[KP 9.1] Euler approximation

SDE

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t$$

~~EM $(Y_n : n \in \mathbb{N})$ is~~

~~EM $\{(Y_n, \tau_n) : n \in \mathbb{N}\}$~~

EM Discrete process $\{(Y_n, \tau_n) : n \in \mathbb{N}\}$ is EM

if

1. $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$.
2. $Y_0 = X_0$
3. $Y_{n+1} = Y_n + a(\tau_n, Y_n)\Delta_n + b(\tau_n, Y_n)\Delta W_n$

   where

   $$\Delta_n = \tau_{n+1} - \tau_n$$

   $$\Delta W_n = W_{n+1} - W_n$$

Goal $Y_n \approx X(\tau_n)$ (in some sense)

Prelim

1. max time step :

$$\delta = \max_n \Delta_n$$

2. If $\delta = \Delta_n = \frac{T}{N}$ $\forall n \le N$, then EM is equidistant one.

3. $\mathbb{E}[\Delta W_n] = 0$, $\mathbb{E}[(\Delta W_n)^2] = \Delta_n$.

$$\Delta W_n \sim N(0, \Delta T_n)$$

ex ① plot EM ~~of~~ BM
~~EM of~~
~~① Implement~~ ~~std BM, plot~~ its curve

② plot EM of GBM.

[KP 9.6] Strong convergence

Given Approx. $\{Y^\delta : \delta > 0\}$ to (X), def

<u>Abs. Err</u>

$$\varepsilon(\delta) = \sup_{0 \le s \le T} \mathbb{E} \left| Y^\delta(s) - X(s) \right|^2$$

<u>Def</u>

① If $\lim_{\delta \to 0} \varepsilon(\delta) = 0$, then $Y^\delta \to X$ (strong)

② If $\varepsilon(\delta) \le C \delta^{2r}$, then conver. order is r.

(1-d, Homo)

Consider. EM of equidistance. with

① $\tau_n = \frac{T}{N}(n-1)$

② $Y_0 = X_0$

~~③ $Y_{n+1} = Y_n + a(\tau_n, Y_n)\delta + b(\tau_n, Y_n)$~~

③ $Y_{n+1} = Y_n + a(Y_n)\delta + b(Y_n)\delta_n W$

for approx. of X of

$$dX_t = a(X_t)\,dt + \sigma(X_t)\,dW_t. \quad X_0$$

[A] a, b Lip.

[prop]

① $\mathbb{E}\left[\sup_{0\le s\le t} |X_s|^2\right] \cancel{\le K\left(|X_0|^2 + e^{Kt}|X_0|^2 \cdot t\right)}$

$\le K|X_0|^2(1 + t\cdot e^{Kt})$

② $\mathbb{E}\left[\sup_{0\le s\le t} |X_t - X_0|^2\right] \le K e^{Kt}|X_0|^2 \cdot t$

Let $n_t = \max\{n = 0, 1, \cdots ; \tau_n \le t\}$

$\bar{Y}^\delta(s) = Y^\delta_{n_s}$

Then $\bar{Y}^\delta(s) = X_0 + \int_0^{\tau_{n_s}} a(\bar{Y}^\delta_s)\,ds + \sigma(\bar{Y}^\delta_s)\,dW_s$

Thm 9.6.2 With [A], $\bar{Y}^{\delta} \to X$ with $r = \frac{1}{2}$.

i.e. $\sup_{0 \le s \le T} \mathbb{E} |\bar{Y}^{\delta}_s - X_s|^2 \le K \delta$

Pf Def.

$$Z(t) = \sup_{0 \le s \le t} \mathbb{E}\left[|\bar{Y}^{\delta}_s - X_s|^2 \right]$$

$$Z(t) \le \sup_{0 \le s \le t} \Big\{ \mathbb{E}\left[\left| \int_0^{\tau_{n_s}} (a(\bar{Y}^{\delta}_r) - a(X_r))\, dr \right|^2 \right]$$

$$+ \mathbb{E}\left[\left| \int_0^{\tau_{n_s}} (b(\bar{Y}^{\delta}_r) - b(X_r))\, dW_s \right|^2 \right]$$

$$+ \mathbb{E}\left[\left| \int_{\tau_{n_s}}^{s} a(X_r)\, dr \right|^2 \right]$$

$$+ \mathbb{E}\left[\int_{\tau_{n_s}}^{s} b(X_r)\, dW_r \Big|^2 \right] \Big\}$$

So

$$Z(t) \le \sup_{0 \le s \le t} \Big\{ \mathbb{E} \int_0 |\bar{Y}^{\delta}_r - X_r|^2 dr \cdot T +$$

