Fourier transform in option pricing

Abstract

Our goal is to use Fourier transform method for European call pricing whenever characteristic function is available for its log price.

1 Fourier transform

There are different types Fourier transforms. If we use wiki, one definition of FT is given by the following:

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{-2\pi i x u} f(x) dx. \tag{1}$$

ex. Write $\hat{f}(-u)$.

Proposition 1 *If $\hat{f}$ is FT of f in the above sense, then f is the inverse FT of $\hat{f}$ in the sense*

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i x u} \hat{f}(u) du = \hat{\hat{f}}(-x).$$

In our context, we will use different definition of Fourier transform.

Definition 2 *FT of f is a function defined by*

$$\mathcal{F}[f](u) = \int_{-\infty}^{\infty} e^{ixu} f(x) dx.$$

If f is a density function of a random variable X, then $\mathcal{F}[f]$ is called characteristic function of X.

ex. Prove $\mathcal{F}[f](u) = \hat{f}(-\frac{u}{2\pi})$.

Proposition 3 *Inverse transform of $\mathcal{F}$ is given by*

$$\mathcal{F}^{-1}[h](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) e^{-iux} du.$$

PROOF: It's enough to show that

$$\mathcal{F}^{-1} \circ \mathcal{F}[f] = f.$$

Setting $h(u) = \mathcal{F}[f](u) = \hat{f}(-\frac{u}{2\pi})$, we have

$$\begin{aligned} \mathcal{F}^{-1}[h](x) &= \tfrac{1}{2\pi} \textstyle\int_{-\infty}^{\infty} \hat{f}(-\frac{u}{2\pi}) e^{-iux} du \\ &= \textstyle\int_{-\infty}^{\infty} \hat{f}(v) e^{2\pi i v x} dv \end{aligned}$$

and the conclusion holds by Propostion 1. □

ex. If f is a real valued function, then prove that

1. The real part of $\mathcal{F}[f]$ is even,
2. The imaginary part of $\mathcal{F}[f]$ is odd.

There are many useful well known Fourier transforms, see https://en.wikipedia.org/wiki/Fourier_transform. For instance,

$$\mathcal{F}[\frac{1}{x}](u) = i\pi\mathrm{sgn}(u).$$

Let's use this to prove the following identity.

ex. Prove

$$\int_0^\infty \frac{\sin x}{x} dx.$$

Proof.

$$\begin{aligned}
\int_0^\infty \tfrac{\sin x}{x} dx \quad &= \tfrac{1}{2}\int_{-\infty}^\infty \tfrac{\sin x}{x} dx \\
&= \tfrac{1}{2}\mathrm{Img}\int_{-\infty}^\infty \tfrac{e^{ix}}{x} dx \\
&= \tfrac{1}{2}\mathrm{Img}\int_{-\infty}^\infty \tfrac{e^{iux}}{x} dx\Big|_{u=1} \\
&= \tfrac{1}{2}\mathrm{Img}\mathcal{F}[1/x](1) = \pi/2.
\end{aligned}$$