§ MC Basics

[Ref]: [Gla03] chap 1

ex1 (comp)

Estimate π by the Algo below:

[Algo(N)]

① Draw N random points uniformly in rectangle $(-1,1)\times(-1,1)$:

$$\{Z_i = (X_i, Y_i) : X_i \sim U(-1,1),\ Y_i \sim U(-1,1)\ \text{iid},\ 1 \le i \le N\}$$

② Identify the number N_i of points inside the unit circle

③ $\hat{\pi} = 4 \cdot \dfrac{N_i}{N}$

Rk π is deterministic,

$\hat{\pi}$ is random to approx. π.

Def A random approx. (estimator) $\hat{\alpha}$ to a deterministic quantity α is called MC.

① $\text{Bias} = \mathbb{E}[\hat{\alpha}] - \alpha$

② $\text{MSE}(\hat{\alpha}) = E[(\hat{\alpha} - \alpha)^2]$

<u>prop</u> $\text{MSE}(\hat{\alpha}) = |\text{Bias}|^2 + \text{Var}(\hat{\alpha})$

<u>pf</u> ....

<u>Rk</u> $\boxed{\text{The smaller MSE, the better MC}}$

<u>Def</u> MC is called <u>unbiased</u>, if $\text{Bias} = 0$

<u>ex2</u>
① Find 1^{st}, 2^{nd} moments of $\hat{\pi}$

② Find Bias & MSE of $\hat{\pi}$.

<u>Def</u> Given a series of estimator of α, say $\{\alpha_n, n \in \mathbb{N}\}$. we say (α_n) is <u>consistent</u> if $\text{p}\lim_{n\to\infty} \alpha_n = \alpha$.

<u>(Rk)</u> p-lim_n and $L_2\text{-lim}_n$ $\longrightarrow$ Next page

<u>prop</u> If $\text{MSE}(\hat{\alpha}_n) \to 0$, then $\hat{\alpha}_n$ is <u>consistent</u>.

<u>ex</u> Prove $(\hat{\pi}_N)$ is consistent estimator to π.

Def "$p\text{-}\lim_n \alpha_n = \alpha$", or "$\alpha_n \xrightarrow[n\to\infty]{} \alpha$ in prob." if

$$\lim_{n\to\infty} \mathbb{P}(|\alpha_n - \alpha| > \varepsilon) = 0 \quad \forall \varepsilon > 0$$

Def "$L_2\text{-}\lim_n \alpha_n = \alpha$" or "$\alpha_n \xrightarrow[n\to\infty]{} \alpha$ in L_2" if

$$\lim_{n\to\infty} \mathbb{E}|\alpha_n - \alpha|^2 = 0$$

ex Justify that

"$p\text{-}\lim_n \alpha_n = \alpha$" implies "$L_2\text{-}\lim_n \alpha_n = \alpha$"

Ans No. b/c counter-example.

Let $\mathbb{P} - U([0,1])$

$\alpha_n : [0,1] \to \mathbb{R}$ s.t.

$$\alpha_n(\omega) = \begin{cases} n & \text{if } 0 < \omega < \frac{1}{n^2} \\ 0 & \text{otherwise} \end{cases}$$

$\alpha : [0,1] \to \mathbb{R}$ s.t.

$$\alpha(\omega) \equiv 0$$

ex Jusity:

"L_2-$\lim_n \alpha_n = \alpha$" implies "P-$\lim_n \alpha_n = \alpha$"

Ans Yes.

$\forall \varepsilon > 0$.

$$\mathbb{P}\left(|\alpha_n - \alpha| > \varepsilon \right) = \mathbb{P}\left(\frac{|\alpha_n - \alpha|}{\varepsilon} > 1 \right)$$

$$\leq \mathbb{E} \frac{|\alpha_n - \alpha|^2}{\varepsilon^2} \xrightarrow[n \to \infty]{} 0.$$

Rk

In the above we use chebyshev inequality:

$$\mathbb{P}(|X| > 1) \leq \mathbb{E}|X|^2.$$

§ Geometric Asian Option for BSM.

