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Problem Solving Tactics - Angelo Di Pasquale, Norman Do, Daniel Matthews

Solutions Manual (by @kishblockpro and Hải Sơn Bách [@jonathanphan])

> mostly just me compiling this fucking thing because homeboi dipped after doing just half of Chapter 1
>
> — @jonathanphan

Table of Contents

1 Methods of Proof

Problem 1

Prove that if $a + b$ is irrational, then at least one a or b is irrational.

Proof. If a, b are both rational, they can be expressed as $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$.

Here, $a = \frac{p}{q}, b = \frac{r}{s}$ $(r, s \in \mathbb{Z}, s \neq 0)$.

We have

$$a + b = \frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}. \tag{1.1}$$

Given that the product of two integers is an integer and the sum of two integers is also an integer, we can conclude that $ps + qr \in \mathbb{Z}$. Hence the numerator and denominator are integers, implying that when $a, b \in \mathbb{Q}$, $a + b \in \mathbb{Q}$.

Therefore if $a + b \notin \mathbb{Q}$, both $a \land b$ cannot be rational; otherwise we'd have a contradiction. It follows that either a or b or both must be irrational. □

Problem 2

Show that any party, there are always at least two people with exactly the same number of friends at the party.

Proof. We'll use the pigeonhole principle.

Assuming there are n people at the party, the maximum number of friends a person can have that are present at the party is $n - 1$. We also know that if one person has no friends the maximum number of friends the rest can have is $n - 2$. Assuming everyone has least one friend, there are thus $n - 1$ holes and n pigeons (excluding 0 as a hole), then clearly at least 2 people have the same number of friends. In the case where someone has no friends, we ave $n - 3$ holes (excluding $0, n - 1$ and n) and $n - 1$ pigeons (excluding the one with zero friends). □

Problem 3

The *equal temperament tuning* of musical instruments is based on the fact that $2^{\frac{19}{12}}$ is very close to 3.

Show that there can be no *perfect tuning* by proving that if $2^x = 3$, then x must be irrational.

Proof. If $x \in \mathbb{Q}$, then it can be written as $x = \dfrac{p}{q}$ $(p, q \in \mathbb{Z}; q \neq 0)$. If

$$2^x = 3 \tag{1.2}$$

then because $3 > 2$ we can conclude that $x > 1$ and it follows that $p > q$ and $p, q \in \mathbb{Z}_+$. We have

$$\begin{aligned} &2^{\frac{p}{q}} = 3 \\ \implies &2^p = 3^q, \end{aligned} \tag{1.3}$$

implying 2^p has the same factorization as 3^q, which is impossible. Hence we've arrived at a contradiction, which means x must be irrational. □

Problem 4

If $m, n \in \mathbb{N}^*$, prove that $\sqrt[m]{n}$ is either a positive integer or irrational.

Proof. If $\sqrt[m]{x} \notin \mathbb{Z}_+ \vee \in \mathbb{I}$, where $m, n \in \mathbb{Z}_+$, it means that $\sqrt[m]{n}$ is rational and not a positive integer. We have

$$\sqrt[m]{n} = \frac{p}{q} \tag{1.4}$$

where p, q are coprime integers. Then,

$$n = \frac{p^m}{q^m}. \tag{1.5}$$

Since p, q share no factors, $\dfrac{p^m}{q^m}$ can't be a positive integer, yet n is, leading to a contradiction. □

Problem 5

Prove that there are infinitely many prime numbers of the form $6n + 5$, where $n \in \mathbb{N}^*$.

Problem 6

$\forall n \in \mathbb{N}^*$, prove that

$$\sum_{i=1}^{n} \frac{1}{n(n-1)} = \frac{n-1}{n}. \tag{1.6}$$

Proof. We'll present two methods of proving this result.

Method 1: Telescoping

We have

$$\begin{aligned} &\frac{1}{1 \cdot 2} + \cdots + \frac{1}{(n-1)n} \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{n} \\ &= \frac{n-1}{n}. \end{aligned} \tag{1.7}$$

Method 2: Induction (left as an exercise to you, the reader because the author's too much of an asshole) □

Problem 7

Prove that $n^2 < 2^n \ \forall n \geq 5$.

