A General characterization of One-Factor Forward rate dependent Volatility Heath-Jarrow-Morton term structure model: Transformation to Markovian Affine form

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ABSTRACT. Heath-Jarrow-Morton (1992) has become the most popular term structure model in interest rate derivatives pricing theory. In the HJM model, the only inputs needed to construct the term structure are the initial yield curve and the volatility structure for all forward rates. But, due to fewer restrictions on the term structure, the HJM model are in general non-Markovian, and contingent claims are difficult to price with the lattice method. To solve this problem some conditions and new parameters would be imposed. In this paper we consider a class of a single factor HJM model with a forward volatility structure depending upon a function of the time and the time to maturity, the instantaneous spot rate and forward rate to a fixed maturity. We demonstrate that under some restrictions the stochastic dynamics determining the prices of interest rate derivatives may be reduced to Markovian affine form. These transformations generalise the Markovian systems obtained by Ritchken and Sankarasubramanian (1995), Bhar and Chiarella (1997), Mercurio and Morelda (1996). Finally, an explicit exponential affine formula for the bond price in terms of the state variables is provided for the model considered.

Keywords: the term structure of interest rates, Volatility Structures, HJM model, Mrkovian and affine models.

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Introduction

Most term structure models exist for pricing interest-rate contingent claims., two approaches may still be distinguished. For one there is the "equilibrium-based" approach, according to which one is to specify one or more factors that are jointly Markov and drive the term structure. Given the process for these factors under the real measure, P , and some specification for the "market price of risk" of each of these factors, one can define the so-called risk-neutral measure, Q, under which all discounted-asset-price processes are martingales (Harrison and Kreps (1979)). Notice that the "market price of risk" specification may either be arbitrarily imposed (Vasicek (1977)) or derived under some restrictive preference and economic-environment assumption (Cox, Ingersoll and Ross (1985)).

More recently, a second strand of literature has developed that avoids the crux of explicitly having to specify the "market price of risk" when pricing interest-rate derivatives. The "arbitrage" approach initiated by Ho and Lee (1986) and generalized by Heath-Jarrow-Morton (1992) (HJM), takes the initial term structure as given and, using the no-arbitrage condition, derives some restrictions on the drift term of the process of the forward rates under the risk-neutral probability measure Q . In essence, HJM show that if there exists a set of traded interest-rate-dependent contracts then the dynamics of the prices of those contracts under the risk-neutral measure is fully specified by their volatility structure. Indeed, absence of arbitrage places restrictions on the drift of those contracts under the Q measure.

Despite the advantages of HJM over the short-rate models it was found to have some drawbacks: some practical, some theoretical. First, those models are non-Markovian in general, and consequently the techniques from the theory of PDEs no longer apply. Second, many volatility term structures σ (t, T) result in dynamics for f (t, T) which are non-Markov (that is, with a finite state space). This introduces path dependency to pricing problems which significantly increases computational times. Third, there are generally no simple formulae or methods for pricing commonly-traded derivatives such as caps and swaptions. Again this is a significant problem from the computational point of view. Finally, if we model forward rates as log-normal processes then the HJM model will explode[1]. This last theoretical problem with the model can be avoided by modelling LIBOR and swap rates as log-normal (market models) rather than instantaneous forward rates.

[1] for example, see Sandmann & Sondermann, 1997.

Suitable restrictions on volatility processes led many researchers to transform the HJM model to finite dimensional Markovian systems. In Ritchken and Sankarasubramanian (95) and Bhar and Chiarella (97) only the one-factor HJM models are considered, while, under a more transparent framework, Inui and Kijima(98) generalise the Ritchken and Sankarasubramanian (95) models to the multifactor case. In Bhar and Chiarella (99), a theoretical framework is introduced for obtaining necessary and sufficient conditions under which HJM models are Markovian, and for constructing minimal realisation in such cases. characteristic of all of these models is that the form of the forward rate volatility processes allows them to be transformed to Markovian form, at the expense, however, of increasing the dimension of the underlying state space.

Although theoretically appealing, the Markovian HJM models obtained in Chiarella and Kwon (2001b) involve a large number of state variables which, at first sight, do not appear to have direct links to market observed quantities. The main aim of this paper is to show under suitable restrictions that the state variables are, in fact, affine functions of a fixed maturity forward interest rates. This observation is useful, for example, in the calibration of model parameters since the state variables for these models are directly observed in the market. This observation also leads to an explicit formula for the bond price in terms of the state variables for these models.

On the other hand, affine class of term structure models as characterized by Duffie and Kan (DK, 1996) has become the dominant class of models because of its analytical tractability. In particular, the affine class possesses closed-form solutions for both bond and bond-option prices (Duffie, Pan, and Singleton (2000)), efficient approximation methods for pricing swaptions (Collin-Dufresne and Goldstein (2002b), Singleton and Umantsev (2002)), and closed-form moment conditions for empirical analysis (Singleton (2001), Pan (2002)). As such, it has generated much attention both theoretically and empirically.

The affine interest rate term structure class become the most important framework for modelling the term structure of interest rates by restricting the spot rate the risk neutral drift and instantaneous covariance matrix of the state vector to be linear in the state vector many desirable features emerge bond prices inherit a simple exponential affine structure Analytic solutions exist for the prices of many fixed income derivatives such as options on zero-coupon bonds, baskets of yields and futures Coupon bond options or swaptions can be priced accurately and efficiently analytic solutions exist for the optimal bond portfolio choice problem However the tractability of the affine framework comes at the potential cost of limiting its flexibility to explain empirical observation For example Duffie, Dai and Singleton document the inability of low dimensional models to capture the predictability of bond risk premium Furthermore Jagannathan, Kaplin and Sun found that low dimensional affine models are unable to capture the joint dynamics of caps, swaptions and bonds.

these models became the focus of a series of papers including Carverhill (1994), Ritchken and Sankarasubramanian (1995), Bhar and Chiarella (1997), Inui and Kijima(1998) and Jong and Santa-Clara (1999). In Chiarella and Kwon (2001b), a common generalisation of these models was obtained in which the components of the forward rate volatility process satisfied ordinary differential equations in the maturity variable. However, the generalised models require the introduction of a large number of state variables which, at first sight, do not appear to have clear links to market observed quantities. In this paper, it is shown that the forward rate curves for these models can often be expressed as affine functions of the state variables, and conversely these state variables can be expressed as function of each other.

By considering a volatility structure depended on a series of fixed maturity forward rate in the HJM framework, expressed by :

$$\sigma(t,T) = g\left[t, T, f(t, x_1),\ldots,f(t, x_m)\right]e^{-\lambda(T-t)} \qquad (1)$$

Chiarella and Kwon (2001) show that the forward rate curves for these models can often be expressed as affine functions of the state variables, and conversely that the state variables in these models can often be expressed as affine functions of a finite number of benchmark forward rates. Consequently, for these models, the entire forward rate curve is not only Markov but affine with respect to a finite number of benchmark forward rates. It is also shown that the forward rate curve can be expressed as an affine function of a finite number of yields which are directly observed in the market. This propriety is useful, for example, in the estimation of model parameters. furthermore, an explicit formula for the bond price in terms of the state variables, generalising the formula given in Inui and Kijima (1998), is provided for the models considered . Finally, they conclude that. Finite dimensional Markovian HJM term structure models provide an ideal setting for the study of term structure dynamics and interest rate derivatives where the flexibility of the HJM framework and the tractability of Markovian models coexist.

In a more recent paper, Bhar, Chiarella El-Hassan and Zheng (2002) consider a single factor Heath-Jarrow-Morton model with a forward rate volatility function depending upon a function of time to maturity, the instantaneous spot rate and a forward rate to a fixed maturity. Formally they suppose that;

$$\sigma(t, T, r(t), f(t,\tau)) = g[r(t), f(t,\tau)]\, e^{-\lambda(T-t)}, \qquad 0 \le t \le \tau < T. \qquad (2)$$

With this specification they found that the stochastic dynamics determining the prices of interest rate derivatives may be reduced to Markov form. Furthermore, the evolution of forward rate curve is completely determined by the two rates specified in the volatility function and it is thus possible to obtain closed form expression for bond prices.

