1 Abstract

- estimation of Fibonacci numbers via golden mean
- eigenvalue and eigenvector and matrix diagonalization

2 Problem

Fibonacci numbers are defined recursively by

$$F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}, \forall n \geq 2.$$

What is F_{100}? Our answer to this problem is surprisingly simple but a big number:

$$F_{100} = [\lambda_1^{100}/\sqrt{5}] \approx 3.54 \cdot 10^{20}$$

where $[a]$ means integer part of a and λ_1 is the golden mean:

$$\lambda_1 = \frac{1+\sqrt{5}}{2}.$$

3 Analysis

We write

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}$$

Then, eigenvectors of A are given by

$$\lambda_1 = \frac{1+\sqrt{5}}{2},\ v_1 = [1, \lambda_1]'$$

and

$$\lambda_2 = \frac{1-\sqrt{5}}{2},\ v_2 = [1, \lambda_2]'.$$

If we denote

$$V = [v_1, v_2],\ \Lambda = diag(\lambda_1, \lambda_2),$$

then we have diagonalization of A as

$$AV = V\Lambda,\ \text{or } A = V\Lambda V^{-1}.$$

Therefore,

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = V\Lambda^n V^{-1} \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}.$$

So we can compute

$$F_{100} = (\lambda_1^{100} - \lambda_2^{100})/(\lambda_1 - \lambda_2).$$