x^2+2x} - \sqrt{x^2-2x} = 2.$

1.4 Continuity

1.4.1 Summary

Defintion 4 (Continuity at an interior point)

We say that a function f is *continuous* at an interior point c of its domain if

$$\lim_{x \longrightarrow c} f(x) = f(c). \tag{1.4}$$

If either $\lim\limits_{x\to c} f(x)$ does not exist or $\lim\limits_{x\to c} f(x) \neq f(c)$ then we say the f is discontinuous at c.

Defintion 5 (Right and left continuity)

We say that f is *right continuous* if $\lim\limits_{x\to c^+} f(x) = f(c)$ and *left continuous* if $\lim\limits_{x\to c^-} f(x) = f(c)$.

Theorem 5 (Continuity from both implies complete continuity)

Function f is continuous at c if and only if it is both left- and right-continuous at c.

Theorem 6 (Intermediate value theorem)

If $f(x)$ is continuous on the interval $[a, b]$ and if $s \in [f(a), f(b)]$, $\exists\, c \in [a, b]$: $f(c) = s$.

Theorem 8 (Max-min theorem)

If $f(x)$ is continuous on the closed, finite interval $[a, b]$, then $\exists\, p, q \in [a, b]$: $\forall x \in [a, b]$,

$$f(u) \leq f(x) \leq f(v). \tag{1.5}$$

1.4.2 Exercises

17. $f(x)$ is continuous $\iff \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) \iff k - 4 = 4 \iff k = 8$.
18. $f(x)$ is continuous $\iff \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) \iff 3 - m = 1 - 3m \iff m = -1$.

1.5 The Formal Definition of Limit

1.5.1 Summary

Defintion 8 (Formal definition of limit)

We say that $f(x)$ *approaches the limit L as x approaches a* if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 : \text{if } 0 < |x - a| < \delta, |f(x) - L| < \varepsilon. \tag{1.6}$$

Defintion 9 (Right limits)

We say that $f(x)$ has a *right limit L* at a and we write

$$\lim_{x\to a^+} f(x) = L \tag{1.7}$$

if $\forall\, \varepsilon > 0, \exists\ > 0$($\delta$ may possibly depend on ε) : $a < x < a + \delta \implies x \in D_f$ and $|f(x) - L| < \varepsilon$.

Defintion 10 (Limits at ∞)

We say that $f(x)$ *approaches the limit L as x approaches infinity*, and we write

$$\lim_{x\to\infty} f(x) = L \tag{1.8}$$

if $\forall\, \varepsilon > 0, \exists\, R$ (possibly depending on ε) : $x > R \implies$
x belongs to the domain of f and

$$|f(x) - L| < \varepsilon. \tag{1.9}$$

Defintion 11 (Infinite limits)

We say that $f(x)$ approaches infinity as x approaches a and write

$$\lim_{x \to a} f(x) = \infty, \tag{1.10}$$

if $\forall\ B > 0$ we can find a positive number δ, possibly depending on B, such that $0 < |x - a| < \delta$, then x belongs to the domain of f and $f(x) > B$.

1.6 Chapter Reiview

1.7 Challenging Problems

Problem 8

a. If f is a continuous function defined on a closed interval $[a, b]$, show that $R(f)$ is a closed interval.
b. What are the possibilities for $R(f)$ if $D(f)$ is an open interval (a, b)?

Problem 10

Find the minimum value of $f(x) = \dfrac{1}{x - x^2}$ on the interval $(0, 1)$. Explain how you know such a minimum value must exist.

Problem 11

a. Suppose that f is a continuous function on the interval $[0, 1]$, and $f(0) = f(1)$. Show that $f(a) = f(a + 0.5)$ for $a \in [0, 0.5]$.
b. If $n > 2$, show that (a + 1/n) for $a \in \left[0, 1 - \frac{1}{n}\right]$.

2 Differentiation

2.1 Tangent Lines and Their Slopes

2.2 The Derivative

2.2.1 Summary

2.2.2 Exercises

Problem 51

Show that the derivative of an odd differerentiable function is even and that derivative of an even differerentiable function is odd.

Proof. If f is an odd differerentiable function then

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} \\ &= \lim\left(h, 0, \frac{-f(x-h)+f(x)}{h}\right) \\ &= f'(x). \end{aligned}$$

If f is an even differerentiable function then

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} \\ &= \lim\left(h, 0, \frac{-f(x-h)-f(x)}{h}\right) \\ &= -f'(x). \end{aligned}$$

□

2.3 Differentiation Rules

2.3.1 Summary

Theorem 2.3.1 (Differentiability implies continuity)

If f is differerentiable at x, then f is continuous at x.

