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Euclidean Geometry in Mathematical Olymipads - Evan Chen

Part I: Fundamentals

Table of Contents

1 Angle Chasing

1.1 Triangles and Circles

1.2 Cyclic quadilaterals

1.3 The Orthic Triangle

Example 1.13

Prove that H is the incenter of ΔDEF.

Lemma 1.14 (The Orthic Triangle)

Suppose ΔDEF is the orthic triangle of acute ΔABC with orthocenter H. Then

(a) Points A, E, F, H lie on a circle with diameter $\overline{AH}$.

(b) Points B, E, F, C lie on a circle with diameter $\overline{BC}$.

(c) H is the incenter of ΔDEF.

Proof. It can easily be proven that $BCEF$, $AFHE$, $HDCE$, $AEDB$, and $HFBD$ are cyclic quadilaterals. Then,

$$\widehat{HFE} = \widehat{HAE} = \widehat{DAE} = \widehat{EBD} = \widehat{HBD} = \widehat{HFD} \tag{1.1}$$

and

$$\widehat{FEH} = \widehat{FAH} = \widehat{FCD} = \widehat{HED} \tag{1.2}$$

$\Longrightarrow H$ is the incenter ΔDEF. ☐

Lemma 1.17 (Reflecting the Orthocenter)

Let H be the the orthocenter ΔABC, as in Figure 1. Let X be the reflection H over $\overline{BC}$ and Y the reflection over the midpoint of $\overline{BC}$.

a. Show that X lies on (ABC).

b. Show that $\overline{AY}$ is a diameter of (ABC)

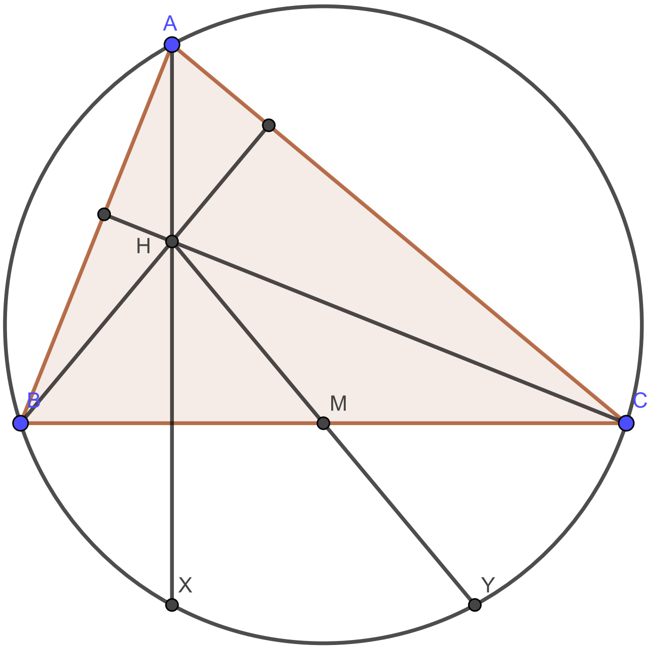

Figure 1.3B: Reflecting the orthocenter.

1.4 The Incenter/Excenter Lemma

Lemma 1.18 (The Incenter/Excenter Lemma)

Let ABC be a triangle with incenter I. Ray AI meets (ABC) again at L. Let I_A be the reflection of I over L. Then,

a. The points I, B, C and I_A lie on a circle with diameter $\overline{II_A}$ and center L. In particular, $LI = LB = LC = LI_A$
b. Rays BI_A and CI_A bisect the exterior angles of ΔABC.

1.4.1 Problem for this Section

Problem 1.19

Fill in the two similar calculations in the proof of Lemma .

1.5 Directed Angles

Defintion 1.20 (Directed angles)

A *directed angle* XYZ is denoted

$$\measuredangle XYZ$$

and its measure is taken mod $180°$.

Given a directed angle $\measuredangle ABC$, it is said to be *positive* if the vertices A, B, C appear in clocwise order and *negative* otherwise.

Theorem 1.22 (Cyclic quadilaterals with Directed Angles)

Points A, B, X, Y are concylic if and only if $\measuredangle AXB = \measuredangle AYB$.

Proposition 1.24 (Directed Angles)

For any distinct points A, B, C, P in the plane, we have the following rules:

a. $\measuredangle APA = 0$.
b. $\measuredangle ABC = -\measuredangle CBA$.
c. $\measuredangle PBA = \measuredangle PBC$ if and only if A, B, C are colinear. Equivalently, if $C \subset \overleftrightarrow{BA}$, then the A in $\measuredangle PBA$ may be replaced by C.
d. If $\overline{AP} \perp \overline{BP}$, then $\measuredangle APB = \measuredangle BPA = 90°$.
e. $\measuredangle APB + \measuredangle BPC = \measuredangle APC$.
f. $\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0$.
g. $\overline{AB} = \overline{BC} \iff \measuredangle ACB = \measuredangle CBA$
h. If (ABC) has center P, then $\measuredangle APB = 2\measuredangle ACB$.
i. If $\overline{AB} \parallel \overline{CD}$, then $\measuredangle ABC + \measuredangle BCD = 0$.

Worked Example 1.26

Let H be the orthocenter of ΔABC, acute or not. Using directed angles, show that $AEHF, BFHD, CDHE, BEFC, CFDA$, and $ADEB$ are cyclic.

Lemma 1.27 (Miquel Point of a Triangle)

Points D, E, F lie on lines BC, CA, and AB of ΔABC, respectively. Then there exists a point lying on all three circles $(AEF), (BFD), (CDE)$.

1.5.1 Problems for this Section

Problem 1.28

We claimed that $\measuredangle FKD + \measuredangle DKE + \measuredangle EKF = 0$ in the above proof. Verify this using Proposition .

Problem 1.29

Show that for any distinct points A, B, C, D, we have $\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB = 0$.