In this paper we extend Bhar, Chiarella El-Hassan and Zheng (2002) model to further generalise the form of the volatility structure to include a new function, α (t, T). In particular we consider the following volatility structure:

$$\sigma(t, T, \alpha(t, T), r(t), f(t, \tau)) = g[r(t), f(t, \tau)]\, \alpha(t, T)\, e^{-\lambda(T-t)}, \quad 0 \leq t \leq \tau < T. \tag{3}$$

The dependency of the volatility to both the instantaneous spot rate and the forward rate was supported by Brennan and Schwartz (1979). They found that the evolution of the term structure is depend not only in the spot interest rate but also in a fixed maturity forward interest rate. however, the consideration of the, α (t, T), function, is motivated by the seems to generalise the term structure volatility to include a class of model developed by Mercurio et Meroleda (1996)[2]. The proprieties of the volatility structure considered will be more discussed in detail in the section 2.

Our general volatility structure conduct us to a general form of bond prices that are in general no Markovian. Restrictions on the volatility specification will be imposed to reduce the general model to Markovian affine form. Theses restrictions concern the function α (t, T) recently introduced over the previous models appeared in the literature.

the remainder of the paper is organized as follows: in Section 2, the HJM framework is briefly reviewed, and the class of Markovian and affine HJM models are defined. The volatility structure considered and the dynamic of the forward rate are determined in Section 3. Reduction of the general one-factor model considered to Markovian affine form, by imposing restrictions on the volatility structure, is insured in section 4. Section 5 applies these results to simple examples, and the paper finally concludes with Section 6.

[2] Mercurio et Merolda (1996) consider the following volatility specification: $\sigma(t, T) = \frac{1+\gamma T}{1+\gamma t} r^{\lambda}(t)\, e^{-\lambda(T-t)}$

2. the HJM framework:

in this section, an overview of the one factor Heath-Jarrow-Morton is given. Then we present definitions of Markovian and Affine term structure model proposed by Kwon (2000).

2.1 the risk neutral Framework:

Fix a trading interval [0, τ], $\tau > 0$, and let (Ω, F , P) be a probability space, where Ω is the set of states of the economy, F is a filtration generated by the standard P-Wiener $W^f(t)$ and P is a probability measure on (Ω, F).

For each maturity $T \in [0, \tau]$, the time t instantaneous forward rate f(t, T,Ω), in the risk-neutral one factor HJM interest rate model, is assumed to satisfy the stochastic integral equation:

$$f(t,T,\omega) = f(0,T) + \int_0^t \sigma^*(s,T,\omega)\ ds + \int_0^t \sigma(s,T,\omega)dW^f(s) \qquad (4)$$

Where $0 \le t \le T \le \tau$, $\sigma^*(s,T,\omega) = \sigma(s,T,\omega)\int_s^T \sigma(s,u,\omega)du$ and ω represents the dependence of the forward rate process on the Wiener path $\omega \in \Omega$. For finite dimensional Markovian specialisations of the HJM model, the path (ω) dependence simplifies to dependence on a finite number of state variables such as the spot rate r(t, ω).

We remark that, Within the single factor HJM framework the initial forward rate curve is exogenously specified and the intertemporal transitions of the whole forward rate curve are dictated by the specified forward rate volatility structure (the volatility of each forward rate with different maturity).

The spot rate process is obtained from equation (1) by setting T = t, so that r(t, ω) = f(t, t, ω).

We obtain:

$$r(t,\omega) = f(0,t) + \int_0^t \sigma^*(s,t,\omega)\ ds + \int_0^t \sigma(s,t,\omega)dW^f(s) \qquad (5)$$

As stochastic differential equation (6) becomes:

$$dr(t) = \left[f_2(0,t) + \frac{\partial}{\partial t}\int_0^t \sigma(s,t,.)\int_s^t \sigma(s,u,.)duds + \int \sigma_2(s,t,.)dW(s) \right]dt + \int_0^t \sigma(t,t,.)dW(t) \qquad (6)$$

Where f_2 (0,t) denotes the partial derivatives of f(0, t) with respect to the second argument.

It can be seen from (4) that the forward rate process f(t, T, ω) is non-Markovian in general, since the volatility processesσ_i(t, T, ω) depend on the path ω, and hence on the past. Even if ω_i(t, T, ω) did not depend on the past, (5) and (6) show that the spot rate process remains non-Markovian in general, due to the path dependent terms in (6) that involve integration over the past. Consequently, the general HJM model does not readily lend itself to practical implementations. If the HJM model can be transformed to a Markovian system, then the resulting system can be tackled more efficiently to obtain the bond price P(t,T,ω), by solving directly, or numerically, the resulting partial differential equation.

Then the price of a T maturity *zero-coupon bon*d at the time ,t, denoted P(t,T,ω), is an adapted process defined by the equation

$$\mathbf{P(t,T,\omega)=e}^{\left(-\int_t^T f(t,u)du\right)} \qquad (7)$$

2.2. the class of Markovian and affine HJM model:

Oh Kang Kown (2000) present a general definition of Markovian and affine of interest rate term structure model developed in the general Heath-Jarrow-Morton framework. His definition of the affine models appears more general of the definition proposed by Duffie and Kan (1996). Thus he don't restrict the diffusion function of interest rate to be a square root of linear function of state variables.

Definition 2.1. let *M* be a term structure model on Ω. Then *M* is said to be *Markov* if there exists a *F*-Markov process z(t, ω) and a function f: $R_+ \times R_+ \times R \rightarrow R$ such that

$$f(t,\tau,Z(t)) = h_0(t,\tau) + h(t,\tau)Z(t) \qquad (8)$$

The vector process z(t, ω) is called the *state vector* for *M* , and is assumed to satisfy the following differential stochastic equation:

$$dz(t) = \mu^z(t,z)dt + \sigma^z(t,z)dW(t) \qquad (9)$$

where $\mu^z : R_+ \times R \rightarrow R$ and $\sigma^z : R_+ \times R \rightarrow R$

The instantaneous forward rates in a Markov model *M* will be written f(t, τ, z(t, ω)), where z(t, ω) is the state vector for M .

Definition 2.2 A Markov model with state vector z(t, ω) is said to be *affine* if there exist functions $a_0 : R_+ \times R_+ \rightarrow R$ and a: $R_+ \times R_+ \rightarrow R$ such that

$$f(t,T,z(t,\omega)) = a_0(t,T) + a(t,T)z(t,\omega) \qquad (10)$$

in conformity with the finding of Duffie and Kan (1996), Kwon show that this definition of the affine term structure have an important effect on the nature of the zero-bond prices., thus he deducts the following lemma:

Lemma 2.3. Let *A* be a d-affine term structure model defined as in (2.5), and assume that $a_i(t, .) \in (R+)$ for all i and t. Then

$$P(t,\tau) = \exp\{-b_0(t,T) - b(t,T)\, z(t,T)\} \tag{11}$$

Where $b_i(t,T) = \int_0^T a_i(t,u)du$ pour tout $0 \leq i \leq n$.

3. the volatility structure and the dynamic of the forward rate:

In this paper we extend Bhar, Chiarella El-Hassan and Zheng (2002) model to further generalise the form of the volatility structure to include a new function, α (t, T). In particular we consider the following volatility structure:

$$\sigma(t,T) = \sigma(r(t), f(t,\tau))\ \alpha(t,T)\, e^{-\lambda(T-t)}, \qquad 0 \leq t \leq \tau < T. \tag{12}$$

Where $\sigma(r(t), f(t, \tau))$ is a function of the instantaneous spot rate, r(t), and of the forward interest rate, $f(t, \tau)$ of a fixed maturity τ. The nature of this function support the empirical findings of Brennan and Schwartz (1979), Ritchken and Sankarasubramanian (1995) and Fan, Gupta and Ritchken (2002), they suggest that the volatility of the forward rate depend not only on the level of the spot rate but also on the level of the forward rate. in particular, Brennan and Schwartz (1979) support strongly that theses two variables explain the evolution of the term structure.

The central component of our volatility structure is the function $\alpha(t, T)$, which is a deterministic function of the time t, and the time to maturity T. we demonstrate that the Markov and affine nature of the HJM considered in this paper will depend on the form of this new function. the incorporation of this function we permit to extend a large class of interest rate model proposed by Mrcurio et Maraleda (1996).

