Analysis Exercise

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Exercise 1:

Let X and Y be two Banach spaces. $C(X,Y)$ is the set of all continuous mappings $f : X \mapsto Y$. For $f, g \in C(X,Y)$, we define

$$\|f - g\| = \sup_{x \in X} \|f(x) - g(x)\|.$$

(i) Prove that $C(X,Y)$ is a Banach space.

(ii) If X and Y are compact, is $C(X,Y)$ compact?

Solution:

(i) We need to show that $C(X,Y)$ is a complete normed vector space. Firstly we show that $C(X,Y)$ is a normed space.

- For any $f \in C(X,Y)$, $\|f\| = \sup_{x\in X} \|f(x)\| \geq 0$. And if $\|f\| = 0$, we have $\sup_{x\in X} \|f\| = 0$, then for any $x \in X$, $f(x) = 0$, thus $f \equiv 0$.
- For any $\lambda \in \mathbb{R}$, $\|\lambda f\| = \sup_{x\in X} \|\lambda f(x)\| = |\lambda| \sup_{x\in X} \|f(x)\| = |\lambda|\|f\|$.
- For any $f, g \in C(X,Y)$, for any $x \in X$, we have $\|f(x) + g(x)\| \leq \|f(x)\| + \|g(x)\|$, and by the definition of the norm in $C(X,Y)$, we have $\|f(x) + g(x)\| \leq \sup_{x\in X} \|f(x)\| + \sup_{x\in X} \|g(x)\| = \|f\| + \|g\|$. By the arbitrary of x, we can get $\|f + g\| = \sup_{x\in X} \|f(x) + g(x)\| \leq \|f\| + \|g\|$.

Next we need to show that $C(X,Y)$ is complete. Suppose $\{f_n\}$ is a Cauchy sequence in $C(X,Y)$, then $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall p > q > N$,

$$\sup_{x\in X} \|f_p(x) - f_q(x)\| < \epsilon.$$

For any $y \in X$, we have

$$\|f_p(y) - f_q(y)\| \leq \sup_{x\in X} \|f_p(x) - f_q(x)\| < \epsilon,$$

thus $f_n(y)$ is a Cauchy sequence in Y. As Y is a Banach space, Y is complete, then $f_n(y)$ converges to some $f(y)$ in Y. From this we can define a function

$$f : X \mapsto Y.$$

Next we show that f is also continuous. Since

$$\|f(x) - f(y)\| \leq \|f(x) - f_n(x)\| + \|f_n(x) - f_n(y)\| + \|f_n(y) - f(y)\|,$$

and $\{f_n\}$ is a continuous function sequence, for the above ϵ, there exists a $N^* \in \mathbb{N}$ and $\delta > 0$, for any $x \in B(y, \delta)$ and $n > N^*$, we have

$$\|f(x) - f(y)\| < 3\epsilon.$$

Hence $f \in C(X, Y)$. And for the above ϵ and $p > q > N$, since $\|f_p(y) - f_q(y)\| < \epsilon$, let $p \to \infty$, we have $\|f(y) - f_q(y)\| \leq \epsilon$. By the arbitrary of $y \in X$, for $q > N$, we can get

$$\sup_{y \in X} \|f(y) - f_q(y)\| \leq \epsilon,$$

which shows that $f_n \to f$ in $C(X, Y)$. Thus $C(X, Y)$ is complete.

(ii) The statement is not true. We can give a counter example as follows. Set $X = [0, 1]$ and $Y = [0, 1]$, and we define a function sequence $f_n : X \mapsto Y$ by

$$f_n(x) = \begin{cases} 0, & x \in [0, \frac{1}{2} - \frac{1}{n}) \\ nx - \frac{n}{2} + 1, & x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1] \end{cases}$$

then we know that X and Y are compact and $\{f_n\}$ is a continuous function sequence from X to Y. And we define

$$f(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1] \end{cases}$$

thus when $n \to \infty$, $f_n(x)$ converges to $f(x)$ almost everywhere. But $f(x)$ is not a continuous function on X, $f(x) \notin C(X, Y)$, thus for any subsequence $\{f_{n_k}\}$ of $\{f_n\}$, we know that $\{f_{n_k}\}$ is not converges in $C(X, Y)$. Hence we know $C(X, Y)$ is not compact.