# 1 Abstract

You will learn

- Euler approximation for the solution of 2-d SDE
- We shall adapt Euler scheme for 2-d for Heston

# 2 Problem

## 2.1 General problem

We will perform Euler scheme for the general 2-d SDE to be considered is given as

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, X_0 = x_0$$

where $b : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a smooth vector field on $\mathbb{R}^2$, $\sigma : \mathbb{R}^2 \mapsto \mathbb{R}^{2\times 2}$ is a smooth matrix-valued function, $W$ is a 2-d standard Brownian motion, and $x_0$ is the initial 2-d vector. It can be written by system of two 1-d SDEs as the following:

$$\begin{cases} dX_{1,t} = b_{1,t}dt + \sigma_{11,t}dW_{1,t} + \sigma_{12,t}dW_{2,t}, & X_{1,0} = x_{1,0} \\ dX_{2,t} = b_{2,t}dt + \sigma_{21,t}dW_{1,t} + \sigma_{22,t}dW_{2,t}, & X_{2,0} = x_{2,0} \end{cases}$$

In the above, we assume $W_1$ and $W_2$ are two independent 1-d Brownian motions.

## 2.2 Heston model

Heston model as a stochastic volatility model belongs to 2-d SDE in the above. However, the domain of the diffusion matrix $\sigma$ is not entire 2-d space.

In the Heston model, the dynamic involves two processes $(S_t, \nu_t)$. More precisely, the asset price $S$ follows generalized geometric Brownian motion with random volatility process $\sqrt{\nu_t}$, i.e.

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_{1,t},$$

while squared of volatility process $\nu$ follows CIR process

$$d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t}(\rho dW_{1,t} + \bar{\rho} dW_{2,t})$$

with $\rho^2 + \bar{\rho}^2 = 1$. Feller condition for its existence of the solution is

$$2\kappa\theta > \xi^2.$$

Our goal is to adapt the above Euler scheme to Heston model with the following parameters:

$$S_0 = 100, \nu(0) = .04, r = .05, \kappa = 1.2, \theta = .04, \xi = .3, \rho = .5.$$

The estimation of Call$(T = 1, K = 100)$ is given as 10.3009, see Page 357 of [1]. We will use this for our comparison to our computation.

# 3 Analysis

## 3.1 Euler scheme

The above SDE can be written as the following integral form:

$$X_t = x_0 + \int_0^t \mu(X_s)ds + \sigma(X_s)dW_s$$

If we denote

$$X_{t,s} = X_s - X_t,$$

then

$$X_{t,t+\delta} = \int_t^{t+\delta} \mu(X_s)ds + \sigma(X_s)dW_s.$$

Ito formula says

$$\mu(X_s) = \mu(X_t) + \int_t^s (\mu'(X_r) + \frac{1}{2}\mu''(X_r))dr + \mu'(X_r)\sigma(X_r)dW_r$$

and

$$\sigma(X_s) = \sigma(X_t) + \int_t^s (\sigma'(X_r) + \frac{1}{2}\sigma''(X_r))dr + \sigma'(X_r)\sigma(X_r)dW_r.$$

If $|s-t| < \delta$ and $\mu, \sigma \in C_b^2$, then [1]

$$\mu(X_s) = \mu(X_t) + O(\delta^{1/2})$$

and

$$\sigma(X_s) = \sigma(X_t) + O(\delta^{1/2}).$$

Thus, $X_{t,t+\delta}$ can be rewritten as

$$X_{t,t+\delta} = \mu(X_t)\delta + \sigma(X_t)W_{t,t+\delta} + O(\delta).$$

or

$$X_{t,t+\delta} \approx \mu(X_t)\delta + \sigma(X_t)W_{t,t+\delta}.$$

With the fact that $W_{i\delta,(i+1)\delta} \sim \sqrt{\delta}Z_i$ are iid normal random variables, we can write the following pseudocode.

pseudocode eulder_1d_path(T, N):

- partition $[0,T]$ equally by $\delta = T/N$;
- set initial $X_0^\delta = x_0$;
- For $i = 0, \ldots, N-1$, with iid standard normal $Z_i$,
  - perform $X_{i+1}^\delta = X_i^\delta + \mu(X_i^\delta)\delta + \sigma(X_i^\delta)\sqrt{\delta}Z_i$.