The volatility that we establish incorporate all current information in the yield curve and has the following properties. First, it express a very general class of HJM term structure model, thus, many models that appear in the literature can be deducted as a special case of our model. In particular, by setting $\alpha(t, T) = \sigma$, where σ is a constant, we deduct the generalized Vasicek (1977) model. The if we suppose that $\sigma(r(t), f(t, \tau)) = r^{1/2}$ we obtain the Hull and White (1990) model. Then if we suppose that $\alpha(t,T) = \frac{1+\gamma T}{1+\gamma t}$ and $\sigma(r(t), f(t,\tau)) = r^{\lambda}(t)$ we obtain the class of model developed by Mercurio et Moraleda (1996). Those three model will be developed in our theoretical framework as special cases in the last section of this paper. Second, considering the restrictions imposed on the form of the function $\alpha(t, T)$, explicit solutions for the zero-coupon bonds prices are permissible, The fact that a formula for the bond price can be obtained is of great utility, since one need only solve the PDE to price interest rate derivatives.

Our object now is to express the general stochastic integral of the forward rate and the instantaneous spot rate in equations (5) and (6) under the volatility structure (12) considered by our model.

With such a volatility specification we ca write:

$$f(t,T) = f(0,T) + \int_0^t \sigma(\alpha(u,T), r(u), f(u,\tau),.)e^{-\lambda(T-u)} \int_u^t \sigma(\alpha(u,s), r(u), f(u,\tau),.)e^{-\lambda(s-u)} ds du \qquad (13)$$
$$+ \int_0^t \sigma(\alpha(u,T), r(u), f(u,\tau),.)e^{-\lambda(T-u)} dW(u)$$

and :

$$r(t) = f(0,t) + \int_0^t \sigma(\alpha(u,t), r(u), f(u,\tau),.)e^{-\lambda(t-u)} \int_u^t \sigma(\alpha(u,s), r(u), f(u,\tau),.)e^{-\lambda(s-u)} ds du \qquad (14)$$
$$+ \int_0^t \sigma(\alpha(u,t), r(u), f(u,\tau),.)e^{-\lambda(t-u)} dW(u)$$

the stochastic differentials equations of the forward and the spot interest rate can be deduced from equation(8) and (9) by generating the stochastic differentials of the integral of theses equations. In **Appendix A** We demonstrate that:

$$df(t,T) = \left[g(r(t), f(t,\tau))^2 \, \alpha(t,T) \; e^{-\lambda(T-t)} \; B(t,T)\right] dt + \sigma(\alpha(t,T), r(t), f(t,\tau)) d\tilde{W}(t) \quad (15)$$

Where $B(t,T) = \int_t^T \alpha(t,s) e^{-\lambda(s-t)} ds$

The term structure of interest rate:

after the determination of the dynamic of the forward rate, we determine now the price at the time t, of T maturity zero-coupon bond. The bond price formula for HJM models derived by separable volatility process was obtained by Ritchken and Sankarasubramanian (1995) for the one dimensional case, and extended to the general case by Inui and Kijima (1998). The formula was then generalised by Chiarella an Kwon (1998,a) to forward rate dependent volatility process. In this subsection, a brief outline of the derivation of the bond price formula given.

we show that under the volatility structure (2) considered, the price of zero-coupon bond is given by:

$$P(t,T)=\frac{P(0,T)}{P(0,t)}e^{-R(t,T)} \quad (16)$$

Where : $$R(t,T)=\pi(t,T)\left[\int_0^t \tilde{\sigma}^*(s,t,.)\,ds+\int_0^t \tilde{\sigma}(s,t,.)\,d\tilde{W}(s)\right]+\delta(t,T,.)\int_0^t \tilde{\sigma}^2(s,t,.)ds \quad (17)$$

Where : $\tilde{\sigma}=\frac{1}{\alpha(s,t)}\sigma(s,t,.)$, $\tilde{\sigma}^*=\frac{1}{\alpha(s,t)}\sigma^*(s,t,.)$

and $\pi(t,T)=\int_t^T \alpha(t,u)e^{-\lambda(u-t)}du$

and $\delta(t,T)=\int_t^T \alpha(t,u)e^{-\lambda(u-t)}\int_t^u \alpha(t,y)e^{-\lambda(y-t)}dydu$

See **Appendix B**.

4. Reduction to Markovian affine form :

the aim of this section is to express the equations (5) and (6) under the volatility specification of equation (2) as a Markovian system of differential equation (SDE), this system will leads to an exponential affine form of the zero-coupon bond price. To do this we impose the following restriction on the form of the function α (t, T).

the next proposition establishes general restriction on the volatility structure to reduce the HJM model considered in this paper to a Markovian affine term structure model.

Proposition 1 : Let *M* be HJM term structure model satisfying equation (12), if $\alpha(t,T)$ is constant or deterministic function that verify :

$$\alpha(s,u)=\alpha(s,t)\alpha(t,u) \quad (18)$$

then the HJM term structure model is a Markovian affine model.

We demonstrate under this restriction that the zero-coupon bond price is an exponential affine function of the state variables, and we determine the additional state variable necessary to make the system Markovian although with a higher dimension, the instantaneous spot rate, the forward interest rate and a new state variable that will be introduced and presented in following.

We show in **Appendix C** that:

$$P(t,T)=\exp\{-\pi(t,T)[f(0,t)-r(t)]-\delta(t,T)\psi(t)\} \quad (19)$$

from this equation it appears that under the restriction imposed the zero-coupon bond price is an exponential affine function of the three state variables; the instantaneous spot rate, the forward interest rate and the $\Psi(t)$.

the bond price has the nearly the same formula as the one derived by Bhar, Chiarella, Al-Hassan (2002) who neglect the function α (t, T). Ritchken and Sankarasubramanian (1995) find the same formula price when they assume a volatility structure independent of the forward rate f(t, τ). Chiarella and Kwon (1999) nave shown that bonds price holds in precisely the same form even when the forward rate volatility depends on a set of discrete forward rates $f(t, \tau_1)$. $f(t, \tau_2)\ldots f(t, \tau_r)$, where $t < \tau_1 < \tau_2 <\ldots<\tau_r \leq T$.

under theses different volatilities structures the state variable evolve differently but the functional relationship remains the same.

we show that $$d\psi(t) = [\sigma(\alpha(t,t), r(t), f(t, \tau))^2 - 2\lambda\psi(t)]dt \qquad (20)$$

<u>Proof :</u>

$$d\psi(t) = \left[\sigma(t, t, r(t), f(t, \tau))^2 + 2\int_0^t \sigma(u, t, r(u), f(u, \tau)).\sigma_2(u, t, r(u)f(u, \tau))du \right]dt$$

$$= \left[\sigma(t, t, r(t), f(t, \tau))^2 - 2\lambda\int_0^t \sigma(u, t, r(u), f(u, \tau))^2 du \right]dt$$

$$= [\sigma(\alpha(t, t), r(t))^2 - 2\lambda\psi(t)]dt$$

we have now reduced the non Markovian stochastic dynamics to a three dimensional Markovian stochastic dynamical system. Where the state variables are expressed one as function of the others.

In particular the subsidiary variable $\Psi(t)$ have an important role in allowing the transformation of the general model to Markovian for. Many other researchers have consider similar variable Chiarella and Kown (2001), Ritchken an sunkarasubramanian (1995) and Inui and Kijma (1998). The definition of $\Psi(t)$ show that this variable resume the path history of the forward rate volatility.

We now go on to determine the dynamic of the spot interest rate model.

$$dr(t) = \left[f_2(0, t) + \frac{\partial}{\partial t}\int_0^t \sigma(u, t,.)\int_u^t \sigma(u,s,.)dsdu + \int \sigma_2(u,t,.)d\tilde{W}(u) \right]dt + \int_0^t \sigma(t, t,.)d\tilde{W}(u) \quad (21)$$

5. Special cases :

In this section, consider some special cases of our general model that correspond to the Vasicek (1977), Cox, Ingersoll and Ross (1985) and Mercurio and Morelda (1996) models.

For the last model we demonstrate that state variables can be expressed as function of each other.

5.1 the Vasicek (1977) generalized model:

The Hull and White (1990) extended Vasicek model M_{Vas} is a 1-dimensional HJM model corresponding to the forward rate volatility:

$$\sigma(t,T)=\sigma_0 e^{-\lambda(T-t)} \tag{22}$$

Where σ_0 and λ are constants from this volatility structure, we remark that $\alpha(t,T)=\sigma_0$ and $\sigma(r(t), f(t,\tau)=1$.

