ftcs_stability_heat_toy

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1 FTCS on a toy problem

1.1 Absttract

FTCS stands for the finite difference operator of Forward in time and central in space. - We will perform FTCS on a simple heat equation - Handwriting computation is possible for FTCS solution - Demonstrate the dangerous instability issue

1.2 Problem

Our heat equation is

$$u_t = u_{xx}, \quad t \geq 0; x \in \mathbb{R}$$

with initial condition

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}.$$

In the above, ϕ is a given function and u is unknown function we seek for. For instance, let's say ϕ is a continuous function given by

$$\phi(x) = \cos(cx) I_{(-\pi/2c, +\pi/2c)}(x).$$

ex. plot the function ϕ

It is known that the above heat equation has unique solution. Now, we are going to use FTCS to approximate the solution with some grid $(\Delta x, \Delta t)$.

1.3 Anal

The **Grid** with $(\Delta x, \Delta t)$ is given by countable points

$$\{(j\Delta x, n\Delta t) : j \in \mathbb{Z}, n \in \mathbb{Z}^+\}.$$

The output of FTCS scheme are numbers u_j^n on each grid point above.

Our objective is to produce FTCS solution (u_j^n), such that their linear interpolation is close to the true solution u on the domain, i.e.

- $u_j^n \sim u(j\Delta x, n\Delta t)$

- If (x, t) is in some grid satisying

$$j\Delta x \leq x < (j+1)\Delta x, \quad n\Delta t \leq t(n+1)\Delta t,$$

then the linear interpolation

$$\hat{u}(x,t) = p_j(x)q_n(t)u_j^n + p_{j+1}(x)q_n(t)u_{j+1}^n + p_j(x)q_{n+1})(t)u_j^{n+1} + p_{j+1}(x)q_{n+1}(t)u_{j+1}^{n+1}$$

with interpolation coefficients

$$p_j(x) = \frac{(j+1)\Delta x - x}{\Delta x}, \quad p_{j+1}(x) = 1 - p_j(x)$$

and

$$q_n(t) = \frac{(n+1)\Delta t - t}{\Delta t}, \quad q_{n+1}(t) = 1 - q_n(t)$$

satisfies

$$\hat{u}(x,t) \sim u(x,t).$$

FTCS scheme suggests the discretization of PDE in the following manner:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

with initialization

$$u_j^0 = \phi(j\Delta x, 0).$$

Then, the above FTCS can be implemented from $n = 0, 1, 2, \ldots$ iteratively.

To see that, we simplify it with the following setup:

$$\Delta = (\Delta x)^2, \frac{\pi}{2c} < \Delta x.$$

Then, FTCS suggests to do the following iterations:

- compute u_j^0 for all j
- given $(u_j^n : j \in \mathbb{Z})$ for some n, compoute $n+1$ level by

$$u_j^{n+1} = u_{j+1}^n - u_j^n + u_{j-1}^n.$$

The comutation yields that

$$\begin{array}{llllllll}
n = 2: & ...0 & 1 & -2 & +3 & -2 & 1 & 0... \\
n = 1: & ...0 & 0 & +1 & -1 & +1 & 0 & 0... \\
n = 0: & ...0 & 0 & +0 & +1 & +0 & 0 & 0...
\end{array}$$

hw

Find out $\max_{j\in\mathbb{Z}} |u_j^{10}|$.