1 Abstract

You will learn

- taking definite integral by ordinary Monte Carlo (OMC)
- exact sampling with python provided random number generators

2 Problem

Example 1 *Our goal is to compute, using OMC by exact sampling*

$$\alpha = \int_0^1 h(x)dx$$

where

$$h(x) = 100 \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x).$$

The exact value shall be

$$\alpha = 1.99.$$

3 Analysis

3.1 OMC by exact sampling

The objective is to

- estimate

  $$\alpha = \mathbb{E}[X], \quad X \sim p(x)$$

  one can use random number generator by computer (if possible)

  $$\{iid\ X_i \sim p(x) : i = 1, 2, \ldots, n, \}.$$

Then, one can compute the approximation of α by

$$\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

We say $\hat{\alpha}_n$ as OMC by *exact sampling*, since the sample X_i produced by random generator has the same distribution as true distribution X, i.e.

$$X_i \sim X, \ \forall i.$$

Example 2 *Prove the properties of the OMC by exact sampling below:*

- X_1 *itself can be treated as an unbiased MC, because*

  $$\mathbb{E}[X_1] = \alpha.$$

  However, MSE is big, ie.

  $$MSE(X_1) = Var(X) = \int x^2 p(x)dx.$$

- *$\hat{\alpha}_n$ is consistent almost surely due to LLN, i.e.*

$$\hat{\alpha}_n \to \alpha, \text{ almost surely as } n \to \infty.$$

  Moreover, $\hat{\alpha}_n$ is unbiased too, and

$$MSE(\hat{\alpha}_n) = Var(\hat{\alpha}_n) = \frac{1}{n} Var(X) \to 0.$$

3.2 Evaluation of integral

Back to our Example 1, we write

$$\alpha = \mathbb{E}[X] = \mathbb{E}[h(Y)],$$

where $X = h(Y)$ and $Y \sim U(0,1)$. In other words, although X-sampling is not directly available in python, one can use $U(0,1)$ random generator (see *numpy.random.uniform*) to produce Y_i, then compute $h(Y_i)$ for the sample X_i.

Algorithm 1 Integral by MC - Example 1

procedure MCINTEGRAL(N) ▷ N is total number of samples
    $s \leftarrow 0$ ▷ s is the sum of samples
    for $i = 1...N$ **do**
        generate two numbers Y from $U(0,1)$ ▷ use *numpy.random.uniform*
        $s \leftarrow s + h(Y)$
    return $\frac{s}{N}$ ▷ return the average