The forward interest rate stochastic integral equation will be :

$$f(t,T)=f(0,T)+\int_0^t \sigma_0 e^{-\lambda(T-u)}\int_u^t \sigma_0 e^{-\lambda(s-u)}ds du+\int_0^t \sigma_0 e^{-\lambda T-u)}d\tilde{W}(u) \tag{23}$$

$$=f(0,T)+\sigma_0^2\int_0^t e^{-\lambda(T-u)}\int_u^t e^{-\lambda(s-u)}ds du+\sigma_0\int_0^t e^{-\lambda T-u)}d\tilde{W}(u) \tag{24}$$

$$=f(0,T)+\sigma_0^2\,\mu(t,T)+\sigma_0\int_0^t e^{-\lambda T-u)}d\tilde{W}(u) \tag{25}$$

where $\mu(t,T)=\frac{1}{k^2}\left[-\frac{1}{2}e^{t\lambda}+e^{T\lambda}\right]e^{-\lambda(2T-t)}+\ (\frac{1}{2}-e^{T\lambda})e^{-2T\lambda}$ (26)

and $df(t,T)=\left[\alpha(t,T)\beta(t,T)e^{-\lambda(T-t)}\right]dt+\sigma_0 e^{-\lambda(T-t)}d\tilde{W}(t)$ (27)

and $\beta(t,T)=\int_t^T \sigma_0 e^{-\lambda(s-t)}ds$

$$=\frac{1}{\lambda}\sigma_0\left[1-e^{-\lambda(T-t)}\right]$$

thus

$$df(t,T)=\frac{1}{\lambda}\sigma_0^2\left[1-e^{-\lambda(T-t)}\right]dt+\sigma_0 e^{-\lambda(T-t)}d\tilde{W}(t) \tag{28}$$

The bond price formula can now be stated.

$$P(t,T)=\exp-\left[\pi'(t,T)\left[r(t)-f(0,t)\right]+\frac{1}{2}\pi'^2(t,T)\psi(t)\right] \tag{29}$$

Thus we demonstrate that in the one dimensional HJM framework the vasicek model is Markovian affine model, and the bond price is exponential affine function of three states variables.

Proof :

From (17) we deduce directly

$$R(t,T)=\pi(t,T)\left[\int_0^t \frac{1}{\alpha(s,t)}\sigma^*(s,t)ds+\int \frac{1}{\alpha(s,t)}\sigma(s,t,.)d\tilde{W}(s)\right]+\delta(t,T)\int_0^t \frac{1}{\alpha^2(s,t)}\sigma^2(s,t)ds$$

where

$$\pi(t,T)=\int_t^T \sigma_0 e^{-\lambda(u-t)}du$$
$$=\frac{1}{\lambda}\sigma_0\left[1-e^{-\lambda(T-t)}\right]$$

$$\delta(t,T)=\sigma_0^2\int_t^T \sigma_0 e^{-\lambda(u-t)}\int_t^u e^{-\lambda(y-t)}dydu$$
$$=-\frac{\sigma_0^2}{2}\left[2e^{-\lambda(T-t)}-e^{-2\lambda(T-t)}-1\right]$$
$$=-\frac{1}{2}\pi(t,T)^2$$

then :

$$R(t,T)=\frac{1}{\sigma_0}\pi(t,T)\left[\int_0^t \sigma^*(s,t)ds+\int_0^t \sigma(s,t,.)d\tilde{W}(s)\right]+\frac{1}{2\sigma_0^2}\pi^2(t,T)\int_0^t \sigma^2(s,t)ds$$

We have showed that:

$$\int_0^t \sigma^*(s,t)ds+\int_0^t \sigma(s,t,.)d\tilde{W}(s)=r(t)-f(0,t)$$

we derive that :

$$R(t,T)=\pi'(t,T)\left[r(t)-f(0,t)\right]+\frac{1}{2}\pi'^2(t,T)\psi(t)$$

where :

$$\psi(t)=\int_0^t \sigma^2(s,t)ds$$

finally we can write :

$$P(t,T)=\exp-\left[\pi'(t,T)\left[r(t)-f(0,t)\right]+\frac{1}{2}\pi'^2(t,T)\psi(t)\right]$$

this formula show that the extended Vasicek (1977) model is a Markovian affine term structure model in the HJM framework

5.2. the CIR (1985) generalized model:

The next example is the generalised CIR model in which the forward rate volatility process is given by $\sigma(t,T)=\sigma_0 r^{\gamma}(t)e^{-\lambda(T-t)}$ (30)

The forward interest rate stochastic integral equation will be :

$$f(t,T) = f(0,T) + \int_0^t \sigma_0 r^{\gamma} e^{-\lambda(T-u)} \int_u^t \sigma_0 r^{\gamma} e^{-\lambda(s-u)} ds du + \int_0^t \sigma_0 r^{\gamma} e^{-\lambda T-u)} d\tilde{W}(u)$$

$$= f(0,T) + \int_0^t \sigma_0 r^{2\gamma} e^{-\lambda(T-u)} \int_u^t \sigma_0 r^{\gamma} e^{-\lambda(s-u)} ds du + \int_0^t \sigma_0 r^{\gamma} e^{-\lambda T-u)} d\tilde{W}(u)$$

$$= f(0,T) + \sigma_0^2 \int_0^t r^{2\gamma}(u) e^{-\lambda(T-u)} \frac{1}{\lambda}\left[1 - e^{-\lambda(T-u)}\right] du + \int_0^t \sigma_0 r^{\gamma}(u) e^{-\lambda(T-u)} d\tilde{W}(u) \qquad (31)$$

thus :

$$df(t,T) = \left[\sigma_0 r^{2\gamma}(t) B(t,T) e^{-\lambda(T-t)}\right] dt + \sigma_0 r^{\gamma}(t) e^{-\lambda(T-t)} d\tilde{W}(t) \qquad (32)$$

Where :

$$B(t,T) = \int_t^T \sigma_0 e^{-\lambda(s-t)} ds$$

$$= \frac{1}{\lambda}\sigma_0\left[1 - e^{-\lambda(T-t)}\right]$$

we show the price, at time t, of zero-coupon bond with maturity T will be:

$$P(t,T) = \exp - \left[\pi'(t,T)\left[r(t) - f(0,t)\right] + \frac{1}{2}\pi'^2(t,T)\psi(t)\right] \qquad (35)$$

<u>Proof:</u>

$$R(t,T) = \pi(t,T)\left[\int_0^t \frac{1}{\alpha(s,t)} \sigma^*(s,t) ds + \int \frac{1}{\alpha(s,t)} \sigma(s,t,.) d\tilde{W}(s)\right] + \delta(t,T)\int_0^t \frac{1}{\alpha^2(s,t)} \sigma^2(s,t) ds$$

where

$$\pi(t,T) = \int_t^T \sigma_0 e^{-\lambda(u-t)} du \quad = \frac{1}{\lambda}\sigma_0\left[1 - e^{-\lambda(T-t)}\right]$$

and

$$\delta(t,T) = \sigma_0^2 \int_t^T \sigma_0 e^{-\lambda(u-t)} \int_t^u e^{-\lambda(y-t)} dy du$$

$$= -\frac{\sigma_0^2}{2\lambda^2}\left[2e^{-\lambda(T-t)} - e^{-2\lambda(T-t)} - 1\right]$$

$$= -\frac{1}{2\lambda^2}\pi(t,T)^2$$

we set :

$$\sigma^*(s,t) = \sigma(s,t,.)\int_s^t \sigma(s,u,.)du$$
$$= \sigma_0^2 r^{2\gamma}(s)e^{-\lambda(t-s)}\int_s^t \sigma(s,u,.)e^{-\lambda(u-t)}du \tag{33}$$

we obtain that : $R(t,T) = \frac{1}{\sigma_0}\pi(t,T)\left[\int_0^t \sigma^*(s,t)ds + \int_0^t \sigma(s,t,.)d\widetilde{W}(s)\right] + \frac{1}{2\sigma_0^2}\pi^2(t,T)\int_0^t \sigma^2(s,t)ds$

from equation (2) of the spot interest rate we can deduce that :

$$\int_0^t \sigma^*(s,t)ds + \int_0^t \sigma(s,t,.)d\widetilde{W}(s) = r(t) - f(0,t)$$

thus, we can write:

$$R(t,T) = \pi'(t,T)\left[r(t) - f(0,t)\right] + \frac{1}{2}\pi'^2(t,T)\psi(t)$$

where :

$$\psi(t) = \int_0^t \sigma^2(s,t)ds \tag{34}$$

Finally we have :

$$P(t,T) = \exp - \left[\pi'(t,T)\left[r(t) - f(0,t)\right] + \frac{1}{2}\pi'^2(t,T)\psi(t)\right] \tag{35}$$

Consequently, the model specified by (4.1) and (4.2) is affine model, since the price of zero-coupon bond is an exponential affine function of the three states variables.

