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Problems and Solutions in Introductory Mechanics - David Morin

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2.1 Problems

Problem 2.8 (Ratio of odd numbers)

An object is dropped from rest. Show that the distances fallen during the first second, the second second, the third second, etc., are in the ratio $1:3:5:7\ldots$.

Solution: Set the original position of the ball as the 0 of our y-axis, $+y$ pointing downward.

Then, the position of the ball is

$$y(t) = \frac{1}{2}gt^2$$

and the distance fallen after each 1-second interval is

$$\Delta y = y(t+1) - y(t) = \frac{g}{2}(2t+1),$$

meaning that the ratio of the distances is $1:3:5:7\ldots$.

Problem 2.9 (Dropped and thrown balls)

A ball is dropped from rest at height h. Directly below on the ground, a second ball is simultaneously thrown upward with speed v_0. If the two balls collide at the moment the second ball is instantaneously at rest, what's the thight of the collision? What's the relative speed of the balls when then collide? Draw the v vs. t plots for both balls.

Solution: The height of the collision is $y_c = \frac{h}{2}$ and the relative speed is v_0.

Problem 2.11 (Two dropped balls)

A ball is dropped from rest at height $4h$. After it has fallen a distance d, a second ball is dropped from rest at height h. What should d be (in terms of h) so that the balls hit the ground at the same time?

Solution: The time for the first ball to hit the ground from rest is $t = \sqrt{\frac{2h}{g}}$.

For the first ball to reach the ground at the same time, d must satisfy

$$
\begin{aligned}
v_0 t + \frac{gt^2}{2} &= 4h - d \quad \left(v_0 = \sqrt{2\frac{h}{g}}\right) \\
\iff 2\sqrt{2hd} + h &= 4h - d \\
\iff d + 2\sqrt{hd} - 3h &= 0 \\
\iff d &= h.
\end{aligned}
$$

3 Kinematics in 2D (and 3D)

3.1 Problems

Problem 3.8 (Projectile and tube)

A projectile is fired horizontally with speed v_0 from the top of a cliff of height h. It immediately enters a fixed tube with length x, as shown in Fig. 3.10. There's friction between the projectile and the tube, the effect of which is to make the projectile decelerate with constant acceleration $-a(a > 0)$. After the projectile leaves the tube, it undergoes normal projectile motion down to the ground.

a. What is the total horizontal distance (let's call it l) that the projectile travels, measured from the base of the cliff? Give your answer in terms of x, h, v_0, g, a.
b. What value of x yields the maximum value of l?

Solution:

a. $$\begin{cases} x = \sqrt{v_0^2 - 2axt} \\ y = \dfrac{gt^2}{2} \end{cases} \implies l = \sqrt{\frac{2h}{g}(v_0^2 - 2ax)}.$$

1. $$\max l = v_0\sqrt{\frac{2h}{g}} \iff x = 0.$$

Problem 3.10 (Clearing a wall)

a. You wish to throw a ball to a friend who is a distance $2l$ away, and you want the ball to just barely clear a wall of height h that's located halfway to your friend, as shown in Fig. 3.12. At what angle θ should you throw the ball?
b. What initial speed v_0 is required? What value of h (in terms of l) yields the minimum v_0? What is the value of θ in this minimum case?

Solution:

a. The range is

$$\frac{2v_0^2 \sin\theta \cos\theta}{g} = 2l$$

and the height is

$$h = \frac{(v_0 \sin\theta)^2}{2g},$$

which means that $\tan\theta = \frac{2h}{l}$.

b. From (a), we can conclude that

$$\sin^2\theta = \frac{4h^2}{4h^2 + l^2} \iff v^2 = g\left(\frac{4h^2 + l^2}{2h}\right).$$

To find $\min v_0$, notice that

$$g\left(\frac{4h^2 + l^2}{2h}\right) \geq 2gl \iff v_0 \geq \sqrt{2gl}.$$

Thus $\min v_0 = \sqrt{2gl} \iff l = 2h$, at which point

$$\theta = \arcsin\sqrt{0.5} = 45°.$$

Problem 3.14 (Throwing to a cliff)

A ball is thrown at angle θ up to the top of a cliff of height L, from a point a distance L from the base, as shown in Fig. 3.15.

a. As a function of θ, what initial speed causes the ball to land right at the edge of the cliff?
b. There are two special values of θ for which you can check your result. Check these.

Solution: We'll call the inital speed v.

a. We have

$$\begin{cases} L = v\cos\theta t \\ L = v\sin\theta t - \frac{gt^2}{2} \end{cases} \Longrightarrow L = L\tan\theta - \frac{gL^2}{2v^2\cos^2\theta}$$

$$\begin{aligned} \Longleftrightarrow L(\tan\theta - 1) &= \frac{gL^2}{2v^2\cos^2\theta} \\ \Longleftrightarrow v^2 &= \frac{gL}{2(\tan\theta - 1)\cos^2\theta} \\ &= \frac{gL}{2(\sin\theta - \cos\theta)\cos\theta.} \end{aligned}$$

1. For $\theta \longrightarrow 45°, v \longrightarrow \infty$. This is to be expected, since the ball is aimed right at the cliff, so if it's not thrown infinitely fast, it will fall down and thus hit below the corner. For $\theta \longrightarrow 90°, v \longrightarrow \infty$. This is also expected, since the ball is aimed straight up, so if the speed isn't infinite, the horizontal component of the velocity won't be large enough to reach the cliff.