$$\mathbb{E} \int_0 |\bar{Y}^{\delta}_r - X_r|^2 dr +$$

$$\mathbb{E}\ \delta \cdot \int_{\tau_{n_s}}^{\tau_{n_s+1}} X_r^2\, dr +$$

$$\mathbb{E} \int_{\tau_{n_s}}^{\tau_{n_s+1}} X_r^2\, dr \Big\}.$$

$$z_t \leq k\int_0^t z_r \, dr + k\delta$$

$$Z(t) \leq k\delta \cdot e^{kt}$$

$$Z(T) \leq k\delta.$$

§1 Num Int. on SDE

Consider 1-D SDE:

$$\begin{cases} dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \\ X(t_0) = X_0 \end{cases}$$

Int. form is

$$X(t) = X_0 + \int_{t_0}^{t} \mu(X_s)\,ds + \sigma(X_s)\,dW_s$$

In interval $[t, t+\delta]$, it implies

$$X_{t,t+\delta} = \int_{t}^{t+\delta} \mu(X_s)\,ds + \sigma(X_s)\,dW_s .$$

where

$$X_{t,s} \triangleq X_s - X_t .$$

Ito formula says

$$\mu(X_s) = \mu(X_t) + \int_t^s \left(\mu'(X_r) + \frac{1}{2}\cancel{\mu}''(X_r)\right) dr$$

$$+ \int_t^s \mu'(X_r)\,\sigma(X_r)\,dW_r .$$

$$\sigma(X_s) = \sigma(X_t) + \int_t^s \left(\cancel{\mu}'(X_r) + \frac{1}{2}\sigma''(X_r)\right) dr$$

$$+ \int_t^s \sigma'(X_r)\,\sigma(X_r)\,dW_r$$

§2. Euler Method

Assume $\delta > 0$. small. then

$$\mu(X_s) \approx \mu(X_t), \quad \sigma(X_s) \approx \sigma(X_t)$$

Then,

$$X_{t,t+\delta} \approx \mu(X_t) \cdot \delta + \sigma(X_t) W_{t,t+\delta}$$

(*) Euler $\hat{X}$ on $[0,T]$ with step size $\delta = \frac{T}{N}$

① partition $[0,T]$ equally, denoted by

$$0 = \tau_0 < \tau_1 < \cdots < \tau_N = T.$$

② $\hat{X}_0 = X_0$

③ $\hat{X}_{n+1} = \hat{X}_n + \mu(\hat{X}_n)\delta + \sigma(\hat{X}_n) \sqrt{\delta} Z_n$.

where $Z_n \sim N(0,1)$ iid.

④ Interpolation

$$I_t^\delta = I_t^\delta(\hat{X}) = \hat{X}_n, \quad \text{if} \quad \tau_n \le t < \tau_{n+1}$$

§3. strong convergence of Euler.

① ~~Abs~~ strong Error

Given approx. Y^δ to X, we define

$$\varepsilon(\delta) = \sup_{0 \le s \le T} \mathbb{E} \left| Y^\delta(s) - X(s) \right|^2$$

① If $\lim_{\delta \to 0} \varepsilon(\delta) = 0$, then we say

$$Y^\delta \to X \quad \text{(strong)}$$

② If $\varepsilon(\delta) \le c\,\delta^{2\gamma}$ for some $c, \gamma > 0$, then γ is conv. order.

Thm $I^\delta(\hat{X}) \to X$ with $\gamma = \frac{1}{2}$, i.e.

$$\sup_{0 \le s \le T} \mathbb{E} \left| I^\delta_s(\hat{X}) - X_s \right|^2 \le K\delta.$$

Pf [see [K.P] §9.6]

§4. Weak conv. of Euler

Weak Error

Given Y^δ to X, we define

$$\mathcal{E}^g(\delta) = \left| \mathbb{E}[g(X_T)] - \mathbb{E}[g(Y^\delta(T))] \right|$$

① We say $Y^\delta(T) \Rightarrow X_T$ (weakly), if

$$\lim_{\delta \to 0} \mathcal{E}^g(\delta) = 0 \quad \forall g \in C_b$$

② We say $Y^\delta_T \Rightarrow X_T$ with order γ, if

$\exists C_g > 0$ s.t. $\mathcal{E}^g(\delta) \le C_g \cdot \delta^\gamma$

for any $g \in C_b$

Thm $I^\delta(\hat{X}) \Rightarrow X$ with $\gamma = 1$.

