1 Abstract

You will learn

- Euler approximation for the solution of 1-d SDE
- Strong and weak convergence rate

2 Problem

Consider 1-d SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, X(0) = x_0$$

We shall find, for some small step size δ

$$X^\delta(t) \approx X(t), \forall t \geq 0$$

in some sense.

3 Analysis

3.1 Euler scheme

The above SDE can be written as the following integral form:

$$X_t = x_0 + \int_0^t \mu(X_s)ds + \sigma(X_s)dW_s$$

If we denote

$$X_{t,s} = X_s - X_t,$$

then

$$X_{t,t+\delta} = \int_t^{t+\delta} \mu(X_s)ds + \sigma(X_s)dW_s.$$

Ito formula says

$$\mu(X_s) = \mu(X_t) + \int_t^s (\mu'(X_r) + \frac{1}{2}\mu''(X_r))dr + \mu'(X_r)\sigma(X_r)dW_r$$

and

$$\sigma(X_s) = \sigma(X_t) + \int_t^s (\sigma'(X_r) + \frac{1}{2}\sigma''(X_r))dr + \sigma'(X_r)\sigma(X_r)dW_r.$$

If $|s - t| < \delta$ and $\mu, \sigma \in C_b^2$, then [1]

$$\mu(X_s) = \mu(X_t) + O(\delta^{1/2})$$

and

$$\sigma(X_s) = \sigma(X_t) + O(\delta^{1/2}).$$

Thus, $X_{t,t+\delta}$ can be rewritten as

$$X_{t,t+\delta} = \mu(X_t)\delta + \sigma(X_t)W_{t,t+\delta} + O(\delta).$$

or

$$X_{t,t+\delta} \approx \mu(X_t)\delta + \sigma(X_t)W_{t,t+\delta}.$$

[1] $f(\delta) = O^{\delta^\gamma}$ means $|f(\delta)| \leq K\delta^\gamma$ for some random variable K and all $\delta \in (0, \epsilon)$.