Viscosity solution and comparison principle

Exercise 1:

Consider the ODE

$$\begin{cases} |u'(x)| - 1 = 0, \text{ on } x \in (-1,1) \\ u(\pm 1) = 0 \end{cases} \tag{1}$$

(1) Is $v(x) = |x| - 1$ a viscosity solution of (1)?

(2) Is $u(x) = 1 - |x|$ a viscosity solution of (1)?

(3) Can you prove comparison principle?

Solution:

(1) For $|u'(x)| - 1 = 0$, we denote

$$F(x, u, p, X) = |p| - 1.$$

By the definition of semi-jets, for $v(x) = |x| - 1$, when $x = 0$, we have

$$J^{2,-}v(0) = ((-1,1) \times \mathbb{R}) \cup (\{1\} \times (-\infty, 0]) \cup (\{-1\} \times (-\infty, 0]).$$

Thus for $x = 0$, there exists $(p, X) \in ((-1,1) \times \mathbb{R}) \subset J^{2,-}v(0)$ such that

$$F(x, v, p, X) = |p| - 1 < 0,$$

so, we know that $v(x) = |x| - 1$ is not a viscosity supersolution of $|u'(x)| - 1 = 0$, then it is not a viscosity solution of (1).

(2) Similarly, by the definition of semi-jets, for $u(x) = 1 - |x|$, for $x = 0$, we have

$$J^{2,+}u(0) = ((-1,1) \times \mathbb{R}) \cup (\{-1\} \times [0, +\infty)) \cup (\{1\} \times [0, +\infty))$$

and

$$J^{2,-}u(0) = \emptyset.$$

When $x \in (0,1)$, we have

$$J^{2,+}u(x) = \{-1\} \times [0, +\infty), \quad J^{2,-}u(x) = \{-1\} \times (-\infty, 0],$$

and when $x \in (-1, 0)$,

$$J^{2,+}u(x) = \{1\} \times [0, +\infty), \quad J^{2,-}u(x) = \{1\} \times (-\infty, 0].$$

Hence we can conclude that for any $x \in (-1,1)$ and $(p, X) \in J^{2,+}u(x)$,

$$F(x, u, p, X) = |p| - 1 \leq 0,$$

then $u(x) = 1 - |x|$ is a viscosity subsolution. And for any $(p, X) \in J^{2,-}u(x)$,

$$F(x, u, p, X) = |p| - 1 \geq 0,$$

then $u(x) = 1 - |x|$ is a viscosity supersolution. For $x = 1$ or $x = -1$, $u(x) = 1 - |x| = 0$, so we know that $u(x) = 1 - |x|$ is a viscosity solution of the ODE (1).

(3) To get the comparison principle for the ODE (1), if we denote $\Omega = (-1, 1)$, we need to show that: let $u \in USC(\bar{\Omega})$ and let $v \in LSC(\bar{\Omega})$ be a viscosity subsolution and supersolution of (1) respectively, if $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\bar{\Omega}$.

Suppose that

$$\max_{\bar{\Omega}}(u - v)(x) = \theta > 0, \tag{2}$$

if we choose $\mu \in (0, 1)$ such that

$$(1 - \mu) \max_{\bar{\Omega}} u \leq \frac{\theta}{2},$$

then easily we can get

$$\max_{\bar{\Omega}}(\mu u - v) =: \tau \geq \frac{\theta}{2}.$$

For $\bar{x} \in \bar{\Omega}$ such that $(\mu u - v)(\bar{x}) = \tau$, we may suppose that $\bar{x} \in \Omega$. Otherwise $\bar{x} \in \partial\Omega$, if we further suppose that $\mu < 1$ is close to 1 such that

$$-(1 - \mu) \min_{\partial\Omega} v \leq \frac{\theta}{4},$$

then as $u \leq v$ on $\partial\Omega$, we have

$$\frac{\theta}{2} \leq \tau = \mu u(\bar{x}) - v(\bar{x}) \leq \mu v(\bar{x}) - v(\bar{x}) = (\mu - 1)v(\bar{x}) \leq \frac{\theta}{4},$$

which is a contradiction.

