Viscosity solution of parabolic equation

Question:

Consider the parabolic equation

$$\begin{cases} \partial_t v + \partial_x v + \frac{1}{2}\partial_{xx} v - cv + f = 0, \quad \text{on } [0,T) \times \mathbb{T} \\ v(T,x) = 0, \quad \text{on } x \in \mathbb{T} \end{cases} \tag{1}$$

Prove that if $c, f \in C([0,T) \times \mathbb{T})$, there exists a viscosity solution $v \in C([0,T) \times \mathbb{T})$. Moreover, the $v(t,x)$ has a probability representation of

$$v(t,x) = \mathbb{E}\Big[\int_t^T \exp\Big\{-\int_t^s c(r, X^{t,x}(r))\, dr\Big\} f(s, X^{t,x}(s))\, ds\Big] \tag{2}$$

where

$$X^{t,x}(s) = x + (s-t) + W(s) - W(t) \tag{3}$$

for some Brownian motion W.

Solution:

Firstly, we show that the value function (2) is a viscosity supersolution of the equation

$$-\partial_t v - (\partial_x v + \frac{1}{2}\partial_{xx} v - cv + f) = 0$$

on $[0,T) \times \mathbb{T}$. As $c, f \in C([0,T) \times \mathbb{T})$, we have $|u|_0 \le e^{|c|_0 T}|f|_0 T$, then v is bounded on $[0,T) \times \mathbb{T}$. Let $(\bar{t}, \bar{x}) \in [0,T) \times \mathbb{T}$ and let $\varphi \in C^2([0,T) \times \mathbb{T})$ be a test function such that

$$0 = (v_* - \varphi)(\bar{t}, \bar{x}) = \min_{(t,x) \in [0,T) \times \mathbb{T}} (v_* - \varphi)(t,x), \tag{4}$$

where $v_*(t,x)$ is the lower-semicontinuous envelope of $v(t,x)$. By the definition of $v_*(\bar{t}, \bar{x})$, there exists a sequence $\{(t_m, x_m)\}$ in $[0,T) \times \mathbb{T}$ such that

$$(t_m, x_m) \to (\bar{t}, \bar{x}) \text{ and } v(t_m, x_m) \to v_*(\bar{t}, \bar{x})$$

when m goes to infinity. By the continuity of φ, when $m \to \infty$, we have

$$\eta_m = v(t_m, x_m) - \varphi(t_m, x_m) \to 0.$$

We denote by $X_s^{t_m, x_m}$ the associated process with the initial data $X_{t_m} = x_m$. Let θ_m be the stopping time given by

$$\theta_m = \inf\{s \ge t_m : (s - t_m, X_s^{t_m, x_m} - x_m) \notin [0, h_m) \times \alpha B\} \tag{5}$$

where $\alpha > 0$ is some given constant, B denotes the unit ball of $\mathbb{T}$, and

$$h_m = \sqrt{\eta_m}\, \mathbb{I}_{\{\eta_m \neq 0\}} + m^{-1}\mathbb{I}_{\{\eta_m = 0\}}.$$

Then we have $\theta_m \to \bar{t}$ as $m \to \infty$ since h_m converges to 0. Applying the dynamic programming principle for $v(t_m, x_m)$ to θ_m and get

$$v(t_m, x_m) \geq \mathbb{E}\Big[\int_{t_m}^{\theta_m} \beta(t_m, s) f(s, X_s^{t_m, x_m})\, ds + \beta(t_m, \theta_m) v(\theta_m, X_{\theta_m}^{t_m, x_m})\Big],$$

where

$$\beta(t_m, s) = \exp\Big\{ -\int_{t_m}^{s} c(r, X^{t_m, x_m}(r))\, dr \Big\}.$$

Since $v \geq v_* \geq \varphi$, then

$$\varphi(t_m, x_m) + \eta_m \geq \mathbb{E}\Big[\int_{t_m}^{\theta_m} \beta(t_m, s) f(s, X_s^{t_m, x_m})\, ds + \beta(t_m, \theta_m) \varphi(\theta_m, X_{\theta_m}^{t_m, x_m})\Big].$$

Applying the Itô's formula to the smooth test function φ, and since $\beta(t_m, s) D\varphi(s, X_s^{t_m, x_m})$ is bounded on the interval $[t_m, \theta_m]$, then we have

$$\frac{\eta_m}{h_m} + \mathbb{E}\Big[\frac{1}{h_m}\int_{t_m}^{\theta_m} \beta(t_m, s)\big(-\frac{\partial \varphi}{\partial t} - L\varphi - f\big)(s, X_s^{t_m, x_m})\, ds\Big] \geq 0,$$

where $L\varphi = \frac{\partial \varphi}{\partial x} + \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2} - c\varphi$. By a.s. continuity of the trajectory $X_s^{t_m, x_m}$, it follows that for m sufficient large, $\theta_m(\omega) = t_m + h_m$ a.s. Thus by mean value theorem, the random variable inside the expectation converges to

$$-\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) - L\varphi(\bar{t}, \bar{x}) - f(\bar{t}, \bar{x})$$

when $m \to \infty$. Moreover, this random variable is bounded by a constant independent of m. We then obtain

$$-\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) - \frac{\partial \varphi}{\partial x}(\bar{t}, \bar{x}) - \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2}(\bar{t}, \bar{x}) + c\varphi(\bar{t}, \bar{x}) - f(\bar{t}, \bar{x}) \geq 0 \tag{6}$$

when m goes to infinity by dominate convergence theorem.