5.3 Mercurio and Moraleda (1996) model:

The final example considered is the Mercurio and Moraleda (1996) model where the volatility of the forward rate is expressed as follow:

$$\sigma(t,T) = \frac{1+\gamma T}{1+\gamma t} r^{\gamma}(t)\ e^{-\lambda(T-t)} \tag{36}$$

this volatility structure imply that :

$\sigma(r(t), f(t,\tau)) = r^{\gamma}(t)$ and $\alpha(t,T) = \frac{1+\gamma T}{1+\gamma t}$, under this form $\alpha(t, T)$ verify our propriety :

$\alpha(t, T) = \alpha(t, u)\times\alpha(u, T)$.

the forward interest rate stochastic integral will be given by :

$$f(t,T) = f(0,T) + \int_0^t \sigma(\alpha(u,T), r(u))e^{-\lambda(T-u)}\int_u^T \sigma(\alpha(u,s), r(u))e^{-\lambda(s-u)}ds du + \int_0^t \sigma(\alpha(u,T), r(u))e^{-\lambda(T-u)}d\widetilde{W}(u)$$
$$= f(0,T) + \int_0^t r^{\gamma}(u)\frac{1+\gamma T}{1+\gamma u}e^{-\lambda(T-u)}\int_u^T r^{\gamma}(u)\frac{1+\gamma s}{1+\gamma u}e^{-\lambda(s-u)}ds du + \int_0^t \sigma(\alpha(u,T), r(u))e^{-\lambda(T-u)}d\widetilde{W}(u)$$

$$= f(0,T) + \int_0^t r^{2\gamma}(u)\frac{1+\gamma T}{\kappa^2(1+\gamma u)^2}(\lambda+\gamma+\lambda\gamma u)e^{-\lambda(T-u)} - (\lambda+\gamma+T\gamma\lambda)e^{-2\gamma(T-u)}du \quad + \int_0^t \sigma(\alpha(u,T), r(u))e^{-\lambda(T-u)}d\tilde{W}(u)$$

(36)

we show that the dynamic of the forward rate will be expressed as:

$$df(t,T) = \left[\frac{r^{2\gamma}(t)}{\lambda^2}\frac{1+\gamma T}{(1+\gamma t)^2}e^{-\lambda(T-t)}\left[(\lambda+\gamma+\gamma\lambda t) - (\lambda+\gamma T\lambda+\gamma)e^{-\lambda(T-t)}\right]\right]dt + \sigma_f d\tilde{W}(t) \quad (37)$$

<u>Proof :</u>

To take the difference of the first integral term in equation (100)

$$\frac{d}{dt}\int_0^t r^{\gamma}(u)\alpha(u,T)e^{-\lambda(T-u)}\int_u^T r^{\gamma}(u)\alpha(u,s)e^{-\lambda(s-u)}dsdu \quad = r^{\gamma}(t)\alpha(t,T)e^{-\lambda(T-t)}\int_t^T r^{\gamma}(t)\alpha(t,s)e^{-\lambda(s-t)}ds$$

$$= r^{2\gamma}(t)\alpha(t,T)e^{-\lambda(T-t)}\int_t^T \alpha(t,s)e^{-\lambda(s-t)}ds$$

$$= \frac{r^{2\gamma}(t)}{\lambda^2}\frac{1+\gamma T}{(1+\gamma t)^2}e^{-\lambda(T-t)}\left[(\lambda+\gamma+\gamma\lambda t) - (\lambda+\gamma T\lambda+\gamma)e^{-\lambda(T-t)}\right]$$

the differential of the stochastic integral term is simply:

$$d\int_0^t \sigma(\alpha(u,T), r(u))e^{-\lambda(T-u)}d\tilde{W}(u) \; = \; \sigma(\alpha(t,T), r(t))e^{-\lambda(T-t)}d\tilde{W}(t)$$

we deduct that : $df(t,T) = \mu_f dt + \sigma_f d\tilde{W}(t)$

where $\mu_f = \frac{r^{2\gamma}(t)}{\lambda^2}\frac{1+\gamma T}{(1+\gamma t)^2}e^{-\lambda(T-t)}\left[(k+\gamma+\gamma\kappa t) - (\lambda+\gamma T\lambda+\gamma)e^{-\lambda(T-t)}\right]$

the same procedure we permit to demonstrate that the differential equation of the instantaneous spot rate will be given by:

$$dr(t) = \left[f_2(0,t) + K(t)\left(r(t) - f(0,t)\right) + \psi(t)\right]dt + \sigma(t,t)d\tilde{W}(t) \quad (38)$$

where : $\psi(t) = \int_0^t \sigma(\alpha(u,t), r(u))^2 du$

and $K(t) = \lambda\left[\lambda - \frac{2\gamma}{1+\gamma t}\right]$

See **Appendix D**.

this equation show that the dynamic of the spot rate is an affine function of three states variables; r (t), f(0,t) et Ψ (t).

the term structure of Interest rate :

Finally, the formula for the bond price can now be obtained in terms of the state variables.

$$P(t,T)=\frac{P(0,T)}{P(0,t)}e^{-R(t,T)} \qquad (39)$$

Where : $R(t,T)=\pi(t,T)[r(t)-f(0,t)]+\delta(t,T)\psi(t)$ (40)

See **Appendix E**.

This result we show that bond price to be expressed as an exponential affine function of the three state variables considered. This formula show further that in a HJM framework, the Mercurio et Moraleda model is a Markovian affine term structure model.

6. Conclusion:

A fairly broad class of forward rate volatility process within the HJM framework has been considered in this paper. The volatility structure described will allow the construction of term structure models with volatility processes that depend on the instantaneous spot interest rate and a forward rate of for fixed maturity and a function $\alpha(t, T)$ which depends on the date, t, and the time to maturity, T. it is shown how the stochastic dynamics of the resulting system can be reduced to a Markovian affine model under three state variables, the instantaneous spot interest rate, the forward rate, and a subsidiary variable $\Psi(t)$ that resumes the forward volatility history. This transformation is insured by a restriction imposed on the function $\alpha(t,T)$. one fundamental propriety of our model that it establish an explicit formula for the bond price in terms of the state variables which generalises the results of Ritchken and Sankarasubramanian (1995) and Bhar, Chiarella, Al-Hassan and Zheng (2000). Finally, we have applied theses results to some examples appeared in the literature, by considering the extended Vasicek (1977), the extended CIR(1985) and the Mercurio and Morelda (1996) models. We demonstrate in this framework that those model are Markovian affine term structure models.

The results obtained in this paper are of significant value in implementing the models in practice, and the research into the practical implementation and evaluation of these models remains an on-going project.