__Bond__ $\quad r$

__stk__ $\quad BSM(\sigma) \sim S_t$

i.e. $S_t = S_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma W_t\}$ w.r.t. $\mathbb{Q}$

$$\sim S_0 \exp\{(r - \tfrac{1}{2}\sigma^2)t + \sigma\sqrt{t}\, Z\}.$$

where $Z \sim N(0,1)$

~~GAC pay~~

__Payoff__ for geometric asian option

① $GAC(T, K, n)$

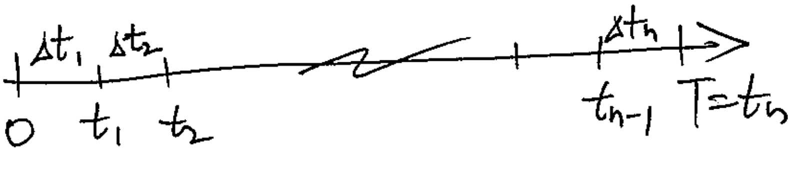
$$\Pi_T^c = (A_T - K)^+$$

② $GAP(T, K, n)$

$$\Pi_T^p = (A_T - K)^-$$

where

$$A_T = (S_{t_1} S_{t_2} \cdots S_{t_n})^{\frac{1}{n}}$$



$$\Delta t_n = \Delta t_{n-1} = \cdots = \Delta t_1 = \frac{T}{n}$$

__Q.__ what's $\Pi_0^c = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\Pi_T^c]$, $\Pi_0^p = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\Pi_T^p]$?

set $\mu = r - \frac{1}{2}\sigma^2$, $\Delta t = \frac{T}{n}$

$$S(t_1) = S_0 \exp\left(\mu\Delta t + \sigma\sqrt{\Delta t}\, Z_1\right).$$

$$S(t_2) = S(t_1) \exp\left(\mu\Delta t + \sigma\sqrt{\Delta t}\, Z_2\right)$$

$$\vdots$$

$$S(t_n) = S(t_{n-1}) \exp\left(\mu\Delta t + \sigma\sqrt{\Delta t}\, Z_n\right)$$

where $\{Z_1, \cdots Z_n\}$ iid $N(0,1)$

$$S(t_i) = S_0 \cdot \exp\left(i\mu\Delta t + \sigma\sqrt{\Delta t} \sum_{j=1}^{i} Z_j\right)$$

$$\prod_{i=1}^{n} S(t_i) = S_0^n \cdot \exp\left(\mu\Delta t \sum_{i=1}^{n} i + \sigma\sqrt{\Delta t} \sum_{i=1}^{n}\sum_{j=1}^{i} Z_j\right)$$

$$= S_0^n \exp\left(\frac{(n+1)n}{2}\mu\Delta t + \sigma\sqrt{\Delta t}\sum_{j=1}^{n}(n+1-j) Z_j\right)$$

Note $\sum_{j=1}^{n}(n+1-j) Z_j \sim$ normal with

$$\mathbb{E}\left[\sum_{j=1}^{n}(n+1-j) Z_j\right] = 0$$

$$\mathrm{Var}\left(\sum_{j=1}^{n}(n+1-j) Z_j\right) = \sum_{j=1}^{n}(n+1-j)^2 = \sum_{j=1}^{n} j^2 = \frac{n(2n+1)(n+1)}{6}$$

i.e. for some $\hat{Z} \sim N(0,1)$

$$\cancel{A(T)^n} = A^n(T) = S_0^n \exp\left(\frac{(n+1)n}{2}\mu\Delta t + \sigma\sqrt{\Delta t}\sqrt{\left(\sum_{j=1}^{n} j^2\right)}\,\hat{Z}\right)$$

$$A(T) = S_0 \exp\left(\underbrace{\boxed{\frac{(n+1)}{2}\mu}}_{\hat{\mu}}\Delta t + \underbrace{\sqrt{\boxed{\frac{\sum_{j=1}^{n} j^2}{n}}}}_{\hat{\sigma}^2}\,\sigma\sqrt{\Delta t}\,\hat{Z}\right)$$

$$A(T) = S_0 \exp\left(\underbrace{\boxed{\frac{(n+1)\,\mu}{2}}}_{\left(\hat{r}-\frac{1}{2}\hat{\sigma}^2\right)} \Delta t + \underbrace{\boxed{\frac{\sigma \sqrt{\sum_{j=1}^{n} j^2}}{n}}}_{\hat{\sigma}} \sqrt{\Delta t}\, Z \right)$$

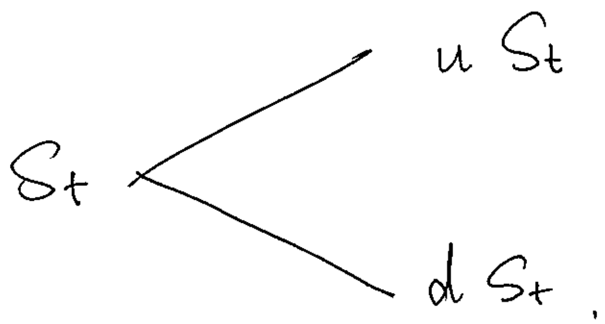
$$\Pi_0^c = e^{-rT}\,\mathbb{E}^{Q}\left[\left(S_0 \exp\left\{\left(\hat{r}-\tfrac{1}{2}\hat{\sigma}^2\right)\Delta t + \hat{\sigma}\sqrt{\Delta t}\, Z\right\} - K\right)^+\right]$$

$$= e^{-rT} e^{\hat{r}\Delta t}\,\mathbb{E}^{Q}\left[e^{-\hat{r}\Delta t}\left(\quad - \quad \right)^+\right]$$

$$= e^{\hat{r}\Delta t - rT}\ \text{BSM-Call}\left(S_0, \hat{r}, \hat{\sigma}, \Delta t, K\right)$$