Theorem 2.3.2 (Differentiation rules for sums, differences, and constant multiples)

2.3.2 Exercises

53. Use induction on $n = 1$. Note that for $x = k + 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x} x^{\frac{n+1}{2}} = \frac{n+1}{2} x^{\frac{n+1}{2}-1}.$$

54. Obviously, the product rule is true for $n = 1$, so assume it's true $\forall\ n = k = \mathbb{N}^*$. Then, for $n = k + 1$,

$$\begin{aligned} (f_1 \cdot \ldots \cdot f_{k+1})' &= f_{k+1}(f_1' \ldots f_k + \ldots + f_1 \ldots f_k') + f_1 \cdot \ldots \cdot f_k f_{k+1}' \\ &= f_1' f_2 \ldots f_{k+1} + f_1 f_2' \ldots f_{k+1} + \ldots + f_1 f_2 \ldots f_{k+1}'. \end{aligned}$$

2.4 The chain rule.

2.4.1 Summary

Theorem 2.4.1 (The Chain Rule)

If $f(u)$ is differerentiable at $u = g(x)$ and $g(x)$ is differerentiable at x, then the composite function $f \circ g(x) = f[g(x)]$ is differerentiable at x, and

$$(f \circ g)'(x) = f'[g(x)]g'(x).$$

2.4.2 Exercises

28. $f'(2f(3f(x))) = 6f'(2f(3f(x)))f'(3f(x))f'(x)$.
29. $f'(2 - 3f(4 - 5t)) = 15f'(4 - 5t)f'(2 - 3f(4 - 5t))$.

2.5 Derivatives of Trigonometric Functions

2.5.1 Summary

Theorem 2.5.1

$\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$.

2.5.2 Exercises

53. $\lim\limits_{x \to 0} \dfrac{\tan 2x}{x} = \lim\limits_{x \to 0} \dfrac{\sin 2x}{2x} \cdot \dfrac{2}{\cos 2x} = 2$.
54. $\lim\limits_{x \to \pi} \sec(1 + \cos x) = 1$
55. $\lim\limits_{x \to 0} x^2 \csc x \cot x = 1$
56. $\lim\limits_{x \to 0} \cos\left(\dfrac{\pi - \pi\cos^2 x}{x^2}\right) = \lim\limits_{x \to 0} \cos\left(\dfrac{\pi \sin^2 x}{x^2}\right) = -1$.
62.

2.6 Higher-order Derivatives

2.6.1 Summary

2.6.2 Exercises

24. $y = \tan kx \implies y' = k\sec^2 kx \implies y'' = 2k^2 \sec kx(\sec kx \tan kx) = 2k^2 y\left(1 + y^2\right)$.
25. $y = \sec kx \implies y'k^2\left(\sec^3 kx - \sec kx \tan^2 kx\right) = k^2 y\left(2y^2 - 1\right)$.
28. $(fg)' = f'g + fg' \implies (fg)'' = f''g + f'g' + f'g' + fg'' - f''g + 2f'g' + fg''$.

14 Multiple Integration

14.1 Double Integrals

14.1.1 Summary

Defintion 1 (The double integral over a rectangle)

We say that f is *integrable* over the rectangle D and has *double integral*

$$I = \iint_D f(x, y)\,\mathrm{d}A$$

if $\forall\ \varepsilon > 0$ there exists a number δ on ε, such that

$$|R(f, P) - I| < \varepsilon$$

hold for every partition P of D satisfying $\|P\| < \varepsilon$ and for all choices of the points (x^*_{ij}, y^*_{ij}) in the subrectangles of P.

14.1.2 Exercises

13. $\displaystyle\int_{-4}^{1}\int_{-1}^{3} \mathrm{d}x\,\mathrm{d}y = 20.$
14. $\displaystyle\iint_D x + 3\,\mathrm{d}A = 0 + 3\iint_D \mathrm{d}A = 6\pi$
15. Because T is symmetric about $x + y = 0$, $\displaystyle\iint_T x + y\,\mathrm{d}A = 0.$
16. $\displaystyle\iint_{|x|+|y|\le 1} \left[x^3\cos\left(y^2\right) + 3\sin y - \pi\right]\mathrm{d}A = 0 + 0 - 4\cdot\tfrac{1}{2}\cdot 1^2 = -2\pi$
17. $\displaystyle\iint_{x^2+y^2\le 1} 4x^2y^3 - x + 5\,\mathrm{d}A = 0 + 0 + 5\pi = 5\pi.$
18. $\displaystyle\iint_{x^2+y^2\le a^2} \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}A = \frac{2\pi a^3}{3}$
19. $\displaystyle\iint_{x^2+y^2\le a^2} a - \sqrt{x^2 + y^2}\,\mathrm{d}A = \frac{\pi a^3}{3}.$