Problem 8

$\forall n \in \mathbb{Z}_+$, prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{1.8}$$

Proof. Use induction on i. □

Problem 9

$\forall n \in \mathbb{Z}_+$, prove that

$$\sum_{i=1}^{n} i^3 = \frac{[n(n+1)]^2}{4} \tag{1.9}$$

and go on to conclude that

$$\sum_{i=1}^{n} i^3 = (1 + 2 + \ldots + n)^2. \tag{1.10}$$

Proof. Use induction, simple as. Note that

$$\frac{n(n+1)}{2} = 1 + \ldots + n. \tag{1.11}$$

□

Problem 10

Recall that the Fibonacci sequence is defined recursively by $F_1 = 1, F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.

Prove the following identity for Fibonacci numbers: $\forall n \geq 1$,

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}. \tag{1.12}$$

Proof. Use induction… again. 'Nuff said.

(Y'all can do this right? Right?!) □

Problem 11

Prove that every positive integer can be uniquely expressed as a sum of different numbers, where each number is of the form 2^n for some nonnegative integer n.

Problem 12

Suppose $x \in \mathbb{R}$ such that $x + x^{-1} \in \mathbb{Z}$. Prove that $x^n + x^{-n}$ is also an integer for any positive integer n.

Problem 13

Any finite collection of lines in the plane divides the plane up into regions.

Prove that it is possible to color each of these regions either black or white in such a way that no two regions which share a common edge have the same color.

Problem 14

Show that if there are 5 points in a square of side length 1 m, then there exist two of them which are less than 75 cm apart.

Problem 15

Four points are given inside a square with length 8 m.

a. Prove that two of them are less than $\sqrt{65}$ m apart.
b. Can you prove, beyond a shadow of a a doubt, that two of them are < 8 m apart?

Proof.

a. Using the pigeonhole principle, let's divide the square into 3 regions: two 7×4 regions and one 8×1 region. The largest diagonals of these regions are $\sqrt{65}$ meters apart. If a point is positioned at the center crossing of all these regions' borders, the 8×1 region cannot be utilized. Thus, placing a point there leaves us with 3 points and 2 regions remaining, which solves the problem.

   If we don't place a point at the central intersection, we are left with casework, which is left to the reader as an exercise (not gonna do that for now - mofo who's editing this). The fundamental difficulty of the problem lies in dividing it into 3 regions.
b. No, you can't. You can four points at the corners of the square and they will all be at least 8 m apart.

□

Problem 16

a. Prove that if x and y can be each be written such that the sum of the square of two integers, then so can xy.

b. Prove that x and y are both of the form $a^2 + 2b^2 (a, b \in \mathbb{Z})$, then so is xy.

c. Let k be a fixed integer. Prove that if x and y are both of the form $a^2 + kb^2 (a, b \in \mathbb{Z})$, then so is xy.

Problem 17

Consider the non-empty subsets $\{1, \ldots, n\}$. For each of these subsets, consider the reciprocal of the products of its elements.

Determine the sum of all of these numbers.

2 Number Theory

3 Diophantine Equations

4 Plane Geometry (yay geo my favorite)

Problem 1

Two parallel lines are tangent to a circle with center O. A third line, also tangent to the circle, meets the two parallel lines at A and B.

Prove that $OA \perp OB$.

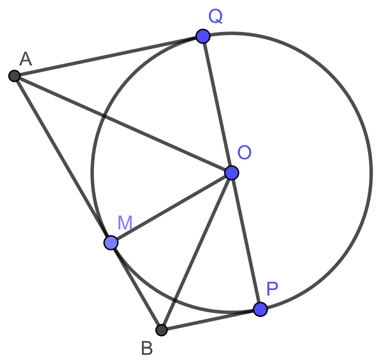

Figure 1: Problem 1.1. Making O the midpoint of $\overline{QP}$ wasn't the brightest idea.

