1 Abstract

- Using Monte Carlo to approximate π;
- Introduce Monte Carlo basics.

2 Problem

Approximate the value π.

3 Analysis

Consider the following question:

- You shoot a square $(-1,1)^2$. Suppose your shot is uniform in this square, then what is the probability you have a successful shot? We say "your shot is successful", if your shot belongs to the unit ball B_1.

The answer is

$$\text{Prob of succesful shot } = \frac{\text{Area of } B_1}{\text{Area of } (-1,1)^2} = \frac{\pi}{4}.$$

This means that, as long as one can approximate probability of successful shot, one can approximate π by multiplying 4. This can be done by computer:

- Simulate many uniform shots, and compute the ratio of successful shots.

4 Implementation

Pseudocode:

- Generate N iid points
$$\{(X_i, Y_i) : i = 1, 2, \ldots, N,\ X_i, Y_i \sim U(-1,1)\};$$
- Count n, the number of points satisfying
$$X_i^2 + Y_i^2 < 1.$$
- Compute
$$\hat{\pi} = 4 \cdot \frac{n}{N}.$$

5 Monte Carlo basics

One can implement above approximation multiple times and observe that

- (random estimator) Target value π is deterministic, but each implementation gives different outcome $\hat{\pi}$;
- (Convergence) Each obtained outcome, as long as N is large enough, gives some close approximation.

We are going to generalize our observations in this below.

- A random estimator $\hat{\alpha}$ to a deterministic value α is called as Monte Carlo (MC) approximation.
- Moreover, we define
$$Bias = \mathbb{E}[\hat{\alpha}] - \alpha$$
and
$$MSE = \mathbb{E}[(\hat{\alpha} - \alpha)^2].$$
- If Bias is zero, then we call this as unbiased MC.

Proposition 1 $MSE(\hat{\alpha}) = |Bias(\hat{\alpha})|^2 + Var(\hat{\alpha})$.

Proof: Ex. □