Fourier transform in option pricing

Abstract

Our goal is to use Fourier transform method for European call pricing whenever characteristic function is available for its log price.

1 Fourier transform

There are different types Fourier transforms. If we use wiki, one definition of FT is given by the following:

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{-2\pi i x u} f(x) dx. \tag{1}$$

ex. Write $\hat{f}(-u)$.

Proposition 1 *If $\hat{f}$ is FT of f in the above sense, then f is the inverse FT of $\hat{f}$ in the sense*

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i x u} \hat{f}(u) du = \hat{\hat{f}}(-x).$$

In our context, we will use different definition of Fourier transform.

Definition 2 *FT of f is a function defined by*

$$\mathcal{F}[f](u) = \int_{-\infty}^{\infty} e^{ixu} f(x) dx.$$

If f is a density function of a random variable X, then $\mathcal{F}[f]$ is called characteristic function of X.

ex. Prove $\mathcal{F}[f](u) = \hat{f}(-\frac{u}{2\pi})$.

Proposition 3 *Inverse transform of $\mathcal{F}$ is given by*

$$\mathcal{F}^{-1}[h](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) e^{-iux} du.$$

Proof: It's enough to show that

$$\mathcal{F}^{-1} \circ \mathcal{F}[f] = f.$$

Setting $h(u) = \mathcal{F}[f](u) = \hat{f}(-\frac{u}{2\pi})$, we have

$$\begin{aligned} \mathcal{F}^{-1}[h](x) &= \tfrac{1}{2\pi} \textstyle\int_{-\infty}^{\infty} \hat{f}(-\frac{u}{2\pi}) e^{-iux} du \\ &= \textstyle\int_{-\infty}^{\infty} \hat{f}(v) e^{2\pi i v x} dv \end{aligned}$$

and the conclusion holds by Propostion 1. □

ex. If f is a real valued function, then prove that

1. The real part of $\mathcal{F}[f]$ is even,
2. The imaginary part of $\mathcal{F}[f]$ is odd.

There are many useful well known Fourier transforms, see https://en.wikipedia.org/wiki/Fourier_transform. For instance,

$$\mathcal{F}[\frac{1}{x}](u) = i\pi \mathrm{sgn}(u).$$

Let's use this to prove the following identity.

ex. Prove

$$\int_0^\infty \frac{\sin x}{x} dx = \pi/2.$$

Proof.

$$\begin{array}{ll} \int_0^\infty \frac{\sin x}{x} dx & = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin x}{x} dx \\ & = \frac{1}{2} \mathrm{Img} \int_{-\infty}^\infty \frac{e^{ix}}{x} dx \\ & = \frac{1}{2} \mathrm{Img} \int_{-\infty}^\infty \frac{e^{iux}}{x} dx \Big|_{u=1} \\ & = \frac{1}{2} \mathrm{Img} \mathcal{F}[1/x](1) = \pi/2. \end{array}$$

2 Fourier transform methods in option pricing

We consider the price of European call with maturity T and strike K underlying a given stock price S_T. In many cases, the characteristic function of the log price $\ln S_T$ can be explicitly written, for instance in BSM and Heston. Our question is:

> If a characteristic function of $\ln S_T$ is given, what is the price of Call with T and K?

We assume that

- interest rate is $r > 0$,
- $s_T = \ln S_T$
- $k = \ln K$
- and the characteristic function of a random variable $s_T = \ln S_T$ is given by

$$\phi(u) = \mathbb{E} \exp(ius_T).$$

2.1 First Fourier transform method

Our objective is to design an engine with

- Input: r, ϕ, T, K
- output: $C = e^{-rT}\mathbb{E}[(S_T - K)^+]$.

Proposition 4 *The price of Call(T, K) is*

$$C = S_0 I_1 - Ke^{-rT} I_2,$$

where

$$I_1(\phi, k) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\Big(\frac{e^{-iuk}\phi(u-i)}{iu\phi(-i)}\Big) du$$

and

$$I_2(\phi, k) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\Big(\frac{e^{-iuk}\phi(u)}{iu}\Big) du.$$

The above presentation actually gives straightforward evaluation for python as long as the characteristic function is available. One may use scipy.integrate.quad for two integrations in the formula.

PROOF: One can verify step by step

- $I(a > b) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin(u(a-b))}{u} du$ from from $\int_0^\infty \frac{\sin x}{x} dx = \pi/2$ in the exercise in the above,
- For $\phi = \phi_X$, we have

  $$\mathbb{P}(X > H) = I_2(\phi, H)$$

  and

  $$\frac{\mathbb{E}[e^X I(X > H)]}{\mathbb{E}[e^X]} = I_1(\phi, H).$$

$$C = \mathbb{E}[e^{-rT} S_T I(\ln S_T > \ln K)] - K e^{-rT} \mathbb{E}[I(\ln S_T > \ln K)].$$

Note that

$$\mathbb{E}[e^{-rT} S_T] = S_0.$$

Therefore, we have

$$C = S_0 \frac{\mathbb{E}[S_T I(\ln S_T > \ln K)]}{\mathbb{E}[S_T]} - K e^{-rT} \mathbb{E}[I(\ln S_T > \ln K)].$$

Now, we conclude the result by utilizing the above lemmas with $X = \ln S_T$.

□

2.2 Fourier transform method by Carr-Madan

We denote

- $$C_T(k) = e^{-rT} \mathbb{E}[(S_T - K)^+]$$
- For some damping factor $\alpha > 0$, we denote

  $$c_T(k) = e^{\alpha k} C_T(k)$$
- $$\psi = \mathcal{F}[c_T].$$

The key is to find a formula of ψ_T in terms of ϕ, and take inverse FT for $c_T(k)$. Our objective is to design an engine with

- Input: r, ϕ, T, k, α
- output: $C_T(K)$

In theory, any $\alpha > 0$ works for this pricing engine. However, the performance is dependent to the choice of α. Often, α is chosen in $[1/2, 2]$ for better performance.

Proposition 5 *The call price is*

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-iwk} \psi(w) dw,$$

where

$$\psi(w) = \frac{e^{-rT}\phi(w - (\alpha+1)i)}{\alpha^2 + \alpha - w^2 + i(2\alpha+1)w}.$$

Proof: First we look for ψ. By definition, we have with p_T the density of s_T

$$\begin{aligned} \psi(w) &= \int_{-\infty}^{\infty} e^{iwk} e^{\alpha k} e^{-rT} \int_k^\infty (e^s - e^k) p_T(s) ds dk \\ &= \int_{-\infty}^{\infty} e^{-rT} p_T(s) \int_{-\infty}^{s} (e^{(iw+\alpha)k+s} - e^{(iw+\alpha+1)k}) dk ds \end{aligned}$$

This leads to the formula of ψ after some calculations.

Next, we write c_T.

- We can prove that $w \mapsto \psi(w)e^{-iwx}$ has even real part and odd imaginary part. Indeed, c_T is real, we have

  $$\psi = \text{even} + i\text{odd}.$$

  Therefore,

  $$\psi(w)e^{-iwx} = (\text{even} + i\text{odd})(\text{even} + i\text{odd}) = \text{even} + i\text{odd}.$$

- $$\begin{aligned} c_T(x) &= \mathcal{F}^{-1}[\psi](x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(w) e^{-iwx} dw \\ &= \frac{1}{\pi} \int_0^\infty \psi(w) e^{-iwx} dw \end{aligned}$$

Finally, we use the definition of $C_T(x) = e^{-\alpha k} c_T(k)$. $\square$