Proof. Let $C = \overleftrightarrow{AB} \cap \{O\}$. Then $\widehat{OBA} = \widehat{OBC} = \widehat{OBD} = \frac{1}{2}\widehat{DBA}$, and similarly, $\widehat{OAB} = \widehat{OAC} = \frac{1}{2}\widehat{BAE}$. Thus $\widehat{OAB} + \widehat{OAC} = \frac{1}{2}\left(\widehat{DBA} + \widehat{BAE}\right) = 90° \Longrightarrow OA \perp OB$. □

Problem 2

Suppose that two circles are externally tangent at P. Let a common tangent touch the circles at A and B.

Prove that ΔAPB is right-angled.

Proof. Since $\overleftrightarrow{AB}$ is the common tangent $\{O_1\}, \{O_2\}$, we have $\widehat{PAB} = 0.5\overset{\frown}{PB}$ and $\widehat{PBA} = 0.5\overset{\frown}{BP}$. Thus

$$\begin{aligned}\widehat{PAB} + \widehat{PBA} &= \frac{1}{2}\left(\overset{\frown}{PA} + \overset{\frown}{PB}\right) \\ &= \frac{1}{2}\left(\widehat{AO_1O_2} + \widehat{BO_2O_1}\right) \\ &= 180°.\end{aligned} \tag{4.1}$$

□

Problem 3

Let ABC be a triangle with incenter I. Suppose that X is the midpoint of $\overset{\frown}{BC}$ not containing A on the circumcircle of ΔABC.

Prove that X is the circumcircle of ΔABC.

Proof. Proving $XB = XC$ is trivial; it now stands that we need to prove $XI = XB$ to complete the proof. $\widehat{BAX} = \frac{\overset{\frown}{BX}}{2} = \frac{\overset{\frown}{BC}}{4} = \frac{\hat{A}}{2} = \widehat{BAI}$, which makes A, I, X colinear.

Then, $\widehat{IBX} = \widehat{IBC} + \widehat{CBX} = \frac{\hat{B} + \overset{\frown}{CX}}{2} = \frac{\hat{B} + \hat{C}}{2}$ and

$$\widehat{BIX} = \widehat{ABI} + \widehat{BAI} = \frac{\hat{A} + \hat{B}}{2}. \tag{4.2}$$

$\Longrightarrow \widehat{BIX} = \widehat{IBX}$. ΔIBX is isosceles, which means that $IB = IX$. Thus X is the circumcircle of ΔIBC. □

Problem 4

Let D, E, F be points on sides AB, BC, CA of ΔABC such that $DE = BE$ and $FE = CE$.

Prove that the circumcenter of ΔADF lies on the bisector of $\angle DEF$.

Proof. We'll show that O, D, E, F are colinear. First, noticing that

$$\begin{aligned} \widehat{DAF} &= \widehat{BAC} \\ &= \frac{\widehat{DF}}{2} \\ \Longrightarrow \hat{A} &= \frac{\widehat{DOF}}{2} \\ \Longleftrightarrow \widehat{DOF} &= 2\hat{A}. \end{aligned} \tag{4.3}$$

Next,

$$\widehat{DEF} = 180° - \left(\widehat{DEB} + \widehat{FEC}\right) = 2\hat{B} + 2\hat{C} - 180°. \tag{4.4}$$

Then $\widehat{DOF} + \widehat{DEF} = 2\hat{B} + 2\hat{C} - 180° + 2\hat{A} = 180°$. This means that O, D, E, F is a cyclic quadrilateral. Thus $\widehat{ODF} = \widehat{OFD} = \widehat{OED} = \widehat{OEF}$, the desired result. □

Problem 5

Given a triangle ABC, let the median from vertex A intersect the circumcircle of the triangle again at K. A circle Γ passes through A, B such that BC is tangent to Γ. Let $L = AK \cap \Gamma (L \neq A)$.

Prove that $BLCK$ is a parallelogram.

Proof. Noticing that $\widehat{LBM} = \frac{\widehat{BL}}{2} = \widehat{BAL} = \widehat{BAK} = \frac{\widehat{BK}}{2} = \widehat{BCK}$, we obtain $\Delta BML \cong \Delta CMK$(a-s-a). From this, we can easily prove that $BLCK$ is a parallelogram. cir
□

Problem 6

Let two circles intersect at A and B. Suppose that a common tangent to the to circles meets them at P and Q.

