GCE August, 2018

Jiamin JIAN

Exercise 1:

Let X and Y be two metric spaces and f a mapping from X to Y.

(i) Show that f is continuous if and only if for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.

(ii) Prove or disprove: assume that f is injective. Then f is continuous if and only if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$.

(iii) Prove or disprove: assume that X is compact. Then f is continuous if and only if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$.

Solution:

(i) Firstly, we show that if f is continuous, then for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$. Since $\overline{f(A)}$ is closed, $f^{-1}(\overline{f(A)})$ is closed as f is continuous, where $f^{-1}(\overline{f(A)})$ is the inverse image of $\overline{f(A)}$. Since $A \subset f^{-1}(f(A))$, then we have $A \subset f^{-1}(\overline{f(A)})$. Since the closure of A is contained in any closed set containing A, so we have $\overline{A} \subset f^{-1}(\overline{f(A)})$. Thus we know that for any $x \in \overline{A}$, we have $f(x) \in \overline{f(A)}$, then we get $f(\overline{A}) \subset \overline{f(A)}$.

Secondly, we show that if for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$, we have f is continuous. To verify that f is continuous, we just need to show that for any closed set $C \subset Y$, the inverse image of the C under the function f is also a closed set. We denote $D = f^{-1}(C)$, then we want to show D is closed in X. Since $f(\overline{D}) \subset \overline{f(D)} = \overline{f(f^{-1}(C))} = \overline{C} = C$, we know that $f(\overline{D}) \subset C$. Thus we have $\overline{D} \subset f^{-1}(C) = D$, then we know that D is a closed set in X. So, f is continuous.

(ii) The statement is not true. We can give a counter example as following. We suppose $X = \mathbb{R}^+, Y = \mathbb{R}^+$ and $\forall x \in X, f(x) = \frac{1}{x}$. Then $f(x)$ is continuous in X. We set $A = [1, +\infty)$, and we have $A \subset X$. So, $\overline{A} = [1, +\infty) = A$, and we know that $f(\overline{A}) = (0, 1]$. Since $f(A) = (0, 1]$, we have $\overline{f(A)} = [0, 1]$. Thus $f(\overline{A}) \subsetneqq \overline{f(A)}$, and we can not say $f(\overline{A}) = \overline{f(A)}$.

(iii) From the question (i), we know that if for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$, we have f is continuous. Then, if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$, we have f is continuous.

Next we should verify if f is continuous, then for every subset A of X, $f(\overline{A}) = \overline{f(A)}$. By the result we get from question (i), we know that if f is continuous, then for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$. We just need to verify $\overline{f(A)} \subset f(\overline{A})$. Since $A \subset \overline{A}$, then $f(A) \subset f(\overline{A})$ and $\overline{f(A)} \subset \overline{f(\overline{A})}$. As $A \subset X$ and X is compact, then $\overline{A}$ is compact. As f is continuous, we have $\overline{f(\overline{A})} = f(\overline{A})$. So we can get $\overline{f(A)} \subset f(\overline{A})$. In summary, when f is continuous, we have $f(\overline{A}) \subset \overline{f(A)}$ and $\overline{f(A)} \subset f(\overline{A})$. Thus if f is continuous, for every subset A of X, we have $f(\overline{A}) = \overline{f(A)}$.

To sum up, we showed that f is continuous if and only if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$.

Exercise 2:

Let $K \subset \mathbb{R}$ have finite measure and let $f \in L^\infty(\mathbb{R})$. Show that the function F defined by

$$F(x) := \int_K f(x+t)\,dt$$

is uniformly continuous on $\mathbb{R}$.

Solution:

We want to show that $\forall \epsilon > 0$, there exists a $\delta > 0$, such that when $|x-y| < \delta$, we have $|F(x) - F(y)| < \epsilon$. We verify the result by definition. Since

$$|F(x) - F(y)| = \Big| \int_K f(x+t)\,dt - \int_K f(y+t)\,dt \Big|,$$

we change the variable and denote $K_1 = \{k + x | k \in K\}$ and $K_2 = \{k + y | k \in K\}$, then we have

$$|F(x) - F(y)| = \Big| \int_{K_1} f(t)\,dt - \int_{K_2} f(t)\,dt \Big|.$$

We denote $\operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| = C$. Since $f \in L^\infty(\mathbb{R})$, then $\forall \epsilon > 0$, there exist a positive number M such that

$$\int_{K_1 \bigcap [-M,M]^c} |f(t)|\,dt < \epsilon.$$

Otherwise, $\exists \epsilon > 0$, and $\forall M > 0$, we have $\int_{K_1 \bigcap [-M,M]^c} |f(t)|\,dt \geq \epsilon$. We set $M \to +\infty$, then $\int_{K_1 \bigcap [-M,M]^c} f(t)\,dt < C\mu\{K_1 \bigcap [-M,M]^c\} \to 0$. It is contradiction. So, for all $\epsilon > 0$, there exist a M, such that

$$\begin{aligned}
|F(x) - F(y)| &= \Big| \int_{K_1} f(t)\,dt - \int_{K_2} f(t)\,dt \Big| \\
&= \Big| \int_{K_1 \bigcap [-M,M]} f(t)\,dt + \int_{K_1 \bigcap [-M,M]^c} f(t)\,dt \\
&\quad - \int_{K_2 \bigcap [-M,M]} f(t)\,dt - \int_{K_2 \bigcap [-M,M]^c} f(t)\,dt \Big| \\
&\leq \Big| \int_{K_1 \bigcap [-M,M]} f(t)\,dt - \int_{K_2 \bigcap [-M,M]} f(t)\,dt \Big| + 2\epsilon.
\end{aligned}$$

