Comparison principle

1. HJB equation:

Consider the HJB equation in the form

$$-\frac{\partial w}{\partial t} + \beta w - H(t, x, D_x w, D_x^2 w) = 0 \qquad \text{on } [0, T) \times \mathbb{R}^n \tag{1}$$

with a Hamitonian

$$H(t, x, p, M) = \sup_{a \in A} \left[b(x, a) \cdot p + \frac{1}{2} tr(\sigma(x, a)\sigma'(x, a)M) + f(t, x, a) \right] \tag{2}$$

for $(t, x, p, M) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times S_n$, where A is a subset of $\mathbb{R}^m$, S_n is $n \times n$ symmetric matrix and $\beta(t, x)$ is a function which has lower bound.

2. Assumptions:

- $b(x, a), \sigma(x, a)$ admit the linear growth condition and satisfy a uniform Lipschitz condition in x,
- $\beta(t, x)$ is lower bounded,
- f is uniformly continuous in (t, x), uniformly in $a \in A$.

3. Comparison principle:

Let U (resp. V) be a u.s.c. viscosity subsolution (resp. l.s.c. viscosity supersolution) with polynomial growth condition to (1), such that $U(T, \cdot) \leq V(T, \cdot)$ on $\mathbb{R}^n$, then $U \leq V$ on $[0, T) \times \mathbb{R}^n$.

Proof: Step 1. Let $\tilde{U}(t, x) = e^{\lambda t} U(t, x)$ and $\tilde{V}(t, x) = e^{\lambda t} V(t, x)$. Then a straightforward calculation shows that $\tilde{U}$ (resp. $\tilde{U}$) subsolution (resp. supersolution) to

$$-\frac{\partial w}{\partial t} + (\beta(t, x) + \lambda) w - \tilde{H}(t, x, D_x w, D_x^2 w) = 0 \qquad \text{on } [0, T) \times \mathbb{R}^n$$

where $\tilde{H}$ has the same form as H with f replaced by $\tilde{f}(t, x) = e^{\lambda t} f(t, x)$. Therefore, as $\beta(t, x)$ is lower bounded, we can take a λ so that $\beta(t, x) + \lambda > 0$, and possibly replacing (U, V) by $(\tilde{U}, \tilde{V})$, we can assume w.l.o.g. that $\beta(t, x) > 0$ for any $(t, x) \in [0, T) \times \mathbb{R}^n$.

Step 2, penalization and perturbation of supersolution. From the polynomial growth condition on U, V, we may choose an integer p greater than 1 such that

$$\sup_{[0,T] \times \mathbb{R}^n} \frac{|U(t, x) + V(t, x)|}{1 + |x|^p} < \infty$$

and we consider the function $\phi(t,x) = e^{-\lambda t}(1+|x|^{2p}) = e^{-\lambda t}\psi(x)$. From linear growth condition on b, σ, there exists positive constant c such that

$$\begin{array}{rl} & -\frac{\partial \phi}{\partial t} + \beta\phi - \sup_{a\in A}\left[b\cdot D_x\phi + \frac{1}{2}tr(\sigma\sigma' D_x^2\phi) + f\right] \\ = & \lambda e^{-\lambda t}\psi(x) + \beta e^{-\lambda t}\psi(x) - \sup_{a\in A}\left[b\cdot e^{-\lambda t}D_x\psi(x) + \frac{1}{2}tr(\sigma\sigma' e^{-\lambda t}D_x^2\psi(x)) + f\right] \\ = & e^{-\lambda t}\Big\{(\beta+\lambda)\psi(x) - \sup_{a\in A}\left[b\cdot D_x\psi(x) + \frac{1}{2}tr(\sigma\sigma' D_x^2\psi(x)) + e^{\lambda t}f\right]\Big\} \\ \geq & e^{-\lambda t}(\beta+\lambda-c)\psi \geq 0 \end{array}$$

by taking $\lambda \geq c - \beta$. This implies that for all $\epsilon > 0$, the function

$$V_\epsilon = V + \epsilon\phi$$

is a supersolution to (1). Futhermore, from the growth condition on U, V, ϕ, we have for all $\epsilon > 0$,

$$\lim_{|x|\to\infty}\sup_{[0,T]}(U - V_\epsilon)(t,x) = -\infty.$$

Step 3. From the step 1 and step 2, we may assume w.l.o.g. that $\beta(t,x) > 0$ for all $(t,x) \in [0,T)\times\mathbb{R}^n$ and the supreme of the u.s.c. function $U - V$ on $[0,T]\times\mathbb{R}^n$ is attained on $[0,T]\times\mathcal{O}$ for some open bounded set $\mathcal{O}$ of $\mathbb{R}^n$.

