1 Abstract

Our goal is to learn

- Euler approximation for the solution of 1-d SDE
- Strong and weak convergence rate

2 Problem

Consider 1-d SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, X(0) = x_0$$

We shall find, for some small step size δ

$$X^\delta(t) \approx X(t), \forall t \geq 0$$

in some sense. For convenience, we assume μ and σ are infinitely smooth and bounded in all its values and derivatives.

3 Euler scheme

The above SDE can be written as the following integral form:

$$X_t = x_0 + \int_0^t \mu(X_s)ds + \sigma(X_s)dW_s$$

If we denote

$$X_{t,s} = X_s - X_t,$$

then

$$X_{t,t+\delta} = \int_t^{t+\delta} \mu(X_s)ds + \sigma(X_s)dW_s.$$

Ito formula says

$$\mu(X_s) = \mu(X_t) + \int_t^s (\mu'(X_r) + \frac{1}{2}\mu''(X_r))dr + \mu'(X_r)\sigma(X_r)dW_r$$

and

$$\sigma(X_s) = \sigma(X_t) + \int_t^s (\sigma'(X_r) + \frac{1}{2}\sigma''(X_r))dr + \sigma'(X_r)\sigma(X_r)dW_r.$$

If $|s - t| < \delta$ is small enough, we expect $\int_t^s ...$ is again small, and write

$$\mu(X_s) \approx \mu(X_t)$$

and

$$\sigma(X_s) \approx \sigma(X_t).$$

Thus, $X_{t,t+\delta}$ can be rewritten as

$$X_{t,t+\delta} \approx \mu(X_t)\delta + \sigma(X_t)W_{t,t+\delta}.$$

With the fact that $W_{i\delta,(i+1)\delta} \sim \sqrt{\delta} Z_i$ are iid normal random variables, Euler method repeats the following recursive formula:

$$X^{\delta}_{i\delta,(i+1)\delta} = \mu(X^{\delta}_{i\delta})\delta + \sigma(X^{\delta}_{i\delta})\sqrt{\delta} Z_i, \forall i = 0, \dots, N-1.$$

We can write the following pseudocode.

Algorithm 1 Euler for SDE 1d to simulate $\hat{X} \approx \langle X, \Pi_{T,N} \rangle$

inputs: drfit $\mu(\cdot)$ and volatility $\sigma(\cdot)$
procedure EULER1D(x, T, N) ▷ x: initial state, T: terminal time,
▷ N: number of meshes
  $\delta \leftarrow T/N$; $X^{\delta}_0 \leftarrow x$; ▷ Init
  for $i = 0 \dots N-1$ **do**
    $t_{i+1} \leftarrow t_i + \delta$
    $Z \leftarrow \mathcal{N}(0, 1)$
    $X^{\delta}_{i+1} \leftarrow X^{\delta}_i + \mu(X^{\delta}_i)\delta + \sigma(X^{\delta}_i)\sqrt{\delta} Z$
  return $\{(t_i, X^{\delta}_i) : i = 0, 1, \dots, N\}$. ▷ output is a discrete path

4 Strong convergence rate

Algo **??** gives a sequence of numbers:

$$(X^{\delta}_0, X^{\delta}_1, \dots, X^{\delta}_N) := X^{\delta}.$$

To compare with continuous true path $(X_t : t \in [0, T])$, we first do the piecewise linear interpolation of X^{δ}, that is

$$L^{\delta}_t = \frac{(i+1)\delta - t}{\delta} X^{\delta}_i + \frac{t - i\delta}{\delta} X^{\delta}_{i+1}, \quad \text{if } i\delta \le t < (i+1)\delta.$$

Theorem 1 *RMSE of Euler approximation under uniform norm has convergence order* $1/2$*, i.e.*

$$\mathbb{E}\Big[\sup_{0 \le t \le T} |X_t - L^{\delta}_t|\Big] \le K\delta^{1/2}.$$

PROOF: see Theorem 2.7.3 of [2]. □

ex. Show that

$$\mathbb{E}[|X_T - L^{\delta}_T|] \le K\delta^{1/2}.$$

4.1 A remark on constant interpolation of Euler solution

If we denote the piecewise constant interpolation by

$$C^{\delta}_t = X^{\delta}_i, \text{ if } i\delta \le t < (i+1)\delta,$$

then the above strong convergence fails. Let's use the following example to illustrate this issue.

Let $X = W$ be the Brownian motion itself. Euler yields

$$C^{\delta}_t = W_{[t/\delta]\delta}.$$

Therefore,

$$RMSE = \mathbb{E}\Big[\sup_{0\le t\le T}|X_t - C_t^\delta|\Big] = \mathbb{E}\Big[\sup_{0\le t\le T}|W_t - W_{[t/\delta]\delta}|\Big] = \mathbb{E}\Big[\sup_{i=0,\dots,N-1} Y_i\Big],$$

where

$$Y_i = \sup_{i\delta\le t<(i+1)\delta} |W_t - W_{i\delta}|.$$

Note $Y_i \ge |W_{i\delta,(i+1)\delta}| := \sqrt{\delta}|Z_i|$, then

$$RMSE \ge \sqrt{\delta}\mathbb{E}\Big[\sup_{i=0,\dots,N-1}|Z_i|\Big] > O(\delta^{1/2}).$$

5 Weak convergence rate

Given Y^δ and X, we define

$$e^g(\delta) = \Big|\mathbb{E}[g(X_T)] - \mathbb{E}[g(Y_T^\delta)]\Big|.$$

Then, we say Y_T^δ converges to X_T weakly if

$$\lim_{\delta\to 0} e^g(\delta) = 0, \ \forall g \in C_b.$$

We say weak convergence rae is γ, if

$$\exists K_g > 0, \ s.t. \ e^g(\delta) \le K_g \delta^\gamma$$

for any $g \in C_b$.

Theorem 2 *C^δ covnverges to X with $\gamma = 1$.*

PROOF: see section 9.7 of [1] □

References

[1] P. E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1992. 3

[2] Xuerong Mao. *Stochastic Differential Equations and Applications.* Horwood Pub Ltd, 2007. 2