1 Abstract

- Understand stability condition on FFD solution of heat equation

2 Problem

Solve $u(x,t)$ from the heat equation using FFD

$$u_t = u_{xx}, \quad t > 0, x \in \mathbb{R}$$

with initial data

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}.$$

parameters

- $\phi(x) = |1 - 10x| \cdot I(|x| < 0.1)$.
- space step size $h = .2$
- time step size $\theta = h^2 = .04$.

3 Analysis

We use FTCS (Forward finite difference in time, Central finite difference in state) to solve the above heat equation. This means that we use finite difference form of

$$u_t(x,t) \simeq \frac{u(x, t+\theta) - u(x,t)}{\theta} := \delta_\theta^t u(x,t)$$

and

$$u_{xx}(x,t) \simeq \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} := \delta_h^{xx} u(x,t).$$

where h and θ are some positive mesh size in space h and in time, respectively.

Discrete domain is accordingly a grid of

$$\{(jh, n\theta) : j + 1 \in \mathbb{N}, j \in \mathbb{Z}\}.$$

We denote by u_j^n is the FTCS solution at grid point $(jh, n\theta)$, then we shall have

$$u_t(jh, n\theta) \simeq \frac{u_j^{n+1} - u_j^n}{\theta}, \quad u_{xx}(jh, n\theta) \simeq \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}.$$

Plug it into heat equation, we obtain discrete heat equation of

$$\frac{u_j^{n+1} - u_j^n}{\theta} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}.$$

For simplicity, we set

$$s = \frac{\theta}{h^2}$$

and isolate u^{n+1} to the left hand side, then

$$u_j^{n+1} = su_{j+1}^n + (1-2s)u_j^n + su_{j-1}^n, \quad \forall j \in \mathbb{Z}, n+1 \in \mathbb{N}. \tag{1}$$

Together with initial condition, we have

$$u_j^0 = \phi(jh), \quad \forall j \in \mathbb{Z}. \tag{2}$$

As a summary,

- By the FTCS solution of heat equation, we mean

  $$\{u_j^n : \forall j \in \mathbb{Z}, n \in \mathbb{N}\}$$

  satisfying equations (2) - (1).

- By L^∞ convergence, we mean that the L^∞ error

  $$\epsilon_{h,\theta} = \sup_{\forall j \in \mathbb{Z}, n \in \mathbb{N}} |u_j^n - u(jh, n\theta)|$$

  goes to zero as $(h,\theta) \to (0^+, 0^+)$.

- By L^∞ stability, we mean the uniform boundedness of the numerical solution, i.e. there exists a constant K such that

  $$\sup_{\forall j \in \mathbb{Z}, n \in \mathbb{N}} |u_j^n| < K, \ \forall h, \theta > 0.$$

For any numerical solution, our ultimate wish is to have its convergence. To have a convergence, it is crucial to examine its stability. Accordingly, the number $s = \theta/h^2$ is defined for simplicity earlier, but it turns out to be crucial.

Implementing FTCS is essentially a sequence of realization of the following , stencil (template) given by (1) line by line in n.

$$\begin{array}{ccc} & * & \\ & & \\ \circ & \circ & \circ \\ (s) & (1-2s) & (s) \end{array}$$

Pseudocode heat_ftcs(h, θ):

- Set initial $\{u_j^0, \forall j \in \mathbb{Z}\}$ by (2);
- For $n - 1 \in \mathbb{N}$, do:

  $$u_j^n \implies u_j^{n+1}, \forall j \in \mathbb{Z} \text{ by (1)}.$$

The above pseudocode is not practical since the grid points are infinitely many. But, if the desired computation is for instance

$$\{u_j^n : j = a, a+1, \ldots, b-1, b\}$$

for some integers n and $a < b$, then one can set initial on finitely many points

$$\{u_j^0, \quad j = a-n, a-n+1, \ldots, b+n-1, b+n\}.$$

4 Numerical result

We use hand computation to demonstrate instability with the parameters given as

- $\phi(x) = |1 - 10x| \cdot I(|x| < 0.1)$.
- space step size $h = .2$
- time step size $\theta = h^2 = .04$.

Note that, $s = 1$ and corresponding stencil is

$$\begin{array}{ccc} & * & \\ \circ & \circ & \circ \\ (1) & (-1) & (1) \end{array}$$

One can easily figure out numerical outcomes

$$\{u_j^3 : j = -4, -3, \ldots, 3, 4\}$$

as follows.

$$\begin{array}{ccccccccc} 0 & 1 & -3 & 6 & -7 & 6 & -3 & 1 & 0 \\ & 0 & 1 & -2 & 3 & -2 & 1 & 0 & \\ & & 0 & 1 & -1 & 1 & 0 & & \\ & & & 0 & 1 & 0 & & & \\ & & & & (u_0^0) & & & & \end{array}$$

It can be seen that $|u_0^n| \to \infty$ as $n \to \infty$, which demonstrates its instability.