A question about the parabolic PDE

Question:

Assume that $b, l \in C_b^{1,2}(\mathbb{R}^+, \mathbb{R})$, consider the parabolic PDE

$$(P1) \quad \begin{cases} \partial_t v = b\partial_x v + \partial_{xx} v + l, & \forall (t,x) \in (\mathbb{R}^+, \mathbb{R}) \\ v(0,x) = 0, & \forall x \in \mathbb{R} \end{cases} \tag{1}$$

It is standard that there is a classical solution. Assume that $\partial_{xxx} v$ exists, by taking derivative to x to the equation (1) on the both side, with $\partial_x v = \hat{v}$, we have

$$\begin{cases} \partial_t \hat{v} = b\partial_x \hat{v} + \partial_{xx} \hat{v} + \partial_x l + \hat{v} \cdot \partial_x b, & \forall (t,x) \in (\mathbb{R}^+, \mathbb{R}) \\ \hat{v}(0,x) = 0, & \forall x \in \mathbb{R} \end{cases}$$

Therefore, by the uniqueness of the solution, we conclude that the solution of

$$(P2) \quad \begin{cases} \partial_t \hat{u} = b\partial_x \hat{u} + \partial_{xx} \hat{u} + \partial_x l + \hat{u} \cdot \partial_x b, & \forall (t,x) \in (\mathbb{R}^+, \mathbb{R}) \\ \hat{u}(0,x) = 0, & \forall x \in \mathbb{R} \end{cases} \tag{2}$$

satisfies $\hat{u}(t,x) = \partial_x v(t,x)$ for any $(t,x) \in (\mathbb{R}^+, \mathbb{R})$.

Does the above conclusion holds without assuming $v \in C^{1,3}$? That is let $b, l \in C_b^{1,3}(\mathbb{R}^+, \mathbb{R})$, does $\hat{u}$ of (2) and v of (1) satisfies

$$\hat{u}(t,x) = \partial_x v(t,x).$$

Solution:

We define

$$u(t,x) = g(t) + \int_0^x \hat{u}(t,y)\, dy, \quad \forall (t,x) \in (\mathbb{R}^+, \mathbb{R})$$

where $g(\cdot)$ is the function we want to find to make $u(t,x)$ is the solution of equation (1). Suppose $u(t,x)$ is the solution of equation (1), for the initial condition, we need

$$u(0,x) = g(0) + \int_0^x \hat{u}(0,y)\, dy = g(0) = 0.$$

And for $(t,x) \in (\mathbb{R}^+, \mathbb{R})$, we have

$$\begin{aligned} \partial_t u &= g'(t) + \int_0^x \partial_t \hat{u}(t,y)\, dy \\ &= g'(t) + \int_0^x (b\partial_x \hat{u} + \partial_{xx} \hat{u} + \partial_x l + \hat{u} \cdot \partial_x b)(t,y)\, dy \\ &= g'(t) + (b\hat{u} + \partial_x \hat{u} + l)|_0^x \\ &= g'(t) + b\hat{u} + \partial_x \hat{u} + l - (b\hat{u} + \partial_x \hat{u} + l)(t,0). \end{aligned}$$

By the definition of $u(t,x)$, we can get that

$$\partial_x u(t,x) = \hat{u}(t,x), \quad \forall (t,x) \in (\mathbb{R}^+, \mathbb{R}),$$

then we have

$$\partial_t u - (b\partial_x u + \partial_{xx} u + l) = g^{'}(t) - (b\hat{u} + \partial_x \hat{u} + l)(t,0).$$

Thus the sufficient condition of $u(t,x)$ is the solution of equation (1) is

$$(C1) \qquad \begin{cases} g^{'}(t) = (b\hat{u} + \partial_x \hat{u} + l)(t,0) \\ g(0) = 0 \end{cases} \tag{3}$$

where $\hat{u}(t,x)$ is the solution of equation (2). We define

$$g(t) = \int_0^t (b\hat{u} + \partial_x \hat{u} + l)(s,0)\, ds, \quad \forall t \in \mathbb{R}^+, \tag{4}$$

which satisfies the formula (3). Thus

$$u(t,x) = \int_0^t (b\hat{u} + \partial_x \hat{u} + l)(s,0)\, ds + \int_0^x \hat{u}(t,y)\, dy, \quad \forall (t,x) \in (\mathbb{R}^+, \mathbb{R})$$

is the solution of (1) and it satisfies

$$\partial_x u(t,x) = \hat{u}(t,x), \quad \forall (t,x) \in (\mathbb{R}^+, \mathbb{R}).$$

What's more, we have $u(t,x) \in C_b^{1,3}(\mathbb{R}^+, \mathbb{R})$ and it is the unique solution of (1).