1 Abstract

- SDE
- and related financial models

2 SDE

2.1 General problem

We will consider the general d-dimensional SDE:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, X_0 = x_0$$

where $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a smooth vector field on $\mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ is a smooth matrix-valued function, W is a d-dimensional standard Brownian motion, and x_0 is the initial d-dimensional vector.

Some theoretical interests are the sufficient condition for the unique solvability, and computations, which can be founded in the literature.

2.2 Example: 2-d SDE

It can be written by system of two 1-d SDEs as the following:

$$\begin{cases} dX_{1,t} = b_{1,t}dt + \sigma_{11,t}dW_{1,t} + \sigma_{12,t}dW_{2,t}, & X_{1,0} = x_{1,0} \\ dX_{2,t} = b_{2,t}dt + \sigma_{21,t}dW_{1,t} + \sigma_{22,t}dW_{2,t}, & X_{2,0} = x_{2,0} \end{cases}$$

In the above, we assume W_1 and W_2 are two independent 1-d Brownian motions.

3 Stock models

3.1 Arithmetic BM

We denote by $BM(\mu, \sigma^2)$ the dynamics

$$dX_t = \mu dt + \sigma dW_t.$$

3.2 Geometric BM

We denote by $GBM(s, r, \sigma^2)$ the dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t, S_0 = s$$

Non-negativity of the GBM process is good for modeling stock price, namely BSM.

Example 1 *Prove that*

- $X_t = \ln S_t$ *has the distribution* $\mathcal{N}(s + (r - \frac{1}{2}\sigma^2)t, \sigma^2 t)$*;*
- *The characteristic function of* X_t *is* $\phi_t = \exp\{iu(s + (r - \frac{1}{2}\sigma^2)t) - \frac{u^2\sigma^2 t}{2}\}$.

3.3 Stochastic volatility model: Local volatility

Due to limit capacity of GBM in calibration, one can extend the asset price as

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t.$$

The difference is that the volatility σ_t is a random process and this model is classified as stochastic volatility model.

If volatility is modelled by $\sigma_t = \hat{\sigma}(t, S_t)$ for some deterministic function $\hat{\sigma}$, then it is called local volatility model, one of the most important case in stochastic volatility models.

3.3.1 CEV

The stock follows

$$dS_t = \mu S_t dt + \sigma S_t^{\gamma} dW_t.$$

- $\gamma = 1$ gives GBM.
- When $\gamma < 1$, we see the so-called leverage effect, commonly observed in equity markets, where the volatility of a stock increases as its price falls.
- Conversely, when $\gamma > 1$, it exhibits so-called inverse leverage effect often observed in commodity markets, whereby the volatility of the price of a commodity tends to increase as its price increases.

3.4 Stochastic volatility model: Heston model

Heston model as a stochastic volatility model belongs to 2-d SDE in the above. However, the domain of the diffusion matrix σ is not entire 2-d space.

In the Heston model, the dynamic involves two processes (S_t, ν_t). More precisely, the asset price S follows generalized geometric Brownian motion with random volatility process $\sqrt{\nu_t}$, i.e.

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_{1,t},$$

while squared of volatility process ν follows CIR process

$$d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t}(\rho dW_{1,t} + \bar{\rho} dW_{2,t})$$

with $\rho^2 + \bar{\rho}^2 = 1$.

- Feller condition for its existence of the solution is

  $$2\kappa\theta > \xi^2.$$

- A Heston model with parameters $(S_0, v_0, r, \kappa, \theta, \xi, \rho)$ has the characteristic function of $\ln S_T$ as

  $$\phi_T(w) = f_1(w) f_2(w) f_3(w),$$

  where

  $$t_1 = \kappa - i\rho\xi u$$
  $$D = \sqrt{\{}t_1^2 + (u^2 + iu)\xi^2,$$

$$
\begin{aligned}
G &= (t_1 - D)/(t_1 + D), \\
t_2 &= 1 - Ge^{-DT} \\
f_1 &= \exp(iu(\ln S_0 + rT)) \\
f_2 &= \exp(v_0(1 - e^{-DT})(t_1 - D)/\xi^2/t_2) \\
f_3 &= \exp(\kappa\theta(T(t_1 - D) - 2\ln(t_2/(1 - G)))/\xi^2).
\end{aligned}
$$

See page 53 of
https://github.com/songqsh/songqsh.github.io/blob/master/paper/Ng05.pdf

Example 2 *A benchmark to Heston model with the following parameters:*

$$S_0 = 100, v_0 = 0.0175, r = 0., \kappa = 1.5768, \theta = 0.0398, \xi = 0.5751, \rho = -0.5751.$$

The estimation of Call$(T = 1, K = [80, 100, 120])$ *is given as*

$$[32.5, 22.3, 14.7].$$

See Page 61 of [2].

4 Short rate models

In general, interest rate r_t is random and the zero bond price $P(0, T)$ follows

$$P(0, T) = \mathbb{E}[\exp\{-\int_0^T r(u)du\}].$$

4.1 Vasicek model

It is a model for short rate r_t given by OU process:

$$dr_t = \alpha(b - r_t)dt + \sigma dW_t.$$

4.2 Ho-Lee model

It is a short rate model given by

$$dr_t = g(t)dt + \sigma dW_t.$$

4.3 Hull-White model

It is short rate model, which extends Vasicek model, given by

$$dr_t = [g(t) + h(t)r_t]dt + \sigma(t)dW_t,$$

where g, h, σ are given deterministic functions.

Example 3

- *determine function g, h, σ for the Vasicek model;*
- *write explicit solution for HW.*

4.4 CIR model

It is short rate of

$$dr_t = \alpha(b - r_t)dt + \sigma\sqrt{r_t}dW_t.$$

Note that, squared volatility in Heston model has the same dynamics.

4.5 Affine term structure: Multifactor model

We say that a model of d-dim factor variable X_t is affine if the zero bond can be written as

$$P(t,T) = \exp\{A(t,T) + B^T(t,T)X_t\}.$$

Example 4 *Verify that Vasicek model is one-factor affine model.*

Indeed, one can have affine class model in more general settings.

4.5.1 Guassian Multifactor models

Let the short rate given by

$$r_t = \mu + \theta^T X_t$$

where $\theta \in \mathbb{R}^d$, $\mu \in \mathbb{R}$, and d-factor process X_t is given by d-dimensional OU process

$$dX_t = BX_tdt + KdW_t.$$

Then, it belongs to affine class, see for explicit $P(t,T)$ in p107 of [1].

References

[1] A. Cairns. *Interest Rate Models: An Introduction.* Princeton University Press, 2004. 4

[2] Ali Hirsa. *Computational methods in finance.* CRC Press, 2012. 3