CRR-model

$$\begin{cases} M^{CRR} = \{(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), \mathbb{T}, (S, B)\}. \\ t \in \{0, \Delta t, \cdots\cdots, M\Delta t = T\} \triangleq \mathbb{T} \\ B_t = e^{-r(T-t)} \\ S_{t+\Delta t} = S_t \cdot m_t = \begin{cases} u\, S_t \\ d\, S_t. \end{cases} \\ \text{where} \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad \sigma > 0 \end{cases}$$



<u>Def</u> $\{X_t : t \in \mathbb{T}\}$ is mtgl. if

$$\mathbb{E}\left[X_{t+1} \mid X_t\right] = X_t$$

<u>ex</u> Find a prob $\mathbb{Q}$ s.t. $\{e^{-rt} S_t,\ t \in \mathbb{T}\}$ is mtgl

<u>Sol:</u>

$$q = \mathbb{Q}(m_t = u) = \frac{e^{r\Delta t} - d}{u - d}$$

$$1 - q = \mathbb{Q}(m_t = d)$$

Q. why $C_0^M \longrightarrow C_0$ when $M \to \infty$?

Def $\mathbb{Q}$ is risk-neutral prob (EMM). if $\{e^{-rt} S_t,\ t \in T\}$ is $\mathbb{Q}$-mtgl.

ex For CRR Find

$$Var^{\mathbb{Q}}(\ln M_t) \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}}(\ln M_t)$$

Soln $V^{\mathbb{Q}} = \sigma^2 \Delta t.$

$\mathbb{E}^{\mathbb{Q}} = \cdots$

ex For what σ, is $\{\ln M_t,\ t \in T\}$ a $\mathbb{Q}$-mtgl ?

Soln ϕ only if $r = \sigma$

Computation

① G_T is known at maturity i.e. $G_T \in \mathcal{F}_T$.

~~② $C_t = e^{-r\Delta T}$~~

② Compute Backwardly, i.e.

$$C_t = e^{-r\Delta t}\, \mathbb{E}_t^{\mathbb{Q}}[C_{t+1}].$$

~~Ex Bin Tree Eu Option Pricing~~

Ex Bin Tree Eu Option Pricing

Given

① Bond: $r = 0.05$

② Stk:

BinTree($S_0 = 50$, $N=2$, $T=0.5$, $P_u = 1.2$, $P_d = 0.8$)

③ Payoff:

ⓐ Call (T, K)

$$\Pi_T = (S_T - K)^+$$

ⓑ Put (T, K)

$$\Pi_T = (S_T - K)^- .$$

Goal

Find Π_0 ~~and Δ, P_0.~~

Class

"Binomial European Option"

Setup-parameters:

$M = N+1$: # of terminal nodes

$u = p_u$: up ~~grow~~ factor

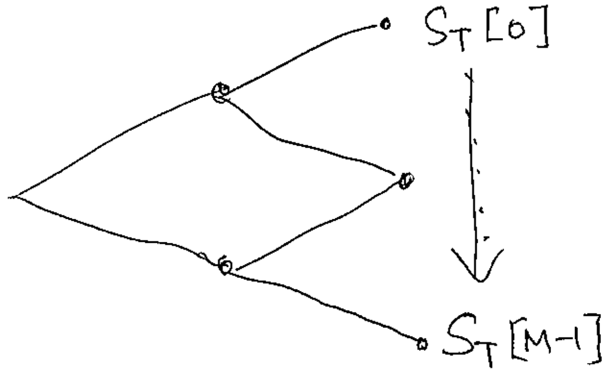
$d = p_d$: down factor

$$\left.\begin{array}{l} q_u = \dfrac{e^{(r-\delta)\Delta t} - d}{u - d} \\ q_d = 1 - q_u \end{array}\right\} : \text{EMM} \qquad (\delta = 0)$$

