Soln for Hw on IS - Digital options

① **OMC**, For $Z_i \sim$ iid $N(0,1)$

$$\mathbb{E}[\hat{V}_{10}^2] = \mathbb{E}\left[\left(\frac{1}{10}\sum_{i=1}^{10} I(Z_i < -2)\right)^2\right]$$

$$= \frac{1}{100}\mathbb{E}\left[\sum_{i=1}^{10} \mathbb{E}\, I(Z_i<-2) + \sum_{i\neq j} \mathbb{E}\, I(Z_i<-2)\, I(Z_j<-2)\right]$$

$$= \frac{1}{100}\left[10\,\Phi(-2) + 90\,\Phi^2(-2)\right]$$

$$\boxed{= \frac{1}{10}\left(\Phi(-2) + 9\,\Phi^2(-2)\right)} = 0.0228$$

② IS $(b=2)$: $Z_i \sim N(0,1)$ iid

$$\hat{V}_{10} = \frac{1}{10}e^{\frac{1}{2}b^2}\sum_{i=1}^{10} e^{b(Z_i-b)}\, I(Z_i - b < -2)$$

$$\mathbb{E}\,\hat{V}_{10}^2 = \frac{e^{b^2}}{100}\mathbb{E}\left[\left(\sum_{i=1}^{10} e^{b(Z_i-b)}\, I(Z_i<-2+b)\right)^2\right]$$

$$= \frac{e^{b^2}}{100}\mathbb{E}\Bigg[\sum_{i=1}^{10} e^{2b(Z_i-b)}\, I(Z_i<-2+b) +$$

$$\sum_{i\neq j} e^{b(Z_i-b)}\, I(Z_i<-2+b)\cdot e^{b(Z_j-b)}\, I(Z_j<b-2)\Bigg]$$

$$= \frac{e^{b^2}}{100}\cdot 10\ \mathbb{E}\, e^{2b(Z-b)}\, I(Z<b-2) +$$

$$\frac{e^{b^2}}{100}\cdot 90\left(\mathbb{E}\, e^{b(Z-b)}\, I(Z<b-2)\right)^2$$

$$\mathbb{E}\,\hat{V}_{10}^2 = \frac{e^{b^2}}{10} f(2,b) + \frac{9e^{b^2}}{10} f^2(1,b)$$

where $f(m,b) = \mathbb{E}\, e^{mb(z-b)}\, I(z<b-2)$

$$= e^{-mb^2} \int_{-\infty}^{b-2} e^{mbz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$= e^{-mb^2} e^{\frac{m^2b^2}{2}} \int_{-\infty}^{-2+b} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-mb)^2}{2}}\, dz$$

$$\boxed{\begin{aligned} x &= (z-mb)\Big|_{z=-2+b} \\ &= -2+b-mb \end{aligned}}$$

$$= e^{\frac{m^2-2m}{2}b^2} \int_{-\infty}^{-2+(1-m)b} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= e^{\frac{m^2-2m}{2}b^2} \cdot \Phi(-2+b-mb)$$

In particular,

$$f(1,b) = e^{-\frac{1}{2}b^2}\, \Phi(-2)$$

$$f(2,b) = \Phi(-2-b)$$

$$\mathbb{E}\,\hat{V}_{10}^2 = \frac{e^{b^2}}{10} f(2,b) + \frac{9e^{b^2}}{10} f^2(1,b)$$

$$= \frac{1}{10} e^{b^2} \Phi(-2-b) + \frac{9}{10} \Phi^2(-2)$$

$$= \frac{1}{10}\left(e^{b^2} \Phi(-2-b) + 9\, \Phi^2(-2)\right)$$

so

$$\mathbb{E}\,\hat{V}_{10}^2 \Big|_{b=2} = \frac{1}{10}\left(e^4\, \Phi(-4) + 9\, \Phi^2(-2)\right)$$

$$= 0.0017$$