We denote $S = (K_1 \bigcap [-M,M]) \Delta (K_2 \bigcap [-M,M])$, then we have

$$|F(x) - F(y)| \leq \int_S |f(t)|\,dt + 2\epsilon \leq C\mu\{S\} + 2\epsilon.$$

As $K_1 \bigcap [-M, M]$ and $K_2 \bigcap [-M, M]$ are finite, and $K_1 = \{k+x | k \in K\}$, $K_2 = \{k+y | k \in K\}$, we can cover the set S by several open sets whose measure is $|y-x|$, then we have

$$|F(x) - F(y)| \leq Cm|y-x| + 2\epsilon,$$

where C is a positive number. We set $\delta = \frac{\epsilon}{Cm}$, then we have

$$|F(x) - F(y)| \leq 3\epsilon,$$

so, $F(x)$ is uniformly continuous on $\mathbb{R}$.

Exercise 3:

Let $\{f_n\}$ be a sequence in $L^1(\mathbb{R})$ such that $f_n \to 0$ a.e.

(i) Show that if $\{f_{2n}\}$ is increasing and $\{f_{2n+1}\}$ is decreasing, then

$$\int f_n \to 0.$$

(ii) Prove or disprove: if $\{f_{kn}\}$ is decreasing for every prime number k, then

$$\int f_n \to 0.$$

(Note on notation: e.g., if $k = 2$, then $\{f_{kn}\} = \{f_{2n}\}$. Note also that 1 is not prime).

Solution:

(i) Firstly, we consider the sequence $\{f_{2n} - f_2\}$. Since $\{f_{2n}\}$ is increasing, $f_{2n} \to 0$ and $\{f_n\} \in L^1(\mathbb{R})$ for all n, then $\{f_{2n} - f_2\}$ is increasing and $f_{2n} - f_2 \to -f_2$ a.e., then by the monotone convergence theorem, we have

$$\lim_{n \to +\infty} \int (f_{2n} - f_2) = \int \lim_{n \to +\infty} (f_{2n} - f_2) = \int -f_2,$$

then we have

$$\lim_{n \to +\infty} \int f_{2n} = 0.$$

Similarly, as $\{f_{2n+1}\}$ is decreasing, we know that $\{f_1 - f_{2n-1}\}$ is a increasing sequence and $f_1 - f_{2n-1} \to f_1$ a.e., by the monotone convergence theorem, we have

$$\lim_{n \to +\infty} \int (f_1 - f_{2n-1}) = \int \lim_{n \to +\infty} (f_1 - f_{2n-1}) = \int f_1,$$

then we have

$$\lim_{n \to +\infty} \int f_{2n-1} = 0.$$

Then we show that for any subsequence of $\{\int f_n\}$, which denoted as $\{\int f_{n_k}\}$, we can find a subsequence of $\{\int f_{n_k}\}$, which is denoted as $\{\int f_{n_{k_l}}\}$, and we have

$$\lim_{n \to +\infty} \int f_{n_{k_l}} = 0.$$

For the subsequence $\{\int f_{n_k}\}$, we take the even number in the indicator set n_k if it is infinite, or we can take the odd number in the indicator set n_k if it is infinite, then we can get the subsequence of $\{\int f_{n_k}\}$, which is denoted as $\{\int f_{n_{k_l}}\}$. Since we have showed that $\lim_{n\to+\infty}\int f_{2n} = 0$ and $\lim_{n\to+\infty}\int f_{2n-1} = 0$, then we know that $\lim_{n\to+\infty}\int f_{n_{k_l}} = 0$. So, we know that

$$\int f_n \to 0.$$

(ii) The statement is not true. We can find a counter example as follows. We define

$$f_p(x) = p\,\mathbb{I}_{[0,\frac{1}{p}]}(x),$$

where p is a prime number and

$$f_m(x) = 2\,\mathbb{I}_{[0,\frac{1}{m}]}(x),$$

where m is a not prime number. Then we know that $\{f_{np}\}$ is decreasing for every prime number p. But we can find a subsequence of $\{f_n\}$, which is denoted as $\{f_p\}$, p is the prime number, and $\lim_{n\to+\infty}\int f_p \neq 0$ as

$$\lim_{p\to+\infty}\int f_p = \lim_{p\to+\infty}\int p\,\mathbb{I}_{[0,\frac{1}{p}]}(x)\,dx = 1.$$