Pf See [K.P] §9.7

§5. Milstein scheme

Assume δ = small.

Recall §1 that.

$$\mu(X_s) \approx \mu(X_t)$$

$$\sigma(X_s) \approx \sigma(X_t) + \sigma'\sigma(X_t)\, W_{t,s}$$

Thus

$$X_{t,t+\delta} \approx \mu\delta + \int_t^{t+\delta} (\sigma(X_t) + \sigma'(X_t)\sigma(X_t) W_{t,s}) dW_s$$

$$= \mu\delta + \sigma W_{t,t+\delta} + \sigma'\sigma \int_t^{t+\delta} W_{t,s}\, dW_s$$

where

$$\int_t^{t+\delta} W_{t,s}\, dW_s = \int_t^{t+\delta} (W_s - W_t)\, dW_s$$

$$= \int_t^{t+\delta} W_s\, dW_s - W_t W_{t,t+\delta}$$

Note

$$\int_t^{t+\delta} W_s\, dW_s = \frac{1}{2}(W_{t+\delta}^2 - W_t^2) - \frac{1}{2}\delta$$

$$= W_t W_{t,t+\delta} + \frac{1}{2}(W_{t,t+\delta}^2 - \delta)$$

Hence,

$$X_{t,t+\delta} \approx \mu\delta + \sigma W_{t,t+\delta} + \frac{1}{2}\sigma'\sigma(W_{t,t+\delta}^2 - \delta).$$

[Algo].

Milstein $\hat{X}$ on $[0,T]$ with $\delta = \frac{T}{N}$

① partition

② Init. $\hat{X}_0 = X_0$

③ $$\hat{X}_{n+1} = \hat{X}_n + \mu(\hat{X}_n)\delta + \sigma(\hat{X}_n)\sqrt{\delta} Z_n + \frac{1}{2}\sigma'(\hat{X}_n)\sigma(\hat{X}_n)(\delta Z_n^2 - \delta)$$

where $Z_n \sim N(0,1)$ iid.

④ Interpolation.

$$I_t^\delta = \hat{X}_n \quad \text{if} \quad \tau_n \le t < \tau_{n+1}$$

Rk Milstein has convergence

strong order $= 1$

weak order $= 1$

see § 10.3. [KP99].

§4. Term Structure

Def

Simple rate R over $[0,T]$ means

ex You have $X_0 = 1$ dollar today.

You're allowed to invest money at a simple rate R over $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ for any $k \in \mathbb{N}$.

What's the max. value you can earn ~~after~~ over $[0,T]$?

Sol $\left(1+\frac{R}{n}\right)^{nT}$

Def

n-compound rate R over $[0,T]$ means

ex $\lim_{n\to\infty}\left(1+\frac{R}{n}\right)^{nT} = ?$

Def

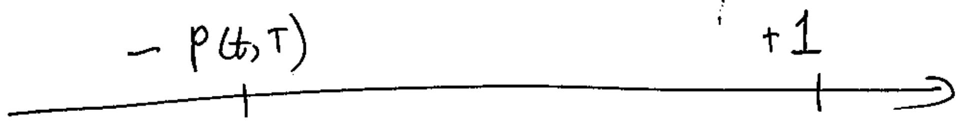
A cont. compounding rate R over $[0,T]$ means

Def

Zero-coupon bond is a secrity with

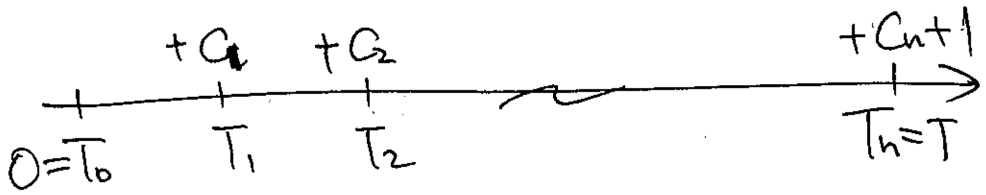
$$\text{Payoff}\Big|_T = 1$$

we use $P(t,T)$ to denote its price at time t.