Consider the mapping $\Phi_\epsilon : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ defined by

$$\Phi_\epsilon(x, y) = \mu u(x) - v(y) - \frac{|x - y|^2}{2\epsilon}. \tag{3}$$

Choose $(x_\epsilon, y_\epsilon) \in (\bar{\Omega}, \bar{\Omega})$ such that

$$\max_{x,y \in \bar{\Omega}} \Phi_\epsilon(x, y) = \Phi_\epsilon(x_\epsilon, y_\epsilon),$$

then

$$\Phi_\epsilon(x_\epsilon, y_\epsilon) \geq \sup_{x \in \bar{\Omega}} \Phi_\epsilon(x, x) = \sup_{x \in \bar{\Omega}}(\mu u - v)(x) = \tau \geq \frac{\theta}{2}.$$

We suppose that $\lim_{\epsilon \to 0}(x_\epsilon, y_\epsilon) = (\hat{x}, \hat{y})$ for some $(\hat{x}, \hat{y}) \in (\bar{\Omega}, \bar{\Omega})$. Also, we have that

$$\frac{|x_\epsilon - y_\epsilon|^2}{2\epsilon} \leq \mu u(x_\epsilon) - v(y_\epsilon) - \tau \leq M_\mu = \mu \max_{\bar{\Omega}} u - \min_{\bar{\Omega}} v,$$

then $|x_\epsilon - y_\epsilon|^2 \leq 2\epsilon M_\mu$. By sending $\epsilon \to 0$, we have $\hat{x} = \hat{y}$. Hence, the above inequality implies that

$$\mu u(\hat{x}) - v(\hat{x}) = \tau,$$

which yield $\hat{x} \in \Omega$ because of the choice of μ. Thus we see that $(x_\epsilon, y_\epsilon) \in \Omega \times \Omega$ for some small $\epsilon > 0$. Moreover, we have

$$\begin{aligned} 0 \leq \liminf_{\epsilon \to 0} \frac{|x_\epsilon - y_\epsilon|^2}{2\epsilon} &\leq \limsup_{\epsilon \to 0} \frac{|x_\epsilon - y_\epsilon|^2}{2\epsilon} \\ &\leq \limsup_{\epsilon \to 0} (\mu u(x_\epsilon) - v(y_\epsilon)) - \tau \\ &\leq (\mu u - v)(\hat{x}) \leq 0, \end{aligned}$$

which implies

$$\lim_{\epsilon \to 0} \frac{|x_\epsilon - y_\epsilon|^2}{2\epsilon} = 0.$$

Taking

$$\phi(x) = \frac{1}{\mu}\Big(v(y_\epsilon) + \frac{|x - y_\epsilon|^2}{2\epsilon}\Big),$$

we see that $u - \phi$ attains its maximum at $x_\epsilon \in \Omega$. By the definition of viscosity subsolution, we have

$$\frac{|x_\epsilon - y_\epsilon|}{\mu\epsilon} \leq 1,$$

which yields $\frac{|x_\epsilon - y_\epsilon|}{\epsilon} \leq \mu$. On the other hand, taking

$$\psi(y) = \mu u(x_\epsilon) - \frac{|y - x_\epsilon|^2}{2\epsilon},$$

we see that $v - \psi$ attains its minimum at $y_\epsilon \in \Omega$. Similarly, by the definition of viscosity supersolution, we have

$$\frac{|x_\epsilon - y_\epsilon|}{\epsilon} \geq 1.$$

Then we can get

$$1 \leq \frac{|x_\epsilon - y_\epsilon|}{\epsilon} \leq \mu,$$

which contradicts with $\mu \in (0, 1)$.

Theorem:

Consider the following PDE:

$$H(x, Du) - f(x) = 0, \qquad x \in \Omega \tag{4}$$

where $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$. We suppose that

- there is a continuous function $\omega_H : [0, \infty) \to [0, \infty)$ such that $\omega_H(0) = 0$ and

$$|H(x, p) - H(y, p)| \leq \omega_H(|x - y|(1 + |p|))$$

  for $x, y \in \Omega$ and $p \in \mathbb{R}^n$,

- H has homogeneous degrees $\alpha > 0$ with respect to the second variable, i.e. there is $\alpha > 0$ such that
$$H(x, \mu p) = \mu^\alpha H(x, p)$$
for $x \in \Omega, p \in \mathbb{R}^n$ and $\mu > 0$,

- there is a $\sigma > 0$ such that
$$\min_{x \in \bar{\Omega}} f(x) = \sigma > 0.$$

Let $u \in USC(\bar{\Omega})$ and let $v \in LSC(\bar{\Omega})$ be a viscosity subsolution and supersolution of (4) respectively, if $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\bar{\Omega}$.