If $\overleftrightarrow{AB} \cap \overline{PQ} = M$, show that M is the midpoint of PQ.

Proof. Since M is on the radical axis of the two circles, we have $MP^2 = MQ^2 \iff \overline{MP} = \overline{MQ}$.

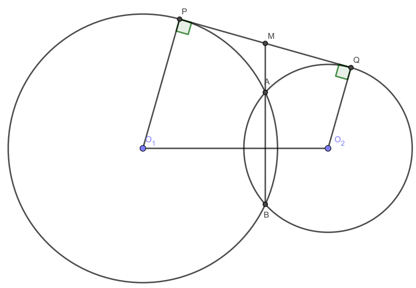

Figure 2: Problem 4.6.

□

Problem 7

ABC is a triangle, right-angled at C. The internal bisectors of $\widehat{BAC}$ and $\widehat{ABC}$ meet BC and CA at P and Q, respectively. Let M and N be the feet of the perpendicular from P and Q to $\overline{AB}$, respectively.

Find the measure of $\widehat{MCN}$.

Solution.

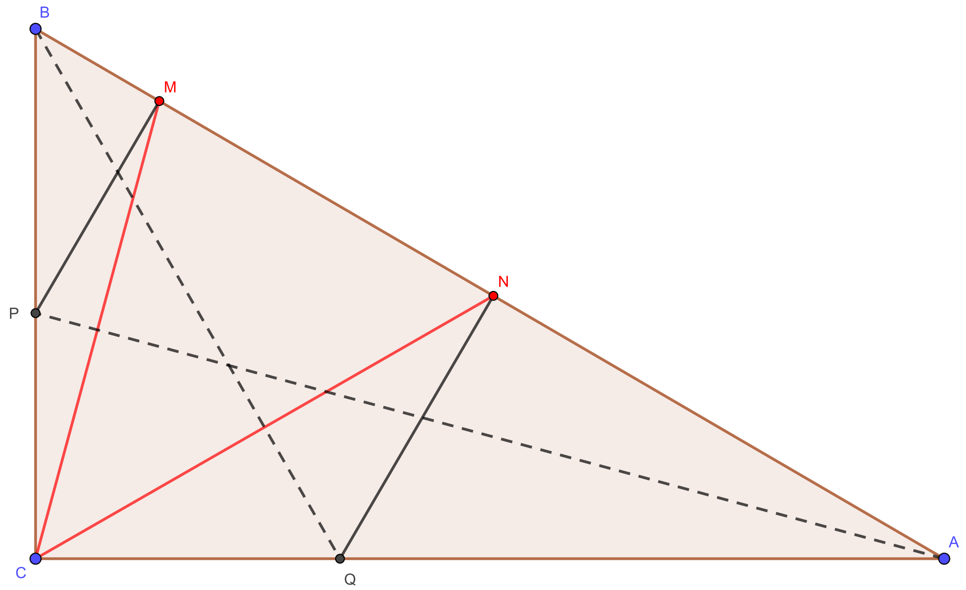

Figure 3: Problem 4.7.

First, it's easy to see that $\Delta PMC = \Delta PCA$ (s-a). Then $\overline{PM} = \overline{PC} \Longrightarrow \widehat{PCM} = \widehat{PMC} = \frac{\hat{C}}{2}$. Following a similar procedure, we have $\widehat{QCN} = \widehat{QNC} = \frac{\hat{B}}{2} \Longrightarrow \widehat{MCN} = 90° - \hat{B}/2 - \hat{C}/2 = 45°$. $\square$

Problem 9

Let $\overline{AD}$ be an altitude and H be the orthocenter of ΔABC. Let $\overleftrightarrow{AD} \cap (ABC) = X$.

a. Prove that $\overline{HD} = \overline{HX}$.
b. Prove that the circumradius of the triangles formed by H and any two vertices is equal to the circumradius of ΔABC.

Problem 10

Two circles of equal radius intersect at A and B. The point C is one of the circles such that B is the midpoint of $\overset{\frown}{AC}$.

Prove that $\overline{AC}$ is tangent to the other circle.

Proof.