Def

A coupon bond with $\{(C_i, T_i) : i = 1, 2 \dots n\} \triangleq c$ is a security with cash flow



ex If $T_{i-1} < t \le T_i$, then prove

~~$P_c(t) = P(t$~~

$$P_c(t) = P(t,T) + \sum_{k=i}^{n} C_i P(t,T_i)$$

Rk coupon bond price can be obtained from a series of zero bonds prices.

Then, ~~how to find~~ what's zero bond prices?

Def

spote rate $R(t,T)$ is the yield to maturity of $P(t,T)$, i.e

$$R(t,T) = -\frac{\log P(t,T)}{T-t}$$

Def For $t < T < s$

Forward rate at t on $[T,s]$ is

$$F(t,T,s) = \frac{1}{s-T}\log\frac{P(t,T)}{P(t,s)}$$

ex (RK)

① $-P(t,T)$ $\qquad +1$

② $-\dfrac{P(t,T)}{P(t,S)}\cdot P(t,S)$ $\qquad + \dfrac{P(t,T)}{P(t,S)}$

③ $\qquad -1 \qquad \exp\{(s-T)\cdot F(t,T,s)\}$

Thus $$\frac{P(t,T)}{P(t,S)} = \exp\{(S-T)F(t,T,s)\}$$

Rk $\quad F(t, t, s) = R(t, s)$

Def

The instantaneous forward rate at time t and $T>t$ is

$$f(t,T) = \lim_{s \to T} F(t, T, s)$$

ex
$$\begin{aligned}
f(t,T) &= \lim_{s\to T} F(t, T, s) \\
&= \lim_{s\to T} \frac{1}{s-T} \log \frac{p(t,T)}{p(t,s)} \\
&= -\lim_{s\to T} \frac{\log p(t,s) - \log p(t,T)}{s-T} \\
&= -\partial_T \log p(t,T) \\
&= -\frac{1}{p(t,T)} \cdot \partial_T p(t,T)
\end{aligned}$$

i.e.,

$$p(t,T) = \exp\left\{-\int_0^T f(t,u)\,du\right\}$$

and

$$f(t,T) = -\frac{\partial_T p(t,T)}{p(t,T)}$$

Def

Short rate $r(t)$ is

$$r(t) = \lim_{T \to t} R(t,T).$$

Facts (need pf)

① $r(t) = f(t,t) = \lim_{T \to t} R(t,T)$

② $P(t,T) = \mathbb{E}^{Q}\left[\exp\{-\int_t^T r(s)\,ds\} \,\middle|\, \mathcal{F}_t\right]$

$$= \exp\{-\int_t^T f(t,u)\,du\}.$$
$$= \exp\{-R(t,T)\,(T-t)\}$$

ex

Given c.c. forwarde rates as

T	1	2	3	4	5
$F(0,T-1,T)$	0.042	0.05	0.055	0.056	0.053

③ Find/Complete the table below

T	1	2	3	4	5
$P(0,T)$					
$R(0,T)$					

Hint: use

$$P(0,T) = \exp\{-\sum_{i=1}^{T} F(0,i-1,i)\}$$

§ SDE Models.

§1. GBM.

<u>Def</u> $dX_t = X_t(\mu\, dt + \sigma\, dW_t)$

<u>Explicit Soln</u>

$$X_t = X_0 \exp\{\hat{\mu} t + \sigma W_t\}$$

where $\hat{\mu} = \mu - \frac{1}{2}\sigma^2$

It can be rewritten as

$$X_{t+\delta} = X_t \exp\{\hat{\mu}\delta + \sigma W_{t,t+\delta}\}.$$

<u>Def</u> An approx. $\hat{X}$ on $\{0 = \tau_0 < \tau_1 < \cdots < \tau_N = T\}$ is called <u>exact</u>, if

$$\text{Law}(\hat{X}_{\tau_0}, \hat{X}_{\tau_1}, \cdots, \hat{X}_{\tau_N}) = \text{Law}(X_{\tau_0}, X_{\tau_1}, \cdots, X_{\tau_N})$$

i.e. for any b.d. cont. func $f: \mathbb{R}^{N+1} \to \mathbb{R}$ it shall have

$$\hat{\mathbb{E}} f(\hat{X}_{\tau_0}, \cdots \hat{X}_{\tau_N}) = \mathbb{E} f(X_{\tau_0}, X_{\tau_1} \cdots X_{\tau_N})$$

Exact simulation $\hat{X}$ on $[0, T]$.