14.2 Iteration of Double Integrals in Cartesian Coordinates

14.2.1 Summary

14.2.2 Exercises 14.2

12. $\displaystyle\int_0^a\int_y^a \sqrt{a^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^a y\sqrt{a^2 - y^2} - a\sqrt{a^2 - y^2}\,\mathrm{d}y = \left(\frac{\pi}{4} - \frac{1}{3}\right)a^3.$
13. $\displaystyle\int_0^1\int_{x^2}^{x} \frac{x}{y}e^y\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_y^{\sqrt{y}} \frac{x}{y}e^y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 e^y/2 - \frac{ye^y}{2}\,\mathrm{d}y = \frac{e}{2} - 1.$
14. $\displaystyle\int_0^1\int_0^x \frac{xy}{1 + x^4}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \frac{x^3}{2(1 + x^4)}\,\mathrm{d}x = \frac{\ln 2}{8}.$
15. $\displaystyle\int_0^1\int_y^1 e^{-x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^x e^{-x^2}\,\mathrm{d}y\,\mathrm{d}x = \frac{1 - e^{-1}}{2}.$

16. $$\int_0^{\pi/2}\int_y^{\pi/2}\frac{\sin x}{x}\,dx\,dy=\int_0^{\pi/2}\int_0^x\frac{\sin x}{x}\,dy\,dx=\int_0^{\frac{\pi}{2}}\sin x\,dx=1.$$
17. $$\int_0^1\int_x^1\frac{y^\lambda}{x^2+y^2}\,dy\,dx=\int_0^1\int_0^y\frac{y^\lambda}{x^2+y^2}\,dx\,dy=\int_0^1\frac{\pi}{4}y^{\lambda-1}\,dy=\frac{\pi y^\lambda}{4\lambda}.$$
18. $$\int_0^1\int_x^{x^{\frac{1}{3}}}\sqrt{1-y^4}\,dy\,dx=\int_0^1\int_{y^3}^y\sqrt{1-y^4}\,dx\,dy=\frac{\pi}{8}-\frac{1}{6}.$$
19. $$\int_0^1\int_0^x 1-x^2\,dy\,dx=\int_0^1 x-x^3\,dx=\frac{1}{2}.$$
20. $$\int_0^1\int_0^y 1-x^2\,dx\,dy=\int_0^1\int_x^1 1-x^2\,dy\,dx=\int_0^1 1-x^2-x+x^3\,dx=\frac{5}{12}.$$
21. $$\int_0^1\int_0^x 1-x^2-y^2\,dy\,dx=\int_0^1 x-\frac{4x^3}{3}\,dx=\frac{1}{6}.$$
22. $$\iint_S 1-y^2-x^2\,dA=\int_0^{2\pi}\int_0^1(1-r^2)r\,dr\,d\theta=2\pi\cdot\frac{\pi}{4}=\frac{\pi}{2}.$$
23. $$\int_1^2\int_0^x\frac{1}{x+y}\,dy\,dx=\ln 2.$$
24. $$\int_0^{\pi^{\frac{1}{4}}}\int_0^y x^2\sin(y^4)\,dx\,dy=\frac{1}{3}\int_0^{\pi^{\frac{1}{4}}}y^3\sin(y^4)\,dy=\frac{1}{6}.$$
25. $$4\int_0^1\int_0^{\sqrt{\frac{1-x^2}{2}}}1-x^2-2y^2\,dy\,dx=\frac{\pi}{2\sqrt{2}}.$$
26. $$\int_0^b\int_0^{-\frac{b}{a}x+b}2-\frac{x}{a}-\frac{y}{b}\,dy\,dx=\frac{b}{2}\int_0^a 3-\frac{4x}{a}+\frac{x^2}{a^2}\,dx=\frac{2}{3}ab.$$
27.
28.

14.3 Improper Integrals and a Mean-Value Theorem

14.3.1 Summary

14.3.2 Exercises 14.3

1. $$\int_0^\infty\int_0^\infty e^{-x-y}\,dx\,dy=\int_0^\infty e^{-y}\,dy=e^{-y}|_\infty^0=1$$
2. $$\int_0^\infty\int_0^\infty\frac{1}{(1+x^2)(1+y^2)}\,dx\,dy=\int_0^\infty\frac{1}{1+x^2}\,dx\int_0^\infty\frac{1}{1+y^2}\,dy=\frac{\pi}{2}\cdot\frac{\pi}{2}=\frac{\pi^2}{4}.$$
3. $$\int_0^1\int_{-\infty}^\infty\frac{y}{1+x^2}\,dx\,dy=\frac{\pi}{2}$$
4. $$\int_0^1\int_x^{2x}\frac{1}{x\sqrt{y}}\,dy\,dx=4\left(\sqrt{2}-1\right).$$
5. $$\int_0^\infty\int_0^\infty\frac{x^2+y^2}{(1+x^2)(1+y^2)}\,dx\,dy=\int_0^\infty\int_0^\infty 1-2\frac{1}{(x^2+1)(y^2+1)}\,dx\,dy=\infty.$$

