1   Abstract

- Using Monte Carlo to approximate π;

- Introduce Monte Carlo basics.

2   Problem

Approximate the value π.

3   Analysis

Consider the following question:

- You shoot a square $(-1,1)^2$. Suppose your shot is uniform in this square, then what is the probability you have a successful shot? We say "your shot is successful", if your shot belongs to the unit ball B_1.

The answer is

$$\text{Prob of succesful shot } = \frac{\text{Area of } B_1}{\text{Area of } (-1,1)^2} = \frac{\pi}{4}.$$

This means that, as long as one can approximate probability of successful shot, one can approximate π by multiplying 4. This can be done by computer:

Algorithm 1 MC estimation of π

1: **procedure** MCPI(N)   ▷ N is total number of samples
2:     $n \leftarrow 0$ ▷ n is number of hits
3:     **for** $i = 1...N$ **do**
4:         generate two numbers X, Y from $U(-1,1)$
5:         **if** $X^2 + Y^2 < 1$ **then** $n \leftarrow n + 1$
6:     **return** $\frac{4n}{N}$

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4   Monte Carlo basics

4.1   Bias and MSE

One can implement above approximation multiple times and observe that

- (random estimator) Target value π is deterministic, but each implementation gives different outcome $\hat{\pi}$;

- (Convergence) Each obtained outcome, as long as N is large enough, gives some close approximation.

We are going to generalize our observations in this below.

- A random estimator $\hat{\alpha}$ to a deterministic value α is called as Monte Carlo (MC) approximation.

- Moreover, we define
$$Bias = \mathbb{E}[\hat{\alpha}] - \alpha$$
and
$$MSE = \mathbb{E}[(\hat{\alpha} - \alpha)^2].$$

- (def) If Bias is zero, then we call this as *unbiased* MC.

Proposition 1 $MSE(\hat{\alpha}) = |Bias(\hat{\alpha})|^2 + Var(\hat{\alpha})$. *In particular, if $\hat{\alpha}$ is unbiased, then MSE is Variance.*

PROOF: ... $\square$

Although seemingly absurd, we consider the above estimator with $N = 1$, which is equivalent to

- Consider
$$\hat{\alpha} = 4I(X_1^2 + Y_1^2 < 1), \ \ X_1, Y_1 \sim U(-1, 1)$$
as MC for π. Then the outcome is either 0 or 4. In any case, it is a bad approximation.

- However, we can show that it's an unbiased MC. (why?)

- Find MSE?

4.2 Ordinary Monte Carlo

Unbiased MC is very desirable, because one can employ crude (ordinary) MC to make it more accurate: [1]

- Suppose $\hat{\alpha}$ is a square integrable unbiased MC;

- Obtain N independent replicates
$$\{\hat{\alpha}_i : i = 1, \ldots, N\}.$$

- Taking their average, it gives a new MC:
$$\beta_N = \frac{1}{N} \sum_{i=1}^{N} \hat{\alpha}_i.$$

- β_N is unbiased again. (why?)

- $MSE(\beta_N) = \frac{1}{N} MSE(\hat{\alpha}) \to 0$. (why?)

- β_N is *almost surely consistent*, (why?) i.e.
$$\beta_N \to \alpha, \ \ \text{almost surely or } \mathbb{P}(\lim_N \beta_N = \alpha) = 1.$$

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[1] We say a random variable X is in L^p, if its pth moment exists, i.e. $\mathbb{E}|X|^p < \infty$. If $X \in L^2$, then we say it's square integrable.

- β_N is L^2-consistent, (why?) i.e.

$$\beta_N \to \alpha \text{ in } L^2 \text{ or } \mathbb{E}(\beta_N - \alpha)^2 \to 0.$$

As a conclusion, one can always use crude MC to make better approximation provided there exists an unbiased MC $\hat{\alpha}$, which is obtainable in sacrifice of higher computational cost. Given a fixed amount of computational cost, to improve the efficiency, it is essential to reduce $Var(\hat{\alpha})$ as much as possible.

Ex. Prove that L^2 consistency implies consistency in probability.