① Let $t_i = \frac{T}{N} \cdot i$, $i = 0, 1, 2 \cdots N$

② $\hat{X}_0 = X_0$

③ Repeat, $n = 0, \cdots N-1$

$$\hat{X}_{n+1} = \hat{X}_n \cdot \exp\{\hat{\mu}\delta + \sigma\sqrt{\delta} Z_n\}.$$

ex Find explifit formula for $\mathbb{E}[f(X_T)]$

① if $f(x) = (x-k)^+$

② if $f(x) = (x-k)^-$

ex prove that Euler approx. is not exact approx.

§2. Gaussian short rate models

§2.1. Vasicek model

<u>def</u> (Vasicek)

$$dr(t) = \alpha(b - r(t))\,dt + \sigma\,dW_t .$$

(Ho-Lee)

$$dr(t) = g(t)\,dt + \sigma\,dW_t$$

(Hull-white)

$$dr(t) = [g(t) + h(t)\cdot r(t)]\,dt + \sigma(s)\,dW_t$$

<u>explicit soln</u> (HW)

$$r(t) = e^{H(t)} r(0) + \int_0^t e^{H(t)-H(s)} g(s)\,ds + \int_0^t e^{H(t)-H(s)} \sigma(s)\,dW_s$$

With $H(t) = \int_0^t h(s)\,ds$.

<u>Rk</u> Gaussian process $r(t)$ is a process which has $(r(t_1), \dots r(t_n))$ being multivariable normal distribution for any $\{t_1, \dots t_n\}$.

Exact simulation of HW

$$r(t+\delta) = e^{H_{t,t+\delta}} r(t) + \int_t^{t+\delta} e^{H_{s,t+\delta}} g(s)\,ds + \int_t^{t+\delta} e^{H_{s,t+\delta}} \sigma(s)\,dW_s$$

$$= \mathrm{I} \cdot r(t) + \mathrm{II} + \mathrm{III}$$

$$\cancel{= \mathrm{I} + \mathrm{II} + \mathrm{III}}$$

$$\cancel{\int_t^{t+\delta} e^{H}}$$

$$\mathrm{III} \sim \hat{\sigma} Z.$$

where $\hat{\sigma}^2 = \int_t^{t+\delta} e^{2H_{s,t+\delta}} \sigma^2(s)\,ds$

Thus, as long as I, II, $\hat{\sigma}^2$ are explicitly computable, one can have exact sim.

~~ex~~ ① write exact sim for Vasicek

~~② Compare Euler/Milstein/Exact simulations for vasicek.~~

Fact: For vasicek.

$$\ln P(t,T) = A(t,T) - B(t,T)\, r(t)$$

where

$$B(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}$$

$$A(t,T) = \left(B(t,T) - (T-t)\right)\left(b - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{4\alpha} B(t,T)^2$$

Rk: $\ln P(t,T) = F_{t,T}(r(t))$ where $F_{t,T}$ is affine function.
So it's called "affine class of rate model".

ex Compare Euler / Milstein / Exact sim.
with the computation of zero Bond $P(t,T)$.
on vasicek.

§ 2.2. Multifactor models.

$$dX_t = C(b - X(t))\,dt + D\,dW(t)$$

where $b, X \in \mathbb{R}^d$. $W \in \mathbb{R}^d$.
$C, D \in \mathbb{R}^{d\times d}$.

Explicit soln is available.

§3. Square ~~root~~ diffusion

§3.1 CIR rate model.

Model $r(0) > 0, \quad \alpha, b \geq 0.$

$$dr(t) = \alpha(b - r(t))\,dt + \sigma\sqrt{r(t)}\,dW_t$$

Fact

① No explicit soln.

② $r(t) \geq 0 \quad \forall t$

③ If $2\alpha b \geq \sigma^2$ then $r(t) > 0 \quad \forall t.$

Euler approx.

$$r(t+\delta) = r_t + \alpha(b - r_t)\delta + \sigma\sqrt{(r_t)^+}\sqrt{\delta}\,Z$$

To make sure positive

§3.2. Heston Stoch. Vol. model.

Model Stock $S(t)$ follows

$$\begin{cases} dS_t = S_t\left(\mu\,dt + \sqrt{V(t)}\,dW_1(t)\right) \\ dV_t = \alpha(b - V(t)) + \sigma\sqrt{V_t}\,dW_2(t) \end{cases}$$

ex Design approx. of Put/Call price under Heston.