Analysis Exercise

Exercise 2:

Let (X, η) and (Y, ρ) be two Polish spaces. $C(X, Y)$ is the set of all continuous mappings $f : X \mapsto Y$. For $f, g \in C(X, Y)$, we define

$$d(f, g) = \sup_{x \in X} \rho(f(x), g(x)).$$

(i) Prove that $(C(X, Y), d)$ is a Polish space.

(ii) If $K \subset Y$ is compact, is $C(X, K)$ compact in $C(X, Y)$?

Solution:

(i) We need to show that $C(X, Y)$ is separable and completely metrizable. Firstly we show that $(C(X, Y), d)$ is a metric space.

- For any $f, g \in C(X, Y)$, if $d(f, g) = 0$, we have $\sup_{x \in X} \rho(f(x), g(x)) = 0$, thus $f(x) = g(x)$ for any $x \in X$. We know that $f \equiv g$. If $f = g$, $d(f, g) = \sup_{x \in X} \rho(f(x), g(x)) = 0$. Hence $d(f, g) = 0 \iff f = g$.

- For any $f, g \in C(X, Y)$,

$$d(f, g) = \sup_{x \in X} \rho(f(x), g(x)) = \sup_{x \in X} \rho(g(x), f(x)) = d(g, f).$$

- For any $f, g, h \in C(X, Y)$, and for any $x \in X$, we have $\rho(f(x), g(x)) \leq \rho(f(x), h(x)) + \rho(h(x), g(x))$. Then we know that

$$\rho(f(x), g(x)) \leq \sup_{x \in X} \rho(f(x), h(x)) + \sup_{x \in X} \rho(h(x), g(x)).$$

  By the arbitrary of $x \in X$, $d(f, g) = \sup_{x \in X} \rho(f(x), g(x)) \leq d(f, h) + d(h, g)$.

Next we show that $C(X, Y)$ is complete. Suppose $\{f_n\}$ is a Cauchy sequence in $C(X, Y)$, then $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall p > q > N$,

$$\sup_{x \in X} \rho(f_p(x), f_q(x)) < \epsilon.$$

For any $y \in X$, we have

$$\rho(f_p(y), f_q(y)) \leq \sup_{x \in X} \rho(f_p(x), f_q(x)) < \epsilon,$$

thus $f_n(y)$ is a Cauchy sequence in Y. As Y is a Polish space, Y is complete, then $f_n(y)$ converges to some $f(y)$ in Y. From this we can define a function

$$f : X \mapsto Y.$$

Next we show that f is also continuous. Since

$$\rho(f(x), f(y)) \leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f_n(y), f(y)),$$

and $\{f_n\}$ is a continuous function sequence, for the above ϵ, there exists a $N^* \in \mathbb{N}$ and $\delta > 0$, for any $x \in B(y, \delta)$ and $n > N^*$, we have

$$\rho(f(x), f(y)) < 3\epsilon.$$

Hence $f \in C(X, Y)$. And for the above ϵ and $p > q > N$, since $\rho(f_p(y), f_q(y)) < \epsilon$, let $p \to \infty$, we have $\rho(f(y), f_q(y)) \leq \epsilon$. By the arbitrary of $y \in X$, for $q > N$, we can get

$$\sup_{y \in X} \rho(f(y), f_q(y)) \leq \epsilon,$$

which shows that $f_n \to f$ in $C(X, Y)$. Thus $C(X, Y)$ is complete.

Next we need to show that $C(X, Y)$ is separable. As X and Y are polish space, we can take a countable dense set $\hat{X}$, $\hat{Y}$ from X, Y. And we define $M(\hat{X}, \hat{Y})$ be the set of all mappings $\hat{X} \mapsto \hat{Y}$ with the metric

$$\hat{d}(f, g) = \sup_{x \in \hat{X}} \rho(f(x), g(x)), \quad \forall f, g \in M(\hat{X}, \hat{Y}).$$

For $f \in C(X, Y)$, we define

$$B(f, \delta) = \{g \in C(X, Y) : d(f, g) < \delta\},$$

where $\delta > 0$ is a constant. To prove that $C(X, Y)$ is separable, we need to show that: $\forall \epsilon > 0$, there exists a sequence of functions $\{f_n\}_{n=1}^{\infty}$ such that

$$\bigcup_{n=1}^{\infty} B(f_n, \epsilon) = C(X, Y).$$

Define the projector $P : M(X, Y) \mapsto M(\hat{X}, Y)$ as

$$Pf(x) = f(x), \quad \forall f \in M(X, Y), \forall x \in \hat{X},$$

where $M(X, Y)$ is the set of all mappings from X to Y. Since $\hat{X}$ is countable, by the Axiom of Choice, $\forall \epsilon > 0$, $\forall f \in C(X, Y)$, there exists $h \in M(\hat{X}, \hat{Y})$ such that

$$\hat{d}(Pf, h) < \frac{\epsilon}{3}.$$

As $M(\hat{X}, \hat{Y})$ is countable, we index it as $M(\hat{X}, \hat{Y}) = \{h_n : n \in \mathbb{N}\}$. For the above ϵ, setting

$$A_n = \{f \in C(X, Y) : \hat{d}(Pf, h_n) < \frac{\epsilon}{3}\},$$

and we have $\bigcup_{n=1}^{\infty} A_n = C(X,Y)$. By the Axiom of Choice, for any $n \in \mathbb{N}$, we can take a $f_n \in A_n$. Now we can prove that for the above ϵ,

$$\bigcup_{n=1}^{\infty} B(f_n, \epsilon) = C(X,Y).$$

By the definition of $B(f_n, \epsilon)$, we have

$$\bigcup_{n=1}^{\infty} B(f_n, \epsilon) \subset C(X,Y).$$

And for any $f \in C(X,Y)$, as $C(X,Y) = \bigcup_{n=1}^{\infty} A_n$, there exists $k \in \mathbb{N}$ such that $f \in A_k$, then we have

$$d(f, f_k) \leq \hat{d}(Pf, h_k) + \hat{d}(h_k, Pf_k) < \frac{2\epsilon}{3} < \epsilon.$$

Thus we also have $C(X,Y) \subset \bigcup_{n=1}^{\infty} B(f_n, \epsilon)$. Then we know that $C(X,Y)$ is separable.

(ii) The statement is not true. We can give a counter example as follows. Set $K = [0,1], Y = \mathbb{R}$ and $X = [0,1]$, and we define a function sequence $f_n : X \mapsto K$ by

$$f_n(x) = \begin{cases} 0, & x \in [0, \frac{1}{2} - \frac{1}{n}) \\ nx - \frac{n}{2} + 1, & x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1] \end{cases}$$

then we know that $K \subset Y$ and K is compact and $\{f_n\}$ is a continuous function sequence from X to K. And we define

$$f(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1] \end{cases}$$

thus when $n \to \infty$, $f_n(x)$ converges to $f(x)$ almost everywhere. But $f(x)$ is not a continuous function on X, $f(x) \notin C(X,K)$, thus for any subsequence $\{f_{n_k}\}$ of $\{f_n\}$, we know that $\{f_{n_k}\}$ is not converges in $C(X,K)$. Hence we know $C(X,K)$ is not compact in $C(X,Y)$.