1 Abstract

- Using Monte Carlo to approximate π;
- Introduce Monte Carlo basics.

2 Problem

Approximate the value π.

3 Analysis

Consider the following question:

- You shoot a square $(-1,1)^2$. Suppose your shot is uniform in this square, then what is the probability you have a successful shot? We say "your shot is successful", if your shot belongs to the unit ball B_1.

The answer is

$$\text{Prob of succesful shot } = \frac{\text{Area of } B_1}{\text{Area of } (-1,1)^2} = \frac{\pi}{4}.$$

This means that, as long as one can approximate probability of successful shot, one can approximate π by multiplying 4. This can be done by computer:

- Simulate many uniform shots, and compute the ratio of successful shots.

Pseudocode ($pi_m c(N)$)

- Generate N iid points

$$\{(X_i, Y_i) : i = 1, 2, \ldots, N,\ X_i, Y_i \sim U(-1,1)\};$$

- Count n, the number of points satisfying

$$X_i^2 + Y_i^2 < 1.$$

- Compute

$$\hat{\pi} = 4 \cdot \frac{n}{N}.$$

4 Monte Carlo basics

One can implement above approximation multiple times and observe that

- (random estimator) Target value π is deterministic, but each implementation gives different outcome $\hat{\pi}$;

- (Convergence) Each obtained outcome, as long as N is large enough, gives some close approximation.

We are going to generalize our observations in this below.

- A random estimator $\hat{\alpha}$ to a deterministic value α is called as Monte Carlo (MC) approximation.
- Moreover, we define
$$Bias = \mathbb{E}[\hat{\alpha}] - \alpha$$
and
$$MSE = \mathbb{E}[(\hat{\alpha} - \alpha)^2].$$
- If Bias is zero, then we call this as unbiased MC.

Proposition 1 *$MSE(\hat{\alpha}) = |Bias(\hat{\alpha})|^2 + Var(\hat{\alpha})$. In particular, if $\hat{\alpha}$ is unbiased, then MSE is Variance.*

Proof: ... $\square$

Although seemingly absurd, we consider the above estimator with $N = 1$, which is equivalent to

- Consider
$$\hat{\alpha} = 4I(X_1^2 + Y_1^2 < 1), \ X_1, Y_1 \sim U(-1, 1)$$
as MC for π. Then the outcome is either 0 or 4. In any case, it is a bad approximation.
- However, we can show that it's an unbiased MC. (why?)
- Find MSE?

Unbiased MC is very desirable, because one can employ crude MC to make it more accurate: [1]

- Suppose $\hat{\alpha}$ is a square integrable unbiased MC;
- Obtain N independent replicates
$$\{\hat{\alpha}_i : i = 1, \dots, N\}.$$
- Taking their average, it gives a new MC:
$$\beta_N = \frac{1}{N} \sum_{i=1}^{N} \hat{\alpha}_i.$$
- β_N is unbiased again. (why?)
- $MSE(\beta_N) = \frac{1}{N} MSE(\hat{\alpha}) \to 0$. (why?)
- β_N is almost surely consistent, (why?) i.e.
$$\mathbb{P}(\lim_N \beta_N = \alpha) = 1.$$

[1] We say a random variable X is in L^p, if its pth moment exists, i.e.
$$\mathbb{E}X^p < \infty.$$
If $X \in L^2$, then we say it's square integrable.

- β_N is L^2-consistent, (why?) i.e.
$$\mathbb{E}(\beta_N - \alpha)^2 \to 0.$$

As a conclusion, one can always use crude MC to make better approximation provided there exists an unbiased MC $\hat{\alpha}$. But this requires higher computational cost. Given a fixed amount of computational cost, to improve the efficiency, it is essential to reduce $Var(\hat{\alpha})$ as much as possible.

Ex. Given i.i.d $\{\alpha_i : i \in 1, 2, \ldots, N\}$, we use

$$\beta_N = \frac{1}{N}\sum_{i=1}^{N}(\alpha_i - \bar{\alpha})^2$$

as the estimator of $Var(\alpha_1)$, where $\bar{\alpha} = \frac{1}{N}\sum_{i=1}^{N}\alpha_i$. Suppose $\alpha_1 \in L^2$, then

- Prove it is biased.
- Prove that it is consistent in L^2.
- Can you propose an unbiased estimator?

Ex. Prove that L^2 consistency implies almost sure consistency.

Ex. Can you improve the MC for π? Hint: antithetic variate method

Ex. Can you propose a deterministic approximation to π?