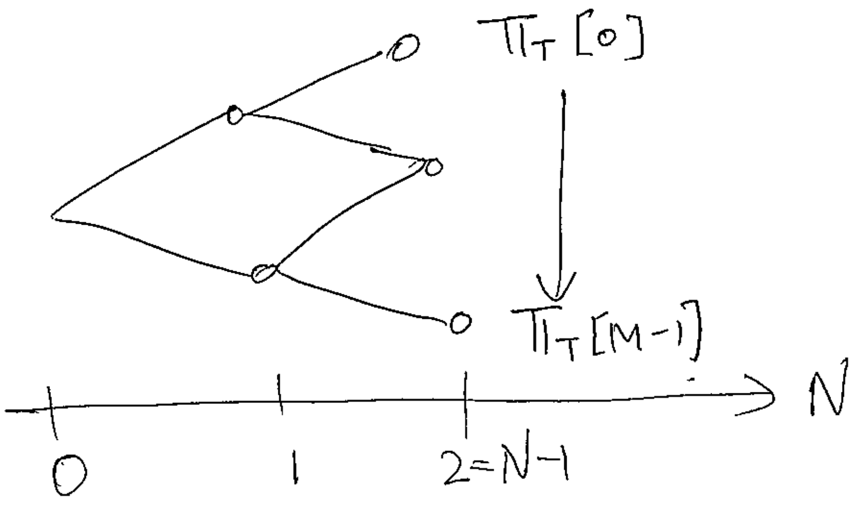
$\Delta t = \dfrac{T}{N}$: Time length of one period (inherited)

$df = e^{-(r-\delta)\Delta t}$: Discount factor for one period.

Initialize-stock-tree: Compute $\{S_T[i],\ i = 0, 1, \cdots M-1\}$



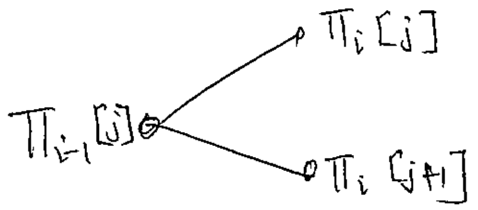
Initialize-payoff-tree: Compute $\{\Pi_T[i],\ i = 0 \cdots M-1\}$



<u>Traverse - tree</u> : (Backward)

$$\Pi_N[:] \rightarrow \Pi_{N-1}[\ ] \rightarrow \sim \longrightarrow \Pi_0[\ ].$$

By EMM averaging repeatly, i.e.



$$\Pi_{i-1}[j] = \left(q_u \cdot \Pi_i[j] + q_d\, \Pi_i[j+1]\right) \cdot d_f .$$

Option Pricing (BinTree + CRR)

★ ex1 BinTree ($S_0 = 50$, $N=2$, $T=0.5$, $u=1.2$, $d=0.8$, $r=0.05$)

$$\left.\begin{array}{l} \text{Eu Call } (T, K) \\ \text{Eu Put } (T, K) \\ \text{Am Call } (T, K) \\ \text{Am Put } (T, K) \end{array}\right\} T = 0.5, \quad K = 50.$$

Goal ① BinTree + Eu Call.

② BinTree + Am Put

★

CRR ($S_0 = 50$, $N=2$, $T=0.5$, $\sigma = 0.3$, $r = 0.05$)

= BinTree ($S_0 = 50$, $N=2$, $T=0.5$, $u = e^{\sigma\sqrt{\Delta t}}$, $d = 1/u$, $r=0.05$)

where $\Delta t = \dfrac{T}{N}$

Q. In ex1, find equivalent σ for CRR.

R. Eu Call (T, K) + CRR($S_0 = 50$, $N=2$, $T=0.5$, $\sigma=0.3$, $r=0.05$)

$$\Big\Downarrow N \to \infty$$

Eu Call (T, K) + BSM (S_0, T, σ, r).

Q. For $Bin(S_0, N, T, u, d, r)$

① Find $\mathbb{Q}$.

② Find

$$\mathbb{E}^{\mathbb{Q}}[S_{t+\Delta t} \mid S_t] = e^{r\Delta t} S_t \triangleq \bar{r} S_t$$

and

$$Var^{\mathbb{Q}}[S_{t+\Delta t} \mid S_t] = \cancel{\sqrt{\quad}}\,(u-\bar{r})(\bar{r}-d) S_t^2$$

Q Do the same for $CRR(S_0, N, T, \sigma, r)$

① Find $\mathbb{Q}$

② Find $\mathbb{E}^{\mathbb{Q}}[S_{t+\Delta t} \mid S_t]$, $Var^{\mathbb{Q}}(S_{t+\Delta t} \mid S_t)$

③ $Var^{\mathbb{Q}}\left(\ln \frac{S_T}{S_0}\right) = \sigma^2 T.$