[1] $f(\delta) = O^{\delta^\gamma}$ means $|f(\delta)| \leq K\delta^\gamma$ for some random variable $K$ and all $\delta \in (0,\epsilon)$.

## 3.2 Strong convergence rate

euler_1d_path(T, N) gives a sequence of numbers:

$$(X_0^\delta, X_1^\delta, \ldots, X_N^\delta) := X^\delta.$$

To compare with continuous true path $(X_t : t \in [0,T])$, we first do the piecewise linear interpolation of $X^\delta$, that is

$$L_t^\delta = \frac{(i+1)\delta - t}{\delta} X_i^\delta + \frac{t - i\delta}{\delta} X_{i+1}^\delta, \quad \text{if } i\delta \le t < (i+1)\delta.$$

**Theorem 1** *RMSE of Euler approximation under uniform norm has convergence order* $1/2$, *i.e.*

$$\mathbb{E}\Big[\sup_{0\le t\le T} |X_t - L_t^\delta|\Big] \le K\delta^{1/2}.$$

PROOF: see Theorem 2.7.3 of [3]. □

**ex.** Show that

$$\mathbb{E}[|X_T - L_T^\delta|] \le K\delta^{1/2}.$$

### 3.2.1 A remark on constant interpolation of Euler solution

If we denote the piecewise constant interpolation by

$$C_t^\delta = X_i^\delta, \text{ if } i\delta \le t < (i+1)\delta,$$

then the above strong convergence fails. Let's use the following example to illustrate this issue.

Let $X = W$ be the Brownian motion itself. Euler yields

$$C_t^\delta = W_{[t/\delta]\delta}.$$

Therefore,

$$RMSE = \mathbb{E}\Big[\sup_{0\le t\le T} |X_t - C_t^\delta|\Big] = \mathbb{E}\Big[\sup_{0\le t\le T} |W_t - W_{[t/\delta]\delta}|\Big] = \mathbb{E}\Big[\sup_{i=0,\ldots,N-1} Y_i\Big],$$

where

$$Y_i = \sup_{i\delta\le t<(i+1)\delta} |W_t - W_{i\delta}|.$$

Note $Y_i \ge |W_{i\delta,(i+1)\delta}| := \sqrt{\delta}|Z_i|$, then

$$RMSE \ge \sqrt{\delta}\mathbb{E}\Big[\sup_{i=0,\ldots,N-1} |Z_i|\Big] > O(\delta^{1/2}).$$

## 3.3 Weak convergence rate

Given $Y^\delta$ and $X$, we define

$$e^g(\delta) = \Big|\mathbb{E}[g(X_T)] - \mathbb{E}[g(Y_T^\delta)]\Big|.$$

Then, we say $Y_T^\delta$ converges to $X_T$ weakly if

$$\lim_{\delta\to 0} e^g(\delta) = 0, \ \forall g \in C_b.$$

We say weak convergence rae is $\gamma$, if

$$\exists K_g > 0,\ s.t.\ e^g(\delta) \leq K_g \delta^\gamma$$

for any $g \in C_b$.

**Theorem 2** *$C^\delta$ covnverges to $X$ with $\gamma = 1$.*

PROOF: see section 9.7 of [2] □

# References


[1] Paul Glasserman. *Monte Carlo Methods In Financial Engineering*. Springer, 2004. 1

[2] P. E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1992. 3

[3] Xuerong Mao. *Stochastic Differential Equations and Applications*. Horwood Pub Ltd, 2007. 3