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Appendix A

We demonstrate that:

$$df(t,T) = \left[g(r(t), f(t,\tau))^2\ \alpha(t,T)\ e^{-\lambda(T-t)}\ B(t,T)\ \right]dt + \sigma(\alpha(t,T), r(t), f(t,\tau))d\tilde{W}(t) \quad (15)$$

Where $B(t,T) = \int_t^T \alpha(t,s)e^{-\lambda(s-t)}ds$

Proof :

We begin by evaluate,

$$\frac{\partial}{\partial t}\int_0^t \sigma(\alpha(u,t), r(u), f(u,\tau),.)e^{-\lambda(T-u)}\int_u^t \sigma(\alpha(u,s), r(u), f(u,\tau),.)e^{-\lambda(s-u)}dsdu$$

$$= \sigma(\alpha(t,T), r(t), f(t,\tau),.)e^{-\lambda(T-t)}\int_t^T \sigma(\alpha(t,s), r(t), f(t,\tau),.)e^{-\lambda(s-t)}ds$$

$$= \sigma(\alpha(t,T), r(t), f(t,\tau),.)e^{-\lambda(T-t)}g(r(t), f(t,\tau))\int_t^T \alpha(t,s)\ e^{-\lambda(s-u)}ds$$

$$= g(\ r(t), f(t,\tau))^2\ \alpha(t,T)e^{-\lambda(T-t)}B(t,T)$$

Where : $B(t,T) = \int_t^T \alpha(t,s)\ e^{-\lambda(s-t)}ds$

Then the differential stochastic equation will be :

$$d\int_0^t \sigma(\alpha(u,T), r(u), f(u,\tau))\ e^{-\lambda(T-u)}d\tilde{W}(t) = \sigma(\alpha(t,T), r(t), f(t,\tau))\ e^{-\lambda(T-t)}d\tilde{W}(t)$$

finally we obtain:

$$df(t,T) = \left[g(r(t), f(t,\tau))^2\ \alpha(t,T)\ e^{-\lambda(T-t)}B(t,T)\ \right]dt + \sigma(\alpha(t,T), r(t), f(t,\tau))d\tilde{W}(t)$$

Appendix B

the price of zero-coupon bond is given by:

$$P(t,T) = \frac{P(0,T)}{P(0,t)}e^{-R(t,T)}$$

Where : $R(t,T) = \pi(t,T)\left[\int_0^t \tilde{\sigma}^*(s,t,.)\ ds + \int_0^t \tilde{\sigma}(s,t,.)\ d\tilde{W}(s)\ \right] + \ \delta(t,T,.)\int_0^t \tilde{\sigma}^2(s,t,.)ds$

Where : $\tilde{\sigma} = \frac{1}{\alpha(s,t)}\sigma(s,t,.)$, $\quad \tilde{\sigma}^* = \frac{1}{\alpha(s,t)}\sigma^*(s,t,.)$

and $\pi(t,T) = \int_t^T \alpha(t,u)e^{-\lambda(u-t)}du$

and $\delta(t,T)=\int_t^T \alpha(t,u)e^{-\lambda(u-t)}\int_t^u \alpha(t,y)e^{-\lambda(y-t)}dydu$

<u>Proof :</u>

By definition of P(t,T) and (3), we can write : : $P(t,T)=\exp\left(-\int_t^T f(t,s)ds\right)$

$$P(t,T)=\frac{P(0,T)}{P(0,t)}\exp\left(-\int_t^T\int_0^t \sigma^*(s,u,.)dsdu-\int_t^T\int_0^t \sigma(s,u,.)dW(s)du\right)$$

Where : $\sigma(s,T,.)=\sigma(r(s),f(s,\tau))\alpha(s,T)e^{-\lambda(T-s)}$

$$\sigma^*(s,T,.)=\sigma(s,T,.)\int_s^T \sigma(s,u,.)du$$

$$=\sigma(r(s),f(s,\tau))^2\alpha(s,T)e^{-\lambda(T-s)}\int_s^T \alpha(s,u,.)e^{-\lambda(u-s)}du$$

thus, $P(t,T)=\frac{P(0,T)}{P(0,t)}\exp(-R(t,T))$

Where :
$$R(t,T)=\int_t^T\int_0^t \sigma^*(s,u,.)dsdu+\int_t^T\int_0^t \sigma(s,u,.)d\tilde{W}(s)du$$

$$=\int_0^t\int_t^T \sigma^*(s,u,.)duds+\int_0^t\int_t^T \sigma(s,u,.)dud\tilde{W}(s)$$

$$=\int_0^t A(s,T)ds+\int_0^t B(s,T)d\tilde{W}(s)$$

firstly,

$$A(s,T)=\int_t^T \sigma^*(s,u)du=\int_t^T \sigma(s,u,.)\int_s^T \alpha(s,u\,.)duds$$

$$=\sigma(r(s),f(s,\tau,.))\int_t^T \alpha(s,u)du\ \left\{\int_s^t \alpha(s,y,.)e^{-\lambda(y-s)}\sigma(r(s),f(s,\tau))dy+\int_t^u \sigma(r(s),f(s,\tau))dy\right\}du$$

$$=\sigma(r(s),f(s,\tau,.))\int_t^T \alpha(s,u)du\ \left\{\int_s^t \alpha(s,y,.)e^{-\lambda(y-s)}\sigma(r(s),f(s,\tau))dy+\int_t^u \sigma(r(s),f(s,\tau))dy\right\}du$$

$$=\sigma(r(s),f(s,\tau,.))^2\int_t^T \alpha(s,u)e^{-\lambda(u-s)}du\ \int_s^t \alpha(s,y,.)e^{-\lambda(y-s)}dy+$$
$$\sigma(r(s),f(s,\tau))^2\int_t^u \alpha(s,u)e^{-\lambda(u-s)}\int_t^u \alpha(s,y,.)e^{-\lambda(y-s)}dydu$$

$$= \sigma(r(s), f(s,\tau,.))^2 e^{-\lambda(t-s)} \int_t^T \alpha(s,u) e^{-\lambda(u-t)} du \int_s^t \alpha(s,y,.) e^{-\lambda(y-s)} dy +$$

$$\sigma(r(s), f(s,\tau))^2 e^{-2\lambda(t-s)} \int_t^u \alpha(s,u) e^{-\lambda(u-t)} \int_t^u \alpha(s,y,.) e^{-\lambda(y-t)} dydu$$

$$= \frac{1}{\alpha(s,t)} \sigma^*(s,t)\, \pi(t,T) + \frac{1}{\alpha(s,t)^2} \sigma^2(s,t,.) \delta(t,T,.)$$

Where : $\pi(t,T) = \int_t^T \alpha(t,u) e^{-\lambda(u-t)} du$

And : $\delta(t,T) = \int_t^T \alpha(t,u) e^{-\lambda(u-t)} \int_t^u \alpha(t,y) e^{-\lambda(y-t)} dydu$

Secondly :

$$B(s,u) = \int_t^T \sigma(s,u,.) du$$

$$= \int_t^T \sigma(r(s), f(s,\tau), \alpha(s,u)) e^{-\lambda(u-s)} du$$

$$= \sigma(r(s), f(s,\tau,.)) e^{-\lambda(t-s)} \int_t^T \alpha(s,u) e^{-\lambda(u-s)} du$$

$$= \frac{1}{\alpha(s,t)} \sigma(s,t) \int_t^T \alpha(s,u) e^{-\lambda(u-s)} du$$

$$= \frac{1}{\alpha(s,t)} \sigma(s,t)\, \pi(t,T)$$

then we can write:

$$R(t,T) = \int_0^t A(s,T) ds + \int_0^t B(s,T) d\tilde{W}(s)$$

$$= \int_0^t [\frac{1}{\alpha(s,t)} \sigma^*(s,t)\, \pi(t,T) + \frac{1}{\alpha(s,t)^2} \sigma^2(s,t,.) \delta(t,T,.)] ds + \int_0^t \frac{1}{\alpha(s,t)} \sigma(s,t)\, \pi(t,T) d\tilde{W}(s)$$

$$= \pi(t,T) \left[\int_0^t \frac{1}{\alpha(s,t)} \sigma^*(s,t)\, ds + \int_0^t \frac{1}{\alpha(s,t)} \sigma(s,t)\, d\tilde{W}(s) \right] + \delta(t,T,.) \int_0^t \frac{1}{\alpha(s,t)^2} \sigma^2(s,t,.) ds$$

We consider that:

$\tilde{\sigma} = \frac{1}{\alpha(s,t)} \sigma(s,t,.)$ and $\tilde{\sigma}^* = \frac{1}{\alpha(s,t)} \sigma^*(s,t,.)$

Finally, we can conclude that :

$$R(t,T)=\pi(t,T)\left[\int_0^t \tilde{\sigma}^*(s,t,.)\,ds+\int_0^t \tilde{\sigma}(s,t,.)\,d\tilde{W}(s)\right]+\ \delta(t,T,.)\int_0^t \tilde{\sigma}^2(s,t,.)ds$$

Appendix C

We show that:

$$P(t,T)=\exp\{-\pi(t,T)[f(0,\ t)-r(t)\ ]-\delta(t,T)\psi(t)\ \}$$

Proof :