To prove $U \leq V$ on $[0,T]\times\mathbb{R}^n$, we argue by contradiction, which yields

$$M = \sup_{[0,T]\times\mathbb{R}^n}(U-V) = \sup_{[0,T]\times\mathcal{O}}(U-V) > 0. \tag{3}$$

We consider a bounded sequence $(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon)_\epsilon$ that maximizes for all $\epsilon > 0$, the function Φ_ϵ on $[0,T]^2\times\mathbb{R}^n\times\mathbb{R}^n$ with

$$\Phi_\epsilon(t,s,x,y) = U(t,x) - V(s,y) - \phi_\epsilon(t,s,x,y) \tag{4}$$

where

$$\phi_\epsilon(t,s,x,y) = \frac{1}{2\epsilon}[|t-s|^2 + |x-y|^2]. \tag{5}$$

Next we show that

$$M_\epsilon \to M \text{ and } \phi_\epsilon(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon) \to 0 \tag{6}$$

as $\epsilon \to 0$. For any $(t,x) \in [0,T]\times\mathcal{O}$, we have

$$\begin{aligned} M_\epsilon &\geq \sup_{s,y}(U(t,x) - V(s,y) - \phi_\epsilon(t,s,x,y)) \\ &\geq U(t,x) - V(t,x) - \phi_\epsilon(t,t,x,x) \\ &= U(t,x) - V(t,x), \end{aligned}$$

then

$$\sup_{t,x} M_\epsilon = M_\epsilon \geq \sup_{t,x}(U(t,x) - V(t,x)) = M.$$

Thus for all $\epsilon > 0$,

$$\begin{aligned} M \le M_\epsilon &= U(t_\epsilon, x_\epsilon) - V(s_\epsilon, y_\epsilon) - \phi_\epsilon(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon) \\ &\le U(t_\epsilon, x_\epsilon) - V(s_\epsilon, y_\epsilon). \end{aligned}$$

The bounded sequence $(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon)_\epsilon$ converges, up to a subsequence, to some $(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in [0,T]^2 \times \bar{\mathcal{O}}^2$. Moreover, since the sequence $(U(t_\epsilon, x_\epsilon) - V(s_\epsilon, y_\epsilon))_\epsilon$ is bounded and the sequence $(\phi_\epsilon(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon))_\epsilon$ is also bounded, which implies $\bar{t} = \bar{s}$ and $\bar{x} = \bar{y}$. By sending ϵ to 0, we have

$$M \le U(\bar{t}, \bar{x}) - V(\bar{t}, \bar{x}) \le M$$

and so $M = (U - V)(\bar{t}, \bar{x})$ with $(\bar{t}, \bar{x}) \in [0,T] \times \mathcal{O}$. By sending ϵ to 0, we can get (6).

With $\phi(t,s,x,y) = \frac{1}{2\epsilon}[|t-s|^2 + |x-y|^2]$, we have

$$\begin{aligned} \frac{\partial \phi}{\partial t}(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon) &= -\frac{\partial \phi}{\partial s}(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon) = \frac{t_\epsilon - s_\epsilon}{\epsilon}, \\ D_x\phi(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon) &= -D_y\phi(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon) = \frac{x_\epsilon - y_\epsilon}{\epsilon}, \\ D^2_{xy}\phi(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon) &= \frac{1}{\epsilon}\begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix} \end{aligned}$$

and

$$\left(D^2_{xy}\phi(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon)\right)^2 = \frac{2}{\epsilon} D^2_{xy}\phi(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon).$$

Furthermore, by choosing $\eta = \epsilon$ in the Ishii's lemma, then we have

$$\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \le \frac{3}{\epsilon}\begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}.$$

This implies that for any $n \times d$ matrices C, D,

$$tr(CC'M - DD'M) \le \frac{3}{\epsilon}|C - D|^2. \tag{7}$$

By the Ishii's lemma, we have

$$\left(\frac{1}{\epsilon}(t_\epsilon - s_\epsilon), \frac{1}{\epsilon}(x_\epsilon - y_\epsilon), M\right) \in \bar{\mathcal{P}}^{2,+}U(t_\epsilon, x_\epsilon)$$

$$\left(\frac{1}{\epsilon}(t_\epsilon - s_\epsilon), \frac{1}{\epsilon}(x_\epsilon - y_\epsilon), N\right) \in \bar{\mathcal{P}}^{2,-}V(s_\epsilon, y_\epsilon)$$

From the viscosity subsolution and supersolution characterization of U and V in terms of superjets and subjets, we then have

$$-\frac{1}{\epsilon}(t_\epsilon - s_\epsilon) + \beta U(t_\epsilon, x_\epsilon) - H\Big(t_\epsilon, x_\epsilon, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon), M\Big) \le 0 \tag{8}$$

$$-\frac{1}{\epsilon}(t_\epsilon - s_\epsilon) + \beta V(s_\epsilon, y_\epsilon) - H\Big(s_\epsilon, y_\epsilon, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon), N\Big) \ge 0 \tag{9}$$

By substracting of (8) and (9), applying the (7) to $C = \sigma(x_\epsilon, a)$ and $D = \sigma(y_\epsilon, a)$, we have

$$\begin{aligned}\beta[U(t_\epsilon, x_\epsilon) - V(s_\epsilon, y_\epsilon)] &\leq H\Big(t_\epsilon, x_\epsilon, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon), M\Big) - H\Big(s_\epsilon, y_\epsilon, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon), N\Big) \\ &\leq \mu\Big(|t_\epsilon - s_\epsilon| + |x_\epsilon - y_\epsilon| + \frac{2}{\epsilon}|x_\epsilon - y_\epsilon|^2\Big),\end{aligned}$$

where $\mu(z) \to 0$ as $z \to 0$. By sending ϵ to 0, we have $\beta M \leq 0$. As $\beta(t, x) > 0$ for any $(t, x) \in [0, T] \times \mathcal{O}$, we can get $M < 0$, which contradicts to (3).