1 Abstract

You will learn Exact sampling of Brownian path and Geometric Brownian path

- Exact sampling of Brownian path
- Exact sampling of Geometric Brownian path

Reference:

[1] Section 3.1 of [Gla03]: Random walk construction

2 Anaysis

2.1 Brownian path

Let time mesh Π be of the form

$$\Pi = \{0 = t_0 \leq t_1 \leq \ldots \leq t_N = T\}.$$

We use

$$\langle W, \Pi \rangle = \{W(t) : t \in \Pi\}$$

the projection of the brownian path on Π. To have a simulation of Brownian path by random walk, one can iterate (3.2) of [1], i.e.

$$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} Z_{i+1}. \tag{1}$$

Example 1 *Let uniform mesh be denoted by*

$$\Pi_{T,N} = \{iT/N : i = 0, ..., N\}.$$

- *Write pseudocode.*

Algorithm 1 Use (1), generate $\hat{W}$ to simulate a discrete path $\langle W, \Pi_{T,N} \rangle$.

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procedure EXACTBM1D(T, N) $\triangleright$ T, N is ...

  ...

  ...

- *Prove that $\hat{W}$ is an exact sampling.*
- *Draw 10 path simulations of $t \mapsto \frac{W(t)}{\sqrt{2t \log \log t}}$ on interval $t = [100, 110]$ with mesh size $h = 0.1$.*

2.2 Geometric Brownian path

$GBM(x_0, r, \sigma)$ is given by

$$X(t) = x_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma W(t)\}.$$

We can replace $W(t)$ by its exact simulation $\hat{W}(t)$ to get exact simulation of $X(t)$, i.e.

$$\hat{X}(t) = x_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma \hat{W}(t)\}. \tag{2}$$

In general, we have

Proposition 1 *Let $\hat{X}$ be an exact sampling of X taking values in a metric space S, i.e. $\hat{X} \sim X$. If $f : S \mapsto S'$ is a continuous function from S to another metric space S', then $f(\hat{X})$ is an exact sampling of $f(X)$.*

Example 2 *Let $\Pi_{T,N}$ be the uniform mesh and X be $GBM(x_0, r, \sigma)$.*

- *Write pseudocode.*

Algorithm 2 Use (2), generate $\hat{X}$ to simulate a discrete path $\langle X, \Pi_{T,N} \rangle$.

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procedure EXACTGBM1D(T, N)                                    ▷ T, N is ...
    ...
    ...
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- *Prove that $\hat{X}$ is exact sampling of $\langle X, \Pi_{T,N} \rangle$ by Proposition 1*

2.3 Application to Arithmetic asian option price

Arithmetic asian call option with maturity T and strick K has its pay off as

$$C(T) = (A(T) - K)^+,$$

where $A(T)$ is arithmetic average of the stock price at times $0 \leq t_1 < t_2, \ldots, < t_n = T$, i.e.

$$A(T) = \frac{1}{n} \sum_{i=1}^{n} S(t_i).$$

The call price can be thus written by

$$C_0 = \mathbb{E}[e^{-rT}(A(T) - K)^+].$$

Unlike the geometric asian option, arithmetic counterpart does not have explicit formula for its price. In this below, we shall use MC. In practice, an arithmetic asian option with a given number n of time steps takes the price average at $n + 1$ points

$$t_i = (i - 1)\frac{T}{n}, \quad i = 1, 2, \ldots, (n + 1).$$

Example 3 *Consider Arithmetic asian option price on BSM by exact sampling.*

- *Write a pseudocode for Arithmetic asian option price on BSM*
- *To the Gbm class, add a method*

  arithmeticasian(otype, strike, maturity, nstep, npath)

  for the price by exact sampling.
- *Use your code to compute Arithmetic asian option of*

$$S_0 = 100.0, \sigma = 0.20, r = 0.0475, K = 110.0, T = 1.0, otype = 1, nstep = 5.$$