By definition of P(t, T) and (3), we can write : : $P(t,T)=\exp\left(-\int_t^T f(t,s)ds\right)$

$$P(t,T)=\frac{P(0,T)}{P(0,t)}\exp\left(-\int_t^T\int_0^t \sigma^*(s,u,.)dsdu-\int_t^T\int_0^t \sigma(s,u,.)dW(s)du\right)$$

Where : $\sigma(s,T,.)=\sigma(r(s),f(s,\tau))\alpha(s,T)e^{-\lambda(T-s)}$

$$\sigma^*(s,T,.)=\sigma(s,T,.)\int_s^T \sigma(s,u,.)du$$

$$=\sigma(r(s),f(s,\tau))^2\alpha(s,T)e^{-\lambda(T-s)}\int_s^T \alpha(s,u,.)e^{-\lambda(u-s)}du$$

thus, $P(t,T)=\dfrac{P(0,T)}{P(0,t)}\exp(-R(t,T))$

Where :

$$R(t,T)=\int_t^T\int_0^t \sigma^*(s,u,.)dsdu+\int_t^T\int_0^t \sigma(s,u,.)d\tilde{W}(s)du$$

$$=\int_0^t\int_t^T \sigma^*(s,u,.)duds+\int_0^t\int_t^T \sigma(s,u,.)dud\tilde{W}(s)$$

$$=\int_0^t A(s,T)ds+\int_0^t B(s,T)d\tilde{W}(s)$$

firstly,

we have : $A(s,T)=\displaystyle\int_t^T \sigma^*(s,u)du=\int_t^T \sigma(s,u,.)\int_s^u \alpha(s,u\ .)duds$

$$=\sigma(r(s),f(s,\tau,.))\int_t^T \alpha(s,u)e^{-\lambda(u-s)}du\left\{\int_s^t \alpha(s,y,.)e^{-\lambda(y-s)}\sigma(r(s),f(s,\tau))dy+\int_t^u \sigma(r(s),f(s,\tau))dy\right\}du$$

$$=\sigma(r(s),f(s,\tau,.))^2\int_t^T \alpha(s,u)e^{-\lambda(u-s)}du\int_s^t \alpha(s,y,.)e^{-\lambda(y-s)}dy+\sigma(r(s),f(s,\tau))^2\int_t^T \alpha(s,u)e^{-\lambda(u-s)}\int_t^u \alpha(s,y,.)e^{-\lambda(y-s)}dydu$$

suppose that *Proposition (1)* is satisfied. Then we can write:

$$A(s,T) = \sigma(r(s), f(s,\tau,.))^2 e^{-\lambda(t-s)} \int_t^T \alpha(s,u)e^{-\lambda(u-t)}du \int_s^t \alpha(s,y,.)e^{-\lambda(y-s)}dy +$$

$$\sigma(r(s), f(s,\tau))^2 \alpha(s,t)^2 e^{-2\lambda(t-s)} \int_t^T \alpha(t,u)e^{-\lambda(u-t)} \int_t^u \alpha(t,y,.)e^{-\lambda(y-t)}dydu$$

$$= \pi(t,T)\,\sigma^*(s,t,.) + \delta(t,T,.)\,\sigma^2(s,t,.)$$

Where :

$$\pi(t,T) = \int_t^T \alpha(t,u)e^{-\lambda(u-t)}du$$

and $\delta(t,T) = \int_t^T \alpha(t,u)e^{-\lambda(u-t)} \int_t^u \alpha(t,y)e^{-\lambda(y-t)}dydu$

in the other hand :

$$B(s,u) = \int_t^T \sigma(s,u,.)du$$

$$= \int_t^T \sigma(r(s), f(s,\tau), \alpha(s,u))e^{-\lambda(u-s)}du$$

$$= \sigma(r(s), f(s,\tau,.))\,\alpha(s,t)\,e^{-\lambda(t-s)} \int_t^T \alpha(t,u)e^{-\lambda(u-s)}du$$

$$= \sigma(s,t)\int_t^T \alpha(t,u)e^{-\lambda(u-t)}du$$

$$= \sigma(s,t)\,\pi(t,T)$$

Thus :

$$R(t,T) = \int_0^t A(s,T)ds + \int_0^t B(s,T)d\tilde{W}(s)$$

$$= \int_0^t \sigma^*(s,t,.)\,\pi(t,T)ds + \int_0^t \sigma^2(s,t,.)\delta(t,T,.)ds + \int_0^t \sigma(s,t)\,\pi(t,T)d\tilde{W}(s)$$

$$= \pi(t,T)\left[\int_0^t \sigma^*(s,t,.)\,ds + \int_0^t \sigma(s,t)\ d\tilde{W}(s)\right] + \delta(t,T)\int_0^t \sigma^2(s,t,.)\,ds$$

or the stochastic integral equation of the spot rate (2) we permit to deduct that :

$$r(t) - f(0,t) = \int_0^t \sigma^*(s,t,.)\,ds + \int_0^t \sigma(s,t)\ d\tilde{W}(s)$$

by substituting this quantity in the equation of R(t, T), and by setting :

$$\psi(t) = \int_0^t \sigma(s,t,.)^2 du$$

we deduct that :

$$R(t,T) = \pi(t,T)[r(t) - f(0,t)] + \delta(t,T)\psi(t)$$

thus, $P(t,T)=\exp\left\{-\pi(t,T)[f(0,\ t)\ \text{-}\ r(t)\ ]-\delta(t,T)\psi(t)\ \right\}$

Appendix D

We demonstrate in the case of Mercurio and Merolda (1996) model that:

$$dr(t)=\left[f_2(0,t)+K(t)\left(r(t)\ \text{-}\ f(0,t)\ \right)+\psi(t)\right]dt+\sigma(t,t)d\tilde{W}(t)$$

where : $\psi(t)=\int_0^t \sigma(\alpha(u,t),\ r(u))^2 du$

and $K(t)=\lambda\left[\lambda\text{-}\frac{2\gamma}{1+\gamma t}\right]$

<u>Proof :</u>

$$dr(t)\ =\left[\ f_2\ (0,\ t)+\frac{\partial}{\partial t}\int_0^t \sigma(u,\ t,.)\int_u^t \sigma(u,s,.)dsdu\ +\frac{\partial}{\partial t}\int_0^t \sigma(u,\ t,.)d\tilde{W}(u)\right]dt$$

firstly : $\frac{\partial}{\partial t}\int_0^t \sigma(u,\ t,.)\int_u^t \sigma(u,s,.)dsdu\ =\int_0^t\left[\ \sigma_2(u,t,r(u))\int_u^t \sigma(u,s,r(u))ds+\sigma(u,t,r(u))^2\ \right]du$

$$\sigma_2(u,t,r(u))=\lambda r^{\gamma}(u)e^{-\lambda(t-u)}[\lambda\alpha(u,t)-2\frac{\gamma}{1+\gamma u}]$$

$$=\frac{\gamma}{1+\gamma u}r^{\gamma}(u)e^{-\lambda(t-u)}[\lambda(1+\gamma t)-2]$$

$$=\int_0^t\left[\ \sigma_2(u,t,r(u))\int_u^t \sigma(u,s,r(u))ds+\sigma(u,t,r(u))^2\ \right]du$$

secondly :

$$d\int_0^t \sigma(u,\ t,.)d\tilde{W}(u)=\left[\int_0^t \sigma_2\ (u,\ t,.)d\tilde{W}(u)\right]dt+\sigma(t,T)d\tilde{W}(t)$$

finally we note that :

$$dr(t)\ =\left[\ f_2(0,\ t)+\int_0^t[\sigma_2(u,\ t,.)\int_u^t \sigma(u,s,.)ds\ \ +\sigma^2(u,t)]du+\sigma_2(u,t)d\tilde{W}(u)\ \right]dt+\ +\sigma(t,t)d\tilde{W}(t)$$

or $\int_0^t[\sigma_2(u,\ t,.)\int_u^t \sigma(u,s,.)ds\ \ +\sigma^2(u,t)]du+\sigma_2(u,t)d\tilde{W}(u)=$

$$\int_0^t \lambda r^{\gamma}(u)[\lambda\alpha(u,t)\ \text{-}\ 2\frac{\gamma}{1+\gamma u}]e^{-\lambda(t-u)}\int_u^t \sigma(u,s,.)dsdu\ \ +\int_0^t\ \lambda r^{\gamma}(u)[\lambda\alpha(u,t)-2\frac{\gamma}{1+\gamma u}]e^{-\lambda(t-u)}d\tilde{W}(u)$$