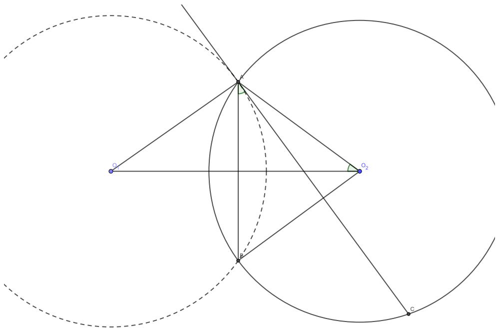

Figure 4: Problem 4.10.

We have

$$\widehat{BAC} = \widehat{BCA} = \widehat{O_2O_1A} \tag{4.5}$$

and

$$O_2AB = O_1AB \tag{4.6}$$

$\Longrightarrow \widehat{CAB} + \widehat{O_2BA} = 90°$. Thus $\overline{AC}$ is tangent to $\{O_2; R\}$. $\square$

Problem 4.11

Two circles C_1 and C_2 intersect at A, B. Let P be a point on C_1 and Q be a point on C_2, such that P, Q lie on opposite sides of the line $\overleftrightarrow{AB}$. Suppose further that $\widehat{APB} + \widehat{AQB} = 90°$.

Prove that if O_1 is the center of C_1 and O_2 is the center of C_2, then ΔO_1AO_2 and O_1BO_2 are right-angled.

Proof.

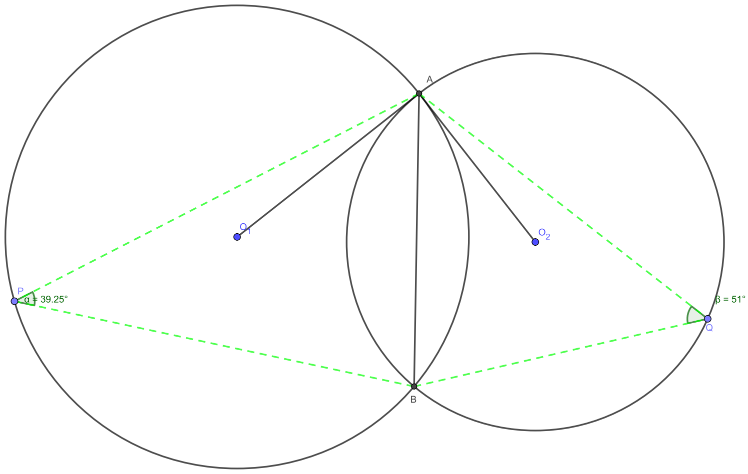

Figure 5: Problem 11.11

Since $\widehat{APB} = \widehat{AO_1O_2}$ and $\widehat{AQB} = \widehat{BO_2O_1}$, $\widehat{AO_1O_2} + \widehat{AO_2O_1} = \widehat{APB} + \widehat{AQB} = 90°$, which implies what we wish to prove. □

Problem 4.12

Let $ABCD$ be a quadilateral such that CD bisects $\widehat{ACB}$. Suppose that

$$\widehat{DAC} + \widehat{DBC} + \widehat{DCB} = 90°.$$

Prove that D is the incenter of ΔABC.

Proof.

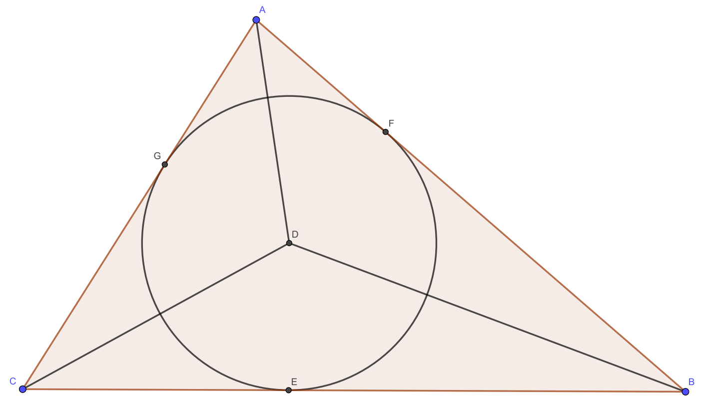

Figure 6: Problem 4.12.

