What are Systems of Linear Equations?

WPI

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Given a linear system,

- write its coefficient matrix,
- write its augmented matrix,
- check a candidate solution.

- Linear algebra is to study system of linear equations via matrix and vector
- They have many applications
    - Machine learning
    - Economics and finance
    - Statistics
    - Image processing
    - and more ...

If you google “Matrix”, you probably get ...

Figure: From movie “The Matrix”

Definitions

- A matrix is a rectangular box of numbers,
  - ex. 2 by 3 matrix

$$A = \begin{bmatrix} 2 & 1 & 15 \\ 1 & 1 & 5 \end{bmatrix}$$

- If a matrix has only one row or only one column it is called a vector,
  - ex. 2 dimensional column vector

$$b_1 = \begin{bmatrix} 15 \\ 5 \end{bmatrix},$$

  - ex. 2 dimensional row vector

$$b_2 = \begin{bmatrix} 15 & 5 \end{bmatrix},$$

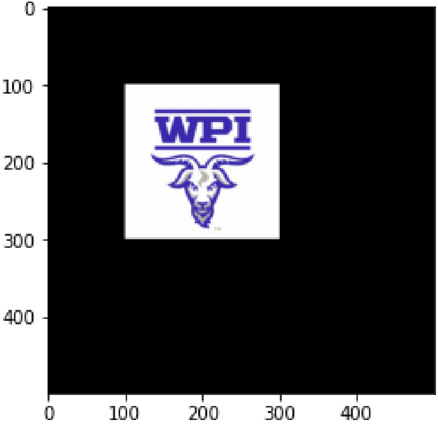

Figure: Before Transormation

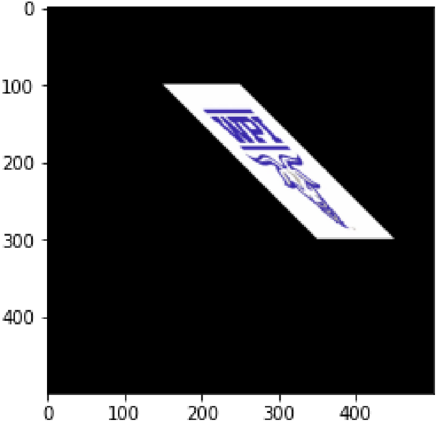

Figure: After Transormation

- Transformation maps each coordinate vector to another coordinate vector by

$$T : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{2}x_1 + x_2 \\ x_2 \end{bmatrix}$$

- (ex.) A coordinate after transformation,

$$T : \begin{bmatrix} 100 \\ 100 \end{bmatrix} \mapsto \begin{bmatrix} 150 \\ 100 \end{bmatrix}$$

- (Q.) Find a corner before the transformation,

$$T : ? \mapsto \begin{bmatrix} 350 \\ 300 \end{bmatrix}$$

- Set the original corner as $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- Solve

$$\begin{cases} \frac{1}{2}x_1 + \quad x_2 = 350 \\ \qquad\qquad x_2 = 300 \end{cases}$$

- The solution is ...

- A linear equation is

$$a_1x_1 + \cdots + a_nx_n \quad = b$$

- Coefficients a_j and b are given.
- $x = (x_1, \cdots x_n)$ is solution if LHS = RHS.

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ex. Identify the followings are linear equations:

$$2x - y = 1 - 3z.$$

$$2x_1 + \cos x_2 = 5.$$

- $m \times n$ linear system is

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n & = b_2 \\ \vdots & \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = b_m. \end{cases}$$

- Coefficients a_{ij} and b_i are given numbers
- $x = (x_1, \cdots x_n)$ is solution if
  LHS = RHS for each equation.

For $m \times n$ linear system,

- coefficient matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

- augmented matrix $[A \mid b]$, i.e.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & | & b_1 \\ a_{21} & \cdots & a_{2n} & | & b_2 \\ \vdots & & & & \\ a_{m1} & \cdots & a_{mn} & | & b_m \end{bmatrix}$$

Consider linear system

$$\begin{cases} x_1 + 2x_2 - x_3 = 4 \\ -x_1 - x_2 + 2x_3 = 1 \\ x_1 + x_3 = 6. \end{cases}$$

- write coefficient matrix
- write augmented matrix
- check the candidate solution $x = (3, 2, 3)$.

Consider linear system

$$\begin{cases} -x_1 - x_2 + 2x_3 & = 1 \\ x_1 \qquad\quad + x_3 & = 6. \end{cases}$$

- write coefficient matrix
- write augmented matrix
- check the candidate solution $x = (3, 2, 3)$.
- Can you say more?

Next,

- How many solutions for a given equation?
- How to solve a linear system systematically?