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The 2023 TST for the High School for the Gifted, Ho Chi Minh National University

Contents

1 Day 1 (3 hours)

Problem 1 (5 points)

Let (u_n) be a set satisfying $u_1 = 1$ and $u_{n+1} = u_n + \dfrac{\ln n}{u_n}\ \forall\ n \geq 1$.

a. Prove that $u_{2023} > \sqrt{2023 \cdot \ln 2023}$.

b. Find

$$\lim_{n \to \infty} \frac{u_n \cdot \ln n}{n}.$$

Problem 2 (5 points)

Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x+1) = f(x) + 1$ and

$$f\left(x^{2024} + x^{2023} + ... + x + 1\right) = [f(x)]^{2024} + [f(x)]^{2023} + ... + f(x) + 1$$

$\forall\ x \in \mathbb{R}$.

Problem 3 (5 points)

Suppose we have a circle (O) and a stationary P that's outside of it. From P, we draw tangents PA, PB to O (A, B are the contact points). C is a varying point on the minor arc $\overset{\frown}{AB}$ of (O). The tangent at C of (O) intersects at D, E, F, respectively.

a. Prove that the line passing through the centers of (PDE) and (PCF) always passes through a stationary point.

b. From O, draw tangents OX, OY to (PDE) and OU, OV to (PCF). Prove that the intersect of $\overleftrightarrow{XY}$ and $\overleftrightarrow{UV}$ lies on a non-moving line.

(ABC) denotes the circumcircle of ΔABC.

Problem 4 (5 points)

Let $n \in \mathbb{Z}^+$ and an $n \times n$ grid.

For each k $(1 \leq k \leq n)$, we pick out a $k \times k$ subgrid of the original $n \times n$ grid.

a. We'll write in numbers $1, 2, 3, ..., 81$ into a 9×9 grid. Prove that there exists a 2×2 subgrid whose sum of its cells is greater than 137.

b. Say n is an odd number. Then, in each cell of the $n \times n$ grid, we input a value in the set $\{-1; 0; 1\}$ such that each 2×2 subgrid has 0 as the sum of its cells. Let S be the sum of all n^2 cells of the greater grid. What's $\max S$?

2 Day 2 (3 hours)

Problem 5 (6 points)

Let a, b, c be real positive numbers such that $ab + bc + ca = 1$. Prove the following inequality:

$$a^{\frac{3}{4}} + b^{\frac{3}{4}} + c^{\frac{3}{4}} \geq 243^{\frac{1}{8}}.$$

Problem 6 (7 points)

Let the acute triangle ABC be circumscribed in (O) and H, D, E, F be the orthocenter and bases of the altitudes going through A, B, C, respectively. L, M, N are the respective intersections of AO with EF, BE, CF. The circle going through A, midpoint I of EF and midpoint P of AH intersects (AEF) again at K.

a. Show that that AK passes through center J of (MHN).

b. The tangents of E and F of (EDL) and (FDL) intersect at T. Prove that TH and AJ intersect on BC.

Problem 7 (7 points)

Let $m, n \geq 2$ be positive integers. Find the greatest $k > 0 \in \mathbb{Z}$ so that there exist real numbers $a_1, ..., a_k$ satisfying both of the following conditions:

i. For $1 \leq l \leq k - m$,

$$\sum_{i=l+1}^{l+m} a_i > 0.$$

ii. For $1 \leq l \leq k - n$,

$$\sum_{i=l+1}^{l+n} a_i < 0.$$