Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth even function satisfying $f(0) = 0$. Our objective is to approximate the second order derivative $f''(0)$.

- Prove that $f'(0) = 0$.
- Chenyu proposes the following estimator for $f''(0)$: for a step size h

$$a_h = \frac{2f(h)}{h^2}.$$

  Please justify that Chenyu's estimation has its convergence $O(h^2)$.
- Is there anyway to improve the above convergence to $O(h^4)$ in the form of

$$b_h = \frac{c_1 f(h) + c_2 f(2h)}{h^2}$$

  for some constants c_1 and c_2?
- If the above function f is odd and other properties remain the same, how do you want to find the $f''(0)$ efficiently?

Solution

- it can be directly shown from the definition of f'.
- $f^n(0) = 0$ for all odd number n. Therefore, taylor expansion gives

$$f(h) = \frac{1}{2}h^2 f''(0) + \frac{1}{24}h^4 f^{(4)}(0) + O(h^6),$$

  and the result follows.
- we can combine the above taylor expansion with

$$f(2h) = 2h^2 f''(0) + \frac{2}{3}h^4 f^{(4)}(0) + O(h^6).$$

  It yields that, with $c_1 = 8/3$ and $c_2 = -1/6$,

$$c_1 f(h) + c_2 f(2h) = h^2 f''(0) + O(h^6).$$

- If f is odd, then $f''(0) = 0$ and no estimate is needed any more.