Problem 3.15 (Throwing from a cliff)

A ball is thrown with speed v at angle θ (w/ respect to horizontal) from the top of a cliff of height h. How far from the base of the cliff does the ball land? (The ground is horizontal below the cliff).

Solution: The equations for the position of the ball is as follows:

$$y = h + v\sin\theta t - g\frac{t^2}{2}$$
$$x = v\cos\theta t.$$

Letting $y = 0$, we have $t = \dfrac{v + \sqrt{(v\sin\theta)^2 + 2hg}}{g}$ and thus

$$R = v\cos t = \frac{v^2\cos\theta}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2hg}{v^2}}\right).$$

Problem 3.16 (Throwing on stairs)

A ball is thrown horizontally with speed v from the floor at thte top of some stairs. The width and height of each step are both equal l.

a. What should v be so that the ball barely clears the corner of the step that is N steps down?
b. How far along the next step does the ball hit?
c. What is d in the limit $N \longrightarrow \infty$?
d. Find the components of the ball's velocity when it grazes the corner, and then explain why their ratio is consistent with your answers to part (c).

Solution:

a. $Nl = 0.5gt^2 = vt \Longleftrightarrow t = \sqrt{\dfrac{2Nl}{g}} \Longrightarrow v\sqrt{\dfrac{NLg}{2}}$

b. $d = v(t_{n+1} - t_n) = \sqrt{\dfrac{NLg}{2}}(\sqrt{\dfrac{2(N+1)l}{g} - \dfrac{2Nl}{g}} = (\sqrt{N(N+1)} - \sqrt{Nl}$

c. $\lim\limits_{N\to\infty} d = \frac{l}{2}$.

d. $\dfrac{v_x}{v_y} = \dfrac{1}{2}$. This ratio is consistent with the answer to (c) because for $N \longrightarrow \infty$, the ball's moving very fast when grazing the corner, ensuring that there's very little downward movement. Also from part (c), we know that the ball travels downward a distance l, and move sideways a distance $\frac{l}{2}$. This would imply that $\frac{v_x}{v_y} = \frac{1}{2}$, exactly the ratio we just derived.

Problem 3.17 (Bullet and sphere)

A bullet is fired horizontally with speed v_0 from the top of a fixed sphere with radius R, as shown in Fig. 3.17. What's the minimum value of v_0 such that the bullet doesn't touch the sphere after it's fired?

Find y as a function of x for the projectile motion, and also find y as a function of x for the sphere near the top where x is small; you'll need to make a Taylor-series approximation. Then compare your two results

Solution: We'll present two solutions to this problem.

Method 1

Choosing the center of the sphere as the origin of our coordinate system, we have

$$y = R - \frac{gx^2}{2v_0^2}.$$

For a point on the sphere (which, for our purposes, can be considered a 2D circle) that's a distance x away from the center, its vertical distance from the center is then $y = \sqrt{R^2 - x^2} = R\sqrt{1 - \frac{x^2}{R^2}} \approx R\left(1 - \frac{x^2}{2R^2}\right) = R - \frac{x^2}{2R}$. Combining our two equations, we get

$$\frac{gx^2}{2v_0^2} = \frac{x^2}{2R} \iff v_0 = \sqrt{gR}.$$

Method 2

To maintain non-contact with the ball, the radial acceleration must be equal to the force of gravity: $\frac{v_0^2}{R} = g \iff v_0 = \sqrt{gR}$.

The range of the bullet is such that $y = -R \implies x = 2R$.

Problem 3.18 (Throwing on an inclined plane)

You throw a ball from a plane is inclined at an angle θ. The inital velocity is perpendicular to the plane, as shown in Fig. 3.18. Consider the point P on the trajectory that is farthest from the plane. For what θ does P have the same height as the starting point? (For the case shown in the figure, P is higher.) Answer this in two steps:

a. Give a contiunity argument that explains why such a θ should in fact exist.
b. Find θ.

In getting a handle on where (and when) P is, it is helpful to use a tilted coordinate system and to isolate what is happening in the direction perpendicular to the plane.

Solution:

a. If $\theta \longrightarrow 90°$, the ball will be flying essentially horizontally, which means that the starting point will be the highest point and P will always be lower than it. If $\theta \longrightarrow 0°$, the ball would be going straight up, so P would be the highest point, which is invalid.

   Therefore, by continuity, θ must lie somewhere between 0 and 90 degrees.
b. To find θ, we'll need to construct two coordinate systems: The first has its y-axis perpendicular to the incline, the second with the x-axis parallel to

the ground. Both have their origin point at the starting position of the ball.

In the first coordinate system, we have

$$y = v_0 t - \frac{gt^2}{\cos\theta}$$
$$\implies y_{\max} \iff t = \frac{v_0}{g\cos\theta},$$

and for the second, the ball reaches the same height as the origin when

$$t = \frac{v_0}{2g\cos\theta}.$$

These two times must be the same, so

$$\frac{2v_0\cos\theta}{g} = \frac{v_0}{g\cos\theta}$$
$$\iff \cos^2\theta = \frac{1}{2} \iff \theta = 45°.$$

4 $F = ma$

5 Energy

6 Momentum

7 Torque

8 Angular momentum

9 Statics

10 Oscillations

11 Gravity

12 Fictious forces