1 Abstract

You will learn

- Euler approximation for the solution of 1-d SDE
- Strong and weak convergence rate

2 Problem

Consider 1-d SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, X(0) = x_0$$

We shall find, for some small step size δ

$$X^\delta(t) \approx X(t), \forall t \geq 0$$

in some sense.

3 Analysis

3.1 Euler scheme

The above SDE can be written as the following integral form:

$$X_t = x_0 + \int_0^t \mu(X_s)ds + \sigma(X_s)dW_s$$

If we denote

$$X_{t,s} = X_s - X_t,$$

then

$$X_{t,t+\delta} = \int_t^{t+\delta} \mu(X_s)ds + \sigma(X_s)dW_s.$$

Ito formula says

$$\mu(X_s) = \mu(X_t) + \int_t^s (\mu'(X_r) + \frac{1}{2}\mu''(X_r))dr + \mu'(X_r)\sigma(X_r)dW_r$$

and

$$\sigma(X_s) = \sigma(X_t) + \int_t^s (\sigma'(X_r) + \frac{1}{2}\sigma''(X_r))dr + \sigma'(X_r)\sigma(X_r)dW_r.$$

If $|s - t| < \delta$ and $\mu, \sigma \in C_b^2$, then [1]

$$\mu(X_s) = \mu(X_t) + O(\delta^{1/2})$$

and

$$\sigma(X_s) = \sigma(X_t) + O(\delta^{1/2}).$$

[1] $f(\delta) = O^{\delta^\gamma}$ means $|f(\delta)| \leq K\delta^\gamma$ for some random variable K and all $\delta \in (0, \epsilon)$.

Thus, $X_{t,t+\delta}$ can be rewritten as

$$X_{t,t+\delta} = \mu(X_t)\delta + \sigma(X_t)W_{t,t+\delta} + O(\delta).$$

or

$$X_{t,t+\delta} \approx \mu(X_t)\delta + \sigma(X_t)W_{t,t+\delta}.$$

With the fact that $W_{i\delta,(i+1)\delta} \sim \sqrt{\delta}Z_i$ are iid normal random variables, we can write the following pseudocode.

pseudocode eulder_1d_path(T, N):

- partition $[0, T]$ equally by $\delta = T/N$;
- set initial $X_0^\delta = x_0$;
- For $i = 0, \ldots, N-1$, with iid standard normal Z_i,
  - perform $X_{i+1}^\delta = X_i^\delta + \mu(X_i^\delta)\delta + \sigma(X_i^\delta)\sqrt{\delta}Z_i$.

3.2 Strong convergence rate

euler_1d_path(T, N) gives a sequence of numbers:

$$(X_0^\delta, X_1^\delta, \ldots, X_N^\delta) := X^\delta.$$

To compare with continuous true path $(X_t : t \in [0, T])$, we first do the piecewise linear interpolation of X^δ, that is

$$L_t^\delta = \frac{(i+1)\delta - t}{\delta} X_i^\delta + \frac{t - i\delta}{\delta} X_{i+1}^\delta, \quad \text{if } i\delta \le t < (i+1)\delta.$$

Theorem 1 *RMSE of Euler approximation under uniform norm has convergence order* $1/2$*, i.e.*

$$\mathbb{E}\Big[\sup_{0\le t\le T} |X_t - L_t^\delta|\Big] \le K\delta^{1/2}.$$

PROOF: see Theorem 2.7.3 of [2]. □

ex. Show that

$$\mathbb{E}[|X_T - L_T^\delta|] \le K\delta^{1/2}.$$

3.2.1 A remark on constant interpolation of Euler solution

If we denote the piecewise constant interpolation by

$$C_t^\delta = X_i^\delta, \text{ if } i\delta \le t < (i+1)\delta,$$

then the above strong convergence fails. Let's use the following example to illustrate this issue.

Let $X = W$ be the Brownian motion itself. Euler yields

$$C_t^\delta = W_{[t/\delta]\delta}.$$

Therefore,

$$RMSE = \mathbb{E}\Big[\sup_{0\le t\le T} |X_t - C_t^\delta|\Big] = \mathbb{E}\Big[\sup_{0\le t\le T} |W_t - W_{[t/\delta]\delta}|\Big] = \mathbb{E}\Big[\sup_{i=0,\ldots,N-1} Y_i\Big],$$

where

$$Y_i = \sup_{i\delta \le t < (i+1)\delta} |W_t - W_{i\delta}|.$$

Note $Y_i \ge |W_{i\delta,(i+1)\delta}| := \sqrt{\delta}|Z_i|$, then

$$RMSE \ge \sqrt{\delta}\mathbb{E}\Big[\sup_{i=0,\dots,N-1} |Z_i|\Big] > O(\delta^{1/2}).$$

3.3 Weak convergence rate

Given Y^δ and X, we define

$$e^g(\delta) = \Big|\mathbb{E}[g(X_T)] - \mathbb{E}[g(Y_T^\delta)]\Big|.$$

Then, we say Y_T^δ converges to X_T weakly if

$$\lim_{\delta \to 0} e^g(\delta) = 0, \ \forall g \in C_b.$$

We say weak convergence rae is γ, if

$$\exists K_g > 0, \ s.t. \ e^g(\delta) \le K_g \delta^\gamma$$

for any $g \in C_b$.

Theorem 2 *C^δ covnverges to X with $\gamma = 1$.*

PROOF: see section 9.7 of [1] □

References

[1] P. E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1992. 3

[2] Xuerong Mao. *Stochastic Differential Equations and Applications.* Horwood Pub Ltd, 2007. 2