Next we show that $v(t, x)$ is a viscosity subsolution of the equation

$$-\partial_t v - (\partial_x v + \frac{1}{2}\partial_{xx} v - cv + f) = 0$$

on $[0, T) \times \mathbb{T}$. Let $(\bar{t}, \bar{x}) \in [0, T) \times \mathbb{T}$ and $\varphi \in C^2([0, T) \times \mathbb{T})$ be such that

$$0 = (v^* - \varphi)(\bar{t}, \bar{x}) > (v^* - \varphi)(t, x) \tag{7}$$

for $(t, x) \in [0, T) \times \mathbb{T}$. In order to prove the required result, we assume that

$$h(\bar{t}, \bar{x}) = \partial_t \varphi(\bar{t}, \bar{x}) + (\partial_x \varphi + \frac{1}{2}\partial_{xx}\varphi - c\varphi + f)(\bar{t}, \bar{x}) < 0.$$

We denote $H(t, x, r, p, M) = p(t, x) + \frac{1}{2}M(t, x) + f(t, x) - c(t, x) r(t, x)$, then we can rewrite the above assumption as

$$h(\bar{t}, \bar{x}) = \partial_t \varphi(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, \varphi(\bar{t}, \bar{x}), \partial_x \varphi(\bar{t}, \bar{x}), \partial_{xx}\varphi(\bar{t}, \bar{x})) < 0.$$

Since H is continuous, then there exists an open neighborhood $\mathcal{N}_r = (\bar{t}-r, \bar{t}+r) \times rB(\bar{x})$ of $(\bar{t}, \bar{x})$, for some $r > 0$ such that

$$h = \partial_t \varphi + H(t, x, \varphi, \partial_x \varphi, \partial_{xx} \varphi) < 0$$

on $\mathcal{N}_r$. By the definition of $\mathcal{N}_r$ and (7), for a constant $\eta > 0$, we have

$$-2\eta e^{r|c|_0} := \max_{\partial \mathcal{N}_\eta} (v^* - \varphi) < 0. \tag{8}$$

Let $\{(t_n, x_n)\}$ be a sequence in $\mathcal{N}_r$ such that

$$(t_n, x_n) \to (\bar{t}, \bar{x}) \text{ and } v(t_n, x_n) \to v^*(\bar{t}, \bar{x}).$$

Since $(v - \varphi)(t_n, x_n) \to 0$, we can assume that the sequence (t_n, x_n) also satisfies

$$|(v - \varphi)(t_n, x_n)| \leq \eta$$

for all $n \geq 1$. We define the stopping time

$$\theta_n = \inf\{s > t_n : (s, X_s^{t_n, x_n}) \notin \mathcal{N}_r\}$$

and we observe that $(\theta_n, X_{\theta_n}^{t_n, x_n}) \in \partial \mathcal{N}_r$ by the pathwise continuity of the process. Then, with $\beta_s = \beta(t_n, s)$, for the constant η and by formula (8), we have

$$\beta_{\theta_n} \varphi(\theta_n, X_{\theta_n}^{t_n, x_n}) \geq 2\eta + \beta_{\theta_n} v^*(\theta_n, X_{\theta_n}^{t_n, x_n}). \tag{9}$$

Since $\beta_{t_n} = 1$ and $|(v - \varphi)(t_n, x_n)| \leq \eta$, by the Itô's formula to the smooth test function φ, we have

$$\begin{aligned} v(t_n, x_n) &\geq -\eta + \varphi(t_n, x_n) \\ &= -\eta + \mathbb{E}\Big[\beta_{\theta_n} \varphi(\theta_n, X_{\theta_n}^{t_n, x_n}) - \int_{t_n}^{\theta_n} \beta_s (\partial_t + \mathcal{L}) \varphi(s, X_s^{t_n, x_n}) \, ds\Big] \\ &\geq -\eta + \mathbb{E}\Big[\beta_{\theta_n} \varphi(\theta_n, X_{\theta_n}^{t_n, x_n}) + \int_{t_n}^{\theta_n} \beta_s (f - h)(s, X_s^{t_n, x_n}) \, ds\Big] \\ &\geq -\eta + \mathbb{E}\Big[\beta_{\theta_n} \varphi(\theta_n, X_{\theta_n}^{t_n, x_n}) + \int_{t_n}^{\theta_n} \beta_s f(s, X_s^{t_n, x_n}) \, ds\Big]. \end{aligned}$$

By the (9), we know that

$$v(t_n, x_n) \geq \eta + \mathbb{E}\Big[\beta_{\theta_n} v^*(\theta_n, X_{\theta_n}^{t_n, x_n}) + \int_{t_n}^{\theta_n} \beta_s f(s, X_s^{t_n, x_n}) \, ds\Big],$$

which is in contradiction with the dynamic programming principle as $\eta > 0$.

Thus by the conclusion we got in the above, we know that the value function (2) is a viscosity solution of the equation

$$\partial_t v + \partial_x v + \frac{1}{2} \partial_{xx} v - cv + f = 0$$

on $[0, T) \times \mathbb{T}$.