Simpson's rule convergence rate

2020年11月4日 星期三 上午10:31

1. Based on the assumption of simpson's rule,
$f(x) = ax^3 + bx^2 + cx + d$. on interval $[a,b]$.
then $f(x) \in C^3$.
Let $F(x) = \int_a^x f(t)\,dt$. denote $x_0 = \frac{a+b}{2}$. , Then by Taylor's Theorem:

$$F(x) = F(x_0) + f(x_0)(x-x_0) + \frac{f'(x_0)}{2!}(x-x_0)^2 + \frac{f''(x_0)}{3!}(x-x_0)^3 + O((x-x_0)^4)$$

$$\int_a^b f(x)\,dx = F(b) - F(a) = f\left(\frac{a+b}{2}\right)\cdot(b-a) + \frac{f''(x_0)}{3}\cdot\frac{(b-a)^3}{8} + O\left(\left(\frac{b-a}{2}\right)^4\right) \quad ①$$

By Taylor's Theorem.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f^3(x_0)}{3!}(x-x_0)^3 + O((x-x_0)^4)$$

then
$$f(a)+f(b) = 2f\left(\frac{a+b}{2}\right) + \frac{f''\left(\frac{a+b}{2}\right)}{2}\cdot\frac{(b-a)^2}{4}\cdot 2 + O\left(\left(\frac{b-a}{2}\right)^4\right).$$

$$f''\left(\frac{a+b}{2}\right) = \frac{4\left(f(a)+f(b) - 2f\left(\frac{a+b}{2}\right) - O\left(\left(\frac{b-a}{2}\right)^4\right)\right)}{(b-a)^2} \quad ②$$

② → ① then we get:

$$\int_a^b f(x)\,dx = f\left(\frac{a+b}{2}\right)(b-a) + \frac{\left(f(a)+f(b)-2f\left(\frac{a+b}{2}\right) - O\left(\left(\frac{b-a}{2}\right)^4\right)\right)(b-a)}{2\times 3} + O\left(\left(\frac{b-a}{2}\right)^4\right)$$

$$= \frac{(b-a)}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right] - \frac{O\left(\left(\frac{b-a}{2}\right)^4\right)(b-a)}{6} + O\left(\left(\frac{b-a}{2}\right)^4\right)$$

$$= \left(\frac{b-a}{6}\right)\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right] + O\left(\left(\frac{b-a}{2}\right)^4\right).$$

Let n is even, $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 \cdots\cdots = x_n - x_{n-1} = h$

Then
$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}\left[f(x_0) + f(x_n) + 4(f(x_1)+f(x_3)\cdots) + 2(f(x_2)+f(x_4)\cdots)\right] + \frac{n}{2}\cdot O(h^4)$$

$$= S[f] + O(h^4).$$