Let the intersects of the lines passing D and perpendicular with BC, AB, AC be E, F, G, respectively. Then, it can be easily be shown that $DECG$, $AGDF$, $DEBF$ are all cyclic quadilaterals □

Problem 4.13

Triangle ABC is a 40°-60°-80° triangle. The angle bisector at A meets BC at D. Let E be the midpoint of $\overline{CD}$ and let F be the foot of the altitude from C to $\overline{AB}$.

Prove that perpendicular bisector of EF intersects AC at its midpoint.

Proof.

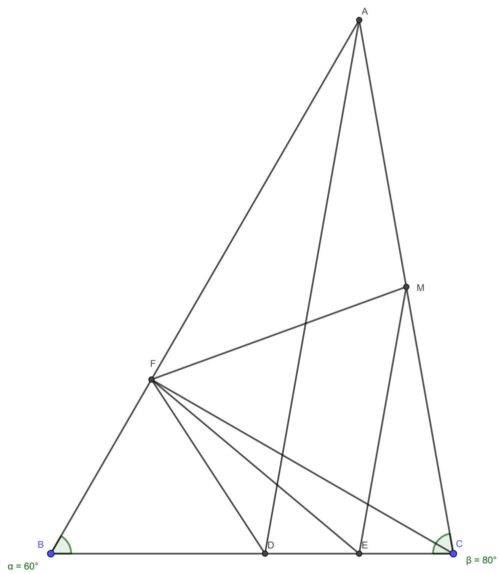

Figure 7: Problem 4.15

Let M be the midpoint of $\overline{AC}$. The demand of the problem is tantamount to proving $AM = MC = MF = ME$. Since $\overline{ME}$ is the medium line, we have $ME \parallel AD \Longrightarrow \widehat{DAC} = \widehat{EMC} = 20° \Longrightarrow ME = MC = MA$. Next, because ΔAFC is right-angled at F, $MA = MC = MF = ME$, which means that the perpendicular bisector of EF intersects AC at its midpoint. $\square$

5 Important Configurations in Geometry

6 Incidence Geometry

7 Transformation Geometry

8 Complex Numbers

9 Polynomials

10 Functional Equations

11 Inequalities

Problem 11.1

Farmer Brown wants a rectangular paddock of area A next to a long straight river, with a fence on the three other sides.

What's the minimum length of fencing that is required?

Problem 11.2

a. Use the AM-GM inequality to find $\max xyz$, where $x, y, z \in \mathbb{R} > 0$ satisfying $x + 2y + 3z = 3$.
b. Use the AM-GM inequality to find $\min x + y + z$, where $x, y, z \in \mathbb{R} > 0$ satisfying $xy^2z^3 = 108$.

Proof.
a. By the AM-GM inequality, we have

$$\begin{aligned}\frac{x+2y+3z}{3} &\geq \sqrt[3]{6xyz} \\ \iff xyz &\leq \frac{(x+2y+3z)^3}{6\cdot 27} \\ &= \frac{1}{6}.\end{aligned}$$

It follows, then, that $\max xyz = \frac{1}{6} \iff x = 2y = 3z \iff (x, y, z) = \left(\frac{1}{2}, 1, \frac{3}{2}\right)$.

b. Also by the AM-GM inequality, we obtain the following:

$$\begin{aligned}x + y + z &= x + 2\left(\frac{y}{2}\right) + 3\left(\frac{z}{3}\right) \\ &\geq 6\sqrt[6]{\frac{xy^2z^3}{27\cdot 4}} \\ &= 6.\end{aligned}$$

Thus $\min x + y + z = 6 \iff x = \frac{y}{2} = \frac{z}{3} \iff (x, y, z) = (1, 2, 3)$.