6. $\displaystyle\int_0^\infty \int_0^1 \frac{1}{1+x+y}\,dy\,dx = \int_0^\infty \ln\left(1+\frac{1}{1+x}\right)dx.$

Since $\displaystyle\lim_{u\to 0^+}\frac{\ln(1+u)}{u} = 1$, $\ln(1+u) \geq \frac{u}{2}$ on some interval $(0, u_0)$. Therefore,

$$\ln\left(1+\frac{1}{1+x}\right) \geq \frac{1}{2(1+x)}$$

and

$$\int_0^\infty \ln\left(1+\frac{1}{1+x}\right)dx \geq \int_0^\infty \frac{1}{2(1+x)}\,dx,$$

which diverges to infinity. Thus the given integral diverges to ∞ by comparison.

7. $\displaystyle\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(|x|+|y|)}\,dx\,dy = \left(\int_{-\infty}^\infty e^{-|x|}\,dx\right)^2 = \left(\int_0^\infty e^{-x}\,dx + \int_{-\infty}^0 e^x\,dx\right)^2 = 2^2 = 4.$

8. On the strip S created by the lines $x+y=0$ and $x+y=0$, $e^{-|x+y|} = e^{-(x+y)} > e^{-1}$, Since S has infinite area,

$$\iint_S e^{-|x+y|}\,dA = \infty$$

Since $e^{-|x+y|} > 0\ \forall\ (x,y) \in \mathbb{R}^2$, we have

$$\iint_{\mathbb{R}^2} e^{-|x+y|}\,dA > \iint_S e^{-|x+y|}\,dA$$

and the given integral diverges to ∞.

9. $\displaystyle\int_1^\infty \int_0^x \frac{1}{x^3}e^{-y/x}\,dy\,dx = \int_\infty^1 \frac{1-e^{-1}}{x^2}\,dx\, 1 - \frac{1}{e}.$

10. $\displaystyle\int_1^\infty \int_0^x \frac{1}{x^2+y^2}\,dy\,dx = \frac{\pi}{4}\int_1^\infty \frac{1}{x}\,dx = \infty$

11. Since $e^{-xy} > 0$ on Q, we have $\displaystyle\iint_Q > \int_1^\infty \int_0^{\frac{1}{x}} e^{-xy}\,dy\,dx > \int_1^\infty \frac{1}{x}\,dx = \infty$, so the integral diverges to ∞.

12. $\displaystyle\int_{\frac{2}{\pi}}^\infty \int_0^{\frac{1}{x}} \frac{1}{x}\sin\left(\frac{1}{x}\right)dy\,dx = \int_{\frac{2}{\pi}}^\infty \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx = \sin\left(\frac{1}{x}\right)\Big|_\infty^{\frac{2}{\pi}} = 1.$

13.
   a. $\displaystyle\int_0^1 \int_0^1 \frac{1}{x+y}\,dy\,dx = \int_0^1 \ln(x+1) - \ln x\,dx = (x+1)\ln(x+1) - x\ln x\big|_0^1 = 2\ln 2 = \ln 4.$

14. $\displaystyle\int_0^1 \int_0^x \frac{1}{x+y}\,dy\,dx = 2\int_0^1 \int_0^x \frac{1}{x+y}\,dy\,dx = 2\ln 2.$

14.4 Double Integral in Polar Coordinates

14.4.1 Summary

14.4.2 Exercises 14.2

2. $\displaystyle\int_0^{2\pi}\int_0^{a} r\cdot r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{2\pi a^3}{3}$

4. $\displaystyle\int_0^{2\pi}\int_0^{a} |r\cos\theta| r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{a^3}{3}\left(\int_0^{\frac{\pi}{2}}\cos\theta\,\mathrm{d}\theta - \int_{\frac{\pi}{2}}^{3\frac{\pi}{2}}\cos\theta\,\mathrm{d}\theta + \int_{3\frac{\pi}{2}}^{2\pi}\cos\theta\,\mathrm{d}\theta\right) = \frac{4a^3}{3}.$

9. $\displaystyle\int_0^{2\pi}\int_0^{a} re^{r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \pi e^{a^2}.$

11. $\displaystyle\int_0^{\frac{\pi}{3}}\int_0^{a} r(\cos\theta+\sin\theta)r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\left(\sqrt{3}-1\right)}{6}a^3.$

19. $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{a} r^2\cos\theta\sin\theta r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{a^4}{16}.$

20. $\displaystyle\int_0^{\pi}\int_0^{1+\cos\theta} r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi}\frac{(1+\cos\theta)^3\sin\theta}{3}\,\mathrm{d}\theta = \frac{(1+\cos\theta)^4}{12}\bigg|_{\pi}^{0} = \frac{4}{3}.$