(*) Given a transform T, How do you find A?

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

$\exists\, A \in \mathbb{R}^{m \times n}$ s.t.

$$[e_1, \cdots e_n] \xrightarrow{T} \underbrace{[Te_1, Te_2 \cdots Te_n]}_{A}$$

(*) Given A, How to reduce it into RREF?

All following questions are equivalent:

Given $A \in \mathbb{R}^{m \times n}$, $A = [a_1 \cdots a_n]$

① Does each row of A have pivot?

② Does $Ax = y$ always have soln $\forall y \in \mathbb{R}^m$?

③ Does $\mathrm{span}\{a_1 \cdots a_n\}$ generats $\mathbb{R}^m$?

④ Is the $Tx = Ax$ <u>onto</u>?

⑤ Are there m columns of A linearly independent?

Given $A = [a_1 \cdots a_n] \in \mathbb{R}^{m \times n}$, all followings are equivalent questions.

① Is $Tx = Ax$ 1-1 mapping?

② Does $Ax = 0$ (homo system) have only trivial soln?

③ Does each column of A have pivot?

④ Is there no free variable for $Ax = 0$?

Midterm (sample)

Name:
ID:
All answers shall be justified properly to get full credits.

Throughout this exam, let $A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} & A & \end{bmatrix} = \boxed{\ A_2\ }$$

(1) Find a matrix E such that EA yields reduced row echelon form of A.

$$A \xrightarrow{E_1\ \boxed{\frac{1}{2}R_1 \to R_1}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = A_1$$

$$\xrightarrow{E_2\ \boxed{R_2 - R_1 \to R_2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = A_2$$

$E_1 A = A_1$

$\underbrace{\boxed{E_2 E_1}}_{E} A = A_2$

(2) Write its solution set of the matrix equation $Ax = v$ into a parametric vector form, if there is any.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_1\ \boxed{\frac{1}{2}R_1 \to R_1}} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_2\ \boxed{R_2 - R_1 \to R_2}} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = E_2$$

$$E = E_2 E_1 = \boxed{?\ \ }$$

(3) Is the transformation defined by $Tx = Bx$ a one-to-one or onto mapping? Explain its geometric meaning.

(4) We denote by $A = [a_1, a_2, a_3, a_4]$, where a_1, a_2, a_3, a_4 are column vectors of A.
   (a) Does v belong to the span generated by $\{a_1, a_2, a_3, a_4\}$? If yes, find a linear combination.

   (b) Are vectors $\{a_1, a_2, a_3, a_4\}$ independent?

(5) (a) Compute $v^T A^{100} v$, $v^T B^{100} v$, respectively, whenever they are well defined.

   (b) Compute BA, BAB, respectively, whenever they are well defined.