$\square$

Problem 11.3

Prove that if $a, b, c > 0 \in \mathbb{R}$ satisfying $abc = 1$, then

$$(a+b)(b+c)(c+a) \geq 8.$$

Proof. Again, use the AM-GM inequality to yield

$$(a+b)(b+c)(c+a) \geq 2\cdot 2\cdot 2\sqrt{ab\cdot bc\cdot ca} = 8abc = 8.$$

Equality holds when $a = b = c = 1$. $\square$

Problem 11.4

Let $x, y > 0 \in \mathbb{R} : x + y = 1$. Show that

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq \frac{9}{4}.$$

Problem 11.5

If $a, b, c > 0$, prove in at least three different ways that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof.
Method 1
Let

$$a = \frac{y+z-x}{2}, \quad b = \frac{x+z-y}{2}, \quad c = \frac{x+y-z}{2}.$$

With these substitutions, we obtain

$$\frac{y+z-x}{2x} + \frac{z+x-y}{2y} + \frac{x+y-z}{2z} \geq \frac{3}{2}$$

Subbing again, this time with

$$\frac{y}{2x} = p, \quad \frac{z}{2x} = q, \quad \frac{x}{2y} = r,$$

we get

$$p + q + r + \frac{1}{4p} + \frac{1}{4q} + \frac{1}{4r} - \frac{3}{2} \geq \frac{3}{2}$$
$$\iff p + \frac{1}{4p} + q + \frac{1}{4q} + r + \frac{1}{4r} \geq 3,$$

which can be easily proven by applying the AM-GM inequality thrice. ☐

Problem 11.6

Use the arrangement inequality to prove the following, where $a, b, c \in \mathbb{R} > 0$:

a. $\sum_{\text{cyc}}^{a,b,c} a^2 b \leq a^3 + b^3 + c^3$

b. $\sum_{\text{cyc}}^{a,b,c} \frac{1}{a} \leq \sum_{\text{cyc}}^{a,b,c} \frac{a}{b^2}$

c. $\sum_{\text{cyc}}^{a,b,c} a^{bc} \leq \sum_{\text{cyc}}^{a,b,c} a^{ab}$

Problem 11.7

Use the Cauchy-Schwarz inequality to find $\min x^2 + y^2 + z^2$ where $x, y, z \in \mathbb{R}$ satisfying $3x + 4y + 5z = 10$.

Solution. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (x^2 + y^2 + z^2)(9 + 16 + 25) &\geq (3x + 4y + 5z)^2 = 100 \\ \iff x^2 + y^2 + z^2 &\geq 2. \end{aligned}$$

Thus $\min x^2 + y^2 + z^2 = 2 \iff \frac{x}{3} = \frac{y}{4} = \frac{z}{5} \iff \begin{cases} x = 1.2 \\ y = 1.6. \\ z = 2 \end{cases}$ □

Problem 11.8

For $a, b, c \in \mathbb{R} > 0$, prove that

$$\frac{a + b + c}{3} \geq \sqrt{\frac{ab + bc + ca}{3}}.$$

Proof. Squaring both sides of the inequality, we get

$$\begin{aligned} (a + b + c)^2 &\geq 3(ab + bc + ca) \\ \iff a^2 + b^2 + c^2 &\geq ab + bc + ca. \end{aligned}$$

Here, we can use either the AM-GM inequality or the rearragnement inequality to prove the last inequality. □

Problem 11.9

Let $a, b, c > 0$ satisfying

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

Prove that

$$(a-1)(b-1)(c-1) \geq 8.$$

Problem 11.10

For positive real numbers a, b, c, d, prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

Proof. Using the Cauchy-Schwarz inequality, we obtain

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d}\right)(a+b+c+d) \geq \left(a \cdot \frac{1}{a} + b \cdot \frac{1}{b} + c \cdot \frac{2}{c} + d \cdot \frac{4}{d}\right)^2$$
$$\Longleftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d},$$

as desired. □

Problem 11.11

Let A, B, C be the angles of a triangle. If the triangle is acute, use Jensen's inequality to prove the following Inequalities. Also determine which of these Inequalities remain true if the triangle is obtuse.

a. $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$;

b. $\cos A + \cos B + \cos C \leq \frac{3}{2}$;

c. $\cos A \cos B \cos C \leq \frac{1}{8}$.

