

## 1 Abstract

- Using Monte Carlo to approximate  $\pi$ ;
- Introduce Monte Carlo basics.

## 2 Problem

Approximate the value  $\pi$ .

## 3 Analysis

Consider the following question:

- You shoot a square  $(-1, 1)^2$ . Suppose your shot is uniform in this square, then what is the probability you have a successful shot? We say “your shot is successful”, if your shot belongs to the unit ball  $B_1$ .

The answer is

$$\text{Prob of succesful shot} = \frac{\text{Area of } B_1}{\text{Area of } (-1, 1)^2} = \frac{\pi}{4}.$$

This means that, as long as one can approximate probability of successful shot, one can approximate  $\pi$  by multiplying 4. This can be done by computer:

- Simulate many uniform shots, and compute the ratio of successful shots.

## 4 Implementation

Pseudocode:

- Generate  $N$  iid points

$$\{(X_i, Y_i) : i = 1, 2, \dots, N, X_i, Y_i \sim U(-1, 1)\};$$

- Count  $n$ , the number of points satisfying

$$X_i^2 + Y_i^2 < 1.$$

- Compute

$$\hat{\pi} = 4 \cdot \frac{n}{N}.$$

## 5 Monte Carlo basics

One can implement above approximation multiple times and observe that

- (random estimator) Target value  $\pi$  is deterministic, but each implementation gives different outcome  $\hat{\pi}$ ;
- (Convergence) Each obtained outcome, as long as  $N$  is large enough, gives some close approximation.

We are going to generalize our observations in this below.

- A random estimator  $\hat{\alpha}$  to a deterministic value  $\alpha$  is called as Monte Carlo (MC) approximation.
- Moreover, we define

$$Bias = \mathbb{E}[\hat{\alpha}] - \alpha$$

and

$$MSE = \mathbb{E}[(\hat{\alpha} - \alpha)^2].$$

- If Bias is zero, then we call this as unbiased MC.

**Proposition 1**  $MSE(\hat{\alpha}) = |Bias(\hat{\alpha})|^2 + Var(\hat{\alpha})$ .

PROOF: ...  $\square$