Uniformly elliptic PDEs

Question:

Consider the PDE

$$\inf_{a\in(-1,1)}\{(1+a^2)u''(x)\}=0 \tag{1}$$

on $x\in(-1,1)$. Is it uniformly elliptic?

Definition:

For $F:\Omega\times\mathbb{R}^n\times S^n\mapsto\mathbb{R}$, where S^n is $n\times n$ symmetric matrices. We say F is uniformly elliptic (with the uniform ellipticity constants $0<\lambda\leq\Lambda$) if

$$\mathcal{P}^-(X-Y)\leq F(x,p,X)-F(x,p,Y)\leq\mathcal{P}^+(X-Y) \tag{2}$$

for $x\in\Omega, p\in\mathbb{R}^n$ and $X,Y\in S^n$, where $\mathcal{P}^+$ and $\mathcal{P}^-$ are Pucci's operators which defined as follows: for $X\in S^n$,

$$\mathcal{P}^+(X):=\max\{-\mathrm{trace}(AX)\,|\,\lambda I\leq A\leq\Lambda I \text{ for } A\in S^n\},$$

$$\mathcal{P}^-(X):=\min\{-\mathrm{trace}(AX)\,|\,\lambda I\leq A\leq\Lambda I \text{ for } A\in S^n\}.$$

Solution:

For the PDE (1), we have

$$F(x,p,X)=\inf_{a\in(-1,1)}\{(1+a^2)X\},$$

then

$$\begin{aligned}F(x,p,X)-F(x,p,Y)&=\inf_{a\in(-1,1)}\{(1+a^2)(X-Y)\}\\&=\sup_{a\in(-1,1)}\{-(1+a^2)(X-Y)\}.\end{aligned}$$

For $0<\lambda<\Lambda$, the Pucci's operators is

$$\mathcal{P}^+(X-Y)=\max\{-A(X-Y)\,|\,\lambda\leq A\leq\Lambda \text{ for } A\in\mathbb{R}\},$$

$$\mathcal{P}^-(X-Y)=\min\{-A(X-Y)\,|\,\lambda\leq A\leq\Lambda \text{ for } A\in\mathbb{R}\}.$$

So, we need the condition for $\lambda\leq A\leq\Lambda$,

$$\mathcal{P}^-(X-Y)\leq\sup_{a\in(-1,1)}\{-(1+a^2)(X-Y)\}\leq\mathcal{P}^+(X-Y).$$

For $\lambda\leq 1$ and $\Lambda\geq 2$, as $(1+a^2)\in[1,2)$, we have

$$\mathcal{P}^-(X-Y)\leq F(x,p,X)-F(x,p,Y)\leq\mathcal{P}^+(X-Y),$$

thus the PDE (1) is uniformly elliptic.