Proof. We'll state the following observations before going into the proofs: $\sin x$ is concave on $\left[0, \frac{\pi}{2}\right]$, while $\cos x$ is convex on the same interval. You, the reader, can verify that for yourself by taking their respective second derivatives.

a. Applying Jensen's inequality, we have

$$\sin A + \sin B + \sin C \le 3\sin\left(\frac{A+B+C}{3}\right) = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}.$$

b. Again, we have

$$\cos A + \cos B + \cos C = \sin\left(\frac{\pi}{2} - A\right) + \sin\left(\frac{\pi}{2} - B\right) + \sin\left(\frac{\pi}{2} - C\right)$$

$$\iff \cos A + \cos B + \cos C \le 3\sin\left(\frac{\pi}{2} - \frac{A+B+C}{3}\right) = 3\sin\frac{\pi}{6} = \frac{3}{2}.$$

c. Notice that $\cos A \cos B \cos C \le \dfrac{\cos^3 A + \cos^3 B + \cos^3 C}{3}$, and using the argument in (b), we get

$$\frac{\cos^3 A + \cos^3 B + \cos^3 C}{3} \le \sin^3\left(\frac{\pi}{2} - \frac{A+B+C}{3}\right) = \sin^3\frac{\pi}{6} = \frac{1}{8}.$$

□

Problem 11.12

If $a_1, \ldots, a_n$ are distinct positive integers, prove that

$$\sum_{i=1}^{n} \frac{a_i}{i^2} \ge \sum_{i=1}^{n} \frac{1}{i}.$$

Problem 11.13

If $x_1, \ldots, x_n$ are positive numbers whose product is 1, prove that

$$\prod_{i=1}^{n} (2 + a_i) \ge 3^n.$$

Problem 11.14

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sum_{\text{cyc}}^{a,b,c} \frac{ab}{a^5 + b^5 + ab} \le 1$$

and determine when equality holds.

Proof. We have $a^5 + b^5 + ab \ge 3\sqrt[3]{a^6 b^6} = 3a^2 b^2 \Longrightarrow \dfrac{ab}{a^5 + b^5 + ab} \le \dfrac{1}{3ab}$

$$\sum_{\text{cyc}}^{a,b,c} \frac{ab}{a^5+b^5+ab} \le \frac{1}{3ab}+\frac{1}{3bc}+\frac{1}{3ca} =$$

□

Problem 11.15

If a, b, c are positive real numbers, prove that

$$\frac{9}{a+b+c} \le \sum_{\text{cyc}}^{a,b,c} \frac{2}{a+b} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Problem 11.16

Prove that if $a, b, c \in \mathbb{R} > 0 : a+b+c < 1$, then

$$\frac{abc(1-a-b-c)}{(a+b+c)(1-a)(1-b)(1-c)} \le \frac{1}{81}.$$

Problem 11.17

Suppose that the numbers $x, y, z \ge 1$ such that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 2.$$

Prove that

$$\sqrt{x+y+z} \ge \sum_{\text{cyc}}^{x,y,z} \sqrt{x-1}.$$

Problem 11.18

For positive real numbers x, y, z, show that

$$\sum_{\text{cyc}}^{x,y,z} \frac{x}{(x+y)(x+z)} \le \frac{9}{4(x+y+z)}.$$

Problem 11.19

Let $x, y, z \in \mathbb{R} : xyz = 1$. Show that

$$\sum_{\text{cyc}}^{x,y,z} \frac{x^3}{(1+y)(1+z)} \ge \frac{3}{4}.$$

Problem 11.20

Let $a_1, \ldots, a_n$ be positive real numbers satisfying

$$\sum_{i=1}^{n} \frac{1}{a_i + 100} = \frac{1}{100}.$$

Prove that $\sqrt[n]{a_1 \ldots a_n} \geq 100(n-1)$.

Problem 11.21

Let $n \geq 3 \in \mathbb{Z}$, and let $a_2, \ldots, a_n$ be positive real numbers such that $a_2 \ldots a_n = 1$. Prove that

$$\prod_{i=2}^{n} (1 + a_i)^i > n^n.$$

12 Geometric Inequalities

13 Combinatorics

14 Graph Theory

15 Games and Invariants

16 Combinatorial Geometry

Problem 8 (Ozzie MO 2019/3)

Let A, B, C, D and E be five points in that order on a circle Ω Suppose that $\overline{AB} = \overline{CD}$ and $\overline{BC} = \overline{DE}$. Let the chords AD and BE intersect at a point P. Prove that (AEF) lies on Ω.