$$= \int_0^t \lambda r^\gamma(u)\, \alpha(u,t)[\lambda - 2\frac{\gamma}{1+\gamma t}]e^{-\lambda(t-u)} \int_u^t \sigma(u,s,.)dsdu \;+ \int_0^t \lambda r^\gamma(u)\, \alpha(u,t)[\lambda - 2\frac{\gamma}{1+\gamma t}]e^{-\lambda(t-u)} d\tilde{W}(u)$$

$$= \lambda\left[\lambda - \frac{2\gamma}{1+\gamma t}\right]\int_0^t r^\lambda(u)\, \alpha(u,t)e^{-\lambda(t-u)}\int_u^t \alpha(u,s,.)e^{-\lambda(s-u)}dsdu \;+ \lambda\left[\lambda - \frac{2\gamma}{1+\gamma t}\right]\int_0^t r^\lambda(u)\, \alpha(u,t)\, e^{-\lambda(t-u)}d\tilde{W}(u)$$

$$= K(t)\left[\int_0^t r^\gamma(u)\, \alpha(u,t)e^{-\lambda(t-u)}\int_u^t \alpha(u,s,.)e^{-\lambda(s-u)}dsdu \;+ \int_0^t r^\gamma(u)\, \alpha(u,t)\, e^{-\lambda(t-u)}d\tilde{W}(u)\right]$$

$$= K(t)\left[\int_0^t \sigma^*(u,t)du \;+ \int_0^t \sigma(u,t)d\tilde{W}(u)\right]$$

$$= K(t)\,[r(t) - f(0,t)]$$

where, $K(t) = \lambda\left[\lambda - \frac{2\gamma}{1+\gamma t}\right]$

thus : $dr(t) = \left[f_2(0,t) + K(t)\big(r(t) - f(0,t)\big) + \psi(t)\right]dt + \sigma(t,t)d\tilde{W}(t)$

and we have demonstrate that :

$$d\psi(t) = [\sigma(\alpha(t,t), r(t))^2 - 2\lambda\psi(t)]dt$$

Appendix E

We deduct in the case of Mercurio and Merolda (1996) model tht zero coupon bond at time t with maturity T can be expressed by:

$$P(t,T) = \frac{P(0,T)}{P(0,t)}e^{-R(t,T)}$$

Where : $R(t,T) = \pi(t,T)[r(t) - f(0,t)] + \delta(t,T)\psi(t)$

<u>Proof :</u>

Using the relationship : $P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$

And equation (8) for the forward f(t,s) we obtain for the bond price the expression:

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left(-\int_t^T\int_0^t \sigma^*(s,u,.)dsdu - \int_t^T\int_0^t \sigma(s,u,.)dW(s)du\right)$$

we set : $\sigma(s,T,.) = r^\lambda(t)\,\alpha(s,T)e^{-\lambda(T-s)}$

$$\sigma^*(s,T,.) = \sigma(s,T,.)\int_s^T \sigma(s,u,.)du$$

$$= r^{2\lambda}(s)\alpha(s,T)e^{-\lambda(T-s)}\int_s^T \alpha(s,u,.)e^{-\lambda(u-s)}du$$

$$P(t,T)=\frac{P(0,T)}{P(0,t)}\exp\left(-R(t,T)\right)$$

Where : $R(t,T)=\int_t^T\int_0^t \sigma^*(s,u,.)dsdu+\int_t^T\int_0^t \sigma(s,u,.)d\widetilde{W}(s)du$

$$=\int_0^t\int_t^T \sigma^*(s,u,.)duds+\int_0^t\int_t^T \sigma(s,u,.)dud\widetilde{W}(s)$$

$$=\int_0^t A(s,T)ds+\int_0^t B(s,T)d\widetilde{W}(s)$$

$$A(s,T)=\int_t^T \sigma^*(s,u)du=\int_t^T \sigma(s,u,.)\int_s^T \alpha(s,u\ .)duds$$

$$=r^{2\lambda}(s)\alpha(s,t)e^{-\lambda(T-s)}\int_t^T \alpha(t,u)e^{-\lambda(u-t)}\times$$

$$\left\{\int_s^t \alpha(s,y,.)e^{-\lambda(y-s)}dy+r^{2\lambda}(s)\alpha^2(s,t)e^{-2\lambda(T-s)}\int_t^u \alpha(t,u)e^{-\lambda(u-t)}\int_t^u \alpha(t,y)e^{-\lambda(y-t)}dydu\right\}du$$

firstly :

$A(s,T)=\sigma^*(s,t,.)\ \pi(t,T)+\sigma^2(s,t,.)\delta(t,T)$

where : $\pi(t,T)=\int_t^T \alpha(t,u)e^{-\lambda(u-t)}du$

$$=\frac{1}{\lambda^2}\frac{1}{(1+\gamma t)}e^{-\lambda(T-t)}\left[(\gamma+\lambda\gamma t)-(\lambda+\gamma T\lambda+\gamma)e^{-\lambda(T-t)}\right]$$

and $\delta(t,T)=\int_t^T \alpha(t,u)e^{-\lambda(u-t)}\int_t^u \alpha(t,y)e^{-\lambda(y-t)}dydu$

$$=\frac{\lambda^2+\gamma^2+2\gamma\lambda+2\gamma^2t\lambda+2\gamma t\lambda^2+\lambda^2t^2\gamma^2}{2\lambda^4(1+\gamma t)^2}+\frac{1}{\lambda^4(1+\gamma t)^2}\Big[\frac{1}{2}e^{t\lambda}\gamma^2T^2\lambda^2+\gamma^2T\lambda e^{t\lambda}+T\gamma\lambda^2e^{t\lambda}-\gamma\lambda^2Te^{T\lambda}+$$

$$\gamma\lambda e^{t\lambda}-\lambda^2e^{T\lambda}-2\gamma\lambda e^{T\lambda}-\gamma t\lambda^2e^{T\lambda}-\gamma^2t\lambda e^{T\lambda}+\frac{1}{2}\lambda^2e^{t\lambda}+\frac{1}{2}\gamma^2e^{t\lambda}-\gamma^2e^{T\lambda}-\gamma^2T\lambda e^{T\lambda}-\gamma^2t\lambda^2Te^{T\lambda}\Big]$$

secondly :

$$B(s,u)=\int_t^T \sigma(s,u,.)du$$

$$=\int_t^T r^{\lambda}(s)\ \alpha(s,u)e^{-\lambda(u-s)}du$$

$$= r^{\lambda}(s)\,\alpha(s,t)e^{-\lambda(t-s)}\int_t^T \alpha(t,u)e^{-\lambda(u-t)}du$$

$$= \sigma(s,t)\int_t^T \alpha(t,u)e^{-\lambda(u-t)}du$$

$$= \sigma(s,t,.)\ \pi(t,T)$$

finally we obtain:

$$R(t,T) = \int_0^t A(s,T)ds + \int_0^t B(s,T)d\tilde{W}(s)$$

$$= \int_0^t \sigma^*(s,t,.)\,\pi(t,T)ds + \int_0^t \sigma^2(s,t,.)\delta(t,T,.)ds + \int_0^t \sigma(s,t)\,\pi(t,T)d\tilde{W}(s)$$

$$= \pi(t,T)\left[\int_0^t \sigma^*(s,t,.)\,ds + \int_0^t \sigma(s,t)\ d\tilde{W}(s)\right] + \delta(t,T)\int_0^t \sigma^2(s,t,.)\,ds$$

or from equation (2) of the spot interest rate we can deduce that :

$$r(t) - f(0,t) = \int_0^t \sigma^*(s,t,.)\,ds + \int_0^t \sigma(s,t)\ d\tilde{W}(s)$$

by subtituting this quantity in the equation of R(t, T) and by setting :

$$\psi(t) = \int_0^t \sigma(s,t,.)^2 du$$

we obtain:

$$R(t,T) = \pi(t,T)[r(t) - f(0,t)] + \delta(t,T)\psi(t)$$

then : $P(t,T) = \exp\{-\pi(t,T)[f(0,t) - r(t)] - \delta(t,T)\psi(t)\}$