# 1 Abstract

- Implement FFD on heat equation

- Understand stability condition

# 2 Problem

Solve $u(x, t)$ from the heat equation using FFD

$$u_t = u_{xx}, \quad t > 0, x \in \mathbb{R}$$

with initial data

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}.$$

**parameters**

- $\phi(x) = |1 - 10x| \cdot I(|x| < 0.1)$.

- space step size $h = .2$

- time step size $\theta = h^2 = .04$.

# 3 Analysis

We use FTCS (Forward finite difference in time, Central finite difference in state) to solve the above heat equation. This means that we use finite difference form of

$$u_t(x, t) \simeq \frac{u(x, t + \theta) - u(x, t)}{\theta} := \delta_\theta^t u(x, t)$$

and

$$u_{xx}(x, t) \simeq \frac{u(x + h, t) - 2u(x, t) + u(x - h, t)}{h^2} := \delta_h^{xx} u(x, t).$$

where $h$ and $\theta$ are some positive mesh size in space $h$ and in time, respectively.

Discrete domain is accordingly a grid of

$$\{(jh, n\theta) : j + 1 \in \mathbb{N}, j \in \mathbb{Z}\}.$$

We denote by $u_j^n$ is the FTCS solution at grid point $(jh, n\theta)$, then we shall have

$$u_t(jh, n\theta) \simeq \frac{u_j^{n+1} - u_j^n}{\theta}, \quad u_{xx}(jh, n\theta) \simeq \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}.$$

Plug it into heat equation, we obtain discrete heat equation of

$$\frac{u_j^{n+1} - u_j^n}{\theta} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}.$$

For simplicity, we set

$$s = \frac{\theta}{h^2}$$

and isolate $u^{n+1}$ to the left hand side, then

$$u_j^{n+1} = su_{j+1}^n + (1-2s)u_j^n + su_{j-1}^n, \quad \forall j \in \mathbb{Z}, n+1 \in \mathbb{N}. \tag{1}$$

Together with initial condition, we have

$$u_j^0 = \phi(jh), \quad \forall j \in \mathbb{Z}. \tag{2}$$

As a summary,

- By the FTCS solution of heat equation, we mean

$$\{u_j^n : \forall j \in \mathbb{Z}, n \in \mathbb{N}\}$$

  satisfying equations (2) - (1).

- By $L^\infty$ convergence, we mean that the $L^\infty$ error

$$\epsilon_{h,\theta} = \sup_{\forall j \in \mathbb{Z}, n \in \mathbb{N}} |u_j^n - u(jh, n\theta)|$$

  goes to zero as $(h, \theta) \to (0^+, 0^+)$.

- By $L^\infty$ stability, we mean the uniform boundedness of the numerical solution, i.e. there exists a constant $K$ such that

$$\sup_{\forall j \in \mathbb{Z}, n \in \mathbb{N}} |u_j^n| < K, \ \forall h, \theta > 0.$$

For any numerical solution, our ultimate wish is to have its convergence. To have a convergence, it is crucial to examine its stability. Accordingly, the number $s = \theta/h^2$ is defined for simplicity earlier, but it turns out to be crucial.

Implementing FTCS is essentially a sequence of realization of the following , stencil (template) given by (1) line by line in $n$.

$$*$$

$$
\begin{array}{ccc}
\circ & \circ & \circ \\
(s) & (1-2s) & (s)
\end{array}
$$

**Pseudocode** heat_ftcs$(h, \theta)$:

- Set initial $\{u_j^0, \forall j \in \mathbb{Z}\}$ by (2);

- For $n - 1 \in \mathbb{N}$, do:

$$u_j^n \implies u_j^{n+1}, \forall j \in \mathbb{Z} \text{ by } (1).$$

The above pseudocode is not practical since the grid points are infinitely many. But, if the desired computation is for instance

$$\{u_j^n : j = a, a+1, \ldots, b-1, b\}$$

for some integers $n$ and $a < b$, then one can set initial on finitely many points

$$\{u_j^0, \quad j = a-n, a-n+1, \ldots, b+n-1, b+n\}.$$

## 3.1 Numerical result

We use hand computation to demonstrate instability with the parameters given as

- $\phi(x) = |1 - 10x| \cdot I(|x| < 0.1)$.

- space step size $h = .2$

- time step size $\theta = h^2 = .04$.

Note that, $s = 1$ and corresponding stencil is

$$*$$

$$
\begin{array}{ccc}
\circ & \circ & \circ \\
(1) & (-1) & (1)
\end{array}
$$

One can easily figure out first a few lines of the numerical outcomes as follows.

$$*$$

$$
\begin{array}{ccc}
\circ & \circ & \circ \\
(1) & (-1) & (1)
\end{array}
$$