1 Abstract

You will learn

- Euler approximation for the solution of 2-d SDE
- We shall adapt Euler scheme for 2-d for Heston

2 Problem

2.1 General problem

We will perform Euler scheme for the general 2-d SDE to be considered is given as

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, X_0 = x_0$$

where $b : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a smooth vector field on $\mathbb{R}^2$, $\sigma : \mathbb{R}^2 \mapsto \mathbb{R}^{2\times 2}$ is a smooth matrix-valued function, W is a 2-d standard Brownian motion, and x_0 is the initial 2-d vector. It can be written by system of two 1-d SDEs as the following:

$$\begin{cases} dX_{1,t} = b_{1,t}dt + \sigma_{11,t}dW_{1,t} + \sigma_{12,t}dW_{2,t}, & X_{1,0} = x_{1,0} \\ dX_{2,t} = b_{2,t}dt + \sigma_{21,t}dW_{1,t} + \sigma_{22,t}dW_{2,t}, & X_{2,0} = x_{2,0} \end{cases}$$

In the above, we assume W_1 and W_2 are two independent 1-d Brownian motions.

2.2 Heston model

Heston model as a stochastic volatility model belongs to 2-d SDE in the above. However, the domain of the diffusion matrix σ is not entire 2-d space.

In the Heston model, the dynamic involves two processes (S_t, ν_t). More precisely, the asset price S follows generalized geometric Brownian motion with random volatility process $\sqrt{\nu_t}$, i.e.

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_{1,t},$$

while squared of volatility process ν follows CIR process

$$d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t}(\rho dW_{1,t} + \bar{\rho} dW_{2,t})$$

with $\rho^2 + \bar{\rho}^2 = 1$. Feller condition for its existence of the solution is

$$2\kappa\theta > \xi^2.$$

Our goal is to adapt the above Euler scheme to Heston model with the following parameters:

$$S_0 = 100, \nu(0) = .04, r = .05, \kappa = 1.2, \theta = .04, \xi = .3, \rho = .5.$$

The estimation of Call$(T = 1, K = 100)$ is given as 10.3009, see Page 357 of [1]. We will use this for our comparison to our computation.

3 Analysis

For the general problem with small $\delta > 0$, we write Euler scheme as

$$\begin{cases} X_{1,t+\delta} \approx X_{1,t} + b_1(X_t)\delta + \sigma_{11}(X_t)W_{1,t,t+\delta} + \sigma_{12}(X_t)W_{2,t,t+\delta} \\ X_{2,t+\delta} \approx X_{2,t} + b_2(X_t)\delta + \sigma_{21}(X_t)W_{1,t,t+\delta} + \sigma_{22}(X_t)W_{2,t,t+\delta} \end{cases}$$

In Heston model, the coefficients corresponds to, with $x = (x_1, x_2)$

$$b_1(x) = rx_1, b_2(x) = \kappa(\theta - x_2), \sigma_{11}(x) = \sqrt{x_2}x_1, \sigma_{12}(x) = 0, \sigma_{21}(x) = \xi\sqrt{x_2}\rho, \sigma_{22}(x) = \xi\sqrt{x_2}\bar{\rho}.$$

However, the above scheme does not directly work out, since $X_{2,t}$ during approximation needs to take non-negative number to have $\sqrt{X_{2,t}}$ makes sense. Instead, we will replace it by $\sqrt{(X_{2,t})^+}$.

References

[1] Paul Glasserman. *Monte Carlo Methods In Financial Engineering.* Springer, 2004. 1

[2] P. E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics (New York).* Springer-Verlag, Berlin, 1992.

[3] Xuerong Mao. *Stochastic Differential Equations and Applications.* Horwood Pub Ltd, 2007.