

## 1 Abstract

- Using Monte Carlo to approximate  $\pi$ ;
- Introduce Monte Carlo basics.

## 2 Problem

Approximate the value  $\pi$ .

## 3 Analysis

Consider the following question:

- You shoot a square  $(-1, 1)^2$ . Suppose your shot is uniform in this square, then what is the probability you have a successful shot? We say “your shot is successful”, if your shot belongs to the unit ball  $B_1$ .

The answer is

$$\text{Prob of succesful shot} = \frac{\text{Area of } B_1}{\text{Area of } (-1, 1)^2} = \frac{\pi}{4}.$$

This means that, as long as one can approximate probability of successful shot, one can approximate  $\pi$  by multiplying 4. This can be done by computer:

- Simulate many uniform shots, and compute the ratio of successful shots.

Pseudocode for `pi_mc(N)`:

- Generate  $N$  iid points

$$\{(X_i, Y_i) : i = 1, 2, \dots, N, X_i, Y_i \sim U(-1, 1)\};$$

- Count  $n$ , the number of points satisfying

$$X_i^2 + Y_i^2 < 1.$$

- Compute

$$\hat{\pi} = 4 \cdot \frac{n}{N}.$$

## 4 Monte Carlo basics

### 4.1 Bias and MSE

One can implement above approximation multiple times and observe that

- (random estimator) Target value  $\pi$  is deterministic, but each implementation gives different outcome  $\hat{\pi}$ ;

- (Convergence) Each obtained outcome, as long as  $N$  is large enough, gives some close approximation.

We are going to generalize our observations in this below.

- A random estimator  $\hat{\alpha}$  to a deterministic value  $\alpha$  is called as Monte Carlo (MC) approximation.
- Moreover, we define

$$Bias = \mathbb{E}[\hat{\alpha}] - \alpha$$

and

$$MSE = \mathbb{E}[(\hat{\alpha} - \alpha)^2].$$

- If Bias is zero, then we call this as unbiased MC.

**Proposition 1**  $MSE(\hat{\alpha}) = |Bias(\hat{\alpha})|^2 + Var(\hat{\alpha})$ . In particular, if  $\hat{\alpha}$  is unbiased, then MSE is Variance.

PROOF: ...  $\square$

Although seemingly absurd, we consider the above estimator with  $N = 1$ , which is equivalent to

- Consider

$$\hat{\alpha} = 4I(X_1^2 + Y_1^2 < 1), \quad X_1, Y_1 \sim U(-1, 1)$$

as MC for  $\pi$ . Then the outcome is either 0 or 4. In any case, it is a bad approximation.

- However, we can show that it's an unbiased MC. (why?)
- Find MSE?

## 4.2 Ordinary Monte Carlo

Unbiased MC is very desirable, because one can employ crude (ordinary) MC to make it more accurate: <sup>1</sup>

- Suppose  $\hat{\alpha}$  is a square integrable unbiased MC;
- Obtain  $N$  independent replicates

$$\{\hat{\alpha}_i : i = 1, \dots, N\}.$$

- Taking their average, it gives a new MC:

$$\beta_N = \frac{1}{N} \sum_{i=1}^N \hat{\alpha}_i.$$

- $\beta_N$  is unbiased again. (why?)

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<sup>1</sup>We say a random variable  $X$  is in  $L^p$ , if its  $p$ th moment exists, i.e.

$$\mathbb{E}X^p < \infty.$$

If  $X \in L^2$ , then we say it's square integrable.

- $MSE(\beta_N) = \frac{1}{N}MSE(\hat{\alpha}) \rightarrow 0$ . (why?)
- $\beta_N$  is almost surely consistent, (why?) i.e.

$$\mathbb{P}(\lim_N \beta_N = \alpha) = 1.$$

- $\beta_N$  is  $L^2$ -consistent, (why?) i.e.

$$\mathbb{E}(\beta_N - \alpha)^2 \rightarrow 0.$$

As a conclusion, one can always use crude MC to make better approximation provided there exists an unbiased MC  $\hat{\alpha}$ , which is obtainable in sacrifice of higher computational cost. Given a fixed amount of computational cost, to improve the efficiency, it is essential to reduce  $Var(\hat{\alpha})$  as much as possible.

**Ex.** Given i.i.d  $\{\alpha_i : i \in 1, 2, \dots, N\}$ , we use

$$\beta_N = \frac{1}{N} \sum_{i=1}^N (\alpha_i - \bar{\alpha})^2$$

as the estimator of  $Var(\alpha_1)$ , where  $\bar{\alpha} = \frac{1}{N} \sum_{i=1}^N \alpha_i$ . Suppose  $\alpha_1 \in L^2$ , then

- Prove it is biased.
- Prove that it is consistent in  $L^2$ .
- Can you propose an unbiased estimator?

**Ex.** Prove that  $L^2$  consistency implies consistency in probability.

**Prj.** Investigate the convergence rate of `pi_mc(N)`.

- Repeat  $m = 100$  times:  
For  $N \in \{2^n : n = 5, \dots, 10\}$ : Run `PiMc(N)`
- approximate their MSE for each  $N$  with  $m$  computations;
- plot log-log chart and approximate its convergence order.

**Ex.** Can you improve the MC for  $\pi$ ? Hint: antithetic variate method

**Ex.** Can you propose a deterministic approximation to  $\pi$ ?