1 Abstract

You will learn

- taking definite integral by ordinary Monte Carlo (OMC)
- exact sampling with python provided random number generators

2 Problem

Example 1 *Our goal is to compute, using OMC by exact sampling*

$$\alpha = \int_0^1 h(x)dx$$

where

$$h(x) = 100 \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x).$$

The exact value shall be

$$\alpha = 1.99.$$

3 Analysis

3.1 OMC by exact sampling

The objective is to

- estimate

  $$\alpha = \mathbb{E}[X], \quad X \sim p(x)$$

  one can use random number generator by computer (if possible)

  $$\{iid\ X_i \sim p(x) : i = 1, 2, \ldots, n, \}.$$

Then, one can compute the approximation of α by

$$\hat{\alpha}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

We say $\hat{\alpha}_n$ as OMC by *exact sampling*, since the sample X_i produced by random generator has the same distribution as true distribution X, i.e.

$$X_i \sim X,\ \forall i.$$

Proposition 1 *Prove the properties of the OMC by exact sampling below:*

- *X_1 itself can be treated as an unbiased MC, because*

  $$\mathbb{E}[X_1] = \alpha.$$

  However, MSE is big, ie.

  $$MSE(X_1) = Var(X) = \int x^2 p(x)dx.$$

- *$\hat{\alpha}_n$ is consistent almost surely due to LLN, i.e.*

$$\hat{\alpha}_n \to \alpha, \text{ almost surely as } n \to \infty.$$

Moreover, $\hat{\alpha}_n$ is unbiased too, and

$$MSE(\hat{\alpha}_n) = Var(\hat{\alpha}_n) = \frac{1}{n} Var(X) \to 0.$$

3.2 Evaluation of integral

Back to our Example 1, we write

$$\alpha = \mathbb{E}[X] = \mathbb{E}[h(Y)],$$

where $X = h(Y)$ and $Y \sim U(0,1)$. In other words, although X-sampling is not directly available in python, one can use $U(0,1)$ random generator (see *numpy.random.uniform*) to produce Y_i, then compute $h(Y_i)$ for the sample X_i.

Algorithm 1 Integral by MC - Example 1

```
procedure MCINTEGRAL(N)                         ▷ N is total number of samples
    s ← 0                                       ▷ s is the sum of samples
    for i = 1...N do
        generate two numbers Y from U(0,1)      ▷ use numpy.random.uniform
        s ← s + h(Y)
    return s/N                                  ▷ return the average
```

Proposition 2 *Suppose*

- *$\hat{Y}$ is exact sampling of Y, ie, $\hat{Y} \sim Y$*
- *h is almost surely continuous w.r.t Y, i.e.*

$$\mathbb{P}Y^{-1}(D_h) = \mathbb{P}(Y \in D_h) = 0,$$

where D_h is the set of discontinuous points of the function h.

Then, $h(\hat{Y})$ is unbiased estimator of $\mathbb{E}h(Y)$, i.e. $\mathbb{E}h(\hat{Y}) = \mathbb{E}h(Y)$.

Example 2 *Prove that $h(\hat{Y})$ is unbiased estimator of α in Example 1.*