Stochastic Control Exercises

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Exercise 1:

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space, where W_t is a Brownian motion. The state process X_t is defined as follows:

$$X_t = \int_0^t a(s)\,ds + W_t,$$

and we define the minimum cost as

$$V = \inf_{a(\cdot)\in\mathcal{A}} \mathbb{E}\Big[\int_0^t a^2(s)\,ds + X_T^2\Big],$$

where $\mathcal{A}$ is the collection of all $\mathcal{F}_t$ progressively measurable process. Find the value V.

Solution:

Firstly we derive the HJB equation of this control problem. Since

$$dX_t = a(t)\,dt + dW_t,$$

then for the state process we have $b(X_t, a(t)) = a(t)$ and $\sigma(X_t, a(t)) = 1$. Then we know that the HJB equation is

$$\frac{\partial v}{\partial t} + \inf_{a\in A}[L^a v + a^2] = 0,$$

where $L^a v = a\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}$, and the domain is $[0,T]\times\mathbb{R}$. Since

$$L^a v + a^2 = a\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + a^2,$$

then when $a = -\frac{1}{2}\frac{\partial v}{\partial x}$, the term $L^a v + a^2$ can get the minimum value. So we get the HJB equation under the domain $[0,T]\times\mathbb{R}$:

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2 = 0 \\ v(T,x) = x^2 \end{cases}$$

And we have $V = v(0,0)$.

Next we need to solve HJB equation. We denote $\tau = T - t$ and $v(t,x) = u(\tau,x) = u(T-t,x)$, then we have

$$\begin{cases} -\frac{\partial u}{\partial \tau} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{4}\left(\frac{\partial u}{\partial x}\right)^2 = 0 \\ u(\tau,x) = x^2 \end{cases}$$

and $V = u(T,0)$. We suppose

$$u(\tau,x) = \phi_1(\tau)x^2 + \phi_2(\tau)x + \phi_3(\tau),$$

then we have $\phi_1(0) = 1, \phi_2(0) = 0, \phi_3(0) = 0$ and since

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{d\phi_1}{d\tau}x^2 + \frac{d\phi_2}{d\tau}x + \frac{d\phi_3}{d\tau} \\ \frac{\partial u}{\partial x} = 2\phi_1 x + \phi_2 \\ \frac{\partial^2 u}{\partial x^2} = 2\phi_1 \end{cases}$$

then we can get the system of ODE as follows

$$\begin{cases} \frac{d\phi_1}{d\tau} + (\phi_1)^2 = 0 \\ \frac{d\phi_2}{d\tau} + \phi_1\phi_2 = 0 \\ \frac{d\phi_3}{d\tau} - \phi_1 + \frac{1}{4}(\phi_2)^2 = 0 \end{cases}$$

with $\phi_1(0) = 1, \phi_2(0) = 0, \phi_3(0) = 0$. By calculation, we have $\phi_1(\tau) = (\tau+1)^{-1}, \phi_2(\tau) = 0$ and $\phi_3(\tau) = \ln(\tau + 1)$, then

$$u(\tau, x) = \frac{1}{\tau + 1}x^2 + \ln(\tau + 1),$$

so we can get $u(T, 0) = \ln(T + 1)$. Since $v(t, x) = u(T - t, x)$, then we have

$$v(t, x) = \frac{1}{T - t + 1}x^2 + \ln(T - t + 1),\ (t, x) \in [0, T] \times \mathbb{R}.$$

So $v(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$. Since $\frac{1}{T-t+1} \leq 1$ and $\ln(T-t+1) \leq \ln(T+1)$ for all $t \in [0, T]$, then we know there exists a constant C such that $v(t, x) \leq C(1 + |x|^2)$. And $v(t, x)$ is the solution of HJB equation with $a(t, x) = \frac{-x}{T-t+1}$, by the verification theorem, we know that $v(t, x)$ is the solution of the control problem. Thus we have $V = \ln(T + 1)$, which means

$$\inf_{a(\cdot)\in\mathcal{A}} \mathbb{E}\Big[\int_0^t a^2(s)\, ds + X_T^2\Big] = \ln(T + 1).$$