1 Abstract

- derive Crank-Nicolson scheme
- prove unconditional stability

2 Problem

We have seen that FTCS scheme is stable for the heat equation

$$u_t = u_{xx}, \quad t > 0, x \in \mathbb{R}$$

with initial data

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}.$$

when $s = \frac{\theta}{h^2} < 1/2$ holds. Next, we are going to present Crank-Nicolson scheme and investigate its sability.

3 Analysis

We recall that FFD in time is

$$u_t(x,t) \simeq \frac{u(x, t+\theta) - u(x,t)}{\theta} := \delta^t_\theta u(x,t)$$

and CFD2 in state is

$$u_{xx}(x,t) \simeq \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} := \delta^{xx}_h u(x,t),$$

where h and θ are some positive mesh size in space h and in time, respectively.

Discrete domain is accordingly a grid of

$$\{(jh, n\theta) : j + 1 \in \mathbb{N}, j \in \mathbb{Z}\}.$$

Recall that FTCS is to find numerical values u^n_j at a grid point $(jh, n\theta)$, such that

$$\delta^t_\theta u(jh, n\theta) \simeq \frac{u^{n+1}_j - u^n_j}{\theta} := (\delta^t_\theta u)^n_j, \quad \delta^{xx}_h u(jh, n\theta) \simeq \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{h^2} := (\delta^{xx}_h u)^n_j.$$

Plug it into the heat equation, we obtain FTCS discrete heat equation

$$u^{n+1}_j = s u^n_{j+1} + (1-2s) u^n_j + s u^n_{j-1}, \quad \forall j \in \mathbb{Z}, n+1 \in \mathbb{N}. \tag{1}$$

The Crank-Nicolson with parameter $\lambda \in (0,1)$ (denoted by CS-λ) is to find numerical values u^n_j at each grid point $(jh, n\theta)$, such that

$$\delta^t_\theta u(jh, n\theta) \simeq (\delta^t_\theta u)^n_j, \quad \delta^{xx}_h u(jh, n\theta) \simeq (\delta^{xx}_h u)^n_j.$$

with the initial condition

$$u^0_j = \phi(jh), \quad \forall j \in \mathbb{Z}. \tag{2}$$

where

$$s = \frac{\theta}{h^2}.$$

By the FTCS solution of heat equation, we mean

$$\{u^n_j : \forall j \in \mathbb{Z}, n \in \mathbb{N}\}$$

satisfying equations (2) - (1).

3.1 Solution

In this below, we solve for FTCS solution in two steps. First, we use the technic of the separation in variable to find all possible solutions satisfying (1). Second step is to choose specific solution by fitting the initial condition (2).

We first search for the solution of (1) given by the product of j-function and n-function:

$$u_j^n = X_j T_n.$$

Of course, any linear combination of such solutions shall give another solution of (1).

Plug above form into (1), it writes

$$\frac{T_{n+1}}{T_n} = 1 - 2s + s\frac{X_{j+1} - X_j}{X_j}.$$

Note that, left hand side is a function of n while right hand side is a function of j for all (n, j). Thus, to be equal, they must be equal to a constant, say ξ, i.e.

$$\frac{T_{n+1}}{T_n} = 1 - 2s + s\frac{X_{j+1} - X_j}{X_j} := \xi.$$

Therefore, we have

$$T_n = \xi^n T_0 = \xi^n, \forall n \in \mathbb{N}$$

and

$$1 - 2s + s\frac{X_{j+1} - X_j}{X_j} := \xi, \forall j \in \mathbb{Z}.$$

In the above, $T_0 = 1$ is assumed w.l.o.g. (why?) Another trick is to postulate X in the form of

$$X_j = (e^{ikh})^j$$

for some $k \in \mathbb{Z}$. The reason is that, we expect the solution X is in L^2, and our solution could be linear combination of the above Fourier basis functions.

Then, we can solve for ξ by plugging into X-equation above, that

$$\xi(k) := \xi = 1 - 2s + 2s\cos khj.$$

At last, we have general representation of the numerical solution:

$$u_j^n = \sum_{k\in\mathbb{Z}} b_k e^{ikhj} \xi^n(k).$$

The rest is to determine coefficients $(b_k : k \in \mathbb{Z})$ from the initial condition using orthorgonal basis functions.

3.2 Stability condition

From the above solution representation, we shall have

if $|\xi(k)| \leq 1$ for all $k \in \mathbb{Z}$, then FTCS is stable.

Since $s > 0$, the inequality $\xi(k) \leq 1$ holds automatically. The lower bound of $\xi(k)$ is $1 - 4s$. So we shall require $-1 \leq 1 - 4s$ for its stability, which at last gives the **sufficient condition of the stability** by

$$s \leq 1/2.$$

The example discussed above indicates that the general procedure to determine stability in a diffusion or wave problem is to separate variables in the difference equation. For the time factor we obtain a simple equation with an amplication factor $\xi(k)$. In the analysis above, we use $|\xi(k)| \leq 1$ for the stability. More precisely, it can be shown that the correct condition necessary for stability is

$$|\xi(k)| \leq 1 + O(\theta), \forall k \in \mathbb{Z}.$$

This is the **von Neumann stability condition**.