Consider ODE

$$-\epsilon u'' + u = x,\ \forall x \in (0,1),\ u(0) = u(1) = 0, \tag{1}$$

with $\epsilon = 10^{-10}$. This examples is taken from Example 5.2 of

- Qingshuo Song, George Yin, Zhimin Zhang, AN epsilon-uniform finite element method for singularly perturbed boundary value problems.

Instead of FEM, we are going to discuss CFD solution of (1). Answer the following questions:

1. Prove that
$$u(x) = x - \frac{\exp(\frac{x-1}{\sqrt{\epsilon}}) - \exp(-\frac{x+1}{\sqrt{\epsilon}})}{1 - \exp(-\frac{2}{\sqrt{\epsilon}})}$$
is the unique solution.
2. Using CFD on (1), find out the matrix L^h and vector $R^h f$, such that the numerical solution satisfies $L^h u^h = R^h f$.
3. Prove the consistency and stability of L^h.
4. Compute CFD solution u^h with $h = 1/5$. Compare with the FEM solution of the paper, which one is better?