Note 1

Linear system

Definitions

A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad (1)$$

where b and the **coefficients** $a_1, \ldots, a_n$ are real or complex numbers, usually known

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables—say, $x_1, \ldots, x_n$. An example is

Ex. Write a linear system and a non-linear system

Ex. Write the general form of m by n linear system

# of eqns

# of unkowns

ex

$$2x_1 + (-1)x_2 = 1$$

$a_1 = 2$ $a_2 = -1$ $b = 1$

ex

$$\begin{cases} 2x_1 - x_2 = 1 \\ x_1 + x_2 = 0 \end{cases}$$

ex

$$\begin{cases} \sin(x_1) + x_2 = 0 \\ x_1 - x_2 = 1 \end{cases}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Representation

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

3 × 3 system
$$\begin{cases} 1\cdot x_1 - 2x_2 + 1\,x_3 = 0 \\ 0\cdot x_1 + 2x_2 - 8x_3 = 8 \\ 5x_1 + 0x_2 - 5x_3 = 10 \end{cases} \tag{3}$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

is called the **coefficient matrix** (or **matrix of coefficients**) of the system (3), and

$$\left[\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] \tag{4}$$

is called the **augmented matrix** of the system. ~~(The second row here contains a zero~~

G.F.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\underbrace{\left[\begin{array}{ccc:c} \overbrace{a_{11} \cdots\cdots a_{1n}}^{\text{coeff. matrix}} & & & b_1 \\ \vdots & & & \vdots \\ a_{m1} \cdots\cdots a_{mn} & & & b_m \end{array}\right]}_{\text{augmented matrix}}$$

Ex. Write coefficient matrix and augmented matrix of general system

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Solution set

$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 0 \end{array}\right)$$

ex $\begin{cases} 2x_1 - x_2 = 1 \\ x_1 + x_2 = 0 \end{cases}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -\frac{1}{3} \end{pmatrix}$$

A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively. For

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

ex Give me linear system with two solns

$$\begin{cases} 2x_1 + 2x_2 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} a \\ -a \end{pmatrix}$$

Ex. Write a linear system with exactly one solution
Ex. Write a linear system with infinite many solutions
Ex. Write a linear system with no solution
Ex. Write a linear system with exactly two solutions?

Fact If a system has two solns, then there exist infinite many soln.

Ex any system with no soln?

$$\text{eq1}\quad \begin{cases} 2x_1 + 2x_2 = b \\ x_1 + x_2 = 0 \end{cases} \iff \left(\begin{array}{cc|c} 2 & 2 & \overset{=1}{b} \\ 1 & 1 & 0 \end{array}\right)$$

(Q) for what b, does it have "no soln"?

(A) any $b \neq 0$

Action

$$R_1/2 \Rightarrow \begin{cases} x_1 + x_2 = b/2 \\ x_1 + x_2 = 0 \end{cases} \iff \left(\begin{array}{cc|c} 1 & 1 & b/2 \\ 1 & 1 & 0 \end{array}\right) \quad \text{eq2}$$

"eq1 is equiv to eq2"

How many solutions?

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ consistent.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Ex. prove there is no linear system with exactly two solutions ✓

Your Fact: How to distinguish a system in terms of number of soln?

ex $\left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b \end{array}\right)$

there exists unique soln for any $\begin{pmatrix} a \\ b \end{pmatrix}$, why?

Rk. Existence and uniqueness of solution is a big question.

Python solver

```
[1]  import numpy as np
```

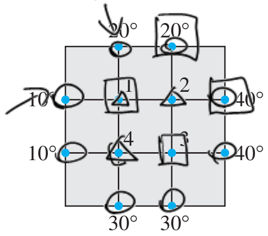
```
[4]  a = np.array([[3,1], [1,2]]) #coefficient matrix
     b = np.array([9,8]) #rhs
     x = np.linalg.solve(a, b) #soln
```

P11

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let $T_1, \ldots, T_4$ denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.[2] For instance,

origin for heat eqn (partial differential equation)

$$T_1 = (10 + 20 + T_2 + T_4)/4, \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



$$T_1 = \frac{20 + 10 + T_2 + T_4}{4}$$

$$T_2 = \frac{20 + 40 + T_3 + T_1}{4}$$

$$T_3 = \frac{\quad}{4}$$

$$T_4 = \frac{\quad}{4}$$

33. Write a system of four equations whose solution gives estimates for the temperatures $T_1, \ldots, T_4$.

34. Solve the system of equations from Exercise 33. [*Hint:* To speed up the calculations, interchange rows 1 and 4 before starting "replace" operations.]

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[2] See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.