

# 1 Abstract

- Implement FFD on heat equation
- Understand stability condition

# 2 Problem

Solve  $u(x, t)$  from the heat equation using FFD

$$u_t = u_{xx}, \quad t > 0, x \in \mathbb{R}$$

with initial data

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}.$$

**parameters**

- $\phi(x) = |1 - 10x| \cdot I(|x| < 0.1)$ .
- space step size  $h = .2$
- time step size  $\theta = h^2 = .04$ .

# 3 Analysis

We use FTCS (Forward finite difference in time, Central finite difference in state) to solve the above heat equation. This means that we use finite difference form of

$$u_t(x, t) \simeq \frac{u(x, t + \theta) - u(x, t)}{\theta} := \delta_\theta^t u(x, t)$$

and

$$u_{xx}(x, t) \simeq \frac{u(x + h, t) - 2u(x, t) + u(x - h, t)}{h^2} := \delta_h^{xx} u(x, t).$$

where  $h$  and  $\theta$  are some positive mesh size in space  $h$  and in time, respectively.

Discrete domain is accordingly a grid of

$$\{(jh, n\theta) : j + 1 \in \mathbb{N}, j \in \mathbb{Z}\}.$$

We denote by  $u_j^n$  is the FTCS solution at grid point  $(jh, n\theta)$ , then we shall have

$$u_t(jh, n\theta) \simeq \frac{u_j^{n+1} - u_j^n}{\theta}, \quad u_{xx}(jh, n\theta) \simeq \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}.$$

Plug it into heat equation, we obtain discrete heat equation of

$$\frac{u_j^{n+1} - u_j^n}{\theta} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}.$$

For simplicity, we set

$$s = \frac{\theta}{h^2}$$

and isolate  $u^{n+1}$  to the left hand side, then

$$u_j^{n+1} = su_{j+1}^n + (1 - 2s)u_j^n + su_{j-1}^n, \quad \forall j \in \mathbb{Z}, n + 1 \in \mathbb{N}. \quad (1)$$

Together with initial condition, we have

$$u_j^0 = \phi(jh), \quad \forall j \in \mathbb{Z}. \quad (2)$$

As a summary,

- By the FTCS solution of heat equation, we mean

$$\{u_j^n : \forall j \in \mathbb{Z}, n \in \mathbb{N}\}$$

satisfying equations (2) - (1).

- By  $L^\infty$  convergence, we mean that the  $L^\infty$  error

$$\epsilon_{h,\theta} = \sup_{\forall j \in \mathbb{Z}, n \in \mathbb{N}} |u_j^n - u(jh, n\theta)|$$

goes to zero as  $(h, \theta) \rightarrow (0^+, 0^+)$ .

- By  $L^\infty$  stability, we mean the uniform boundedness of the numerical solution, i.e. there exists a constant  $K$  such that

$$\sup_{\forall j \in \mathbb{Z}, n \in \mathbb{N}} |u_j^n| < K, \quad \forall h, \theta > 0.$$

For any numerical solution, our ultimate wish is to have its convergence. To have a convergence, it is crucial to examine its stability. Accordingly, the number  $s = \theta/h^2$  is defined for simplicity earlier, but it turns out to be crucial.

Implementing FTCS is essentially a sequence of realization of the following , stencil (template) given by (1) line by line in  $n$ .

$$\begin{array}{ccc} & * & \\ \circ & & \circ \\ (s) & (1 - 2s) & (s) \end{array}$$

**Pseudocode** heat\_ftcs( $h, \theta$ ):

- Set initial  $\{u_j^0, \forall j \in \mathbb{Z}\}$  by (2);
- For  $n - 1 \in \mathbb{N}$ , do:

$$u_j^n \implies u_j^{n+1}, \forall j \in \mathbb{Z} \text{ by (1)}.$$

The above pseudocode is not practical since the grid points are infinitely many. But, if the desired computation is for instance

$$\{u_j^n : j = a, a + 1, \dots, b - 1, b\}$$

for some integers  $n$  and  $a < b$ , then one can set initial on finitely many points

$$\{u_j^0, \quad j = a - n, a - n + 1, \dots, b + n - 1, b + n\}.$$

### 3.1 Numerical result

We use hand computation to demonstrate instability with the parameters given as

- $\phi(x) = |1 - 10x| \cdot I(|x| < 0.1)$ .
- space step size  $h = .2$
- time step size  $\theta = h^2 = .04$ .