# 1 Abstract


- We will introduce implied volatility for a given call/put option
- volatility smile will be demonstrated with a given data sets


# 2 Problem

Our goal is to compute

$$\alpha = \int_0^1 h(x)dx$$

where

$$h(x) = 100 \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x).$$

Of course, the exact value shall be

$$\alpha = 1.98.$$

Pretended not to know the exact value, we have used OMC with exact sampling of uniform random variable, denoted by omc_integral(n).

Next, we are going to improve the efficiency of omc_integral(n) by using importance sampling. We also extend our skill on exact sampling by using inverse transform.

# 3 Analysis

## 3.1 Importance sampling

Recall that, to estimate the above integral $\alpha$, we use the uniform random variable $X$, whose density is $p(x) = I_{(0,1)}(x)$, and write

$$\alpha = \mathbb{E}[h(X)|X \sim p] = \int_0^1 h(x)p(x)dx.$$

Naturally, one can sample iid uniform random numbers by computer, denoted by

$$\{\ X_i \sim p : i = 1, 2, \ldots, n\},$$

then taking their average for its approximation of $\alpha$, i.e.

$$\hat{\alpha}_n = \frac{1}{n} \sum_{i=1}^{n} h(X_i).$$

**Example 1** *Compute MSE of $\hat{\alpha}_n$.*

**Solution**. Since it is unbiased, MSE is the same as Variance of $\hat{\alpha}_n$, and it is again equal to $1/n$ of

$$Var[h(X)|X \sim p].$$

Therefore, it is $\frac{100.99}{n}$. □

IS considers, with a smart choice of a pdf $p_1$,

$$\alpha = \int_0^1 h(x)\frac{p(x)}{p_1(x)}p_1(x)dx = \mathbb{E}\Big[h(X)\frac{p(X)}{p_1(X)}\Big|X \sim p_1(x)\Big]$$

Since we observe that the interval $(0, 1/100)$ is much more *important* than $(1/100, 1)$, our choice of $p_1$ is the following:

$$p_1(x) = \frac{1}{C}(2 \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x)),$$

where $C = 101/100$ is the normalizing constant to make $p_1$ to be a valid pdf.

Pseudocode is_integral(n):

- Generate iid $p_1$ samples, denoted by

$$\{X_i : i = 1, 2, \ldots, n\}.$$

- Compute the average of the integrand $h$ adjusted by likelyhood ratio (also referred to radon-nikodym derivative) $p/p_1$, i.e.

$$\bar{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} h(X_i) \cdot \frac{p(X_i)}{p_1(X_i)}.$$

**Example 2** *Prove that MSE of* $\hat{\alpha}_n$ *is* $51.4999/n$.

## 3.2 Inverse transform method

To implement the is_integral($n$), we shall generate $p_1$ samples. But this is not directly available by python. Inverse transform method provides exact sampling as long as the inverse of CDF is explicitly available. Its theoretic basis is given next.

**Proposition 1** *Suppose* $X$ *has its CDF* $F$ *and* $F^{-1}$ *exists, then* $F^{-1}(U) \sim X$*, where* $U \sim U(0, 1)$.

Proof:

$$\mathbb{P}(F^{-1}(U) \le x) = \mathbb{P}(U \le F(x) = F(x).$$

□

Pseudocode it_sampling($F^{-1}, n$):

- Generate iid $U(0, 1)$ random variables

$$\{Y_i : i = 1, \ldots, n\}.$$

- Compute

$$\{X_i = F^{-1}(Y_i) : i = 1, \ldots, n\}.$$

## 4 Exercises

1. Find $F_1$, the cdf of $p_1$.
2. Find $F_1^{-1}$.
3. Implement it_sampling()
4. Implement is_integral()
5. Demonstrate the convergence rate of is_integral()
6. Could you find a pdf $p_2$ better than $p_1$?